Wrocław University of Science and Technology

Doctoral Dissertation

Boundary problems for nonlocal operators

Damian Fafuła

Supervisor: dr hab. inż. Paweł Sztonyk

Field of Science: Natural science
Discipline of Science: Mathematics
Keywords: nonlocal problems, fractional Laplacian, Markov process, Hardy inequality

Wrocław, March 2024

POLITECHNIKA WROCŁAWSKA

ROZPRAWA DOKTORSKA

Warunki brzegowe dla operatorów nielokalnych

DAMIAN FAFUŁA

PROMOTOR: dr hab. inż. Paweł Sztonyk

DZIEDZINA: Dziedzina nauk ścisłych i przyrodniczych

DYSCYPLINA: Matematyka

SŁOWA KLUCZOWE: problemy nielokalne, ułamkowy laplasjan, procesy Markowa, nierówność Hardy'ego

Wrocław, marzec 2024

Podziękowania

Pragnę serdecznie podziękować osobom, które przyczyniły się do powstania niniejszej rozprawy doktorskiej.

Promotowori mojej rozprawy, dr hab. inż. Pawłowi Sztonykowi, dziękuję za okazaną pomoc w prowadzeniu badań, cenne uwagi i sugestie, które wpłynęły w znaczący sposób na jakość otrzymanych wyników, jak również za zaangażowanie i życzliwość okazane mi podczas kilkuletniej opieki naukowej w trakcie studiów doktoranckich, oraz przede wszystkim za wnikliwe uwagi, które przyczyniły się do ulepszenia ostatecznej wersji rozprawy.

Dziękuję serdecznie prof. dr hab. inż. Krzysztofowi Bogdanowi za zainteresowanie mnie teorią potencjału i tematyką operatorów nielokalnych, którą zajmowałem się w trakcie studiów magisterskich, i której kontynuację stanowi niniejsza praca doktorska, za długą współpracę, podczas której nauczyłem się wiele nowego, za spotkania i dyskusje, w trakcie których dzielił się ze mną swoją olbrzymią wiedzą.

Serdeczne podziękowania składam wszystkim tym, którzy przyczynili się do poszerzenia moich matematycznych horyzontów, w szczególności dla: dr Michała Kijaczko, dr Artura Rutkowskiego oraz dr Wojciecha Słomiana.

Pragnę podziękować wszystkim pozostałym osobom obecnym w moim życiu, przede wszystkim: Mamie i Babci Irenie za nieustające wsparcie i pomoc, za zaszczepienie we mnie zamiłowania do nauki — gdyby nie one, to pewnie nie byłbym w tym miejsu, w którym jestem teraz, moim teściom, Iwonie i Grzegorzowi, za nieustanną pomoc i za to, że zawsze mogę na nich liczyć, oraz koleżankom i kolegom. W szczególności w tym miejscu chciałbym podziękować moim przyjaciołom: Sebastianowi Cisińskiemu, Krzysztofowi Gniewaszewskiemu, Karolinie Karpińskiej, Natalii i Krzysztofowi Rudnik oraz Marii Wita.

Osobne, ale równie ważne i szczególne podziękowania kieruję do mojej żony Arlety, za ogromne wsparcie i cierpliwość do mojej osoby oraz za lekturę tej rozprawy na różnych etapach jej przygotowania.

Niektóre wyniki przedstawione w tej rozprawie zostały otrzymane w trakcie badań realizowanych w ramach projektu Narodowego Centrum Nauki: 2017/27/B/ST1/01339.

Contents

Streszczenie

W całej rozprawie pracujemy z jednowymiarową przestrzenią euklidesową $\mathbb{R}$ wyposażoną w metrykę indukowaną przez normę $|\cdot|$. Miarę Lebesgue'a na $\mathbb{R}$ oznaczamy przez $\mathrm{d}x$. Przyjmujemy również, że $D = (0, \infty)$ jest dodatnią półprostą.

Zgodnie z tytułem rozprawy, naszym punktem zainteresowania są tzw. *operatory nielokalne*. Operator $\mathcal{L}$, przekształcający funkcje określone na pewnej przestrzeni topologicznej na inne funkcje, nazywany jest *nielokalnym*, jeśli w celu obliczenia $\mathcal{L}u(x)$, dla pewnej funkcji u, w pewnym punkcie x, potrzebujemy znać również wartości funkcji u w punktach, które są odległe od punktu x. Właśnie ten fakt jest powodem, dla którego używamy nazwy *nielokalny*.

Przeciwieństwem operatorów nielokalnych są powszechnie znane operatory *lokalne*. Jak sugeruje nazwa, wartości operatora lokalnego $\mathcal{K}$ na funkcji u, w punkcie x, zależą jedynie od wartości funkcji u w dowolnie małym otoczeniu punktu x. Najpewniej najbardziej znanymi przykładami operatorów lokalnych są operatory różniczkowe: $u'(x)$, $\nabla u(x)$ oraz operator Laplace'a $\Delta u(x)$. Od teraz skupmy się jednak na operatorach nielokalnych.

Najbardziej znanym przykładem operatora nielokalnego jest *ułamkowy laplasjan* $(-\Delta)^{\alpha/2}$ dla $\alpha \in (0, 2)$ zdefiniowany następująco:

$$(-\Delta)^{\alpha/2}u(x) := \text{p.v.} \int_{\mathbb{R}} (u(x) - u(y))\nu(x, y)\,\mathrm{d}y, \tag{1}$$

gdzie ν jest *miarą Lévy'ego*. Po więcej informacji na temat ν, w szczególności po dokładną postać stałej normującej w jej definicji, odsyłamy do podrozdziału 2.2. Powyższe wyrażenie jest skończone dla wszystkich $x \in \mathbb{R}$ (a więc i poprawnie określone) na przykład dla wszystkich funkcji $u \in C_c^2(\mathbb{R})$, tj. dla funkcji na $\mathbb{R}$ dwukrotnie różniczkowalnych w sposób ciągły i o zwartym nośniku. Warto w tym miejscu podkreślić, że nie precyzujemy tutaj dziedziny ułamkowego laplasjanu, ponieważ dla naszych potrzeb wystarczy, że będziemy rozumieć go punktowo, czyli tak jak w równaniu (1). Po więcej informacji na temat ułamkowego laplasjanu (w tym po szerszy kontekst i po dalsze szczegóły dotyczące np. stałej normalizującej) odsyłamy do prac Kwaśnickiego [47], Nezza i innych [28] oraz Silvestre'a [62]. Podkreślmy jednak, że w tej pracy dokładna wartość stałej normującej w definicji ułamkowego laplasjanu odgrywa znikomą rolę.

Operatory nielokalne w różnych formach obecne są w wielu dziedzinach nauki. Po pierwsze, operatory te są ściśle powiązane z teorią procesów stochastycznych, tzn. dla odpowiednio regularnych funkcji u, $(-\Delta)^{\alpha/2}$ jest generatorem czysto-skokowego procesu α-stabilnego (zob. np. Sato [56, Theorem 31.5]), którego intensywność skoków jest dana przez miarę Lévy'ego ν. Ogólniej, dla dowolnego czysto-skokowego procesu Lévy'ego, jego generator dany jest przez pewien operator nielokalny.

Skokowe procesy Lévy'ego odgrywają ważną rolę w opisywaniu przeróżnych zjawisk świata rzeczywistego. Jednym z najpopularniejszych zastosowań tych procesów są różnego rodzaju modele finansowe, zob. prace Barndorffa-Nielsena, Mikoscha i Resnicka [3], Conta i Tankova [25] oraz Schoutensa [58]. W szczególności, z tych monografii wynika, że z punktu widzenia danych historycznych, skokowe procesy Lévy'ego są bardziej odpowiednie do stosowania w modelach finansowych niż modele oparte, na przykład, na ruchu Browna. Innym z zastosowań tego rodzaju procesów stochastycznych jest genetyka, zob. np. prace Blomberga, Rathnayake'a i Moreau'a [5], Gjessinga, Aalena i Hjorta [37] oraz Landisa, Schraibera i Lianga [48]. Ostatnim z omawianych przez nas zastosowań skokowych procesów Lévy'ego są różne działy fizyki, takie jak mechanika płynów, fizyka ciała stałego czy chemia polimerów (zob. pracę Barndorffa-Nielsena i innych [3]).

Jedną z głównych motywacji do podjęcia naszych badań zawartych w tej rozprawie jest następujący *problem brzegowy Neumanna dla ułamkowego laplasjanu* wprowadzony po raz pierwszy przez Dipierro, Ros-Otona i Valdinoci'ego w pracy [29]:

$$\begin{cases} (-\Delta)^{\alpha/2}u = f, & \text{na } D, \\ \mathcal{N}_{\alpha/2}u = 0, & \text{na } \mathbb{R}\setminus\overline{D}. \end{cases} \tag{2}$$

Powyżej $\mathcal{N}_{\alpha/2}$ jest tzw. *nielokalną pochodną normalną* zdefiniowaną przez wyrażenie

$$\mathcal{N}_{\alpha/2}u(x) := \int_D (u(x)-u(y))\nu(x,y)\,\mathrm{d}y, \qquad x\in\mathbb{R}\setminus\overline{D}. \tag{3}$$

Zwróćmy tutaj uwagę, że zagadnienie (2) składa się z dwóch równań, które zwykle nazywają się *równaniami nielokalnymi*, z uwagi na ich charakter. Zainteresowanie nielokalnymi problemami brzegowymi Neumanna postaci (2) jest całkiem nowe. Koncept ten ma zaledwie kilka lat.

Najbardziej popularnymi i najlepiej zbadanymi nielokalnymi problemami brzegowymi są problemy Dirichleta dotyczące operatora $(-\Delta)^{\alpha/2}$. Są one postaci

$$\begin{cases} (-\Delta)^{\alpha/2}u = f, & \text{na } \Omega, \\ u = g, & \text{na } \mathbb{R}\setminus\Omega, \end{cases} \tag{4}$$

gdzie $\Omega \subset \mathbb{R}$ jest pewnym, odpowiednio regularnym, zbiorem. Były one szeroko badane np. w pracach Ros-Otona [54], Bucura i Valdinoci'ego [18], Felsingera, Kassmanna i Voigta [33], Rutkowskiego [55] oraz Servadei i Valdinociego [59, 60].

Zauważmy, że zdefiniowane wcześniej nielokalne problemy brzegowe dla operatorów nielokalnych często nazywane są *problemami brzegowymi* (Dirichleta lub Neumanna) pomimo, że nie mamy w ich definicji klasycznego warunku brzegowego. Wynika to bezpośrednio z nielokalnego charakteru tych równań. W przeciwieństwie do teorii równań różniczkowych cząstkowych, w naszym podejściu klasyczny warunek brzegowy byłby niewystarczający. Aby to w pewnym stopniu uzasadnić zauważmy, że dla funkcji stałej u dostajemy $(-\Delta)^{\alpha/2}u = 0$. W związku z tym, jeśli chcemy, aby zagadnienie (4) miało jedno rozwiązanie, musimy podać warunki brzegowe. Ponadto, z nielokalnego charakteru ułamkowego laplasjanu wynika, że te warunki brzegowe muszą być zdefiniowane nie tylko na brzegu zbioru Ω, ale również na jego dopełnieniu. Fakt ten jest powodem, dla którego (4) jest naturalnym zagadnieniem brzegowym Dirichleta dla ułamkowego laplasjanu. Analogiczna sytuacja ma miejsce w przypadku zagadnienia brzegowego typu Neumanna.

W dalszej części pracy rozważamy tylko problemy brzegowe Neumanna (2) dla ułamkowego laplasjanu. Naszym głównym celem jest znalezienie rozwiązania tego problemu, jak i odpowiadającemu mu równania ciepła z jednorodnymi warunkami Neumanna. W tym miejscu po więcej informacji odsyłamy do pracy [29], w której autorzy przedstawiają następującą interpretację probabilistyczną równania ciepła Neumanna

$$\begin{cases} u_t + (-\Delta)^{\alpha/2}u = 0, & \text{na } D, \quad t > 0, \\ \mathcal{N}_{\alpha/2}u = 0, & \text{na } \mathbb{R} \setminus \overline{D}, \quad t > 0, \\ u(x,0) = u_0(x), & \text{na } D, \quad t = 0, \end{cases} \tag{5}$$

które odpowiada zagadnieniu (2):

(1) *Rozwiązanie* $u(x,t)$ *równania ciepła Neumanna* (5) *jest rozkładem prawdopodobieństwa pozycji cząstki poruszającej się losowo wewnątrz zbioru* D.

(2) *Jeśli cząstka opuszcza zbiór* D*, to natychmiast do niego powraca.*

(3) *Sposób, w który cząstka ta wraca do zbioru* D *jest następujący: jeśli cząstka uciekła do punktu* $x \in \mathbb{R} \setminus \overline{D}$*, to może powrócić do dowolnego punktu* $y \in D$*. Prawdopodobieństwo przeskoku z* x *do* y *jest proporcjonalne do* $\nu(x,y)$.

Vondraček [66] słusznie zauważył, że powyższa probabilistyczna interpretacja rozwiązania jest nie do końca poprawna. Przedstawimy teraz pokrótce wyniki Vondračka, ponieważ są one ściśle powiązane z wynikami zawartymi w tej rozprawie. Na początek przedstawimy kilka dodatkowych pojęć.

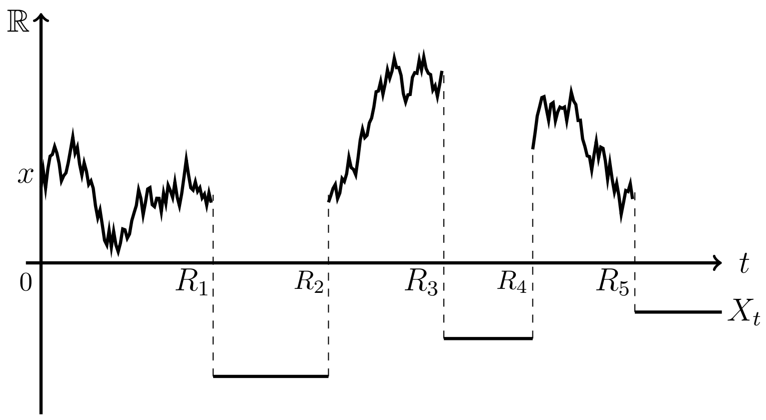
Struktura wariacyjna zagadnienia brzegowego Neumanna (2) sprawia, że w rozważaniach nad nimi istotną rolę pełni forma dwuliniowa $\mathcal{E}_D$ dana wzorem

$$\mathcal{E}_D(u,v) := \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c}(u(x)-u(y))(v(x)-v(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y,$$

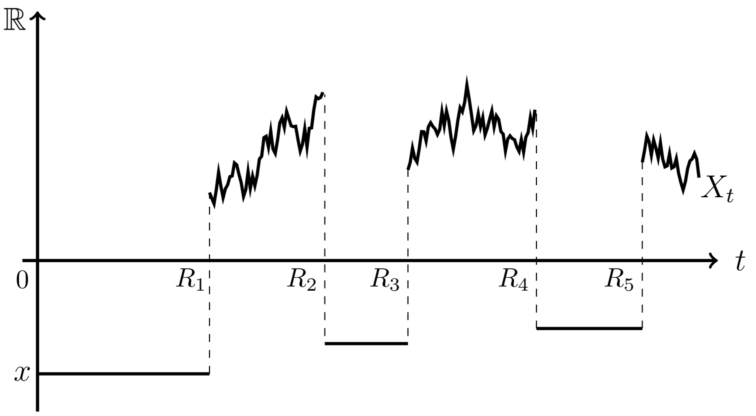
gdzie $u, v : \mathbb{R} \to \mathbb{R}$ jest pewną funkcją, zob. np. [29]. Forma ta była ostatnio punktem zainteresowania w przeróżnych badaniach, zob. np. prace [12, 33, 51, 59, 60, 65], w których analizowano różne własności tej formy. Odsyłamy również w tym miejscu do ostatniej pracy Grube i Hensiek [38] po więcej uwag na temat zagadnienia Neumanna (2).

Jak już było wspomniane wcześniej, naszym punktem zainteresowania jest wymiar $d = 1$, więc przedstawimy teraz wyniki Vondračka [66] tylko dla tego wymiaru. Autor rozważa przestrzeń $L^2(\mathbb{R}, m(\mathrm{d}x))$ z miarą m zdefiniowaną jako $m(\mathrm{d}x) := \mathbb{1}_D(x)\mathrm{d}x + \mathbb{1}_{D^c}(x)\nu(x,D)$ (po definicję wyrażenia $\nu(x,D)$ odsyłamy do (2.9)) i bada własności formy $(\mathcal{E}_D, \widehat{\mathcal{F}})$, gdzie $\widehat{\mathcal{F}} := \{u \in L^2(\mathbb{R}, m(\mathrm{d}x)) : \mathcal{E}_D(u,u) < \infty\}$. Dowodzi, że forma ta jest quasi-regularną formą Dirichleta na $L^2(\mathbb{R}, m(\mathrm{d}x))$, a więc istnieje proces Markova $\widehat{X}$ na $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ ściśle związany z formą $(\mathcal{E}_D, \widehat{\mathcal{F}})$. Na podstawie [66], zachowanie procesu $\widehat{X}$ może być opisane następująco: *startując z D, proces porusza się jak izotropowy proces stabilny do momentu pierwszego wyjścia ze zbioru D. W momencie wyjścia skacze ze zbioru D do punktu y na podstawie jądra $\nu(x,y)$. Następnie przebywa w punkcie y przez losowy czas z rozkładu wykładniczego o średniej jeden, a następnie skacze z powrotem do D na podstawie rozkładu prawdopodobieństwa danego przez $\nu(y,x)/\nu(y,D)$. Następnie sytuacja rozpoczyna się od nowa.* Vondraček dowodzi, że tak określony proces $\widehat{X}$ zachowuje się prawidłowo z punktu widzenia rozwiązania równania ciepła Neumanna (5) przedstawionego w pracy [29]. Zauważa również, że przez usunięcie części procesu $\widehat{X}$, która żyje poza zbiorem D, otrzymujemy proces $\widehat{Z}$ o przestrzeni stanów D, który spełnia opis procesu przedstawionego przez Dipierro i innych w pracy [29]. Ponadto, forma dwuliniowa takiego procesu różni się od formy $\mathcal{E}_D$ — po więcej informacji w tym temacie odsyłamy do pracy [66].

W tej rozprawie przedstawiamy konstrukcję procesu stochastycznego, który jest bardzo podobny do procesu z pracy Vondračka, jednak pomimo to, w naszych badaniach stosujemy inne metody niż Vondraček. Koncentrujemy się na bezpośredniej konstrukcji procesu i badaniu jego własności. Bardziej precyzyjnie, naszym celem jest rozwiązanie następujących problemów: przedstawienie procesu stochastycznego X, podobnego do procesu $\widehat{X}$, który, w pewnym sensie, rozwiązuje zagadnienie brzegowe Naumanna; zbadanie czasu życia i granicy procesu X; udowodnienie, że forma tego procesu odpowiada formie $\mathcal{E}_D$ (przy pewnych założeniach); rozwiązanie zagadnienia (2) przy wykorzystaniu procesu X. W tym



Rys. 1: Trajektoria procesu X przy starcie z $x > 0$.



Rys. 2: Trajektoria procesu X przy starcie z $x < 0$.

miejscu odsyłamy do prac Bogdana i Kunze [15] oraz Kima, Songa i Vondračka [45, 46] po więcej analogicznych rozważań.

Omówimy teraz pokrótce zawartość niniejszej rozprawy. Rozdział 2 poświęcony jest wprowadzeniu wszystkich notacji i konwencji wykorzystywanych w dalszej części pracy. Przedstawiamy w nim podstawową wiedzę na temat α-stabilnych procesów Lévy'ego i teorii potencjału dla zabitych procesów Lévy'ego. Ponadto badamy własności półgrup przedstawionych wcześniej procesów. Na końcu rozdziału wprowadzamy podstawowe definicje i pojęcia związane z ogólną teorią procesów Markowa, w tym mocną własność Markowa.

Rozdział 3 zawiera konstrukcję procesu X, który zachowuje się niemal identycznie do procesu przedstawionego przez Vondračka z tylko jedną zmianą. Proces $\widehat{X}$ przebywa w $\overline{D}^c$ przez czas wykładniczy o średniej jeden, a w przypadku procesu X, czas, jaki spędzamy w dopełnieniu zbioru D, zależy od położenia po wyskoku z D. Bardziej precyzyjnie, postulujemy, że: *startując z D proces X porusza się jak izotropowy α-stabilny proces Lévy'ego do momentu τ_D pierwszego wyjścia z D. W momencie wyjścia proces skacze z pozycji $X_{\tau_D -}$ do* *punktu $y \in \overline{D}^c$ zgodnie z rozkładem prawdopodobieństwa proporcjonalnym do $\nu(X_{\tau_D-}, y)$. Następnie, proces ten przebywa w punkcie y przez czas wykładniczy o średniej $1/\nu(y, D)$ i po upływie tego czasu powraca do punktu $z \in D$ zgodnie z rozkładem $\nu(y, z)/\nu(y, D)$, i sytuacja rozpoczyna się od nowa.* Przykładowa trajektoria procesu X zaprezentowana jest na Rys. 1 i Rys. 2.

Aby przeprowadzić konstrukcję procesu X wykorzystujemy metodę *konkatenacji* procesów Markowa przedstawioną przez Wernera [67], w wyniku której otrzymany proces jest mocnym procesem Markowa. Nie będziemy tutaj przedstawiać szczegółów konstrukcji, aby w tym momencie nie zakłócać uwagi czytelnika. Po szczegóły odsyłamy do omawianego właśnie rozdziału. Następnie, po przeprowadzeniu konstrukcji procesu X, wyznaczamy jego półgrupę operatorów $K = (K_t)_{t\geq 0}$ i badamy jej własności. W szczególności identyfikujemy funkcje ekscesywne dla K, które będą naszym kluczowym narzędziem w dalszych rozważaniach, zob. Twierdzenie 3.20 i Wniosek 3.21 poniżej.

Obliczenia związane z czasem życia oraz granicznym położeniem procesu X zawarte są w Rozdziale 4. Na początku badamy własności zmiennej losowej opisującej pierwszą pozycję powrotu do D. Następnie rozważamy ciąg kolejnych pozycji po powrocie do D i dowodzimy, że granica tego ciągu istnieje z prawdopdobieństwem jeden oraz obliczamy ją, zob. Twierdzenie 4.6 poniżej. Wykorzystując otrzymane wyniki, w następnej kolejności obliczamy sumę wszystkich przyrostów czasów pomiędzy kolejnymi powrotami do D, zob. Twierdzenie 4.10. Łącząc otrzymane wyniki otrzymujemy główne twierdzenie tego rozdziału, które brzmi następująco: dla $x \neq 0$ następujące własności zachodzą $\mathbb{P}_x$-p.w.

(a) Jeśli $\alpha \in (0, 1)$, to czas życia ξ procesu X jest nieskończony i $\lim\limits_{t\to\infty} |X_t| = \infty$.

(b) Jeśli $\alpha \in (1, 2)$, to czas życia ξ procesu X jest skończony i $\lim\limits_{t\nearrow\xi} X_t = 0$.

(c) Jeśli $\alpha = 1$, to czas życia ξ procesu X jest nieskończony i $\lim\limits_{t\to\infty} X_t$ nie istnieje.

Po dokładne sformułowanie tego twierdzenia odsyłamy do Twierdzenia 4.11 poniżej.

Dla porównania, odsyłamy czytelnika do pracy Bogdana, Burdzego i Chena [9, Proposition 4.2], w której autorzy otrzymują analogiczny wynik w przypadku stabilnego procesu cenzurowanego na półprostej, który, w skrócie mówiąc, jest procesem α-stabilnym „zmuszonym” do pozostania wewnątrz D.

Na końcu Rozdziału 4 dowodzimy, że dla $\alpha \in (1, 2)$ półgrupa K procesu X ma własność Fellera, zob. Twierdzenie 4.14 poniżej.

Rozdział 5 jest najbardziej technicznym fragmentem rozprawy. Zdecydowana większość rozdziału poświęcona jest twierdzeniom pomocniczym, które będą nam potrzebne w dalszej części pracy. Zawiera on dowody różnych oszacowań obiektów (jak i ich granice) występujących głównie we wzorze perturbacyjnym dla półgrupy K (po dokładną postać wzoru perturbacyjnego dla półgrupy K odsyłamy do Wniosku 3.4 poniżej). Głównym twierdzeniem tego rozdziału jest dokładna postać generatora punktowego (a dokładniej, minus generatora) dla półgrupy K na funkcjach ekscesywnych, zob. Twierdzenie 5.15.

W rozdziale 6 dla $\alpha \neq 1$ dowodzimy nierówności Hardy'ego dla półgrupy K oraz związanej z nią formy $\mathcal{E}$, zob. Twierdzenie 6.1 poniżej. Przypadek $\alpha = 1$, w momencie pisania tej rozprawy, w dalszym ciągu jest problemem otwartym. Jesteśmy przekonani, że w tym przypadku nierówność Hardy'ego nie zachodzi, ale nie potrafimy wskazać odpowiedniego kontrprzykładu, który potwierdziłby nasze przypuszczenia. Wierzymy, że ten przypadek jest, w pewnym sensie, analogiczny do wyniku Dydy [30]. W dalszej części rozdziału dowodzimy, że forma dwuliniowa odpowiadająca półgrupie K (a więc odpowiadająca procesowi X) pokrywa się z formą $\mathcal{E}_D$ przedstawioną powyżej. Co więcej, przedstawioamy również różne charakteryzacje dziedziny tej formy. Na sam koniec rozdziału otrzymujemy wniosek, że forma odpowiadająca półgrupie K z jej naturalną dziedziną na $L^2(\mathbb{R})$ jest regularna, zob. Twierdzenie 6.9 i Wniosek 6.10 poniżej.

Ostatni Rozdział 7 poświęcony jest znalezieniu rozwiązania zagadnienia brzegowego typu Neumanna w przypadku $\alpha \neq 1$, które w naszym przypadku przyjmuje postać

$$\begin{cases} (-\Delta)^{\alpha/2} u &= f, \quad \text{na } D, \\ \mathcal{N}_{\alpha/2} u &= f, \quad \text{na } \mathbb{R} \setminus \overline{D}. \end{cases}$$

gdzie $f : \mathbb{R}^* \to \mathbb{R}$ jest dowolną ustaloną funkcją ciągłą o zwartym nośniku. Okazuje się, że rozwiązanie tego problemu dane jest przez funkcję $u = Gf$, gdzie G jest 0-*potencjałem* półgrupy K, tzn. $Gf(x) := \int_0^\infty K_t f(x)\,\mathrm{d}t$.

Wyniki zawarte w tej rozprawie doktorskiej zostaną wkrótce opublikowane w formie artykułu w czasopiśmie matematycznym.

Chapter 1

Introduction

In this chapter we present the main results of the thesis. In the whole dissertation we work with the one-dimensional Euclidean space $\mathbb{R}$ equipped with the Euclidean metric induced by the standard norm $|\cdot|$. The Lebesgue measure on $\mathbb{R}$ will be denoted by $\mathrm{d}x$. Throughout the dissertation we assume that $D = (0, \infty)$ is a positive half-line.

As the title of the dissertation indicates, we are mainly interested in so called nonlocal operators. An operator $\mathcal{L}$ which maps functions defined on some topological space to another functions is called *nonlocal* if in order to evaluate $\mathcal{L}u(x)$, for some function u, at some point x, we need to know the values of u at points, that are far away from the point x. Hence, this is why we use the name *nonlocal*.

An opposite to nonlocal operators are commonly known *local* operators. As the name suggests, the value of the local operator $\mathcal{K}$ on the function u at point x depends only on the values of u in an arbitrary small neighborhood of the point x. The most common examples of local operators are differential operators $u'(x)$, $\nabla u(x)$ and the Laplace operator $\Delta u(x)$. Let us focus now on the nonlocal operators.

One of the most popular example of nonlocal operator is the *fractional Laplacian* $(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2)$ defined as follows:

$$(-\Delta)^{\alpha/2}u(x) := \text{p.v.} \int_{\mathbb{R}} (u(x) - u(y))\nu(x, y)\,\mathrm{d}y, \tag{1.1}$$

where ν is a *Lévy measure*. More information about ν, especially the direct form of normalization constant, can be found in Section 2.2. The above expression is finite (and hence well-defined) for all $x \in \mathbb{R}$, e.g. for all $u \in C_c^2(\mathbb{R})$, i.e. twice continuously differentiable functions on $\mathbb{R}$ with compact support. It is worth to underline that we do not precise the domain of the fractional Laplacian, because throughout the dissertation it is understood pointwise, as in (1.1). For more information about the fractional Laplacian, i.e. for a wider context and for further details concerning a normalization constant (which comes from the definition of ν in

Section 2.2), we refer to Kwaśnicki [47], Nezza et al. [28] and Silvestre [62]. We want to emphasise that in this paper the explicit value of normalization constant plays negligible role.

Nonlocal operators in various forms are present in many branches of science. At first nonlocal operators are strictly related with the theory of stochastic processes, i.e. for sufficiently regular function u, $(-\Delta)^{\alpha/2}$ coincides with the infinitesimal generator of pure-jump α-stable Lévy process (see e.g. Sato [56, Theorem 31.5]), which intensity of jumps is given by the Lévy measure ν. More general, for any pure-jump Lévy process, the infinitesimal generator is given by a nonlocal operator.

Jump Lévy processes play an important role in describing real-world phenomena. One of the most popular applications of Lévy processes are models of financial markets, see Barndorff-Nielsen et al. [3], Cont and Tankov [25], and Schoutens [58]. In particular, from these monographs and the references therein it follows that, in terms of historical data, the jump Lévy processes are more appropriate to be used in financial models than the models based e.g. on Brownian motion. Another application of these kinds of processes are genetics, see. e.g. Blomberg et al. [5], Gjessing et al. [37] and Landis et al. [48], and numerous branches of physics (such as fluid mechanics, solid state physics, polymer chemistry), see. e.g. Barndorff-Nielsen et al. [3].

One of the main motivations for the studies included in this thesis is the following *Neumann problem for the fractional Laplacian* first introduced by Dipierro et al. [29]:

$$\begin{cases} (-\Delta)^{\alpha/2}u &= f, \quad \text{in } D, \\ \mathcal{N}_{\alpha/2}u &= 0, \quad \text{in } \mathbb{R}\setminus\overline{D}. \end{cases} \tag{1.2}$$

Here $\mathcal{N}_{\alpha/2}$ denotes so-called *nonlocal normal derivative*, defined by

$$\mathcal{N}_{\alpha/2}u(x) := \int_D (u(x)-u(y))\nu(x,y)\,\mathrm{d}y, \qquad x\in\mathbb{R}\setminus\overline{D}. \tag{1.3}$$

We emphasize that the problem (1.2) is created from two equations usually called *nonlocal equations*. The interest in nonlocal Neumann problem of the form (1.2) is quite new. This concept is only a few years old.

More popular and more studied problems are Dirichlet problems for nonlocal operator $(-\Delta)^{\alpha/2}$, which are of the form

$$\begin{cases} (-\Delta)^{\alpha/2}u = f, \quad \text{in } \Omega, \\ \qquad\quad u = g, \quad \text{in } \mathbb{R}\setminus\Omega, \end{cases} \tag{1.4}$$

for some sufficiently regular domain $\Omega\subset\mathbb{R}$. They have been studied widely in the literature for many years, see e.g. the surveys by Ros-Oton [54], Bucur, Valdinoci [18] and papers by Felsinger et al. [33], Rutkowski [55] and Servadei, Valdinoci [59, 60].

Let us note that nonlocal problems specified above often are called *Dirichlet* or *Neumann boundary problems* for nonlocal operators, even if we do not have classical boundary conditions. This is due to nonlocal nature of these equations. Unlike the theory of partial differential equations, in our approach, a classical boundary condition would be insufficient. It is obvious that for a constant function u we get $(-\Delta)^{\alpha/2}u = 0$. Hence, if we expect uniqueness of a solution to nonlocal problem (1.4), we have to state boundary values, but from nonlocal nature of the operator $(-\Delta)^{\alpha/2}$, the boundary values have to be defined not only on the boundary of the set Ω, but also on the whole complement of Ω. Thus, this is the reason why (1.4) is the natural Dirichlet boundary problem for the fractional Laplacian. The situation is analogous in the case of Neumann boundary problems.

Throughout this dissertation we consider only the Neumann boundary problem (1.2) for the fractional Laplacian. Our aim is to find a solution of this problem as well as the corresponding heat equation with homogeneous Neumann conditions. For a wider discussion, see [29]. The authors introduce the following probabilistic interpretation of the Neumann heat equation

$$\begin{cases} u_t + (-\Delta)^{\alpha/2}u = 0, & \text{in } D, \quad t > 0, \\ \mathcal{N}_{\alpha/2}u = 0, & \text{in } \mathbb{R} \setminus \overline{D}, \quad t > 0, \\ u(x,0) = u_0(x), & \text{in } D, \quad t = 0, \end{cases} \tag{1.5}$$

corresponding to the Neumann problem (1.2):

(1) *The solution* $u(x,t)$ *of the Neumann heat equation* (1.5) *is the probability distribution of the position of a particle moving randomly inside* D.

(2) *When the particle exits* D*, it immediately comes back into* D.

(3) *The way in which it comes back inside* D *is the following: If the particle has gone to* $x \in \mathbb{R} \setminus \overline{D}$*, it may come back to any point* $y \in D$*, the probability of jumping from* x *to* y *being proportional to* $\nu(x,y)$.

Vondraček [66] rightly noticed that this probabilistic interpretation of the solution is somewhat ambiguous. In what follows, we present the results of Vondraček, because they are strictly connected with research included in this dissertation. At first let us introduce some basic notions.

The variational structure of the Neumann problem (1.2) involves the symmetric bilinear form

$$\mathcal{E}_D(u,v) := \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} (u(x)-u(y))(v(x)-v(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y,$$

where $u, v : \mathbb{R} \to \mathbb{R}$, see e.g. [29]. It was recently in point of interest and different properties of this form were studied, cf. e.g. [12, 33, 51, 59, 60, 65]. See also Grube and Hensiek [38] for more discussion about the Neumann problem (1.2).

As our point of interest is dimension one, we will introduce the results by Vondraček [66] only for $d = 1$. The author considers the space $L^2(\mathbb{R}, m(\mathrm{d}x))$, where the measure m is defined as $m(\mathrm{d}x) := \mathbb{1}_D(x)\mathrm{d}x + \mathbb{1}_{D^c}(x)\nu(x, D)$ (for a definition of $\nu(x, D)$ see (2.9)) and study properties of the form $(\mathcal{E}_D, \widehat{\mathcal{F}})$, where $\widehat{\mathcal{F}} := \{u \in L^2(\mathbb{R}, m(\mathrm{d}x)) : \mathcal{E}_D(u, u) < \infty\}$. He proves that such form is quasi-regular Dirichlet form on $L^2(\mathbb{R}, m(\mathrm{d}x))$ and hence there is a Markov process $\widehat{X}$ on $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ properly associated with $(\mathcal{E}_D, \widehat{\mathcal{F}})$. The behaviour of the process $\widehat{X}$ may be described, followed by [66], as follows: *starting in D, the process moves as the isotropic stable process until the first exit time from D. At the exit time, it jumps out of D according to the kernel $\nu(x, y)$. It sits at the exit point y for an exponential time with mean one, then jumps back to D according to probability distribution $\nu(y, x)/\nu(y, D)$ and starts afresh.* The author proves that in fact the process $\widehat{X}$ has proper behaviour from the viewpoint of the solution of the Neumann heat equation (1.5). He also states that by deleting the part of the process $\widehat{X}$ which lives outside D, we obtain the process $\widehat{Z}$ with state space D which satisfies the description of the process from [29] that after it jumps from D, it immediately returns to D. It turns out that its bilinear form differs from the form $\mathcal{E}_D$, see [66] for more details.

In this thesis we make a construction of a stochastic process which is quite similar to the one proposed by Vondraček. Despite that, in our research we use different methods than Vondraček. We focus on a direct construction of the process and study its properties. To be more precise, our aim is to find solutions to the following problems: propose the stochastic process X, similar to the process $\widehat{X}$ of Vondraček, which solves, in some sense, the Neumann boundary problem; investigate the lifetime and limit of this process X; show that in fact the bilinear form of this process corresponds with the form $\mathcal{E}_D$; and at the end find a direct solution of the Neumann boundary problem (1.2). For an analogous research we refer the reader to Bogdan and Kunze [15] and Kim et al. [45, 46].

The outline of the dissertation is as follows. Chapter 2 is devoted to providing the background for the notation proposed above and used in the further part of the thesis. We introduce a basic theory of α-stable Lévy process and potential theory of killed Lévy process. Furthermore, we investigate properties of the semigroup of both processes. At the end of this chapter we discuss a general notion of the Markov processes including the strong Markov property.

Chapter 3 deals with a direct construction of the stochastic process X. The construction is ad hoc. We postulate that the behaviour of this process mainly is the same as the process given by Vondraček with only one change. Unlike the process by Vondraček, our process does not

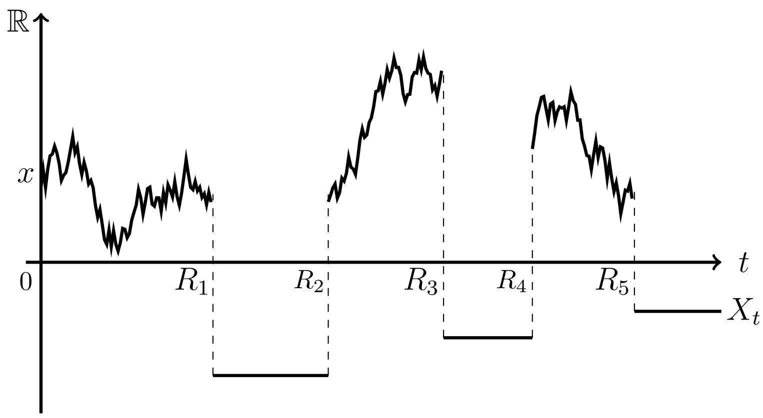


Fig. 1.1: Trajectory of the process X starting from $x > 0$.

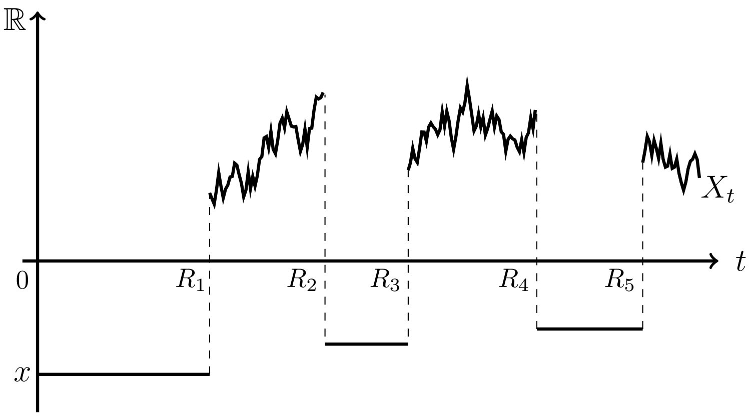


Fig. 1.2: Trajectory of the process X starting from $x < 0$.

stay in $\overline{D}^c$ for exponential time with mean one. In our case, the time which we spend in the complement of D will depend on the position after jump beyond D. More precisely, we state that: *starting from D, the process X moves as the isotropic α-stable Lévy process until the first exit time τ_D from D. At this time, the process X jumps from X_{τ_D-} out of the set D to the point $y \in \overline{D}^c$ according to the kernel $\nu(X_{\tau_D-}, y)$. It sits at the exit point y for an exponential time with mean $1/\nu(y, D)$, then jumps back to $z \in D$ according to probability distribution $\nu(y, z)/\nu(y, D)$ and starts afresh.* An example of the trajectory of the process X is presented in the Figures 1.1 and 1.2.

To make the construction of this process, we use the method of concatenation of the right processes by Werner [67] to obtain a strong Markov process. We will not present the details of the construction here to not disrupt the reader's attention at the moment. We refer for the details to the chapter itself. After the construction of the process X, we determine the explicit form of the semigroup $K = (K_t)_{t \geq 0}$ of the process and study its properties. In particular, we identify excessive functions for the semigroup K, which will be crucial in our further considerations, see Theorem 3.20 and Corollary 3.21 below.

The lifetime and the limit behaviour of the process X is investigated in Chapter 4. First, we look into the random variable describing the first return position of the process X to D and study its properties. Afterwards, we consider the sequence of the consecutive return positions of the process X to D and we prove that the limit of this sequence exists a.e. and we calculate it, see Theorem 4.6 below. Subsequently, we calculate the sum of increments of the time between the consecutive return positions and evaluate its limit, see Theorem 4.10. Then combining the received results we get the main result of this chapter which states that: for $x \neq 0$ the following statements hold $\mathbb{P}_x$-a.s.

(a) If $\alpha \in (0,1)$, then the lifetime ξ of the process X is infinite and $\lim_{t\to\infty} |X_t| = \infty$.

(b) If $\alpha \in (1,2)$, then the lifetime ξ of the process X is finite and $\lim_{t\nearrow\xi} X_t = 0$.

(c) If $\alpha = 1$, then the lifetime ξ of the process X is infinite and $\lim_{t\to\infty} X_t$ does not exist.

For more precise statement of this result see Theorem 4.11 below.

For a comparison, we refer the reader to Bogdan et al. [9, Proposition 4.2], where the authors obtain the analogous result for a censored stable process on the half-line D, which, in short, is a α-stable process "forced" to stay inside D.

Further, at the end of the chapter, we prove that for $\alpha \in (1,2)$ the semigroup K has the Feller property (see Theorem 4.14 below).

Chapter 5 is the most technical part of the whole dissertation. The vast majority of the chapter collects auxiliary theorems that we need in further part of the thesis. It consists of proofs of various estimates of the objects occurring mainly in the perturbation formula for the semigroup K (for the exact form of the perturbation formula for K see Corollary 3.4). Apart from the estimates, the chapter also contains some limit calculations. The main theorem of this chapter is the exact form of the pointwise generator (more precisely, minus generator) for the semigroup K on the excessive functions, see Theorem 5.15 below.

In Chapter 6 we consider $\alpha \neq 1$ and prove the Hardy inequality for K and the form $\mathcal{E}$ of K (see Theorem 6.1 below). In case $\alpha = 1$ we are certain that the Hardy inequality for K does not hold, but for now we cannot propose any counterexample which will justify our claim. We believe that this case will be analogous to the similar result by Dyda [30]. Furthermore, we verify that, in fact, the symmetric bilinear form corresponding to the semigroup K (and hence, corresponding to the process X) coincides with the form $\mathcal{E}_D$ and we propose various characterizations of the domains of this form. At the end of this chapter, it turns out that the bilinear form corresponding to K with its natural domain over $L^2(\mathbb{R})$ is regular (see Theorem 6.9 and Corollary 6.10 for more insight).

Last but not least, Chapter 7 is devoted to finding the direct solution of the Neumann boundary problem for $\alpha \neq 1$,

$$\begin{cases} (-\Delta)^{\alpha/2} u &= f, \quad \text{in } D, \\ \mathcal{N}_{\alpha/2} u &= f, \quad \text{in } \mathbb{R} \setminus \overline{D}. \end{cases}$$

where $f : \mathbb{R}^* \to \mathbb{R}$ is some continuous function with compact support. It turns out that the solution of this problem is given by $u = Gf$, where G is the 0-*potential* of K, i.e. $Gf(x) := \int_0^\infty K_t f(x)\,\mathrm{d}t$.

The results contained in this doctoral dissertation will be soon submitted as an article in a mathematical journal.

Chapter 2

Preliminaries

In this chapter, we propose the basic facts and notations which will be used in this thesis. They are commonly known and widely used. In Section 2.1 we introduce all the notations and conventions which will be used throughout the paper. In Section 2.2 we describe a basic theory concerning an isotropic α-stable Lévy process and the Lévy measure ν. In Section 2.3 we turn to potential theory of killed Lévy process and investigation of the properties of its semigroup. At the end, in Section 2.4, we discuss a general notion of the Markov processes including the strong Markov property.

2.1 Notation

In what follows, $\mathbb{R}$ denotes the real line and $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}$. We also let $D = (0, \infty)$. We emphasize that all sets considered in this dissertation are Borel. For any set $A \subset \mathbb{R}$ we define its *complement* $A^c := \mathbb{R} \setminus A$. The *open ball* centered at a point $x \in \mathbb{R}$ of radius $r > 0$, denoted by $B(x, r)$, is defined by $B(x, r) := \{y \in \mathbb{R} : \ |x - y| < r\}$.

Moreover, for $a, b \in \mathbb{R}$ we set $a \wedge b := \min(a, b)$ and $a \vee b = \max(a, b)$. By $\ln x$ we will understand the natural logarithm of $x > 0$, i.e $\ln x := \log_e x$.

For functions $f, g \geq 0$, we write $f(x) \lesssim g(x)$ when there exists a constant $C \in (0, \infty)$ such that $f(x) \leq Cg(x)$ for all considered arguments x. Similarly, we define the inequality $f(x) \gtrsim g(x)$. We write $f(x) \approx g(x)$ when $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$. We denote constants by $C, C_1, C_2, \ldots$. In most of the dissertation, we are not interested in the exact values of constants, but we write $C = C(a_1, \ldots, a_n)$ if the constant C is chosen to depend only on $a_1, \ldots, a_n$ or we write C_a when C depends only on a.

Let $\mathscr{X} \subset \mathbb{R}$ be an open set. By $\mathcal{B}(\mathscr{X})$ we denote the class of all Borel measurable functions on $\mathscr{X}$. By $\mathcal{B}^+(\mathscr{X})$ and $\mathcal{B}_b(\mathscr{X})$ we denote the subclasses of $\mathcal{B}(\mathscr{X})$ which consist of, respectively, non-negative and bounded functions. Moreover $\mathcal{B}_b^+(\mathscr{X}) := \mathcal{B}_b(\mathscr{X}) \cap \mathcal{B}^+(\mathscr{X})$.

The class of all continuous functions on $\mathscr{X}$ will be denoted by $C(\mathscr{X})$. We also consider subclasses of $C(\mathscr{X})$: $C_b(\mathscr{X})$ – the class of all bounded and continuous functions on $\mathscr{X}$, $C_c(\mathscr{X})$ – the class of all compactly supported functions on $\mathscr{X}$, $C_c^\infty(\mathscr{X})$ – the class of all smooth compactly supported functions on $\mathscr{X}$, $C_0(\mathscr{X})$ – the class of all continuous functions *vanishing at infinity*, i.e.

$$C_0(\mathscr{X}) = \{f \in C(\mathscr{X}) : \forall \varepsilon > 0 \ \exists K_\varepsilon \subset \mathscr{X} \text{ compact such that} |f(x)| < \varepsilon \text{ for } x \notin K_\varepsilon\}.$$

We want to point out that in what follows we will consider mainly the space $C_0(\mathbb{R}^*)$, which from the viewpoint of the above definition is the space of all continuous functions f on $\mathbb{R}^*$ such that $\lim_{|x|\to\infty} f(x) = 0$ and $\lim_{x\to 0} f(x) = 0$.

As usual, we equip $C_c^\infty(\mathscr{X})$ and $C_0(\mathscr{X})$ with the supremum norm $\|f\| = \sup_{x\in\mathscr{X}} |f(x)|$ to make them the Banach spaces. By $L^2(\mathscr{X})$ we understand the space of all square integrable functions on $\mathscr{X}$ equipped with the norm $\|u\|_{L^2(\mathscr{X})} = \sqrt{\int_{\mathscr{X}} |u(x)|^2 \, \mathrm{d}x}$.

Let $(\mathscr{X}, \mathcal{A})$ and $(\mathscr{Y}, \mathcal{B})$ be two measurable spaces. As usual (see e.g. Getoor [36]), a (*probability* or *Markov*) *transition kernel* from $(\mathscr{X}, \mathcal{A})$ to $(\mathscr{Y}, \mathcal{B})$ is a function $K : \mathscr{X} \times \mathcal{B} \to [0, 1]$ such that

(i) for every (fixed) $B \in \mathcal{B}$, the function $x \mapsto K(x, A)$ is $\mathcal{A}$-measurable,

(ii) for every (fixed) $x \in \mathscr{X}$, the function $B \mapsto K(x, B)$ is a probability measure on $(\mathscr{Y}, \mathcal{B})$.

If for every $x \in \mathscr{X}$, $K(x, \mathscr{Y}) \leq 1$ then K is called *subprobability* (or *sub-Markov*) transition kernel.

For any subprobability transition kernel K,

$$Kf(x) := \int_{\mathscr{Y}} f(y) K(x, dy),$$

for $x \in \mathscr{X}$ and $f \in \mathcal{B}_b(\mathscr{Y})$, defines a corresponding *integral operator* from $\mathcal{B}_b(\mathscr{Y})$ to $\mathcal{B}_b(\mathscr{X})$, which is linear and bounded. Hence, with a slight abuse of notation, by K we denote the transition kernel and the integral operator, depending on the context. If the kernel K has density k with respect to the measure $\mathrm{d}y$ on $\mathscr{Y}$, that is $K(x, A) = \int_A k(x, y)\mathrm{d}y$, $A \in \mathcal{B}$, then we call k the kernel, too.

Let $(\mathscr{X}, \mathcal{A}), (\mathscr{Y}, \mathcal{B})$ and $(\mathscr{Z}, \mathcal{C})$ be measurable spaces. Assume that K is a transition kernel from $(\mathscr{X}, \mathcal{A})$ to $(\mathscr{Y}, \mathcal{B})$ and L is a transition kernel from $(\mathscr{Y}, \mathcal{B})$ to $(\mathscr{Z}, \mathcal{C})$. Then a *composition* of integral kernels K and L is defined as the integral kernel KL from $(\mathscr{X}, \mathcal{A})$ to $(\mathscr{Z}, \mathcal{C})$ where

$$(KL)(x, C) := \int_{\mathscr{Y}} K(x, \mathrm{d}y) L(y, C),$$

for $x \in \mathscr{X}$, $C \in \mathcal{C}$. If K and L are probability (resp. subprobability) transition kernels, then KL is probability (resp. subprobability) transition kernel. By a kernel on $(\mathscr{X}, \mathcal{A})$ we understand a kernel from $(\mathscr{X}, \mathcal{A})$ to $(\mathscr{X}, \mathcal{A})$.

A family $(T_t)_{t\geq 0}$ of probability (resp. subprobability) transition kernels on $(\mathscr{X}, \mathcal{A})$ is called a *probability* (resp. *subprobability*) *semigroup* if $T_{t+s} = T_t T_s$, $t, s \geq 0$. A corresponding semigroup $(T_t)_{t\geq 0}$ of operators on $C_0(\mathscr{X})$ is called a *Feller semigroup* (see e.g. Kallenberg [44, p. 369]) if for $0 \leq f \leq 1$ we have $0 \leq T_t f \leq 1$, $t \geq 0$ and it has the additional regularity properties

(P1) for $f \in C_0(\mathscr{X})$ and $t \geq 0$, $T_t f \in C_0(\mathscr{X})$,

(P2) for $f \in C_0(\mathscr{X})$ and $x \in \mathscr{X}$, $T_t f(x) \to f(x)$, as $t \to 0^+$.

It can be shown (see [44, Chapter 19]), that from (P1), (P2) and the semigroup property we obtain the strong continuity of $(T_t)_{t\geq 0}$, i.e. the convergence $\lim_{t\to 0^+} \|T_t f - f\|_\infty = 0$, where $f \in C_0(\mathscr{X})$.

We also consider a *strong Feller* semigroups of operators $(T_t)_{t\geq 0}$, i.e. the semigroups with property $T_t \mathcal{B}_b(\mathscr{X}) \subset C_b(\mathscr{X})$, $t \geq 0$. Furthermore, by a *double Feller semigroups* of operators we will understand the semigroups with both Feller and strong Feller property (see e.g. Chen and Kuwae [21]).

Occasionally we will use the notation of the *Euler beta* function $\mathfrak{B}(x, y)$ which is defined by the following equality

$$\mathfrak{B}(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\, \mathrm{d}t, \qquad x > 0,\ y > 0,$$

and satisfies the equation

$$\mathfrak{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where Γ denotes the Euler gamma function.

2.2 Isotropic α-stable Lévy process

In this section, we introduce a basic knowledge about an isotropic α-stable Lévy process. For a general reference for this section, see Kwaśnicki [47].

Let $\alpha \in (0, 2)$. For $t > 0$, let p_t be the real continuous function on $\mathbb{R}$ with the Fourier transform

$$\int_{\mathbb{R}} p_t(x) e^{i\xi x}\, \mathrm{d}x = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}. \tag{2.1}$$

Thus,

$$p_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x} e^{-t|\xi|^\alpha}\,\mathrm{d}\xi, \quad t>0,\ x\in\mathbb{R}. \tag{2.2}$$

From the equation (2.2) we conclude that the function $x \mapsto p_t(x)$ is continuous (and even smooth), as the Fourier transform of the rapidly decreasing function. Additionally, the function p_t satisfies the following scaling property

$$p_t(x) = t^{-1/\alpha} p_1(xt^{-1/\alpha}), \quad t>0,\ x\in\mathbb{R}, \tag{2.3}$$

and moreover

$$p_t(x) \approx t^{-1/\alpha} \wedge \frac{t}{|x|^{1+\alpha}}, \quad x\in\mathbb{R},\ t>0. \tag{2.4}$$

The above estimations follow directly from Blumenthal and Getoor [6] or Pòlya [52], see also [43]. For $x, y \in \mathbb{R}$ we write $p_t(x,y) := p_t(y-x)$.

Let

$$\mathcal{A}_{1,\alpha} := \frac{2^\alpha \Gamma((\alpha+1)/2)}{\sqrt{\pi}|\Gamma(-\alpha/2)|}.$$

Recall that the semigroup of operators corresponding to p_t is defined by

$$P_t f(x) = \int_{\mathbb{R}} f(y) p_t(x,y)\,\mathrm{d}y, \quad f\in\mathcal{B}_b(\mathbb{R}),$$

and has the infinitesimal generator of the form

$$\begin{aligned} \Delta^{\alpha/2}\varphi(x) &= \mathcal{A}_{1,\alpha}\ \text{p.v.}\int_{\mathbb{R}} \frac{\varphi(x+y)-\varphi(x)}{|y|^{\alpha+1}}\,\mathrm{d}y \\ &:= \mathcal{A}_{1,\alpha} \lim_{\varepsilon\to 0^+}\int_{|y|>\varepsilon} \frac{\varphi(x+y)-\varphi(x)}{|y|^{\alpha+1}}\,\mathrm{d}y, \quad x\in\mathbb{R}, \end{aligned} \tag{2.5}$$

at least for $\varphi \in C_c^\infty(\mathbb{R})$, see Kwaśnicki [47].

The *isotropic α-stable Lévy process* $(Y_t, \mathbb{P}_x^Y)_{t\ge 0}$, $x\in\mathbb{R}$, on $\mathbb{R}$ may be obtained as a càdlàg Markov process with the following transition probability:

$$P_t^Y(y, A) := \int_A p_t(y,z)\,\mathrm{d}z, \quad t>0,\ y\in\mathbb{R},\ A\subset\mathbb{R},$$

and satisfying $\mathbb{P}_x^Y(Y_0 = x) = 1$. Thus, here and in what follows, $\mathbb{P}_x^Y$ and $\mathbb{E}_x^Y$ denote the distribution and expectation for the process Y starting from $x\in\mathbb{R}$. It is well known that $(Y_t, \mathbb{P}_x^Y)$ is a strong Markov process with respect to the standard filtration [7]. Moreover, for $a>0$, $aY_t \overset{\mathrm{d}}{=} Y_{a^\alpha t}$ with respect to $\mathbb{P}_0$, which follows from the fact that

$$\mathbb{E}_0^Y e^{i\xi a Y_t} = \mathbb{E}_0^Y e^{i(a\xi)Y_t} = e^{-t|a\xi|^\alpha} = e^{-(ta^\alpha)|\xi|^\alpha} = \mathbb{E}_0^Y e^{i\xi Y_{a^\alpha t}}.$$

Furthermore, for $B \subseteq \mathbb{R}$, let $\tau_B := \inf\{t \geq 0 : \; Y_t \notin B\}$ and assume that $r > 0$ and $t > 0$. Then, with respect to $\mathbb{P}_0$,

$$\begin{aligned}(Y_t, \tau_{B(0,r)}) &= \big(Y_t, \, \inf\{s \geq 0 : \; |Y_s| \geq r\}\big) \\ &= \big(r \cdot \tfrac{1}{r} Y_t, \, \inf\{s \geq 0 : \; |\tfrac{1}{r} Y_s| \geq 1\}\big) \\ &\overset{\mathrm{d}}{=} \big(r \cdot Y_{r^{-\alpha}t}, \, \inf\{s \geq 0 : \; |Y_{r^{-\alpha}s}| \geq 1\}\big) \\ &= \big(r \cdot Y_{r^{-\alpha}t}, \, r^{\alpha} \inf\{r^{-\alpha}s \geq 0 : \; |Y_{r^{-\alpha}s}| \geq 1\}\big) \\ &= \big(r \cdot Y_{r^{-\alpha}t}, \, r^{\alpha} \tau_{B(0,1)}\big). \end{aligned} \tag{2.6}$$

Similarly we prove that, with respect to $\mathbb{P}_0$,

$$Y_{\tau_{B(0,r)}} \overset{\mathrm{d}}{=} r Y_{\tau_{B(0,1)}}. \tag{2.7}$$

Let $\nu(\mathrm{d}x) := \nu(x)\,\mathrm{d}x$, where

$$\nu(x) := \mathcal{A}_{1,\alpha} |x|^{-1-\alpha}, \quad x \in \mathbb{R}^*. \tag{2.8}$$

It is the Lévy measure of Y, in particular $\int_{\mathbb{R}} (1 \wedge |x|^2) \nu(\mathrm{d}x) < \infty$. Similarly, as in the case of p_t, we slightly abuse the notation and write $\nu(x,y) := \nu(y-x)$. For $A \subset \mathbb{R}$ we also let

$$\nu(x, A) := \nu(A - x) = \int_A \nu(x,y)\,\mathrm{d}y, \qquad x \notin A. \tag{2.9}$$

Recall that $D = (0, \infty) \subset \mathbb{R}$. For further use, we note that

$$\nu(kx, ky) = k^{-\alpha-1} \nu(x,y), \quad k > 0, \; x, y \in \mathbb{R}, \; x \neq y, \tag{2.10}$$

$$\nu(kx, D) = k^{-\alpha} \nu(x, D), \quad k > 0, \; x < 0, \tag{2.11}$$

$$\nu(x, D) = \mathcal{A}_{1,\alpha} |x|^{-\alpha} \int_0^{\infty} \frac{\mathrm{d}y}{(1+y)^{\alpha+1}} = \mathcal{A}_{1,\alpha} \alpha^{-1} |x|^{-\alpha}, \qquad x < 0, \tag{2.12}$$

and

$$\nu(x, D^c) = \nu(-x, D) = \mathcal{A}_{1,\alpha} \alpha^{-1} x^{-\alpha}, \qquad x > 0. \tag{2.13}$$

Of course, $\nu(0, D) = \infty$.

2.3 Potential theory

Let $\tau_D := \inf\{t \geq 0 : \; Y_t \leq 0\}$ and define the transition density of the *process killed* when leaving $D = (0, \infty)$:

$$p_t^D(x,y) = p_t(x,y) - \mathbb{E}_x^Y \big[\tau_D < t; \; p_{t-\tau_D}(Y_{\tau_D}, y)\big], \quad t > 0, \; x, y \in \mathbb{R}. \tag{2.14}$$

The above formula is often called the *Hunt formula*. For reference in this section and for more details in the subject, we refer the reader to [11], [20] or [24, Chapter 2].

The heat kernel (2.14) for D also has the scaling property

$$p_t^D(x,y) = t^{-1/\alpha} p_1^D(t^{-1/\alpha}x, t^{-1/\alpha}y), \quad x,y \in \mathbb{R},\ t>0, \tag{2.15}$$

or, equivalently,

$$p_{k^\alpha t}^D(kx,ky) = k^{-1} p_t^D(x,y), \quad x,y \in \mathbb{R},\ k,t>0. \tag{2.16}$$

It turns out that the function p^D is jointly continuous in t,x,y. Indeed, it follows from the scaling property (2.15), continuity of the function $(x,y) \mapsto p_t(x,y)$ and by the same proof as in Theorem 2.4 in Chung and Zhao [24]. Moreover, for $x,y \in \mathbb{R}$ and $t>0$, the heat kernel p^D is *symmetric*, i.e. $p_t^D(x,y) = p_t^D(y,x)$. It also satisfies

$$0 \le p_t^D(x,y) \le p_t(x,y), \tag{2.17}$$

and $p_t^D(x,y) = 0$ whenever $x \le 0$ or $y \le 0$. It is the Dirichlet heat kernel of the half-line D for the fractional Laplacian, or transition density of the isotropic α-stable process killed when leaving D. Thus,

$$p_t^D(x,A) := \int_A p_t^D(x,y)\,\mathrm{d}y = \mathbb{P}_x^Y(Y_t \in A, \tau_D > t), \quad t>0,\ x\in\mathbb{R},\ A \subset \mathbb{R}, \tag{2.18}$$

and for all $x \in \mathbb{R}$, $t>0$, and bounded functions f,

$$P_t^D f(x) := \int_{\mathbb{R}} f(y) p_t^D(x,y)\,\mathrm{d}y = \mathbb{E}_x^Y[f(Y_t); \tau_D > t].$$

We set $P_0^D = I$, where I denotes the identity operator. Furthermore, the kernel p^D satisfies the Chapman–Kolmogorov equation:

$$p_{t+s}^D(x,y) = \int_{\mathbb{R}} p_t^D(x,z) p_s^D(z,y)\,\mathrm{d}z, \quad x,y\in\mathbb{R},\ s,t>0. \tag{2.19}$$

It also satisfies (see e.g. Bogdan and Grzywny [11])

$$p_t^D(x,y) \approx \mathbb{P}_x^Y(\tau_D > t)\mathbb{P}_y^Y(\tau_D > t) p_t(x,y), \quad t>0,\ x,y>0, \tag{2.20}$$

where

$$\mathbb{P}_x^Y(\tau_D > t) \approx 1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}. \tag{2.21}$$

From (2.4) it follows that

$$p_t^D(x,y) \approx \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{|y|^{\alpha/2}}{\sqrt{t}}\Big)\Big(t^{-1/\alpha} \wedge \frac{t}{|x-y|^{\alpha+1}}\Big), \quad t>0,\ x,y>0. \tag{2.22}$$

The *Green function* of D is

$$G_D(x,y) := \int_0^\infty p_t^D(x,y)\,\mathrm{d}t, \quad x,y\in\mathbb{R}. \tag{2.23}$$

Obviously, $G_D(x,y)=0$ whenever $x\le 0$ or $y\le 0$. For each function $f\ge 0$, from Tonelli's theorem and (2.18), we have

$$\int_D G_D(x,y)f(y)\,\mathrm{d}y = \mathbb{E}_x^Y\int_0^{\tau_D} f(Y_t)\,\mathrm{d}t, \quad x>0,$$

hence, $G_D(x,y)$ is the occupation time density of the process Y prior to the first exit from D. Using (2.16) it can be easily calculated that G_D has the following scaling property:

$$G_D(x,y) = x^{\alpha-1}G_D(1,y/x), \qquad x,y>0. \tag{2.24}$$

The joint distribution of the triple $(\tau_D, Y_{\tau_D-}, Y_{\tau_D})$ is given by the *Ikeda–Watanabe formula*, see [41] (we also refer to [16]):

$$\mathbb{P}_x^Y\big(\tau_D\in I,\ Y_{\tau_D-}\in A, Y_{\tau_D}\in B\big) = \int_I \mathrm{d}t\int_A \mathrm{d}y\int_B \mathrm{d}z\, p_t^D(x,y)\nu(y,z), \quad x>0. \tag{2.25}$$

From (2.18) and (2.25) we have

$$p_t^D(x,D) := \int_D p_t^D(x,y)\,\mathrm{d}y = \mathbb{P}_x^Y(\tau_D>t) = \int_t^\infty \mathrm{d}r\int_D \mathrm{d}a\int_{D^c}\mathrm{d}b\, p_r^D(x,a)\nu(a,b). \tag{2.26}$$

Subsequently, we define the *Poisson kernel* of D for the fractional Laplacian:

$$P_D(x,y) = \int_D G_D(x,z)\nu(z,y)\,\mathrm{d}z, \quad x>0,\ y\le 0. \tag{2.27}$$

Then, for $A\subset(-\infty,0]$,

$$\mathbb{P}_x^Y(Y_{\tau_D}\in A) = \int_A P_D(x,y)\,\mathrm{d}y, \quad x>0.$$

Using (2.24), we obtain the following scaling property of the Poisson kernel:

$$P_D(x,y) = x^{-1}P_D(1,y/x), \qquad x>0,\ y\le 0. \tag{2.28}$$

In fact, the Poisson kernel of $D=(0,\infty)$ is known explicitly (see [8, (3.40)]):

$$P_D(x,y) = \frac{1}{\pi}\sin\Big(\frac{\pi\alpha}{2}\Big)\frac{x^{\alpha/2}}{|y|^{\alpha/2}|x-y|}, \quad x>0,\ y\le 0. \tag{2.29}$$

The following theorems state the Feller property of P^D, both on $C_b(D)$ and $C_0(D)$. These theorems are well known, but we give their proof for convenience of the reader.

Theorem 2.1. *For every $t>0$ we have $P_t^D\mathcal{B}_b(D)\subset C_b(D)$.*

Proof. Let $f \in \mathcal{B}_b(D)$. Then without loss of generality we may assume that $f = 0$ on D^c. For $t > 0$, the function $D \ni x \mapsto P_t f(x) = p_t * f(x)$ is continuous, because $p_t \in L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$. From the Vitali's theorem (see Schilling [57, Theorem 16.6]) it follows that $y \mapsto p_t(x,y)f(y)$ is uniformly integrable with respect to the measure $\mathbb{1}_D(y)\,\mathrm{d}y$. From (2.17) it follows that $y \mapsto p_t^D(x,y)f(y)$ is also uniformly integrable with respect to the measure $\mathbb{1}_D(y)\,\mathrm{d}y$ (see [57, Definition 16.1 or Theorem 16.8]). Hence, because the function $D \ni x \mapsto p_t^D(x,y)$ is continuous, from the Vitali's theorem we conclude the continuity of the function $D \ni x \mapsto P_t^D f(x)$.

Moreover, there exists $M > 0$ such that $|f| \le M$, hence

$$|P_t^D f(x)| \le \int_D p_t^D(x,y)|f(y)|\,\mathrm{d}y \le M \int_D p_t^D(x,y)\,\mathrm{d}y \le M,$$

which implies that the function $P_t^D f$ is bounded. □

Theorem 2.2. *For every $t \ge 0$ we have $P_t^D C_0(D) \subset C_0(D)$.*

Proof. The continuity of the function $D \ni x \mapsto P_t^D f(x)$ follows directly from Theorem 2.1.

We will show that $P_t^D f(x) \to 0$, as $x \to +\infty$. Note that, from (2.17)

$$|P_t^D f(x)| \le \int_D p_t^D(x,y)|f(y)|\,\mathrm{d}y \le \int_D p_t(x,y)|f(y)|\,\mathrm{d}y \to 0,$$

as $x \to \infty$, which follows from the Feller property of P_t (see [44, Theorem 19.10]).

Moreover, $P_t^D f(x) \to 0$, as $x \to 0^+$. Indeed, from (2.26) and (2.21),

$$|P_t^D f(x)| \le \int_D p_t^D(x,y)|f(y)|\,\mathrm{d}y \le \|f\|_\infty\, p_t^D(x,D) = \|f\|_\infty\, \mathbb{P}_x^Y(\tau_D > t) \approx 1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}} \to 0,$$

as $x \to 0^+$. This proves the theorem. □

Theorem 2.3. *For every $\alpha \in (0,2)$, $(P_t^D)_{t\ge 0}$ is a Feller semigroup on $C_0(D)$.*

Proof. Let $t > 0$ and $x > 0$. For $0 \le f \le 1$ from (2.17),

$$0 \le P_t^D f(x) = \int_D p_t^D(x,y)f(y)\,\mathrm{d}y \le \int_D p_t(x,y)f(y)\,\mathrm{d}y \le \int_D p_t(x,y)\,\mathrm{d}y \le 1.$$

Assume that $f \in C_0(D)$. From theorem 2.2, $P_t^D f \in C_0(D)$. We will show that $P_t^D f(x) \to f(x)$, as $t \to 0^+$. Note that from (2.17),

$$\begin{aligned}
\left|P_t^D f(x) - f(x)\right| &= \left| \int_D (f(y) - f(x)) p_t^D(x,y)\,\mathrm{d}y - f(x)(1 - p_t^D(x,D)) \right| \\
&\le \int_D |f(y) - f(x)| p_t^D(x,y)\,\mathrm{d}y + |f(x)|\left|1 - p_t^D(x,D)\right| \\
&\le \int_D |f(y) - f(x)| p_t(x,y)\,\mathrm{d}y + \|f\|_\infty \left|1 - p_t^D(x,D)\right|.
\end{aligned}$$

From (2.26) it follows that $p_t^D(x, D) \to 1$, as $t \to 0^+$. Hence, it suffices to show the convergence to zero of the latter integral.

The proof of the convergence $\int_D |f(y) - f(x)| p_t(x, y)\, \mathrm{d}y \to 0$, as $t \to 0^+$, is the same as in the proof of Theorem 1.7 in [24]. We will present its details here. From the fact that f is continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|y - x| < \delta$ we have $|f(y) - f(x)| < \varepsilon$. Therefore, from (2.4) we get

$$\begin{aligned}
&\int_D |f(y) - f(x)| p_t(x, y)\, \mathrm{d}y \\
&\quad \le \varepsilon \int_{D \cap \{|y-x|<\delta\}} p_t(x, y)\, \mathrm{d}y + \int_{D \cap \{|y-x|\ge\delta\}} |f(y) - f(x)| p_t(x, y)\, \mathrm{d}y \\
&\quad \le \varepsilon + 2 \|f\|_\infty \int_{D \cap \{|y-x|\ge\delta\}} p_t(x, y)\, \mathrm{d}y \\
&\quad \le \varepsilon + 2C \|f\|_\infty t \int_{|w|\ge\delta} |w|^{-\alpha-1}\, \mathrm{d}w \\
&\quad = \varepsilon + \frac{4C}{\alpha} \|f\|_\infty t\delta^{-\alpha}.
\end{aligned}$$

Hence, $\limsup_{t\to 0^+} \int_D |f(y) - f(x)| p_t(x, y)\, \mathrm{d}y \le \varepsilon$ and the claim follows from the arbitrariness of the $\varepsilon > 0$. □

From Theorem 2.1 and Theorem 2.3 it follows that $(P_t^D)_{t\ge 0}$ is the double Feller semigroup.

2.4 Markov processes and the strong Markov property

In what follows, we will work with Markov processes, so we introduce the basic notion for such processes. We will use notation similar to Werner [67], as we use his results in this paper.

A *Markov process* X on $\mathbb{R}$ is defined as a following sextuple (see e.g. [7], [61]):

$$X = (\Omega, \widetilde{\mathcal{F}}, (\mathcal{F}_t)_{t\ge 0}, (X_t)_{t\ge 0}, (\theta_t)_{t\ge 0}, (\mathbb{P}_x)_{x\in\mathbb{R}}),$$

where $(X_t)_{t\ge 0}$ is a right continuous, $\mathbb{R}$-valued stochastic process on a measurable space $(\Omega, \widetilde{\mathcal{F}})$, adapted to the filtration $(\mathcal{F}_t)_{t\ge 0}$ and equipped with the shift operators $(\theta_t)_{t\ge 0}$ on Ω. Moreover, $(\mathbb{P}_x)_{x\in\mathbb{R}}$ is a family of probability measures such that $\mathbb{P}_x(X_0 = x) = 1$ for all $x \in \mathbb{R}$ and $\mathbb{E}_x$ are the corresponding expectations. Furthermore, for arbitrary $t \ge 0$ and $A \in \widetilde{\mathcal{F}}$, the function $x \mapsto \mathbb{P}_x(X_t \in A)$ is measurable, and we have the following *Markov property*: for bounded and measurable function f,

$$\mathbb{E}_x[f(X_{t+s})|\mathcal{F}_s] = \mathbb{E}_{X_s}[f(X_t)], \quad x \in \mathbb{R},\ s, t \ge 0.$$

In our considerations we need a few additional assumptions. We assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is augmented by the null sets and right continuous, and that there exists an isolated, absorbing cemetery point Δ, for which we set $\mathbb{R}_\Delta := \mathbb{R} \cup \{\Delta\}$, such that for the lifetime of the process,

$$\zeta := \inf\{t \geq 0 : \ X_t = \Delta\},$$

we have $X_t = \Delta$ if $t \geq \zeta$. Moreover, there exists a path $[\Delta] \in \Omega$ for which $\zeta([\Delta]) = 0$. We assume also that $X_\infty = \Delta$, $\theta_\infty = [\Delta]$ and for any measurable function f, $f(\Delta) = 0$.

The semigroup $(T_t)_{t\geq 0}$ corresponding (or associated) to the stochastic process X is defined as

$$T_t f(x) = \mathbb{E}_x[f(X_t)], \quad x \in \mathbb{R}, \ t \geq 0,$$

where f is bounded or non-negative.

We say that a function f is *α–excessive* (see e.g. Sharpe [61, (4.11)]) if it is non-negative, measurable and for $\alpha \geq 0$, $e^{-\alpha t} T_t f \uparrow f$ pointwise as $t \to 0^+$. Let $\mathcal{S}_\alpha$ be the set of all such functions. Then the process X is called *right process* if it is a Markov process equipped with an augmented and right continuous filtration and for all $\alpha > 0$, $f \in \mathcal{S}_\alpha$, the map $t \mapsto f(X_t)$ is a.s. right continuous.

It is well known that the killed isotropic α-stable Lévy process

$$Y_t^D = \begin{cases} Y_t, & t < \tau_D, \\ \Delta, & t \geq \tau_D, \end{cases}$$

is a right process with lifetime $\xi = \tau_D$, since it is a Hunt process (as a strong Feller process, see Chung [23]) and every Hunt process is, by definition, a right process (see Getoor [36, Section 9]).

With this convention, right processes have the following *strong Markov property* (see [67]): for every stopping time τ (with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$) and bounded $\mathcal{F}_0$-measurable function f, where $\mathcal{F}_0$ is the universal completion of $\sigma(X_s : \ s \geq 0)$, we have:

$$\mathbb{E}_x[f(X_t \circ \theta_\tau) | \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[f(X_t)], \quad x \in \mathbb{R}. \tag{2.30}$$

We call X a *Feller process* or a *strong Feller process* if its corresponding semigroup $(T_t)_{t\geq 0}$ is Feller or strong Feller, respectively.

Chapter 3

The Servadei–Valdinoci process

In this chapter, we make a construction of the stochastic process on $\mathbb{R}^*$, which will be our main point of interest through the rest of the thesis. Our method of construction relies on a concatenation of right Markov processes by Werner [67]. Further, we derive a transition kernel and a corresponding semigroup of operators for this process. Another main goal of this chapter is to study properties of presented semigroup. More precisely, we prove e.g. its boundedness, strong continuity, strong Feller property, and we introduce the excessive functions for this semigroup.

3.1 Construction of the process

Let us discuss here in detail a construction of the stochastic process, which behaves differently depending on the starting point: starting from $x > 0$ the process behaves as the right process Z^1, while starting from $x < 0$ it behaves as the right process Z^2. The processes Z^1 and Z^2 are defined below.

Let

$$Z^1 = (\Omega^1, \mathcal{F}^1, (\mathcal{F}_t^1)_{t\geq 0}, (Z_t^1)_{t\geq 0}, (\theta_t^1)_{t\geq 0}, (\mathbb{P}_x^1)_{x>0}),$$

be the isotropic α-stable Lévy process killed when leaving $D = (0, \infty)$. We denote the transition probability of Z^1 by

$$P_t^1(x, A) := \int_A p_t^D(x, y)\,\mathrm{d}y, \quad x > 0,\ A \subset \mathbb{R},\ t > 0.$$

Of course, the lifetime of this process is $\zeta_1^Z := \tau_D = \inf\{t \geq 0 :\ Z_t^1 \leq 0\}$.

The second process that we consider is a compound–Poisson type process Z^2 on $(-\infty, 0)$. The process stays in a starting point $x < 0$ for an exponential time with mean $1/\nu(x, D)$ and afterwards it is killed. This process will be denoted by

$$Z^2 = (\Omega^2, \mathcal{F}^2, (\mathcal{F}^2_t)_{t\geq 0}, (Z^2_t)_{t\geq 0}, (\theta^2_t)_{t\geq 0}, (\mathbb{P}^2_x)_{x<0}).$$

The transition probability of the process is

$$P^2_t(x, A) := \delta_x(A)e^{-\nu(x,D)t}, \quad x < 0,\ A \subset \mathbb{R},\ t > 0,$$

and its lifetime ζ^Z_2 is the exponential random variable with mean $1/\nu(x, D)$.

For $t > 0$, $A \subset \mathbb{R}$ we define the integral kernel

$$\widehat{P}_t(x, A) = \begin{cases} P^1_t(x, A), & \text{if } x > 0, \\ P^2_t(x, A), & \text{if } x < 0. \end{cases}$$

We note that $\widehat{P}_t(x, \overline{D}^c) = 0$ if $x > 0$ and $\widehat{P}_t(x, D) = 0$ if $x < 0$.

Lemma 3.1. *For $t > 0$, $\widehat{P}_t$ is a subprobability transition kernel.*

Proof. We clearly have $\widehat{P}_t(x, A) \geq 0$ for all $x \in \mathbb{R}$ and $A \subset \mathbb{R}$. Moreover,

$$\widehat{P}_t(x, \mathbb{R}) = \int_{\mathbb{R}} p^D_t(x, y)\,\mathrm{d}y \leq \int_{\mathbb{R}} p_t(x, y)\,\mathrm{d}y = 1, \quad x > 0,$$

and

$$\widehat{P}_t(x, \mathbb{R}) = e^{-t\nu(x,D)} \leq 1, \quad x < 0.$$

It remains to verify the Chapman–Kolmogorov equation. Let $s, t > 0$ and $A \subset \mathbb{R}$. For $x > 0$ this equality follows directly from (2.19) and for $x < 0$,

$$\begin{aligned}\int_{\mathbb{R}} \widehat{P}_t(x, \mathrm{d}y)\widehat{P}_s(y, A) &= e^{-t\nu(x,D)}\widehat{P}_s(x, A) = e^{-t\nu(x,D)}e^{-s\nu(x,D)}\delta_x(A) \\ &= \delta_x(A)e^{-(s+t)\nu(x,D)} = \widehat{P}_{s+t}(x, A),\end{aligned}$$

which completes the proof. □

Recall that for a transition kernel $\widehat{P}_t$, $t > 0$, by the same symbol we denote the integral operator

$$\widehat{P}_t f(x) := \int_{\mathbb{R}} f(y)\widehat{P}_t(x, \mathrm{d}y),$$

where f is any function for which the above formula makes sense. Additionally, we set $\widehat{P}_0 := I$.

Lemma 3.2. *$(\widehat{P}_t)_{t\geq 0}$ is a Feller semigroup on $C_0(\mathbb{R}^*)$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.*

Proof. From Lemma 3.1 it follows that for any $f \in C_0(\mathbb{R}^*)$, $0 \le f \le 1$ we have $0 \le \widehat{P}_t f \le 1$, $t \ge 0$. Hence $\widehat{P}_t$, $t \ge 0$, is a semigroup of positive contraction operators on $C_0(\mathbb{R}^*)$.

Let $f \in C_0(\mathbb{R}^*)$. We will show that $\widehat{P}_t f \in C_0(\mathbb{R}^*)$. The function $D \ni x \mapsto \widehat{P}_t f(x) = P_t^D f(x)$ is continuous, which follows directly from Theorem 2.3, and of course, $\overline{D}^c \ni x \mapsto \widehat{P}_t f(x) = e^{-\nu(x,D)t} f(x)$ is continuous.

Afterwards, we claim that $\widehat{P}_t f(x) \to 0$, as $x \to 0$. Let $t \ge 0$. From Theorem 2.3, $\widehat{P}_t f(x) = P_t^D f(x) \to 0$ as $x \to 0^+$. For $x \to 0^-$ (see (2.12)),

$$|\widehat{P}_t f(x)| \le \|f\|_\infty \widehat{P}_t \mathbf{1}(x) = \|f\|_\infty e^{-\nu(x,D)t} \to 0.$$

Hence, we proved the claim.

Furthermore, $\widehat{P}_t f(x) \to 0$ as $|x| \to \infty$. Indeed, from Theorem 2.3, $\widehat{P}_t f(x) \to 0$ as $x \to +\infty$. Moreover, for $x < 0$, from (2.12) and the assumption,

$$\widehat{P}_t f(x) = f(x) e^{-\nu(x,D)t} \to 0,$$

as $x \to -\infty$.

To finish the proof, we will show that for $f \in C_0(\mathbb{R}^*)$, $\widehat{P}_t f(x) \to f(x)$, as $t \to 0^+$. For $x > 0$,

$$|\widehat{P}_t f(x) - f(x)| = |p_t^D f(x) - f(x)| \to 0,$$

as $t \to 0^+$, which follows from Theorem 2.3. In the case $x < 0$, we have,

$$|\widehat{P}_t f(x) - f(x)| \le |f(x)| \cdot |e^{-\nu(x,D)t} - 1| \to 0,$$

as $t \to 0^+$. This proves the lemma. □

It follows from Kallenberg [44, Chapter 19] or Getoor and Blumenthal [7, Theorem I.9.4] that there exists a Hunt process (hence, a right process [61, (47.3)]) Z on $\mathbb{R}^*$, with transition function given by $\widehat{P}_t$. The lifetime of this process will be denoted by ζ. The process Z starting from $x > 0$ is the isotropic α-stable Lévy process $Z^1 = Y$ killed when leaving D, while starting from $x < 0$ it is the compound–Poisson type process Z^2.

A concatenation of functions x_1, defined on $[0, z_1)$, x_2, defined on $[0, z_2)$, etc., is a function x defined on $[0, z_1 + z_2 + \ldots)$, by letting $x(t) = x_n\big(t - (z_1 + \ldots + z_n)\big)$ for $z_1 + \ldots + z_n \le t < z_1 + \ldots + z_n + z_{n+1}$. Here $z_1, z_2, \ldots \ge 0$ and if $z_i = 0$ then the function x_i has no affect on x.

Next, we will concatenate a countable sequence of stochastic processes. For $n \in \mathbb{N}$ suppose that the processes $X^n = (\Omega^{(n)}, \mathcal{F}^{(n)}, (\mathcal{F}_t^{(n)})_{t\ge0}, (X_t^{(n)})_{t\ge0}, (\theta_t^{(n)})_{t\ge0}, (\mathbb{P}_x^{(n)})_{x\in\mathbb{R}^*})$ on $E^n := \mathbb{R}^*$ are independent copies of the right process Z. By $\zeta^{(n)}$ we will denote the lifetime of each process X^n, $n \in \mathbb{N}$. Moreover, we also assume that for each from the processes X^n there exists, so called, a dead path $[\Delta^n] \in \Omega^{(n)}$ with the property $\zeta^{(n)}([\Delta^n]) = 0$. We constitute that $\theta_\infty^{(n)}(\omega) := [\Delta^n]$ for all $\omega \in \Omega^{(n)}$. As usual, for any measurable function f we assume that $f(\Delta) = 0$.

We define the transfer kernel k on $\mathbb{R}^*$ (also called *instantaneous kernel*) which will specify jumps between D and $\overline{D}^c$ in the following way: for $A \subset \mathbb{R}$,

$$k(x, A) = \begin{cases} \dfrac{\nu(x, A \cap D^c)}{\nu(x, D^c)}, & x > 0, \\ \dfrac{\nu(x, A \cap D)}{\nu(x, D)}, & x < 0. \end{cases}$$

If $x > 0$, then $k(x, \cdot)$ gives the starting distribution of the process Z^2 initiated after Z^1 died at x. Similarly, for $x < 0$, $k(x, \cdot)$ is the initial distribution of Z^1 initiated after Z^2 died. To be more precise, for $k = 1, 2, \ldots$, we define a transfer kernel $K^k : \Omega^{(k)} \times \mathcal{B} \to [0, 1]$ from X^k to $(X^{k+1}, \mathbb{R}^*)$ with

$$K^k(\omega_k, A) := k(X^k_{\zeta^{(k)}-}(\omega_k), A), \quad \omega_k \in \Omega^{(k)}, \ A \subset \mathbb{R}^*, \tag{3.1}$$

where $\mathcal{B}$ denotes the σ-algebra of Borel measurable subsets of $\mathbb{R}^*$.

Following Werner [67] (see also Sharpe [61, Chapter 14]), we define process X on $E = \mathbb{R}^*$ which is called a *concatenation* of the processes $(X^n)_{n\in\mathbb{N}}$ on $(E^n)_{n\in\mathbb{N}}$. Here are the details.

We set $\Omega = \prod_{n\in\mathbb{N}} \Omega^{(n)}$ and $\widetilde{\mathcal{F}} = \bigotimes_{n\in\mathbb{N}} \mathcal{F}^{(n)}$. For $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$, $t \geq 0$ and $n \geq 1$ we define $R_n(\omega) := \sum_{k=1}^n \zeta^{(k)}(\omega)$ and $R_0(\omega) = 0$, $R_\infty(\omega) = \lim_{n\to\infty} R_n(\omega) = \sum_{n=1}^\infty \zeta^{(n)}(\omega)$. From the construction, for $n \in \mathbb{N}$, $\zeta^{(n)}$ is finite a.s. In order not to abuse notation, we make a convention that $\zeta^{(k)}(\omega) = \zeta(\omega_k)$, where ζ denotes the lifetime of the process Z. From the definition, for $n \in \mathbb{N}$ it is obvious that $R_n < \infty$ a.s.

Following the construction proposed in [67] for all $t \geq 0$, $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ we let

$$X_t(\omega) = \begin{cases} X^1_t(\omega_1), & R_0(\omega) \leq t < R_1(\omega), \\ X^2_{t-R_1(\omega)}(\omega_2), & R_1(\omega) \leq t < R_2(\omega), \\ X^3_{t-R_2(\omega)}(\omega_3), & R_2(\omega) \leq t < R_3(\omega), \\ \vdots & \vdots \\ \Delta, & t \geq R_\infty(\omega). \end{cases}$$

Here we assume that there exists an isolated, absorbing cemetery state denoted by Δ such that for the lifetime R_∞ of the process X we have

$$R_\infty = \inf\{t \geq 0 : \ X_t = \Delta\}.$$

Moreover, $X_t = \Delta$ for all $t \geq R_\infty$. We set $\mathbb{R}_\Delta := \mathbb{R}^* \cup \{\Delta\}$.

For all $t \geq 0$ and $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ we then define the *shift operator* of the process X as follows:

$$\theta_t(\omega) = \begin{cases} (\theta_t^{(1)}(\omega_1), \omega_2, \omega_3, \omega_4, \ldots), & R_0(\omega) \leq t < R_1(\omega), \\ ([\Delta^1], \theta^{(2)}_{t-R_1(\omega)}(\omega_2), \omega_3, \omega_4, \ldots), & R_1(\omega) \leq t < R_2(\omega), \\ ([\Delta^1], [\Delta^2], \theta^{(3)}_{t-R_2(\omega)}(\omega_3), \omega_4, \ldots), & R_2(\omega) \leq t < R_3(\omega), \\ \vdots & \vdots \\ ([\Delta^1], \ldots, [\Delta^{n-1}], \theta^{(n)}_{t-R_{n-1}(\omega)}(\omega_n), \omega_{n+1}, \ldots), & R_{n-1}(\omega) \leq t < R_n(\omega), \\ \vdots & \vdots \end{cases}$$

We will assume that $X_\infty = \Delta$ and $\theta_\infty = [\Delta]$. From the above definitions, it follows immediately that

$$R_n = R_{n-1} + \zeta^{(1)} \circ \theta_{R_{n-1}}, \qquad n \geq 1.$$

The *law* of the concatenated process, i.e. the family of measures $(\mathbb{P}_x, x \in \mathbb{R}^*)$ on $(\Omega, \widetilde{\mathcal{F}})$, is defined as the connection of the distributions $(\mathbb{P}_x^{(n)}, x \in \mathbb{R}^*)$, $n \in \mathbb{N}$, via the transfer kernel K. We will not present the details of this construction here. All the details are presented in Werner [67].

From [67] it follows that $X = (\Omega, \widetilde{\mathcal{F}}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^*})$, constructed above, is a right process on $\mathbb{R}^*$ with the lifetime R_∞ and moreover for $n \in \mathbb{N}$, $x \in \mathbb{R}^*$ and $f \in \mathcal{B}_b(\mathbb{R}^*)$,

$$\mathbb{E}_x\big[f(X_{R_n}) \,\big|\, \mathcal{F}_{R_n-}\big] = K^n f \circ \pi^n, \tag{3.2}$$

where π^n denotes the projection on the n-th coordinate of $\omega = (\omega_1, \omega_2, \ldots)$. In particular, from (3.1), for $\omega \in \Omega$, $A \subset \mathbb{R}^*$ and $f(x) = \mathbb{1}_A(x)$, we have

$$\mathbb{P}_x\big[X_{R_n} \in A \,\big|\, \mathcal{F}_{R_n-}\big](\omega) = K^n(\omega_n, A) = k(X^n_{\zeta^{(n)}-}(\omega_n), A), \tag{3.3}$$

In particular, X has the strong Markov property. More details of this construction, especially the construction of the filtration $(\mathcal{F}_t)_{t\geq 0}$, the shift operators $(\theta_t)_{t\geq 0}$ and the family of measures $(\mathbb{P}_x, x \in \mathbb{R}^*)$ can be found in [67] (see also Ikeda et al. [40], Meyer [50] and Sharpe [61, Chapter II.14]).

3.2 Transition semigroup of the process

Let

$$\widehat{\nu}(x,y) = \begin{cases} \nu(x,y), & \text{if } x \in D,\ y \in D^c \text{ or } x \in D^c,\ y \in D, \\ 0, & \text{otherwise.} \end{cases}$$

By a small abuse of notation we also use $\widehat{\nu}$ to denote the corresponding integral kernel on $\mathbb{R}$:

$$\widehat{\nu}(x, A) = \int_A \widehat{\nu}(x, y)\,\mathrm{d}y, \quad x \in \mathbb{R},\ A \subset \mathbb{R}.$$

Of course, $\widehat{\nu}(x, A) = 0$ if $x \in D$, $A \subset D$. Similarly, $\widehat{\nu}(x, A) = 0$ if $x \in D^c$, $A \subset D^c$.

With the second meaning of $\widehat{\nu}$, we consider the following perturbation series: $K_0 := I$ and for $t > 0$,

$$K_t := \widehat{P}_t + \int_0^t \widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t-t_1}\,\mathrm{d}\,t_1 + \int_0^t \int_{t_1}^t \widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\widehat{P}_{t-t_2}\,\mathrm{d}t_2\,\mathrm{d}t_1 + \ldots = \sum_{n=0}^{\infty} K_{t,n}, \tag{3.4}$$

where $K_{t,0} = \widehat{P}_t$ and, for $n \geq 1$,

$$K_{t,n} := \int \cdots \int_{0<t_1<t_2<\ldots<t_n<t} \widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{t-t_n}\,\mathrm{d}t_1\ldots\mathrm{d}t_n. \tag{3.5}$$

From Bogdan and Sydor [17] it follows that for all $t \geq 0$, K_t is in fact a transition kernel. In what follows we need the stochastic properties of $(K_t)_{t\geq 0}$, so we will not use [17], but the reader interested in purely analytic approach to $(K_t)_{t\geq 0}$ may find [17] instructive. The deficiency of [17] for our goals is that the strong Markov property of the process given by $(K_t)_{t\geq 0}$ was not studied there. For the direct proof of the fact that K_t, $t \geq 0$, is a transition kernel see the proof of Lemma 3.10.

Note that in the further part of the dissertation, in case of non-negative integrands, we will often use the Tonelli's theorem without mentioning.

Lemma 3.3. *For $t > 0$, $n = 0, 1, \ldots$,*

$$K_{t,n+1} = \int_0^t \widehat{P}_r\widehat{\nu}K_{t-r,n}\,\mathrm{d}r = \int_0^t K_{r,n}\widehat{\nu}\widehat{P}_{t-r}\,\mathrm{d}r. \tag{3.6}$$

Proof. By the substitution $u_1 = t_1 + r$, $u_2 = t_2 + r, \ldots, u_n = t_n + r$,

$$\begin{aligned}
\int_0^t \widehat{P}_r\widehat{\nu}K_{t-r,n}\,\mathrm{d}r &= \int_0^t \mathrm{d}r \int_0^{t-r} \mathrm{d}t_1 \int_{t_1}^{t-r} \mathrm{d}t_2 \ldots \int_{t_{n-1}}^{t-r} \mathrm{d}t_n\, \widehat{P}_r\widehat{\nu}\widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{t-r-t_n} \\
&= \int_0^t \mathrm{d}r \int_r^t \mathrm{d}u_1 \int_{u_1}^t \mathrm{d}u_2 \ldots \int_{u_{n-1}}^t \mathrm{d}u_n\, \widehat{P}_r\widehat{\nu}\widehat{P}_{u_1-r}\widehat{\nu}\widehat{P}_{u_2-u_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{t-u_n} \\
&= \int \cdots \int_{0<r<u_1<u_2<\ldots<u_{n-1}<u_n<t} \widehat{P}_r\widehat{\nu}\widehat{P}_{u_1-r}\widehat{\nu}\widehat{P}_{u_2-u_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{t-u_n} \\
&= K_{t,n+1}.
\end{aligned}$$

The second equality in (3.6) arises similarly. □

By (3.4) and (3.6) we get the following perturbation formula for K_t.

Corollary 3.4. *For* $t > 0$,

$$K_t = \widehat{P}_t + \int_0^t \widehat{P}_r \widehat{\nu} K_{t-r} \,\mathrm{d}r = \widehat{P}_t + \int_0^t K_r \widehat{\nu} \widehat{P}_{t-r} \,\mathrm{d}r. \tag{3.7}$$

From the definition, for $x > 0$, $t > 0$, $K_{t,0}(x, \mathrm{d}y)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with density $K_{t,0}(x, y) := p_t^D(x, y)$. Hence, $K_{t,0}(x, y) = K_{t,0}(y, x)$.

Lemma 3.5. *For* $t > 0$, $f, g \in \mathcal{B}_+(\mathbb{R})$,

$$\int_{\mathbb{R}} (K_t f)(x) g(x) \,\mathrm{d}x = \int_{\mathbb{R}} f(x)(K_t g)(x) \,\mathrm{d}x. \tag{3.8}$$

Proof. Note that

$$\begin{aligned}\int_{\mathbb{R}} (K_t f)(x) g(x) \,\mathrm{d}x &= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y) g(x) K_t(x, \mathrm{d}y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y) g(x) K_{t,n}(x, \mathrm{d}y).\end{aligned}$$

It suffices to show that for every $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y) g(x) K_{t,n}(x, \mathrm{d}y) = \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x) g(y) K_{t,n}(x, \mathrm{d}y). \tag{3.9}$$

We will prove it by induction. By definition, symmetry of p^D and Tonelli's theorem, we have

$$\begin{aligned}&\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y) g(x) K_{t,0}(x, \mathrm{d}y) \\ &= \int_D \mathrm{d}x \int_D \mathrm{d}y\, f(y) g(x) p_t^D(x, y) + \int_{D^c} \mathrm{d}x \int_{D^c} \delta_x(\mathrm{d}y) e^{-\nu(x,D)t} f(y) g(x) \\ &= \int_D \mathrm{d}x \int_D \mathrm{d}y\, f(y) g(x) p_t^D(y, x) + \int_{D^c} f(x) g(x) e^{-\nu(x,D)t} \,\mathrm{d}x \\ &= \int_D \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}x\, f(y) g(x) p_t^D(y, x) + \int_{D^c} \mathrm{d}y \int_{\mathbb{R}} \delta_y(\mathrm{d}x) e^{-\nu(y,D)t} f(y) g(x) \\ &= \int_D \mathrm{d}y \int_{\mathbb{R}} f(y) g(x) K_{t,0}(y, \mathrm{d}x) + \int_{D^c} \mathrm{d}y \int_{\mathbb{R}} f(y) g(x) K_{t,0}(y, \mathrm{d}x) \\ &= \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} f(y) g(x) K_{t,0}(y, \mathrm{d}x) \\ &= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x) g(y) K_{t,0}(x, \mathrm{d}y).\end{aligned}$$

Assume that (3.9) holds for some $n \geq 0$. We will prove that (3.9) holds for $n + 1$. From

Lemma 3.3, the Tonelli's theorem, the assumption on n and symmetry of $\widehat{\nu}$,

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y)g(x)K_{t,n+1}(x, \mathrm{d}y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y)g(x)K_{r,n}\widehat{\nu}\widehat{P}_{t-r}(x, \mathrm{d}y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} g(x)\big(\widehat{\nu}\widehat{P}_{t-r}f\big)(y)K_{r,n}(x, \mathrm{d}y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} g(y)\big(\widehat{\nu}\widehat{P}_{t-r}f\big)(x)K_{r,n}(x, \mathrm{d}y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \,\big(\widehat{\nu}\widehat{P}_{t-r}f\big)(x)\big(K_{r,n}g\big)(x) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y\, \widehat{\nu}(x, \mathrm{d}y)\big(\widehat{P}_{t-r}f\big)(y)\big(K_{r,n}g\big)(x) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y\, \widehat{\nu}(x, \mathrm{d}y)\big(\widehat{P}_{t-r}f\big)(x)\big(K_{r,n}g\big)(y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x \,\big(\widehat{P}_{t-r}f\big)(x)\big(\widehat{\nu}K_{r,n}g\big)(x).
\end{aligned}$$

From the case for $n = 0$ and the fact that $K_{t,0} = \widehat{P}_t$ we get that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y)g(x)K_{t,n+1}(x, \mathrm{d}y) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x\, f(x)\big(\widehat{P}_{t-r}\widehat{\nu}K_{r,n}g\big)(x) \\
&= \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}x\, f(x)\big(\widehat{P}_r\widehat{\nu}K_{t-r,n}g\big)(x),
\end{aligned}$$

and again from Lemma 3.3 we obtain,

$$\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(y)g(x)K_{t,n+1}(x, \mathrm{d}y) = \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} f(x)g(y)K_{t,n+1}(x, \mathrm{d}y),$$

which completes the proof. □

From the above lemma we have the following corollary.

Corollary 3.6. *For $t > 0$ and $f, g \in \mathcal{B}_b(\mathbb{R}) \cap L^2(\mathbb{R})$,*

$$\int_{\mathbb{R}} (K_t f)(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)(K_t g)(x)\,\mathrm{d}x.$$

Proof. The result follows directly from Lemma 3.5 by decomposition of a function into positive and negative parts. Indeed, let $f = f^+ - f^-$ and $g = g^+ - g^-$. Then, of course, all the functions f^+, f^-, g^+ and g^- are non-negative. Hence, from Lemma 3.5 we get

$$\begin{aligned}
&\int_{\mathbb{R}} (K_t f)(x) g(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \big(K_t(f^+ - f^-)\big)(x)(g^+(x) - g^-(x))\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \big[(K_t f^+)(x) g^+(x) - (K_t f^+)(x) g^-(x) - (K_t f^-)(x) g^+(x) + (K_t f^-)(x) g^-(x)\big]\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \big[f^+(x)(K_t g^+)(x) - f^+(x)(K_t g^-)(x) - f^-(x)(K_t g^+)(x) + f^-(x)(K_t g^-)(x)\big]\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \big[(f^+(x) - f^-(x))(K_t(g^+ - g^-))(x)\big]\,\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)(K_t g)(x)\,\mathrm{d}x. \qquad \square
\end{aligned}$$

Theorem 3.7. *For $x \neq 0$, $t > 0$ and $f \in \mathcal{B}_b^+(\mathbb{R}^*)$, we have*

$$\mathbb{E}_x f(X_t) = K_t f(x). \tag{3.10}$$

In order to prove Theorem 3.7 we need the following auxiliary results.

For $n = 1, 2, \ldots$, $x \in \mathbb{R}$ and Borel sets $R \subset [0, \infty)$ and $A \subset \mathbb{R} \setminus \{0\}$, we define

$$P_n(x, R, A) = \mathbb{P}_x(R_n \in R, X_{R_n} \in A),$$

the distribution of the pair (R_n, X_{R_n}).

Lemma 3.8. *For $x \neq 0$, $R \subset [0, \infty)$ and $A \subset \mathbb{R} \setminus \{0\}$,*

$$\begin{aligned}
P_1(x, R, A) &= \int_R \mathrm{d}s \int_A (\widehat{P}_s \widehat{\nu})(x, \mathrm{d}a), \\
P_{n+1}(x, R, A) &= \iint P_n(x, \mathrm{d}s, \mathrm{d}v) \int_R \mathrm{d}r \int_A (\widehat{P}_{r-s} \widehat{\nu})(v, \mathrm{d}a) \mathbb{1}_{\{s<r\}}, \quad n \geq 1.
\end{aligned}$$

Proof. Let $f(x) := \mathbb{1}_A(x)$ and note that $Kf(x) = K(x, A)$. From the construction of the process X, $R_n < \infty$ a.e., $n = 1, 2, \ldots$. Moreover, from the definition it is obvious that $R_1 \in \mathcal{F}_{R_1^-}$ (see e.g. Chung and Walsh [22, p. 16] or Sharpe [61, Exercise 6.19]) and therefore from (3.2),

$$\begin{aligned}
P_1(x, R, A) &= \mathbb{P}_x(R_1 \in R, X_{R_1} \in A) \\
&= \mathbb{E}_x\big[\mathbb{E}_x\big[\,\mathbb{1}_R\,(R_1)\,\mathbb{1}_A\,(X_{R_1})\,\big|\,\mathcal{F}_{R_1^-}\big]\big] \\
&= \mathbb{E}_x\big[\,\mathbb{1}_R\,(R_1)\mathbb{E}_x\big[f(X_{R_1})\,\big|\,\mathcal{F}_{R_1^-}\big]\big] \\
&= \mathbb{E}_x\big[\,\mathbb{1}_R\,(R_1) K f \circ \pi^1\big] \\
&= \mathbb{E}_x\big[\,\mathbb{1}_R\,(R_1(\omega)) K(\pi^1(\omega), A)\big].
\end{aligned}$$

For $x > 0$, from (3.1) and from the Ikeda–Watanabe formula (2.25),

$$\begin{aligned}
P_1(x,R,A) &= \mathbb{E}_x\big[\,\mathbb{1}_R\,(\zeta^{(1)}(\omega))k(X^1_{\zeta^{(1)}-}(\omega_1),A)\big]\\
&= \mathbb{E}^1_x\big[\,\mathbb{1}_R\,(\tau_D)k(Z^1_{\tau_D-},A)\big]\\
&= \int_R \mathrm{d}s \int_D \mathrm{d}y\, p^D_s(x,y)\nu(y,D^c)k(y,A)\\
&= \int_R \mathrm{d}s \int_D \mathrm{d}y\, p^D_s(x,y)\nu(y,A\cap D^c)\\
&= \int_R \mathrm{d}s \int_D \mathrm{d}y \int_A \mathrm{d}a\, p^D_s(x,y)\widehat{\nu}(y,a)\\
&= \int_R \mathrm{d}s \int_A (\widehat{P}_s\widehat{\nu})(x,\mathrm{d}a).
\end{aligned}$$

Similarly, for $x < 0$,

$$\begin{aligned}
P_1(x,R,A) &= \mathbb{E}_x\big[\,\mathbb{1}_R\,(\zeta^{(1)}(\omega))k(X^1_{\zeta^{(1)}-}(\omega_1),A)\big]\\
&= \mathbb{E}^2_x\big[\,\mathbb{1}_R\,(\zeta^Y_2)k(Z^2_{\zeta^Z_2-},A)\big]\\
&= k(x,A)\mathbb{P}^2_x\big[\zeta^Z_2\in R\big]\\
&= \int_R \mathrm{d}s \int_A \mathrm{d}a\, e^{-\nu(x,D)s}\widehat{\nu}(x,a)\\
&= \int_R \mathrm{d}s \int_A \mathrm{d}a\, (\widehat{P}_s\widehat{\nu})(x,\mathrm{d}a).
\end{aligned}$$

Furthermore, for $n \geq 1$,

$$\begin{aligned}
P_{n+1}(x,R,A) &= \mathbb{P}_x(R_{n+1}\in R, X_{R_{n+1}}\in A)\\
&= \mathbb{E}_x\big[\mathbb{E}_x\big[\,\mathbb{1}_{\{R_n+\zeta^{(1)}\circ\theta_{R_n}\in R\}}\,\mathbb{1}_{\{X_{R_{n+1}}\in A\}}\big|\mathcal{F}_{R_n}\big]\big]\\
&= \int_\Omega \mathbb{E}_x\big[\,\mathbb{1}_R\,(R_n+\zeta^{(1)}\circ\theta_{R_n})\,\mathbb{1}_A\,(X_{R_{n+1}})\big|\mathcal{F}_{R_n}\big](\omega)\mathbb{P}_x(\mathrm{d}\,\omega).
\end{aligned}$$

Let $G(\omega,\omega') := \mathbb{1}_R(R_n(\omega)+\zeta^{(1)}(\omega'))\,\mathbb{1}_A\,(X_{\zeta^{(1)}(\omega')}(\omega'))$, $\omega,\omega'\in\Omega$. Note that for $\omega' := \theta_{R_n}\omega$,

$$X_{\zeta^{(1)}(\omega')}(\omega') = X_{\zeta^{(1)}\circ\theta_{R_n}(\omega)}(\theta_{R_n}\omega) = X_{R_{n+1}(\omega)}(\omega).$$

Moreover, for $H(\omega) := G(\omega,\theta_{R_n}\omega)$ we have,

$$\mathbb{E}_x[H|\mathcal{F}_{R_n}](\omega) = \mathbb{E}_x\big[\,\mathbb{1}_R\,(R_n+\zeta^{(1)}\circ\theta_{R_n})\,\mathbb{1}_A\,(X_{R_{n+1}})\big|\mathcal{F}_{R_n}\big](\omega).$$

Hence,

$$P_{n+1}(x,R,A) = \int_\Omega \mathbb{E}_x[H|\mathcal{F}_{R_n}](\omega)\mathbb{P}_x(\mathrm{d}\,\omega).$$

From the strong Markov property [7, Exercise 8.16], we obtain that

$$\begin{aligned}
P_{n+1}(x,R,A) &= \int_\Omega \int_\Omega G(\omega,\omega')\mathbb{P}_{X_{R_n}(\omega)}(\mathrm{d}\,\omega')\mathbb{P}_x(\mathrm{d}\,\omega) \\
&= \int_\Omega \int_\Omega \mathbb{1}_R(R_n(\omega)+\zeta^{(1)}(\omega'))\,\mathbb{1}_A\,(X_{\zeta^{(1)}(\omega')}(\omega'))\mathbb{P}_{X_{R_n}(\omega)}(\mathrm{d}\,\omega')\mathbb{P}_x(\mathrm{d}\,\omega) \\
&= \int_\Omega \mathbb{E}_{X_{R_n}(\omega)}\big[\,\mathbb{1}_R\,(R_n(\omega)+\zeta^{(1)})\,\mathbb{1}_A\,(X_{\zeta^{(1)}})\big]\mathbb{P}_x(\mathrm{d}\,\omega) \\
&= \mathbb{E}_x\big[\mathbb{E}_{X_{R_n}}\big[\,\mathbb{1}_R\,(s+\zeta^{(1)})\,\mathbb{1}_A\,(X_{\zeta^{(1)}})\big]\big|_{s=R_n}\big] \\
&= \mathbb{E}_x\big[\mathbb{P}_{X_{R_n}}\big(\zeta^{(1)}+s\in R, X_{\zeta^{(1)}}\in A\big)\big|_{s=R_n}\big] \\
&= \iint P_n(x,\mathrm{d}s,\mathrm{d}v)\mathbb{P}_v(\zeta^{(1)}\in(R-s)\cap(0,\infty), X_{\zeta^{(1)}}\in A).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathbb{P}_v(\zeta^{(1)}\in(R-s)\cap(0,\infty), X_{\zeta^{(1)}}\in A) &= P_1(v,(R-s)\cap(0,\infty),A) \\
&= \int_{(R-s)\cap(0,\infty)} \mathrm{d}r \int_A (\widehat{P}_r\widehat{\nu})(v,\mathrm{d}a) \\
&= \int_R \mathrm{d}r \int_A (\widehat{P}_{r-s}\widehat{\nu})(v,\mathrm{d}a)\,\mathbb{1}_{\{s<r\}}\,.
\end{aligned}$$

Therefore,

$$P_{n+1}(x,R,A) = \iint P_n(x,\mathrm{d}s,\mathrm{d}v)\int_R \mathrm{d}r \int_A (\widehat{P}_{r-s}\widehat{\nu})(v,\mathrm{d}a)\mathbb{1}_{\{s<r\}},$$

which is the desired conclusion. □

Lemma 3.9. *For $x\neq 0$ and $n\geq 2$.*

$$P_n(x,\mathrm{d}r,\mathrm{d}a) = \underset{0<t_1<\ldots<t_{n-1}<r}{\int\cdots\int} (\widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{r-t_{n-1}}\widehat{\nu})(x,\mathrm{d}a)\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}\,\mathrm{d}r. \tag{3.11}$$

Proof. From Lemma 3.8 we have

$$\begin{aligned}
P_2(x,\mathrm{d}r,\mathrm{d}a) &= \iint P_1(x,\mathrm{d}s,\mathrm{d}v)(\widehat{P}_{r-s}\widehat{\nu})(v,\mathrm{d}a)\,\mathbb{1}_{\{s<r\}}\,\mathrm{d}r \\
&= \iint (\widehat{P}_s\widehat{\nu})(x,\mathrm{d}v)\,\mathrm{d}s(\widehat{P}_{r-s}\widehat{\nu})(v,\mathrm{d}a)\,\mathbb{1}_{\{s<r\}}\,\mathrm{d}r \\
&= \int_0^r (\widehat{P}_s\widehat{\nu}\widehat{P}_{r-s}\widehat{\nu})(x,\mathrm{d}a)\,\mathrm{d}s\,\mathrm{d}r.
\end{aligned}$$

For $n>2$ we use induction. Assume that the equality (3.11) holds for some $n\geq 2$. From

Lemma 3.8 we then have

$$
\begin{aligned}
&P_{n+1}(x, \mathrm{d}r, \mathrm{d}a)\\
&= \iint P_n(x, \mathrm{d}s, \mathrm{d}v)(\widehat{P}_{r-s}\widehat{\nu})(v, \mathrm{d}a)\, \mathbb{1}_{\{s<r\}}\, \mathrm{d}r\\
&= \int_0^r \mathrm{d}s \underset{0<t_1<\ldots<t_{n-1}<s}{\int\cdots\int} \int (\widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{s-t_{n-1}}\widehat{\nu})(x, \mathrm{d}v)\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}(\widehat{P}_{r-s}\widehat{\nu})(v, \mathrm{d}a)\mathrm{d}r\\
&= \int_0^r \mathrm{d}s \underset{0<t_1<\ldots<t_{n-1}<s}{\int\cdots\int} (\widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{s-t_{n-1}}\widehat{\nu}\widehat{P}_{r-s}\widehat{\nu})(x, \mathrm{d}a)\mathrm{d}t_1\ldots\mathrm{d}t_{n-1}\mathrm{d}r\\
&= \underset{0<t_1<\ldots<t_n<r}{\int\cdots\int} (\widehat{P}_{t_1}\widehat{\nu}\widehat{P}_{t_2-t_1}\widehat{\nu}\ldots\widehat{\nu}\widehat{P}_{t_n-t_{n-1}}\widehat{\nu}\widehat{P}_{r-t_n}\widehat{\nu})(x, \mathrm{d}a)\mathrm{d}t_1\ldots\mathrm{d}t_n\mathrm{d}r,
\end{aligned}
$$

which ends the proof. □

Proof of Theorem 3.7. Let $x \neq 0$, $t > 0$ and $f \in \mathcal{B}_b^+(\mathbb{R})$. Then we have

$$\mathbb{E}_x f(X_t) = \mathbb{E}_x[f(X_t), 0 \le t < R_1] + \sum_{n=1}^{\infty} \mathbb{E}_x[f(X_t), R_n \le t < R_{n+1}].$$

Obviously, for $x > 0$,

$$\mathbb{E}_x[f(X_t), 0 \le t < R_1] = \mathbb{E}_x^1[f(Z_t^1), 0 \le t < \tau_D] = \int_D f(y)p_t^D(x, y)\, \mathrm{d}y = \widehat{P}_t f(x).$$

Similarly, for $x < 0$,

$$
\begin{aligned}
\mathbb{E}_x[f(X_t), 0 \le t < R_1] = \mathbb{E}_x^2[f(Z_t^2), 0 \le t < \zeta_2^Z] &= \int_{D^c} \delta_x(\mathrm{d}y)e^{-\nu(x,D)t} f(y)\\
&= f(x)e^{-\nu(x,D)t} = \widehat{P}_t f(x).
\end{aligned}
$$

Now, let $I_n(t, x) := \mathbb{E}_x[f(X_t), R_n \le t < R_{n+1}]$, $n \ge 1$. We will show that $I_n(t, x) = K_{t,n} f(x)$. Indeed, using the same method as in the proof of Lemma 3.8, involving the use of [7, Exercise 8.16], we obtain that

$$
\begin{aligned}
I_n(t, x) &= \mathbb{E}_x\big[\mathbb{E}_x[f(X_t), R_n \le t < R_{n+1} \big| \mathcal{F}_{R_n}]\big]\\
&= \mathbb{E}_x\big[\mathbb{E}_x[f(X_{t-R_n} \circ \theta_{R_n}), t < R_n + \zeta^{(1)} \circ \theta_{R_n} \big| \mathcal{F}_{R_n}], R_n \le t\big]\\
&= \mathbb{E}_x\big[\mathbb{E}_{X_{R_n}}[f(X_{t-s}, t - s < \zeta^{(1)}]\big|_{s=R_n}, R_n \le t\big]\\
&= \iint P_n(x, \mathrm{d}s, \mathrm{d}v)\mathbb{E}_v[f(X_{t-s}), t - s < \zeta^{(1)}]\mathbb{1}_{\{s \le t\}}\\
&= \iint P_n(x, \mathrm{d}s, \mathrm{d}v)\widehat{P}_{t-s} f(v)\, \mathbb{1}_{\{s \le t\}}\,. \qquad (3.12)
\end{aligned}
$$

From Lemma 3.8 and (3.5),

$$I_1(t, x) = \int_0^t \mathrm{d}s \int (\widehat{P}_s\widehat{\nu})(x, \mathrm{d}v)\widehat{P}_{t-s} f(v) = \int_0^t (\widehat{P}_s\widehat{\nu}\widehat{P}_{t-s} f)(x)\, \mathrm{d}s = K_{t,1} f(x).$$

It remains to show our claim for $n \geq 2$. From (3.12), Lemma 3.9 and (3.5) it is easy to verify that

$$\begin{aligned} I_n(t,x) &= \iint P_n(x, \mathrm{d}s, \mathrm{d}v) \widehat{P}_{t-s} f(v) \mathbb{1}_{\{s \leq t\}} \\ &= \int_0^t \mathrm{d}s \underset{0<t_1<\ldots<t_{n-1}<s}{\int \cdots \int} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} \int (\widehat{P}_{t_1} \widehat{\nu} \widehat{P}_{t_2-t_1} \widehat{\nu} \ldots \widehat{\nu} \widehat{P}_{s-t_{n-1}} \widehat{\nu})(x, \mathrm{d}v) \widehat{P}_{t-s} f(v) \\ &= \int_0^t \mathrm{d}s \underset{0<t_1<\ldots<t_{n-1}<s}{\int \cdots \int} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1} (\widehat{P}_{t_1} \widehat{\nu} \widehat{P}_{t_2-t_1} \widehat{\nu} \ldots \widehat{\nu} \widehat{P}_{s-t_{n-1}} \widehat{\nu} \widehat{P}_{t-s} f)(x) \\ &= \underset{0<t_1<t_2<\ldots<t_n<t}{\int \cdots \int} (\widehat{P}_{t_1} \widehat{\nu} \widehat{P}_{t_2-t_1} \widehat{\nu} \ldots \widehat{\nu} \widehat{P}_{t-t_n} f)(x) \, \mathrm{d}t_1 \ldots \mathrm{d}t_n \\ &= K_{t,n} f(x), \end{aligned}$$

which completes the proof. □

Lemma 3.10. *The family $(K_t)_{t\geq 0}$ is a semigroup of subprobability transition kernels on $\mathbb{R}^*$, i.e. for $t \geq 0$, K_t is a transition kernel such that for $x \neq 0$, $K_t(x, \mathbb{R}^*) = K_t \mathbf{1}(x) \leq 1$ and $K_{t+s} = K_t K_s$, $s, t \geq 0$.*

Proof. At first, we will prove the subprobability property. It suffices to show (by induction) that for any $N = 0, 1, 2, \ldots$ and $t > 0$, $S_N(x,t) := \sum_{n=0}^N K_{t,n} \mathbf{1}(x) \leq 1$. For $x > 0$ from (2.18) it follows that for any $t > 0$, $S_0(x,t) = K_{t,0} \mathbf{1}(x) = p_t^D(x, D) = \mathbb{P}_x^Y(\tau_D > t) \leq 1$. For $x < 0$ it is obvious that for any $t > 0$ we have $S_0(x,t) = K_{t,0}\mathbf{1}(x) = e^{-\nu(x,D)t} \leq 1$. Hence, $S_0(x,t) \leq 1$ for $x \neq 0$, $t > 0$.

Assume that for some $N \in \mathbb{N}$ and all $t > 0$, $x \neq 0$ we have $S_N(x,t) \leq 1$. Then, from Lemma 3.3,

$$S_{N+1}(x,t) = \widehat{P}_t \mathbf{1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} S_N(x, t-r) \, \mathrm{d}r \leq \widehat{P}_t \mathbf{1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \mathbf{1}(x) \, \mathrm{d}r.$$

For $x > 0$ from (2.26) and (2.25) we have

$$S_{N+1}(x,t) \leq \mathbb{P}_x^Y(\tau_D > t) + \mathbb{P}_x^Y(\tau_D \leq t) = 1.$$

In case $x < 0$ we have

$$\begin{aligned} S_{N+1}(x,t) &\leq e^{-\nu(x,D)t} + \int_0^t \mathrm{d}s \int_D \mathrm{d}y \, e^{-\nu(x,D)s} \nu(x,y) \\ &= \int_t^\infty \nu(x,D) e^{-\nu(x,D)s} \mathrm{d}s + \int_0^t \nu(x,D) e^{-\nu(x,D)s} \mathrm{d}s = 1, \end{aligned}$$

which ends the induction.

The fact that the transition kernels $(K_t)_{t\geq 0}$ satisfy the Chapman–Kolmogorov equation follows directly from Theorem 3.7. Indeed, from the Markov property and from mentioned theorem we have the following Chapman–Kolmogorov equation for K_t: for $f \in \mathcal{B}_b^+(\mathbb{R}^*)$, $s, t > 0$ and $x \neq 0$ we have

$$\begin{aligned} K_{t+s}f(x) &= \mathbb{E}_x f(X_{t+s}) = \mathbb{E}_x\big[\mathbb{E}_x\big(f(X_s)\circ\theta_t \mid \mathcal{F}_t\big)\big] \\ &= \mathbb{E}_x\big[\mathbb{E}_{X_t} f(X_s)\big] = \mathbb{E}_x\big[K_s f(X_t)\big] = K_t K_s f(x). \end{aligned}$$

Further, for $f = \mathbb{1}_A$, $A \subset \mathbb{R}^*$, we get the equality

$$K_{t+s}(x, A) = K_{t+s}\,\mathbb{1}_A\,(x) = K_t K_s\,\mathbb{1}_A\,(x) = \int_{\mathbb{R}} K_t(x, \mathrm{d}y) K_s(y, A), \tag{3.13}$$

which proves the lemma. □

For a direct proof of equality (3.13) see methods proposed in [17].

The following Corollary is a more general version of Theorem 3.7. We extend here the equality (3.10) to the space $\mathcal{B}_b(\mathbb{R}^*)$.

Corollary 3.11. *For $x \neq 0$, $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^*)$, we have*

$$\mathbb{E}_x f(X_t) = K_t f(x).$$

Proof. Let $f \in \mathcal{B}_b(\mathbb{R}^*)$. Then there exist $u, v \in \mathcal{B}_b^+(\mathbb{R}^*)$ such that $f = u - v$. From Theorem 3.7 and Lemma 3.10 we have

$$\mathbb{E}_x u(X_t) = K_t u(x) \leq \|u\|_\infty K_t \mathbf{1}(x) < \infty.$$

Therefore, from Theorem 3.7 we get

$$\mathbb{E}_x f(X_t) = \mathbb{E}_x u(X_t) - \mathbb{E}_x v(X_t) = K_t u(x) - K_t v(x) = K_t f(x). \qquad \square$$

Lemma 3.12. *For $u \in \mathcal{B}_b^+(\mathbb{R})$,*

$$(K_t u)^2(x) \leq (K_t u^2)(x), \qquad x \in \mathbb{R}^*,\ t > 0.$$

Proof. From the Cauchy–Schwarz inequality and Lemma 3.10 we get

$$\begin{aligned} (K_t u)^2(x) &= \Big[\int_{\mathbb{R}} u(y) K_t(x, \mathrm{d}y)\Big]^2 \leq \Big[\int_{\mathbb{R}} u^2(y) K_t(x, \mathrm{d}y)\Big]\Big[\int_{\mathbb{R}} K_t(x, \mathrm{d}y)\Big] \\ &\leq \int_{\mathbb{R}} u^2(y) K_t(x, \mathrm{d}y) = (K_t u^2)(x), \end{aligned}$$

which completes the proof. □

Lemma 3.13. *For $t > 0$ and $u \in \mathcal{B}_b(\mathbb{R}) \cap L^2(\mathbb{R})$,*

$$\|K_t u\|_{L^2(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}.$$

Proof. From Lemma 3.12, Corollary 3.6 and Lemma 3.10 we obtain

$$\|K_t u\|_{L^2(\mathbb{R})}^2 \le \int_{\mathbb{R}} |(K_t|u|)(x)|^2 \,\mathrm{d}x \le \int_{\mathbb{R}} (K_t u^2)(x)\,\mathrm{d}x = \int_{\mathbb{R}} u^2(x) K_t \mathbf{1}(x)\,\mathrm{d}x \le \|u\|_{L^2(\mathbb{R})}^2 ,$$

which proves the lemma. □

The above result implies that for a fixed $t > 0$ operator K_t is a bounded linear operator on the space $\mathcal{B}_b(\mathbb{R}) \cap L^2(\mathbb{R})$ which is a dense subspace of $L^2(\mathbb{R})$. Hence, K_t can be uniquely extended to a linear contraction on $L^2(\mathbb{R})$. Such extension we will also denote by K_t.

Theorem 3.14. *$(K_t)_{t\ge 0}$ is a strongly continuous semigroup on $L^2(\mathbb{R})$, i.e. for $t, s \ge 0$, $K_{t+s}u = K_t K_s u$ in $L^2(\mathbb{R})$ and*

$$\lim_{t\to 0^+} \|K_t u - u\|_{L^2(\mathbb{R})} = 0, \qquad u \in L^2(\mathbb{R}).$$

Proof. Let $u \in L^2(\mathbb{R})$ and assume that a sequence $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^*)$ is convergent to u in $L^2(\mathbb{R})$, i.e. $\|u - u_n\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$. Then from Lemma 3.10 for $x \ne 0$ we have

$$K_{t+s}u_n(x) = \int_{\mathbb{R}} K_{t+s}(x, \mathrm{d}y) u_n(y) = \int_{\mathbb{R}} (K_t K_s)(x, \mathrm{d}y) u_n(y) = K_t K_s u_n(x).$$

From Lemma 3.13 we know that K_{t+s}, K_t and K_s are bounded operators on $L^2(\mathbb{R})$, hence continuous. Therefore, $\|K_{t+s}u_n - K_{t+s}u\|_{L^2(\mathbb{R})} \to 0$ and $\|K_t K_s u_n - K_t K_s u\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$. As a result $K_{t+s}u = K_t K_s u$ in $L^2(\mathbb{R})$.

Now we will prove the strong continuity. Since the family $(K_t)_{t\ge 0}$ forms a contraction semigroup of operators, then from Proposition 1.3 in Engel and Nagel [32, Section 1] it suffices to show a desired convergence only for $u \in C_c^\infty(\mathbb{R}^*)$, because $C_c^\infty(\mathbb{R}^*)$ is a dense subspace of $L^2(\mathbb{R})$.

First, we will show that for $u \in \mathcal{B}_+(\mathbb{R}) \cap C_c^\infty(\mathbb{R}^*)$ and $x \ne 0$,

$$\lim_{t\to 0^+} K_t u(x) = u(x). \tag{3.14}$$

Let $t \to 0^+$. From Theorem 2.3 it is obvious that

$$K_{t,0}u(x) = \mathbf{1}_D(x) \int_D p_t^D(x, \mathrm{d}y) u(y)\,\mathrm{d}y + \mathbf{1}_{D^c}(x) u(x) e^{-\nu(x,D)t} \to u(x).$$

Furthermore, from Corollary 3.4 and Lemma 3.10,

$$\sum_{n=1}^\infty K_{t,n}u(x) \le \|u\|_\infty \int_0^t \widehat{P}_r \widehat{\nu} \mathbf{1}(x)\,\mathrm{d}r.$$

Then for $x > 0$,

$$\int_0^t \widehat{P}_r \widehat{\nu} \mathbf{1}(x)\,\mathrm{d}r = \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x, y) \nu(y, z) = \mathbb{P}_x(\tau_D^Y \le t) \to 0.$$

Similarly, for $x < 0$,

$$\int_0^t \widehat{P}_r \widehat{\nu}\mathbf{1}(x)\,\mathrm{d}r = \int_0^t \nu(x,D)e^{-\nu(x,D)r}\,\mathrm{d}r \to 0.$$

Using decomposition of the signed function into positive and negative parts, we obtain convergence $K_t u(x) \to u(x)$, $t \to 0^+$, for arbitrary (signed) function $u \in C_c^\infty(\mathbb{R}^*)$.

Now let $u \in C_c^\infty(\mathbb{R}^*)$. From Lemma 3.13,

$$\begin{aligned}
\|K_t u - u\|^2_{L^2(\mathbb{R})} &= \|K_t u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})} - 2\langle K_t u, u\rangle \\
&\le 2\big[\,\|u\|^2_{L^2(\mathbb{R})} - \langle K_t u, u\rangle\big] \\
&= 2\langle u - K_t u, u\rangle \\
&= 2\int_{\mathbb{R}\cap \operatorname{supp} u} \big[u(x) - K_t u(x)\big] u(x)\,\mathrm{d}x.
\end{aligned}$$

From the dominated convergence theorem, we obtain the desired convergence. □

Lemma 3.15. *For $t > 0$, $x \neq 0$, $k > 0$, $A \subset \mathbb{R}$,*

(a) $K_{t,n}(kx, kA) = K_{tk^{-\alpha},n}(x, A)$, $n = 0, 1, 2, \ldots$,

(b) $K_t(kx, kA) = K_{tk^{-\alpha}}(x, A)$.

Proof. It suffices to show the equality (a). For $n = 0$, from (2.16), it follows that for $x > 0$,

$$\begin{aligned}
K_{t,0}(kx, kA) &= \int_{kA\cap D} p_t^D(kx, y)\,\mathrm{d}y = \int_{kA\cap D} k^{-1} p^D_{tk^{-\alpha}}(x, y/k)\,\mathrm{d}y \\
&= \int_{A\cap D} p^D_{tk^{-\alpha}}(x, z)\,\mathrm{d}z = K_{tk^{-\alpha},0}(x, A).
\end{aligned}$$

Similarly, for $x < 0$, from (2.11),

$$K_{t,0}(kx, kA) = \delta_{kx}(kA)e^{-\nu(kx,D)t} = \delta_x(A)e^{-\nu(x,D)tk^{-\alpha}} = K_{tk^{-\alpha},0}(x, A).$$

Assume that for some $n \in \{0, 1, \ldots\}$ we have $K_{t,n}(kx, kA) = K_{tk^{-\alpha},n}(x, A)$, $x \neq 0$, $t > 0$, $k > 0$.

Let $x > 0$. From Lemma 3.3, (2.16) and (2.10), for $n = 0, 1, \ldots$, we have the following equality

$$\begin{aligned}
K_{t,n+1}(kx, kA) &= \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(kx, y)\nu(y, z)K_{t-r,n}(z, kA) \\
&= \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, k^{-1} p^D_{rk^{-\alpha}}(x, y/k) k^{-\alpha-1}\nu(y/k, z/k)K_{t-r,n}(z, kA).
\end{aligned}$$

Using the substitution $s = rk^{-\alpha}$, $w = y/k$ and $v = z/k$ we get

$$K_{t,n+1}(kx, kA) = \int_0^{tk^{-\alpha}} \mathrm{d}s \int_D \mathrm{d}w \int_{D^c} \mathrm{d}v\, p_s^D(x, w)\nu(w, v)K_{t-sk^{\alpha},n}(kv, kA).$$

From the assumption,

$$\begin{aligned} K_{t,n+1}(kx, kA) &= \int_0^{tk^{-\alpha}} \mathrm{d}s \int_D \mathrm{d}w \int_{D^c} \mathrm{d}v\, p_s^D(x,w)\nu(w,v) K_{tk^{-\alpha}-s,n}(v,A) \\ &= K_{tk^{-\alpha},n+1}(x,A). \end{aligned}$$

Now, consider the case $x < 0$. Again, from Lemma 3.3, (2.10) and (2.11), for $n = 0, 1, \ldots$, we have the following equality

$$\begin{aligned} K_{t,n+1}(kx, kA) &= \int_0^t \mathrm{d}r \int_D \mathrm{d}z\, e^{-\nu(kx,D)r} \nu(kx,z) K_{t-r,n}(z, kA) \\ &= \int_0^t \mathrm{d}r \int_D \mathrm{d}z\, e^{-\nu(x,D)rk^{-\alpha}} k^{-\alpha-1} \nu(x, z/k) K_{t-r,n}(z, kA). \end{aligned}$$

Using the substitution $s = rk^{-\alpha}$ and $w = z/k$ we get

$$K_{t,n+1}(kx, kA) = \int_0^{tk^{-\alpha}} \mathrm{d}s \int_D \mathrm{d}w\, e^{-\nu(x,D)s} \nu(x,w) K_{t-sk^{\alpha},n}(kw, kA).$$

From the assumption,

$$K_{t,n+1}(kx, kA) = \int_0^{tk^{-\alpha}} \mathrm{d}s \int_D \mathrm{d}w\, e^{-\nu(x,D)s} \nu(x,w) K_{tk^{-\alpha}-s,n}(w, A) = K_{tk^{-\alpha},n+1}(x,A),$$

which is our claim. □

Theorem 3.16. *For $\alpha \in (0,2)$, $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^*)$, $K_t f \in C_b(\mathbb{R}^*)$.*

Proof. Let $f \in \mathcal{B}_b(\mathbb{R}^*)$ and $t > 0$. The boundedness of the function $K_t f$ follows immediately from Lemma 3.10. Therefore, it suffices to show that $K_t f$ is continuous on $\mathbb{R}^*$.

Assume that $x > 0$. Then $x \mapsto \widehat{P}_t f(x) = P_t^D f(x)$ is a continuous function on D, which follows directly from Theorem 2.1. From Corollary 3.4 it suffices to show a continuity of the function

$$D \ni x \mapsto F_t(x) := \int_0^t \widehat{P}_s \widehat{\nu} K_{t-s} f(x)\, \mathrm{d}s = \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_s^D(x,y)\nu(y,z) K_{t-s} f(z).$$

From Lemma 3.10, for $0 \le s \le t$, $y \in D$ and $z \in D^c$,

$$p_s^D(x,y)\nu(y,z)K_{t-s}f(z) \le \|f\|_\infty\, p_s^D(x,y)\nu(y,z). \tag{3.15}$$

Moreover, from Ikeda–Watanabe formula (2.25) and (2.26),

$$G_t(x) := \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_s^D(x,y)\nu(y,z) = 1 - p_t^D(x, D) = 1 - P_t^D \mathbf{1}(x).$$

Therefore, from Theorem 2.1, a function $D \ni x \mapsto G_t(x)$ is continuous. From the Vitali's theorem (see [57, Theorem 16.6]) it follows that a family $D \ni x \mapsto \|f\|_\infty\, p_t^D(x,y)\nu(y,z)$ is uniformly integrable with respect to the measure $\mathbb{1}_{[0,t]}(s)\,\mathbb{1}_D(y)\,\mathbb{1}_{D^c}(z)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z$. From (3.15) we obtain that a family $D \ni x \mapsto p_s^D(x,y)\nu(y,z)K_{t-s}f(z)$ is also uniformly integrable with respect to the measure $\mathbb{1}_{[0,t]}(s)\,\mathbb{1}_D\,(y)\,\mathbb{1}_{D^c}\,(z)\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}z$ (see [57, Definition 16.1]). Hence, because the function $D \ni x \mapsto p_s^D(x,y)$ is continuous, again from the Vitali's theorem, the function $D \ni x \mapsto F_t(x)$ is continuous.

Now let $x < 0$. Then, from the assumption, $\overline{D}^c \ni x \mapsto \widehat{P}_t f(x) = f(x)e^{-\nu(x,D)t}$ is a continuous function. We will show continuity of a function

$$\overline{D}^c \ni x \mapsto F_t(x) := \int_0^t \widehat{P}_s \widehat{\nu} K_{t-s} f(x)\,\mathrm{d}s = \int_0^t \mathrm{d}s \int_D \mathrm{d}y\, e^{-\nu(x,D)s}\nu(x,y)K_{t-s}f(y).$$

Again from Lemma 3.10, $K_{t-s}f(y) \le \|f\|_\infty$ and a function

$$\overline{D}^c \ni x \mapsto \|f\|_\infty \int_0^t \nu(x,D)e^{-\nu(x,D)s}\,\mathrm{d}s = \|f\|_\infty \left[1 - e^{-\nu(x,D)t}\right]$$

is continuous. Hence, from the Vitali's theorem, a family $\overline{D}^c \ni x \mapsto \|f\|_\infty e^{-\nu(x,D)s}\nu(x,y)$ is uniformly integrable with respect to the measure $\mathbb{1}_{[0,t]}(s)\,\mathbb{1}_D\,(y)\,\mathrm{d}s\,\mathrm{d}y$. Therefore, a family $\overline{D}^c \ni x \mapsto e^{-\nu(x,D)s}\nu(x,y)K_{t-s}f(y)$ is uniformly integrable with respect to the same measure and then, again from the Vitali's theorem, we obtain the continuity of the function $\overline{D}^c \ni x \mapsto F_t(x)$. □

3.3 Excessive functions

For $x \in \mathbb{R}$ and $\beta \in \mathbb{R}$, we define function $h_\beta(x) = |x|^\beta$.

Lemma 3.17. *For $x > 0$ and $t > 0$, $\widehat{P}_t h_{\alpha-1}(x) \le h_{\alpha-1}(x)$.*

Proof. Recall that $\alpha \in (0,2)$ and let $\gamma \in (0,1)$ and $\delta \in (\gamma + \alpha/2, 1 + \alpha/2)$. Let

$$g(x) := \int_0^\infty \mathrm{d}t \int_D \mathrm{d}y\, p_t^D(x,y)f(t)y^{-\delta}, \quad x > 0,$$

where $f(t) := \mathcal{C}^{-1}t^{(-\alpha/2-\gamma+\delta)/\alpha}\,\mathbb{1}_{(0,\infty)}\,(t)$ and

$$\mathcal{C} = \int_0^\infty \mathrm{d}t \int_D \mathrm{d}y\, p_t^D(1,y)t^{(-\alpha/2-\gamma+\delta)/\alpha}y^{-\delta}.$$

From Jakubowski and Maciocha [42, Lemma 3.6], for $x > 0$, we have $g(x) = x^{\alpha/2-\gamma}$ and by taking $\gamma = 1 - \alpha/2 \in (0,1)$ we obtain that $g(x) = h_{\alpha-1}(x)$, $x > 0$. Moreover, we have

$$\begin{aligned}
\widehat{P}_t h_{\alpha-1}(x) &= \int_0^\infty p_t^D(x, \mathrm{d}y) h_{\alpha-1}(y) \\
&= \int_0^\infty p_t^D(x, \mathrm{d}y) \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}z\, f(s) p_s^D(y,z) z^{-\delta} \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}z\, f(s) z^{-\delta} \int_0^\infty p_t^D(x, \mathrm{d}y) p_s^D(y,z) \\
&= \int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}z\, f(s) p_{t+s}^D(x,z) z^{-\delta} \\
&= \int_t^\infty \mathrm{d}u \int_0^\infty \mathrm{d}z\, f(u-t) p_u^D(x,z) z^{-\delta} \\
&= \mathcal{C}^{-1} \int_t^\infty \mathrm{d}u \int_0^\infty \mathrm{d}z\, (u-t)^{(\delta-1)/\alpha} p_u^D(x,z) z^{-\delta} \\
&\le \mathcal{C}^{-1} \int_t^\infty \mathrm{d}u \int_0^\infty \mathrm{d}z\, u^{(\delta-1)/\alpha} p_u^D(x,z) z^{-\delta} \\
&= \int_t^\infty \mathrm{d}u \int_0^\infty \mathrm{d}z\, f(u) p_u^D(x,z) z^{-\delta} \\
&\le h_{\alpha-1}(x).
\end{aligned}$$

which completes the proof. □

Lemma 3.18. *For $x \neq 0$,*

$$\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\, \mathrm{d}r = h_{\alpha-1}(x).$$

Proof. Let $x > 0$. Note that

$$\begin{aligned}
&\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\, \mathrm{d}r = \int_0^\infty \mathrm{d}r \int_0^\infty \mathrm{d}a \int_{-\infty}^0 \mathrm{d}y\, p_r^D(x,a) \nu(a,y) h_{\alpha-1}(y) \\
&= \int_{-\infty}^0 P_D(x,y) h_{\alpha-1}(y)\, \mathrm{d}y.
\end{aligned}$$

From (2.29) we have

$$\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\, \mathrm{d}r = \frac{\sin(\pi\alpha/2)}{\pi} x^{\alpha/2} \int_{-\infty}^0 \frac{|y|^{\alpha/2-1}}{|x-y|}\, \mathrm{d}y = \frac{\sin(\pi\alpha/2)}{\pi} x^{\alpha/2} \int_0^\infty \frac{y^{\alpha/2-1}}{x+y}\, \mathrm{d}y.$$

By changing variables $y = xz$ we obtain that

$$\begin{aligned}
\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\, \mathrm{d}r &= \frac{\sin(\pi\alpha/2)}{\pi} x^{\alpha-1} \int_0^\infty \frac{z^{\alpha/2-1}}{1+z} \mathrm{d}z = \frac{\sin(\pi\alpha/2)}{\pi} x^{\alpha-1} \mathfrak{B}(\alpha/2, 1-\alpha/2) \\
&= \frac{\sin(\pi\alpha/2)}{\pi} x^{\alpha-1} \Gamma(\alpha/2)\Gamma(1-\alpha/2) = h_{\alpha-1}(x).
\end{aligned}$$

Now, assume that $x < 0$. Then from (2.12),

$$\begin{aligned}
\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\, \mathrm{d}r &= \int_0^\infty e^{-\nu(x,D)r}\, \mathrm{d}r \int_0^\infty \nu(x,y) y^{\alpha-1}\, \mathrm{d}y = \frac{1}{\nu(x,D)} \int_0^\infty \nu(x,y) y^{\alpha-1}\, \mathrm{d}y \\
&= \alpha |x|^\alpha \int_0^\infty \frac{y^{\alpha-1}}{|x-y|^{\alpha+1}}\, \mathrm{d}y.
\end{aligned}$$

Using the substitution $y = |x|z$ we get

$$\int_0^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r = \alpha|x|^{\alpha-1} \int_0^\infty \frac{z^{\alpha-1}}{(z+1)^{\alpha+1}}\,\mathrm{d}z = \alpha|x|^{\alpha-1}\mathfrak{B}(\alpha,1) = h_{\alpha-1}(x),$$

which is our claim. □

Lemma 3.19. *For $x < 0$,*

$$\widehat{\nu} h_{\alpha-1}(x) = \nu(x,D) h_{\alpha-1}(x).$$

Proof. Note that

$$\widehat{\nu} h_{\alpha-1}(x) = \int_0^\infty \nu(x,y) h_{\alpha-1}(y)\,\mathrm{d}y = \mathcal{A}_{1,\alpha} \int_0^\infty \frac{y^{\alpha-1}}{\big|y+|x|\big|^{\alpha+1}}\,\mathrm{d}y.$$

By the substitution $y = |x|z$ and from (2.12) we get

$$\begin{aligned} \widehat{\nu} h_{\alpha-1}(x) &= \mathcal{A}_{1,\alpha}|x|^{\alpha-1}|x|^{-\alpha} \int_0^\infty \frac{z^{\alpha-1}}{(z+1)^{\alpha+1}}\,\mathrm{d}z = \mathcal{A}_{1,\alpha}|x|^{-\alpha}|x|^{\alpha-1}\mathfrak{B}(\alpha,1) \\ &= \mathcal{A}_{1,\alpha}\alpha^{-1}|x|^{-\alpha}|x|^{\alpha-1} = \nu(x,D) h_{\alpha-1}(x). \end{aligned}$$

This completes the proof. □

The next result is an analogue of Lemma 3.10.

Theorem 3.20. *For $\alpha \in (0,2)$, the function $h_{\alpha-1}(x) = |x|^{\alpha-1}$ is excessive for K_t, i.e.*

$$K_t h_{\alpha-1}(x) \le h_{\alpha-1}(x), \qquad t > 0,\ x \ne 0.$$

Proof. Let $x > 0$. We will show by induction that for any $N = 0, 1, 2, \ldots$ and $t > 0$,

$$S_N(x,t) := \sum_{n=0}^{N} K_{t,n} h_{\alpha-1}(x) \le h_{\alpha-1}(x). \tag{3.16}$$

Let $N = 0$. From Lemma 3.17 it follows that

$$S_0(x,t) = \widehat{P}_t h_{\alpha-1}(x) \le h_{\alpha-1}(x).$$

Using Lemma 3.18 we obtain

$$\widehat{P}_t h_{\alpha-1}(x) = \int_0^\infty \widehat{P}_t \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r = \int_0^\infty \widehat{P}_{t+r} \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r = \int_t^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r, \tag{3.17}$$

and then

$$\begin{aligned} S_1(x,t) &= K_{t,0} h_{\alpha-1}(x) + K_{t,1} h_{\alpha-1}(x) \\ &= \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r} h_{\alpha-1}(x)\,\mathrm{d}r \\ &\le \int_t^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r + \int_0^t \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r \\ &= h_{\alpha-1}(x). \end{aligned}$$

We will show that

$$\begin{aligned}\mathcal{G}(x,t) &:= \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r} h_{\alpha-1}(x)\,\mathrm{d}r + \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s\, \widehat{P}_r \widehat{\nu} \widehat{P}_{s-r} \widehat{\nu} h_{\alpha-1}(x) \\ &= h_{\alpha-1}(x). \qquad (3.18)\end{aligned}$$

Note that for $b < 0$ and $s > 0$,

$$\widehat{P}_s h_{\alpha-1}(b) = e^{-\nu(b,D)s} h_{\alpha-1}(b) = h_{\alpha-1}(b) \int_s^\infty \nu(b,D) e^{-\nu(b,D)u}\,\mathrm{d}u. \qquad (3.19)$$

From (3.17), (3.19), Lemma 3.19 and Lemma 3.18,

$$\begin{aligned}\mathcal{G}(x,t) &= \int_t^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r \\ &\quad + \int_0^t \mathrm{d}r \int_D \widehat{P}_r(x,\mathrm{d}a) \int_{D^c} \widehat{\nu}(a,\mathrm{d}b) h_{\alpha-1}(b) \int_t^\infty \nu(b,D) e^{-\nu(b,D)(s-r)}\,\mathrm{d}s \\ &\quad + \int_0^t \mathrm{d}r \int_D \widehat{P}_r(x,\mathrm{d}a) \int_{D^c} \widehat{\nu}(a,\mathrm{d}b) \int_r^t \mathrm{d}s\, e^{-\nu(b,D)(s-r)} \widehat{\nu} h_{\alpha-1}(b) \\ &= \int_t^\infty \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r + \int_0^t \widehat{P}_r \widehat{\nu} h_{\alpha-1}(x)\,\mathrm{d}r = h_{\alpha-1}(x).\end{aligned}$$

Assume that for some $N \in \{0,1,2,\ldots\}$ and all $t > 0$ we have $S_N(x,t) \le h_{\alpha-1}(x)$. We will show that $S_{N+2}(x,t) \le h_{\alpha-1}(x)$. Using (3.6) and (3.18), we obtain

$$\begin{aligned}&S_{N+2}(x,t) \\ &= K_{t,0} h_{\alpha-1}(x) + K_{t,1} h_{\alpha-1}(x) + \sum_{n=0}^N K_{t,n+2} h_{\alpha-1}(x) \\ &= \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r} h_{\alpha-1}(x)\,\mathrm{d}r + \sum_{n=0}^N \int_0^t \mathrm{d}r \int_0^{t-r} \mathrm{d}s\, \widehat{P}_r \widehat{\nu} \widehat{P}_s \widehat{\nu} K_{t-r-s,n} h_{\alpha-1}(x) \\ &= \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r} h_{\alpha-1}(x)\,\mathrm{d}r + \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s\, \widehat{P}_r \widehat{\nu} \widehat{P}_{s-r} \widehat{\nu} S_N(x,t-s) \\ &\le \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r} h_{\alpha-1}(x)\,\mathrm{d}r + \int_0^t \mathrm{d}r \int_r^t \mathrm{d}s\, \widehat{P}_r \widehat{\nu} \widehat{P}_{s-r} \widehat{\nu} h_{\alpha-1}(x) \\ &= h_{\alpha-1}(x).\end{aligned}$$

Then from the fact that the sequence $(S_N(x,t))_{N\ge 0}$ is non-decreasing and bounded, it follows that

$$K_t h_{\alpha-1}(x) \le h_{\alpha-1}(x), \quad x > 0.$$

Now let $x < 0$. Then from Corollary 3.4, (3.19) and Lemma 3.19,

$$\begin{aligned} K_t h_{\alpha-1}(x) &= \widehat{P}_t h_{\alpha-1}(x) + \int_0^t \widehat{P}_s \widehat{\nu} K_{t-s} h_{\alpha-1}(x)\,\mathrm{d}s \\ &= h_{\alpha-1}(x) e^{-\nu(x,D)t} + \int_0^t \mathrm{d}s \int_{D^c} \widehat{P}_s(x,\mathrm{d}y) \int_D \widehat{\nu}(y,\mathrm{d}z) K_{t-s} h_{\alpha-1}(z) \\ &\le h_{\alpha-1}(x) e^{-\nu(x,D)t} + \int_0^t \mathrm{d}s \int_{D^c} \widehat{P}_s(x,\mathrm{d}y) \widehat{\nu} h_{\alpha-1}(y) \\ &= h_{\alpha-1}(x) e^{-\nu(x,D)t} + \int_0^t \mathrm{d}s \int_{D^c} \widehat{P}_s(x,\mathrm{d}y) \nu(y,D) h_{\alpha-1}(y) \\ &= h_{\alpha-1}(x) \int_t^\infty \nu(x,D) e^{-\nu(x,D)s}\,\mathrm{d}s + h_{\alpha-1}(x) \int_0^t \nu(x,D) e^{-\nu(x,D)s}\,\mathrm{d}s \\ &= h_{\alpha-1}(x), \end{aligned}$$

which ends the proof. □

Corollary 3.21. *For $\alpha \in (0,2)$ and $\beta(\alpha-\beta-1) \ge 0$,*

$$K_t h_\beta(x) \le h_\beta(x), \qquad t > 0,\ x \ne 0.$$

Proof. The case $\beta = 0$ and $\beta = \alpha - 1$ follows directly from Lemma 3.10 and Theorem 3.20. Hence, it suffices to prove the claim for $\beta \ne 0$ and $\beta \ne \alpha - 1$.

Let $\gamma := (\alpha-1)/\beta$. Of course $\gamma > 1$ and from Jensen's inequality we have

$$\begin{aligned} K_t h_\beta^\gamma(x) &= K_t(x,\mathbb{R}) \int_{\mathbb{R}} h_\beta^\gamma(y) \frac{K_t(x,\mathrm{d}y)}{K_t(x,\mathbb{R})} \ge K_t(x,\mathbb{R}) \Big[\int_{\mathbb{R}} h_\beta(y) \frac{K_t(x,\mathrm{d}y)}{K_t(x,\mathbb{R})} \Big]^\gamma \\ &= \big[K_t(x,\mathbb{R})\big]^{1-\gamma} \big[K_t h_\beta(x)\big]^\gamma \ge \big[K_t h_\beta(x)\big]^\gamma. \end{aligned} \tag{3.20}$$

Hence, from (3.20) and from Theorem 3.20,

$$K_t h_\beta(x) \le \big[K_t h_\beta^\gamma(x)\big]^{1/\gamma} = \big[K_t h_{\alpha-1}(x)\big]^{1/\gamma} \le h_{\alpha-1}^{1/\gamma}(x) = h_\beta(x).$$ □

Note that the assumption $\alpha \in (0,2)$ and $\beta(\alpha-\beta-1) \ge 0$ in the previous corollary is equivalent to the following:

1. $0 < \alpha \le 1$ and $\alpha - 1 \le \beta \le 0$,

or

2. $1 \le \alpha < 2$ and $0 \le \beta \le \alpha - 1$.

Chapter 4

The lifetime and limit of the process

Here we study the lifetime of the process $X = (X_t)_{t\geq 0}$. We consider three cases. In case $\alpha \in (0,1)$, we prove that the lifetime of the process X is infinite a.s. and $|X_t| \to \infty$ as $t \to \infty$. For $\alpha \in (1,2)$, we show that the process X hits the origin in finite time. In case $\alpha = 1$, we prove that the lifetime of X is infinite and the limit of $\lim_{t\to\infty} X_t$ does not exist. The precise statement of this fact is given in Section 4.4. At the end of this chapter, we use the obtained results to prove the Feller property of the semigroup K for $\alpha > 1$.

4.1 The first return positions to D

Let $W = X_{R_2}$. It is the first return position to $D = (0,\infty)$ for the process $X = (X_t)_{t\geq 0}$ starting from $x > 0$. By Lemma 3.9,

$$R(x,w) := \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}y \int_{-\infty}^0 \mathrm{d}z\, p_t^D(x,y)\nu(y,z)\frac{\nu(z,w)}{\nu(z,D)}, \quad w \in D,$$

is the density function of W. By (2.23) and (2.27),

$$R(x,w) = \int_{-\infty}^0 P_D(x,z)\frac{\nu(z,w)}{\nu(z,D)}\,\mathrm{d}z.$$

By changing variables $z = xs$, (2.28) and (2.11) we get

$$\begin{aligned} R(x,w) &= x\int_{-\infty}^0 P_D(x,xs)\frac{\nu(xs,w)}{\nu(xs,D)}\,\mathrm{d}s = x^{1-1-\alpha-1+\alpha}\int_{-\infty}^0 P_D(1,s)\frac{\nu(s,w/x)}{\nu(s,D)}\,\mathrm{d}s \\ &= x^{-1}R(1,w/x). \end{aligned}$$

Therefore, $\mathbb{P}_x(W \in \mathrm{d}w) = \mathbb{P}_1(xW \in \mathrm{d}w)$.

Lemma 4.1. *For $\alpha \in (0,1)\cup(1,2)$,*

$$\rho := \mathbb{E}_1 W^{(\alpha-1)/2} < 1.$$

Proof. Let $\alpha \in (0,2)$. Using (2.29) and (2.12) we obtain

$$\mathbb{E}_1 W^{(\alpha-1)/2} = C \int_0^\infty \mathrm{d}w \int_{-\infty}^0 \mathrm{d}z\, |z|^{-\alpha/2} |1-z|^{-1} |z-w|^{-1-\alpha} |z|^\alpha w^{(\alpha-1)/2},$$

where $C = \pi^{-1}\alpha \sin(\pi\alpha/2)$. By Tonelli's theorem and by changing variables $y = -z \in (0,\infty)$, $w = yv$, we get

$$\begin{aligned}
\mathbb{E}_1 W^{(\alpha-1)/2} &= C \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}v\, y^{1-\alpha/2-1-\alpha+\alpha+\alpha/2-1/2} (1+y)^{-1} (v+1)^{-1-\alpha} v^{(\alpha-1)/2} \\
&= C \int_0^\infty \frac{1}{\sqrt{y}(1+y)}\,\mathrm{d}y \int_0^\infty \frac{v^{(\alpha-1)/2}}{(v+1)^{1+\alpha}}\,\mathrm{d}v \\
&= \frac{\alpha}{\pi} \sin\frac{\pi\alpha}{2} \cdot \pi \cdot B\big(\alpha/2+1/2, \alpha/2+1/2\big) \\
&= \sin\frac{\pi\alpha}{2} \cdot \frac{(\Gamma(\alpha/2+1/2))^2}{\Gamma(\alpha)} \\
&\le \frac{(\Gamma(\alpha/2+1/2))^2}{\Gamma(\alpha)} =: G(\alpha).
\end{aligned}$$

We will show that $G(\alpha) < G(1) = 1$ for $\alpha \in (0,1) \cup (1,2)$. Let $\psi(x) := \Gamma'(x)/\Gamma(x)$ be the digamma function. It is well known that $x \mapsto \psi(x)$ is continuous and increasing for $x > 0$ (see e.g. Andrews et al. [1, Theorem 1.2.5]).

In case $\alpha \in (0,1)$ we have $\alpha/2 + 1/2 > \alpha$ and then $\psi(\alpha/2+1/2) > \psi(\alpha)$. Then, of course, $\Gamma'(\alpha/2+1/2)\Gamma(\alpha) - \Gamma(\alpha/2+1/2)\Gamma'(\alpha) > 0$. Therefore,

$$G'(\alpha) = \frac{\Gamma(\alpha/2+1/2)}{\Gamma^2(\alpha)} \big[\Gamma'(\alpha/2+1/2)\Gamma(\alpha) - \Gamma(\alpha/2+1/2)\Gamma'(\alpha)\big] > 0,$$

and so $G(\alpha) < G(1) = 1$.

In case $\alpha \in (1,2)$ we have $\alpha/2 + 1/2 < \alpha$ and then $\psi(\alpha/2+1/2) < \psi(\alpha)$, or $\Gamma'(\alpha/2 + 1/2)\Gamma(\alpha) - \Gamma(\alpha/2+1/2)\Gamma'(\alpha) < 0$. Therefore, $G'(\alpha) < 0$ and again $G(\alpha) < G(1) = 1$. □

Proposition 4.2. *For* $\alpha \in (0,2)$, $\mathbb{E}_1 |\ln W| < \infty$ *and*

$$\mathbb{E}_1 \ln W > 0, \quad \text{for } \alpha \in (0,1), \tag{4.1}$$

$$\mathbb{E}_1 \ln W = 0, \quad \text{for } \alpha = 1, \tag{4.2}$$

$$\mathbb{E}_1 \ln W < 0, \quad \text{for } \alpha \in (1,2). \tag{4.3}$$

Proof. First, we prove that $\mathbb{E}_1 |\ln W| < \infty$. From (2.29) and (2.12) we get

$$\mathbb{E}_1 |\ln W| = C \int_0^\infty \mathrm{d}w \int_{-\infty}^0 \mathrm{d}z\, |z|^{-\alpha/2} |1-z|^{-1} |z-w|^{-1-\alpha} |z|^\alpha |\ln w|,$$

where $C = \pi^{-1}\alpha \sin(\pi\alpha/2)$. By Tonelli's theorem and by changing variables $y = -z \in (0,\infty)$ we get

$$\mathbb{E}_1 |\ln W| = C \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}w\, \frac{y^{\alpha/2} |\ln w|}{(y+1)(w+y)^{\alpha+1}}.$$

By changing variables $w = yv$,

$$\begin{aligned}
\mathbb{E}_1|\ln W| &= C\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}v\, \frac{|\ln(yv)|}{y^{\alpha/2}(y+1)(v+1)^{\alpha+1}} \\
&\le C\int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}v\, \frac{|\ln y| + |\ln v|}{y^{\alpha/2}(y+1)(v+1)^{\alpha+1}} \\
&= C\int_0^\infty \frac{1}{(v+1)^{\alpha+1}}\,\mathrm{d}v \int_0^\infty \frac{|\ln y|}{y^{\alpha/2}(y+1)}\,\mathrm{d}y \\
&+ C\int_0^\infty \frac{|\ln v|}{(v+1)^{\alpha+1}}\,\mathrm{d}v \int_0^\infty \frac{y^{-\alpha/2}}{y+1}\,\mathrm{d}y < \infty.
\end{aligned}$$

Thus, $\mathbb{E}_1|\ln W| < \infty$.

Let $\alpha \in (0,1) \cup (1,2)$. Then, from Jensen's inequality for concave functions and from Lemma 4.1, we get

$$\tfrac{\alpha-1}{2}\mathbb{E}_1 \ln W = \mathbb{E}_1 \ln W^{(\alpha-1)/2} \le \ln \mathbb{E}_1 W^{(\alpha-1)/2} < \ln 1 = 0.$$

Hence, for $\alpha \in (0,1)$, $\mathbb{E}_1 \ln W > 0$, and for $\alpha \in (1,2)$, $\mathbb{E}_1 \ln W < 0$.

Now assume that $\alpha = 1$. Then

$$\begin{aligned}
C^{-1}\mathbb{E}_1(\ln W) &= \int_0^\infty \mathrm{d}y \int_0^1 \mathrm{d}w\, \frac{y^{1/2}\ln w}{(y+1)(w+y)^2} + \int_0^\infty \mathrm{d}y \int_1^\infty \mathrm{d}w\, \frac{y^{1/2}\ln w}{(y+1)(w+y)^2} \\
&=: I + II.
\end{aligned} \tag{4.4}$$

Using substitution $w = 1/v$ and integrating by parts we get

$$\begin{aligned}
I &= \int_0^\infty \mathrm{d}y \int_0^1 \mathrm{d}w\, \frac{\sqrt{y}\ln w}{(y+1)(w+y)^2} = -\int_0^\infty \frac{\sqrt{y}}{y+1}\int_1^\infty \frac{\ln v}{(1+yv)^2}\,\mathrm{d}v\,\mathrm{d}y \\
&= -\int_0^\infty \frac{\ln(1+1/y)}{\sqrt{y}(y+1)}\,\mathrm{d}y = -\int_0^\infty \frac{\ln(y+1)}{\sqrt{y}(y+1)}\,\mathrm{d}y + \int_0^\infty \frac{\ln(y)}{\sqrt{y}(y+1)}\,\mathrm{d}y.
\end{aligned}$$

Moreover, by changing variables $y = 1/v$,

$$\int_0^1 \frac{\ln(y)}{\sqrt{y}(y+1)}\,\mathrm{d}y = -\int_1^\infty \frac{\ln v}{\sqrt{v}(v+1)}\,\mathrm{d}v,$$

hence

$$I = -\int_0^\infty \frac{\ln(y+1)}{\sqrt{y}(y+1)}\,\mathrm{d}y. \tag{4.5}$$

By changing variables $w = yz$ in the integral II, we obtain

$$\begin{aligned}
II &= \int_0^\infty \mathrm{d}y \int_{1/y}^\infty \mathrm{d}z\, \frac{\ln y + \ln z}{\sqrt{y}(y+1)(z+1)^2} \\
&= \int_0^\infty \frac{\ln y}{\sqrt{y}(y+1)}\int_{1/y}^\infty \frac{\mathrm{d}z}{(z+1)^2}\,\mathrm{d}y + \int_0^\infty \frac{1}{\sqrt{y}(y+1)}\int_{1/y}^\infty \frac{\ln z}{(z+1)^2}\,\mathrm{d}z\,\mathrm{d}y \\
&= \int_0^\infty \frac{\sqrt{y}\ln y}{(y+1)^2}\,\mathrm{d}y + \int_0^\infty \frac{1}{\sqrt{y}(y+1)}\Big(\ln(y+1) - \frac{y\ln y}{y+1}\Big)\,\mathrm{d}y \\
&= \int_0^\infty \frac{\ln(y+1)}{\sqrt{y}(y+1)}\,\mathrm{d}y.
\end{aligned} \tag{4.6}$$

From (4.4), (4.5) and (4.6) we get $\mathbb{E}_1(\ln W) = 0$. □

Lemma 4.3. *For $\alpha = 1$ we have $\sigma^2 := \mathbb{E}_1\big[\ln^2 W\big] = 4\pi^2/3$.*

Proof. We observe that

$$\mathbb{E}_1\big[\ln^2 W\big] = \int_0^\infty R(1,w)\ln^2 w \,\mathrm{d}w = \frac{1}{\pi}\int_0^\infty \mathrm{d}w \int_0^\infty \mathrm{d}z\, \frac{z^{1/2}\ln^2 w}{(1+z)(z+w)^2}.$$

By the substitution $y = z^{1/2}$, we get

$$\mathbb{E}_1\big[\ln^2 W\big] = \frac{1}{\pi}\int_0^\infty \mathrm{d}w \int_0^\infty \mathrm{d}y \frac{2y^2\ln^2 w}{(y^2+1)(y^2+w)^2} = \frac{1}{\pi}\int_0^\infty \mathrm{d}w \int_{-\infty}^\infty \mathrm{d}y \frac{y^2\ln^2 w}{(y^2+1)(y^2+w)^2}.$$

Using the Cauchy's residue theorem, we obtain that

$$\int_{-\infty}^\infty \frac{y^2}{(y^2+1)(y^2+w)^2}\,\mathrm{d}y = \frac{\pi(\sqrt{w}-1)^2}{2\sqrt{w}(w-1)^2}, \qquad w \neq 0,\ w \neq 1.$$

Hence, also by the substitution $u = \sqrt{w}$,

$$\begin{aligned}\mathbb{E}_1\big[\ln^2 W\big] &= \int_0^\infty \frac{(\sqrt{w}-1)^2}{2\sqrt{w}(w-1)^2}\ln^2 w\,\mathrm{d}w = \int_0^\infty \frac{(u-1)^2}{(u^2-1)^2}\ln^2(u^2)\,\mathrm{d}u \\ &= 4\int_0^\infty \frac{\ln^2 u}{(u+1)^2}\,\mathrm{d}u.\end{aligned}$$

By the substitution $u = e^z$, we get

$$\mathbb{E}_1\big[\ln^2 W\big] = 4\int_{-\infty}^\infty \frac{z^2 e^z}{(e^z+1)^2}\,\mathrm{d}z = 8\int_0^\infty \frac{z^2 e^z}{(e^z+1)^2}\,\mathrm{d}z.$$

Integrating by parts we obtain the equality

$$\mathbb{E}_1\big[\ln^2 W\big] = 16\int_0^\infty \frac{z}{e^z+1}\,\mathrm{d}z.$$

From Bateman [4, (1.12.5), p. 32] it follows that

$$\mathbb{E}_1\big[\ln^2 W\big] = 8\zeta(2) = \frac{4}{3}\pi^2.$$

□

4.2 Consecutive return positions to D and their limit

For $i = 1, 2, \ldots$ we define random variables

$$W_i = \frac{X_{R_{2i}}}{X_{R_{2i-2}}}. \tag{4.7}$$

Recall that $R_0 = 0$ and then $W_1 = X_{R_2}/X_0$. Note that

$$W_i = W_1 \circ \theta_{R_{2i-2}}. \tag{4.8}$$

Moreover, for $j < i$ we have

$$W_i = W_{i-j} \circ \theta_{R_{2j}}. \tag{4.9}$$

Indeed,

$$W_i = W_1 \circ \theta_{R_{2i-2}} = W_1 \circ \theta_{R_{2(i-j)-2}} \circ \theta_{R_{2j}} = W_{i-j} \circ \theta_{R_{2j}}.$$

Lemma 4.4. *Under $\mathbb{P}_x$, $x > 0$, every random variable W_i, $i = 1, 2, \ldots$, has the same distribution and the density function of W_i is given by $R(1, w)$. In particular, the distribution does not depend on x.*

Proof. For $i = 1, 2, \ldots$ and a bounded function f we have

$$\begin{aligned}
\mathbb{E}_x[f(W_i)] &= \mathbb{E}_x\big[f\big(W_1 \circ \theta_{R_{2i-2}}\big)\big] = \mathbb{E}_x\Big[\mathbb{E}\Big[f\big(W_1 \circ \theta_{R_{2i-2}}\big)\Big|\mathcal{F}_{R_{2i-2}}\Big]\Big] \\
&= \mathbb{E}_x\Big[\mathbb{E}_y\Big[f\big(W_1\big)\Big]\Big|_{y=X_{R_{2i-2}}}\Big] = \mathbb{E}_x\Big[\mathbb{E}_y\Big[f\Big(\frac{X_{R_2}}{X_0}\Big)\Big]\Big|_{y=X_{R_{2i-2}}}\Big] \\
&= \mathbb{E}_x\Big[\mathbb{E}_y\Big[f\Big(\frac{W}{y}\Big)\Big]\Big|_{y=X_{R_{2i-2}}}\Big] = \mathbb{E}_x\big[\mathbb{E}_1[f(W)]\big] \\
&= \mathbb{E}_1[f(W)],
\end{aligned}$$

which proves the lemma. □

Lemma 4.5. *The random variables $\{W_i\}_{i\in\mathbb{N}}$ are independent under $\mathbb{P}_x$, $x > 0$.*

Proof. Let f_i, $i = 1, 2, \ldots$, be bounded functions. From Lemma 4.4 it suffices to show that for $x > 0$, $n \in \mathbb{N}$, $i_1 < i_2 < \ldots < i_n$,

$$\mathbb{E}_x\big[f_1(W_{i_1})f_2(W_{i_2})\ldots f_n(W_{i_n})\big] = \mathbb{E}_x[f_1(W_1)]\mathbb{E}_x[f_2(W_1)]\ldots\mathbb{E}_x[f_n(W_1)]. \tag{4.10}$$

We first show that for $i_1 < i_2$,

$$\mathbb{E}_x\big[f_1(W_{i_1})f_2(W_{i_2})\big] = \mathbb{E}_x\big[f_1(W_1)]\mathbb{E}_x[f_2(W_1)]. \tag{4.11}$$

Indeed, from Lemma 4.4 and the strong Markov property we get

$$\begin{aligned}
\mathbb{E}_x\big[f_1(W_{i_1})f_2(W_{i_2})\big] &= \mathbb{E}_x\Big[\mathbb{E}\Big[f_1\big(W_{i_1}\big)f_2\big(W_{i_2-i_1} \circ \theta_{R_{2i_1}}\big)\Big|\mathcal{F}_{R_{2i_1}}\Big]\Big] \\
&= \mathbb{E}_x\Big[f_1\big(W_{i_1}\big)\mathbb{E}\Big[f_2\big(W_{i_2-i_1} \circ \theta_{R_{2i_1}}\big)\Big|\mathcal{F}_{R_{2i_1}}\Big]\Big] \\
&= \mathbb{E}_x\Big[f_1(W_{i_1})\mathbb{E}_y\big[f_2(W_{i_2-i_1})\big]\Big|_{y=X_{R_{2i_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1(W_{i_1})\mathbb{E}_y\big[f_2(W_1)\big]\Big|_{y=X_{R_{2i_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1(W_{i_1})\mathbb{E}_y\Big[f_2\Big(\frac{X_{R_2}}{y}\Big)\Big]\Big|_{y=X_{R_{2i_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1(W_{i_1})\mathbb{E}_1[f_2(X_{R_2})]\Big] \\
&= \mathbb{E}_1[f_2(X_{R_2})]\mathbb{E}_x[f_1(W_{i_1})] \\
&= \mathbb{E}_x[f_1(W_1)]\mathbb{E}_x[f_2(W_1)].
\end{aligned}$$

Assume now that the equality (4.10) holds for $n \geq 2$. We will prove it for $n+1$,

$$
\begin{aligned}
&\mathbb{E}_x[f_1(W_{i_1})f_2(W_{i_2})\dots f_{n+1}(W_{i_{n+1}})]\\
&= \mathbb{E}_x\big[f_1(W_{i_1})\mathbb{E}\big[f_2(W_{i_2-i_1}\circ\theta_{R_{2i_1}})\dots f_{n+1}(W_{i_{n+1}-i_1}\circ\theta_{R_{2i_1}})\big|\mathcal{F}_{R_{2i_1}}\big]\big]\\
&= \mathbb{E}_x\big[f_1(W_{i_1})\mathbb{E}_y\big[f_2(W_{i_2-i_1})\dots f_{n+1}(W_{i_{n+1}-i_1})\big]\big|_{y=X_{R_{2i_1}}}\big]\\
&= \mathbb{E}_x\big[f_1(W_{i_1})\mathbb{E}_y[f_2(W_1)]\dots\mathbb{E}_y[f_{n+1}(W_1)]\big|_{y=X_{R_{2i_1}}}\big]\\
&= \mathbb{E}_x\Big[f_1(W_{i_1})\mathbb{E}_y\Big[f_2\Big(\frac{X_{R_2}}{y}\Big)\Big]\dots\mathbb{E}_y\Big[f_{n+1}\Big(\frac{X_{R_2}}{y}\Big)\Big]\Big|_{y=X_{R_{2i_1}}}\Big]\\
&= \mathbb{E}_x\big[f_1(W_{i_1})\mathbb{E}_1[f_2(X_{R_2})]\dots\mathbb{E}_1[f_{n+1}(X_{R_2})]\big]\\
&= \mathbb{E}_x[f_1(W_{i_1})]\mathbb{E}_1[f_2(X_{R_2})]\dots\mathbb{E}_1[f_{n+1}(X_{R_2})]\\
&= \mathbb{E}_x[f_1(W_1)]\mathbb{E}_x[f_2(W_1)]\dots\mathbb{E}_x[f_{n+1}(W_1)],
\end{aligned}
$$

which is our claim. □

From Lemma 4.4 and Lemma 4.5 it follows that $\{W_i\}_{i\in\mathbb{N}}$ are i.i.d.

Now we consider consecutive return positions V_n, $n = 0, 1, 2, \dots$, of the process X to D, starting from $x > 0$. Thus, $V_0 = X_0 = x$ and for $n \geq 1$, $V_n = X_{R_{2n}}$. From (4.7) it follows that for $n \geq 1$, $V_n = W_n V_{n-1}$. In other words, there exist i.i.d. random variables W_i, $i = 1, 2, \dots$, with the density function $R(1, w)$ such that for $n \geq 1$, $V_n = X_0 \prod_{i=1}^n W_i$.

Theorem 4.6. *The following statements hold $\mathbb{P}_x$-a.s. for every $x > 0$.*

1. *If $\alpha \in (0, 1)$, then $\lim_{n\to\infty} X_{R_{2n}} = \lim_{n\to\infty} V_n = +\infty$.*
2. *If $\alpha \in (1, 2)$, then $\lim_{n\to\infty} X_{R_{2n}} = \lim_{n\to\infty} V_n = 0$.*
3. *If $\alpha = 1$, then*

$$\liminf_{n\to\infty} X_{R_{2n}} = 0, \qquad \limsup_{n\to\infty} X_{R_{2n}} = +\infty.$$

Proof. Let $\alpha \in (0, 1)$. By the Strong Law of Large Numbers and Proposition 4.2 it follows that

$$\ln \prod_{i=1}^n W_i = \sum_{i=1}^n \ln W_i \to +\infty \ \text{ a.s.},$$

as $n \to \infty$, which implies the desired convergence.

Now let $\alpha \in (1, 2)$. Again, by the Strong Law of Large Numbers and Proposition 4.2 it follows that

$$\ln \prod_{i=1}^n W_i = \sum_{i=1}^n \ln W_i \to -\infty \ \text{ a.s.},$$

as $n \to \infty$, which implies the desired convergence.

Now assume that $\alpha = 1$. Let $S'_n := \sum_{i=1}^n \ln W_i$. From Proposition 4.2 we know that $\mathbb{E}_1 \ln W = 0$ and from Lemma 4.3, $\mathbb{E}_1 |\ln W|^2 = \sigma^2 \in (0, \infty)$. Hence, from the Law of the Iterated Logarithm (see Hartman and Wintner [39] or Acosta [27]), it follows that

$$\limsup_{n\to\infty} \frac{S'_n}{\sqrt{2n \ln\ln n}} = \sigma > 0, \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{S'_n}{\sqrt{2n \ln\ln n}} = -\sigma < 0.$$

Therefore,

$$\limsup_{n\to\infty} \sum_{i=1}^n \ln W_i = +\infty, \qquad \liminf_{n\to\infty} \sum_{i=1}^n \ln W_i = -\infty,$$

and then $\limsup_{n\to\infty} V_n = +\infty$, and $\liminf_{n\to\infty} V_n = 0$. □

4.3 The lifetime of the process X

Let $T = R_2$ be the random time of the first return to $D = (0, \infty)$ of the process $X = (X_t)_{t\geq 0}$ starting from $x > 0$. Then, by Lemma 3.9,

$$S(x,t) := \int_0^t \mathrm{d}r \int_0^\infty \mathrm{d}a \int_{-\infty}^0 \mathrm{d}b \, p_r^D(x,a)\nu(a,b)\nu(b,D)e^{-\nu(b,D)(t-r)}$$

is the density function of T. By changing variables $a = xc$, $b = xe$, $r = x^\alpha s$ and using (2.16), (2.10) and (2.11) we get

$$\begin{aligned}
S(x,t) &= \int_0^{tx^{-\alpha}} \mathrm{d}s \int_0^\infty \mathrm{d}c \int_{-\infty}^0 \mathrm{d}e \, x^{\alpha+1+1} p_{x^\alpha s}^D(x, xc)\nu(xc, xe)\nu(xe, D)e^{-\nu(xe,D)(t-x^\alpha s)} \\
&= \int_0^{tx^{-\alpha}} \mathrm{d}s \int_0^\infty \mathrm{d}c \int_{-\infty}^0 \mathrm{d}e \, x^{\alpha+2} x^{-1} p_s^D(1,c) x^{-1-\alpha}\nu(c,e)x^{-\alpha}\nu(e,D)e^{-\nu(e,D)(tx^{-\alpha}-s)} \\
&= x^{-\alpha} \int_0^{tx^{-\alpha}} \mathrm{d}s \int_0^\infty \mathrm{d}c \int_{-\infty}^0 \mathrm{d}e \, p_s^D(1,c)\nu(c,e)\nu(e,D)e^{-\nu(e,D)(tx^{-\alpha}-s)} \\
&= x^{-\alpha} S(1, tx^{-\alpha}).
\end{aligned}$$

Therefore, $\mathbb{P}_x(T \in dt) = \mathbb{P}_1(x^\alpha T \in dt)$.

For $n = 1, 2, \ldots$ we define random variables

$$T_n = \frac{R_{2n} - R_{2n-2}}{X^\alpha_{R_{2n-2}}}. \tag{4.12}$$

Note that for $n \geq 1$, $R_{2n} - R_{2n-2} = R_2 \circ \theta_{R_{2n-2}}$, hence

$$T_n = T_1 \circ \theta_{R_{2n-2}}. \tag{4.13}$$

Moreover, for $k < n$,

$$T_n = T_{n-k} \circ \theta_{R_{2k}}. \tag{4.14}$$

Indeed,

$$T_n = T_1 \circ \theta_{R_{2n-2}} = T_1 \circ \theta_{R_{2(n-k)-2}} \circ \theta_{R_{2k}} = T_{n-k} \circ \theta_{R_{2k}}.$$

Lemma 4.7. *Under $\mathbb{P}_x$, $x > 0$, every random variable T_n, $n = 1, 2, \ldots$, has the same distribution and the density function of T_n is given by $S(1,t)$. In particular, the distribution does not depend on x.*

Proof. For a bounded or non-negative function f we have

$$\begin{aligned}
\mathbb{E}_x[f(T_n)] &= \mathbb{E}_x[\mathbb{E}[f(T_1 \circ \theta_{R_{2n-2}})|\mathcal{F}_{R_{2n-2}}]] = \mathbb{E}_x\left[\mathbb{E}_y[f(T_1)]\big|_{y=X_{R_{2n-2}}}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_y\left[f\Big(\frac{R_2}{X_0^\alpha}\Big)\right]\Big|_{y=X_{R_{2n-2}}}\right] = \mathbb{E}_x\left[\mathbb{E}_y\left[f\Big(\frac{T}{y^\alpha}\Big)\right]\Big|_{y=X_{R_{2n-2}}}\right] \\
&= \mathbb{E}_x\big[\mathbb{E}_1[f(T)]\big] = \mathbb{E}_1[f(T)],
\end{aligned}$$

which proves the lemma. □

Lemma 4.8. *The random variables $\{T_n\}_{n\in\mathbb{N}}$ are independent under $\mathbb{P}_x$, $x > 0$.*

Proof. Let f_i, $i = 1, 2, \ldots$, be bounded or non-negative functions. From Lemma 4.7 it suffices to show that for $x > 0$, $k \in \mathbb{N}$, $n_1 < n_2 < \ldots < n_k$,

$$\mathbb{E}_x\big[f_1(T_{n_1})f_2(T_{n_2})\ldots f_k(T_{n_k})\big] = \mathbb{E}_x[f_1(T_1)]\mathbb{E}_x[f_2(T_1)]\ldots\mathbb{E}_x[f_k(T_1)]. \tag{4.15}$$

First, we will show that for $n_1 < n_2$,

$$\mathbb{E}_x\big[f_1(T_{n_1})f_2(T_{n_2})\big] = \mathbb{E}_x[f_1(T_1)]\mathbb{E}_x[f_2(T_1)]. \tag{4.16}$$

Indeed, from Lemma 4.7,

$$\begin{aligned}
\mathbb{E}_x\big[f_1(T_{n_1})f_2(T_{n_2})\big] &= \mathbb{E}_x\Big[f_1\big(T_{n_1}\big)\mathbb{E}\Big[f_2\big(T_{n_2-n_1} \circ \theta_{R_{2n_1}}\big)\Big|\mathcal{F}_{R_{2n_1}}\Big]\Big] \\
&= \mathbb{E}_x\Big[f_1\big(T_{n_1}\big)\mathbb{E}_y\big[f_2\big(T_{n_2-n_1}\big)\big]\big|_{y=X_{R_{2n_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1\big(T_{n_1}\big)\mathbb{E}_y\big[f_2\big(T_1\big)\big]\big|_{y=X_{R_{2n_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1\big(T_{n_1}\big)\mathbb{E}_y\Big[f_2\Big(\frac{T}{y^\alpha}\Big)\Big]\Big|_{y=X_{R_{2n_1}}}\Big] \\
&= \mathbb{E}_x\Big[f_1\big(T_{n_1}\big)\mathbb{E}_1\big[f_2\big(T\big)\big]\Big] \\
&= \mathbb{E}_x\big[f_1\big(T_{n_1}\big)\big]\mathbb{E}_1\big[f_2\big(T\big)\big] \\
&= \mathbb{E}_x\big[f_1\big(T_1\big)\big]\mathbb{E}_x\Big[f_2\Big(\frac{R_2}{X_0^\alpha}\Big)\Big] \\
&= \mathbb{E}_x\big[f_1(T_1)\big]\mathbb{E}_x\big[f_2(T_1)\big].
\end{aligned}$$

Assume now that the equality (4.15) holds for some $k \geq 2$ and all $x > 0$. We will show it for $k+1$. We have,

$$\begin{aligned}
&\mathbb{E}_x\big[f_1(T_{n_1})f_2(T_{n_2})\dots f_{k+1}(T_{n_{k+1}})\big] \\
&= \mathbb{E}_x\big[f_1(T_{n_1})\mathbb{E}\big[f_2(T_{n_2-n_1}\circ\theta_{R_{2n_1}})\dots f_{k+1}(T_{n_{k+1}-n_1}\circ\theta_{R_{2n_1}})\big|\mathcal{F}_{R_{2n_1}}\big]\big] \\
&= \mathbb{E}_x\big[f_1(T_{n_1})\mathbb{E}_y\big[f_2(T_{n_2-n_1})\dots f_{k+1}(T_{n_{k+1}-n_1})\big]\big|_{y=X_{R_{2n_1}}}\big] \\
&= \mathbb{E}_x\big[f_1(T_{n_1})\mathbb{E}_y[f_2(T_1)]\dots\mathbb{E}_y[f_{k+1}(T_1)]\big|_{y=X_{R_{2n_1}}}\big] \\
&= \mathbb{E}_x\Big[f_1(T_{n_1})\mathbb{E}_y\Big[f_2\Big(\frac{T}{y^\alpha}\Big)\Big]\dots\mathbb{E}_y\Big[f_{k+1}\Big(\frac{T}{y^\alpha}\Big)\Big]\Big|_{y=X_{R_{2n_1}}}\Big] \\
&= \mathbb{E}_x\big[f_1(T_{n_1})\mathbb{E}_1\big[f_2(T)\big]\dots\mathbb{E}_1\big[f_{k+1}(T)\big]\big] \\
&= \mathbb{E}_x\big[f_1(T_{n_1})\big]\mathbb{E}_1\big[f_2(T)\big]\dots\mathbb{E}_1\big[f_{k+1}(T)\big] \\
&= \mathbb{E}_x\big[f_1(T_1)\big]\mathbb{E}_x\big[f_2(T_1)\big]\dots\mathbb{E}_x\big[f_{k+1}(T_1)\big],
\end{aligned}$$

which is the desired conclusion. □

From Lemma 4.7 and Lemma 4.8 it follows that the family $\{T_n\}_{n\in\mathbb{N}}$ is i.i.d.

Now we consider the increments of times S_n, $n = 0, 1, 2, \dots$, of consecutive returns of the process X to D starting from $x > 0$. Thus, $S_0 := 0$ and for $n \geq 1$, $S_n := R_{2n} - R_{2n-2}$. From (4.12) it follows that for $n \geq 1$, $S_n = V_{n-1}^\alpha T_n$. In other words, there exist i.i.d. random variables T_n with the density function $S(1,t)$ and i.i.d. random variables W_i with the density function $R(1,w)$ such that for $n \geq 1$, $S_n = X_0^\alpha\big(\prod_{0<k<n} W_k^\alpha\big)T_n$.

Lemma 4.9. *For $n \geq 1$, $x > 0$ and bounded or non-negative functions f, g,*

$$\mathbb{E}_x\big[f(V_{n-1})g(T_n)\big] = \mathbb{E}_x\big[f(V_{n-1})]\mathbb{E}_x[g(T_n)\big].$$

Proof. Note that $V_{n-1} = X_{R_{2n-2}}$ and $T_n = T_1 \circ \theta_{R_{2n-2}}$. Then from Lemma 4.7 we have

$$\begin{aligned}
\mathbb{E}_x\big[f(V_{n-1})g(T_n)\big] &= \mathbb{E}_x\big[f(X_{R_{2n-2}})\mathbb{E}\big[g(T_1\circ\theta_{R_{2n-2}})\big|\mathcal{F}_{R_{2n-2}}\big]\big] \\
&= \mathbb{E}_x\big[f(X_{R_{2n-2}})\mathbb{E}_y\big[g(T_1)\big]\big|_{y=X_{R_{2n-2}}}\big] \\
&= \mathbb{E}_x\Big[f(X_{R_{2n-2}})\mathbb{E}_y\Big[g\Big(\frac{R_2}{y^\alpha}\Big)\Big]\Big|_{y=X_{R_{2n-2}}}\Big] \\
&= \mathbb{E}_x\big[f(V_{n-1})\big]\mathbb{E}_1\big[g(R_2)\big] \\
&= \mathbb{E}_x\big[f(V_{n-1})\big]\mathbb{E}_x\Big[g\Big(\frac{R_2}{x^\alpha}\Big)\Big] \\
&= \mathbb{E}_x\big[f(V_{n-1})\big]\mathbb{E}_x\big[g(T_1)\big] \\
&= \mathbb{E}_x\big[f(V_{n-1})\big]\mathbb{E}_x\big[g(T_n)\big],
\end{aligned}$$

which completes the proof. □

Recall that $R_\infty = \lim_{n\to\infty} R_{2n} = \sum_{n=1}^\infty S_n = X_0^\alpha \sum_{n=1}^\infty \big(\prod_{0<k<n} W_k^\alpha\big) T_n$ is the lifetime of the process X.

Theorem 4.10. *The following statements hold $\mathbb{P}_x$-a.s. for every $x \neq 0$.*

1. *If $\alpha \in (0,1]$, then $R_\infty = \infty$.*

2. *If $\alpha \in (1,2)$, then $R_\infty < \infty$.*

Proof. Let $x > 0$. Assume that $\alpha \in (0,1)$. We have $\ln S_n = \alpha \ln V_{n-1} + \ln T_n$. By Theorem 4.6, $V_{n-1} \to \infty$ a.e. as $n \to \infty$. Moreover, for any $n = 1, 2, \ldots$, $\mathbb{P}_x(T_n > 1) = c > 0$. Hence, $\sum_{n=1}^\infty \mathbb{P}_x(T_n > 1) = \infty$. Therefore by Lemma 4.8 and the Borel–Cantelli lemma we have $\mathbb{P}_x\big(\limsup_{n\to\infty}\{T_n > 1\}\big) = 1$, which means that with probability one there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $T_{n_k} > 1$ and then $\ln T_{n_k} > 0$. Therefore, $\ln S_{n_k} \to \infty$ a.e., as $k \to \infty$. As a result, $R_\infty = \sum_{n=1}^\infty S_n = \infty$ $\mathbb{P}_x$-a.s.

Assume that $\alpha \in (1,2)$. From the subadditivity of the function $r^{\frac{\alpha-1}{2\alpha}}$ for $r > 0$, the Tonelli's theorem, Lemma 4.9 and Lemma 4.7,

$$\begin{aligned}
\mathbb{E}_x\Big[\sum_{n=1}^\infty S_n\Big]^{\frac{\alpha-1}{2\alpha}} &\le \mathbb{E}_x\Big[\sum_{n=1}^\infty S_n^{\frac{\alpha-1}{2\alpha}}\Big] = \mathbb{E}_x\Big[\sum_{n=1}^\infty V_{n-1}^{\frac{\alpha-1}{2}} T_n^{\frac{\alpha-1}{2\alpha}}\Big] = \sum_{n=1}^\infty \mathbb{E}_x\Big[V_{n-1}^{\frac{\alpha-1}{2}} T_n^{\frac{\alpha-1}{2\alpha}}\Big] \\
&= \sum_{n=1}^\infty \mathbb{E}_x\Big[V_{n-1}^{\frac{\alpha-1}{2}}\Big]\mathbb{E}_x\Big[T_n^{\frac{\alpha-1}{2\alpha}}\Big] = \mathbb{E}_x\Big[T_1^{\frac{\alpha-1}{2\alpha}}\Big]\sum_{n=1}^\infty \mathbb{E}_x\Big[V_{n-1}^{\frac{\alpha-1}{2}}\Big] \\
&= \mathbb{E}_1\Big[T^{\frac{\alpha-1}{2\alpha}}\Big]\sum_{n=1}^\infty \mathbb{E}_x\Big[V_{n-1}^{\frac{\alpha-1}{2}}\Big].
\end{aligned}$$

Moreover, from Lemma 4.5 and Lemma 4.1,

$$\begin{aligned}
\sum_{n=1}^\infty \mathbb{E}_x\Big[V_{n-1}^{\frac{\alpha-1}{2}}\Big] &= \sum_{n=1}^\infty \mathbb{E}_x\Big[X_0 \prod_{i=1}^{n-1} W_i\Big]^{\frac{\alpha-1}{2}} = x^{\frac{\alpha-1}{2}} \sum_{n=1}^\infty \prod_{i=1}^{n-1} \mathbb{E}_1\big[W_1^{\frac{\alpha-1}{2}}\big] \\
&= x^{\frac{\alpha-1}{2}} \sum_{n=1}^\infty \prod_{i=1}^{n-1} \mathbb{E}_1\big[W^{\frac{\alpha-1}{2}}\big] = x^{\frac{\alpha-1}{2}} \sum_{n=1}^\infty \rho^{n-1} = x^{\frac{\alpha-1}{2}} \frac{1}{1-\rho} < \infty.
\end{aligned}$$

It suffices to show that $\mathbb{E}_1\big[T^{\frac{\alpha-1}{2\alpha}}\big] < \infty$. The random variable $T = R_2$ describes the time of the first return to D, so $T = \zeta^{(1)} + \zeta^{(2)} = \zeta^{(1)} + \zeta^{(1)} \circ \theta_{\zeta^{(1)}}$. Using this observation, we get

$$\begin{aligned}
\mathbb{E}_1 T^{\frac{\alpha-1}{2\alpha}} &= \mathbb{E}_1[\zeta^{(1)} + \zeta^{(1)} \circ \theta_{\zeta^{(1)}}]^{\frac{\alpha-1}{2\alpha}} \\
&\le \mathbb{E}_1\big[\zeta^{(1)}\big]^{\frac{\alpha-1}{2\alpha}} + \mathbb{E}_1\big[\zeta^{(1)} \circ \theta_{\zeta^{(1)}}\big]^{\frac{\alpha-1}{2\alpha}} \\
&= \mathbb{E}_1^Y \tau_D^{\frac{\alpha-1}{2\alpha}} + \mathbb{E}_1\Big[\mathbb{E}_y\big(\zeta^{(1)}\big)^{\frac{\alpha-1}{2\alpha}}\Big|_{y=X_{R_1}}\Big],
\end{aligned}$$

where τ_D is the first exit time from D of the process Y. Recall that $R_1 < \infty$ a.s.

By Bañuelos and Bogdan [2, Exercise 3.2 and Theorem 4.1], $\mathbb{E}_1^Y \tau_D^{\frac{\alpha-1}{2\alpha}} < \infty$, because $\frac{\alpha-1}{2\alpha} < \frac{1}{2}$, for $\alpha \in (1,2)$. Alternatively, the estimates of $\mathbb{E}_1^Y \tau_D^{\frac{\alpha-1}{2\alpha}}$ can be obtained from the estimates of survival probability in Bogdan et al. [13] and the equality

$$\mathbb{E}_x^Y \tau_D^p = p \int_0^\infty t^{p-1} \mathbb{P}_x^Y(\tau_D > t)\,\mathrm{d}t, \quad p > 0,\ x > 0.$$

Furthermore, since $\zeta^{(1)}$ for the starting point $y < 0$ has the exponential distribution with mean $1/\nu(y,D)$, then from (2.12) we have

$$\begin{aligned}\mathbb{E}_y\big(\zeta^{(1)}\big)^{\frac{\alpha-1}{2\alpha}} &= \int_0^\infty t^{\frac{\alpha-1}{2\alpha}} \nu(y,D) e^{-\nu(y,D)t}\,\mathrm{d}t = \big[\nu(y,D)\big]^{\frac{1-\alpha}{2\alpha}} \int_0^\infty s^{\frac{\alpha-1}{2\alpha}} e^{-s}\,\mathrm{d}s \\ &\approx |y|^{\frac{\alpha-1}{2}} \int_0^\infty s^{\frac{\alpha-1}{2\alpha}} e^{-s}\,\mathrm{d}s = \Gamma\big(\tfrac{3\alpha-1}{2\alpha}\big)|y|^{\frac{\alpha-1}{2}} \approx |y|^{\frac{\alpha-1}{2}}.\end{aligned}$$

Hence,

$$\mathbb{E}_1\Big[\mathbb{E}_y\big(\zeta^{(1)}\big)^{\frac{\alpha-1}{2\alpha}}\Big|_{y=X_{R_1}}\Big] \approx \mathbb{E}_1\big[|X_{R_1}|^{\frac{\alpha-1}{2}}\big] = \mathbb{E}_1^Y\big[|Y_{\tau_D}|^{\frac{\alpha-1}{2}}\big],$$

and from (2.29),

$$\mathbb{E}_1^Y |Y_{\tau_D}|^{\frac{\alpha-1}{2}} = \int_{-\infty}^0 |y|^{\frac{\alpha-1}{2}} P_D(1,y)\,\mathrm{d}y \approx \int_{-\infty}^0 |y|^{-1/2}|1-y|^{-1}\,\mathrm{d}y = \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}(1+y)} < \infty.$$

Let $\alpha = 1$. From Theorem 4.6, $\limsup_{n\to\infty} V_n = +\infty$ $\mathbb{P}_x$-a.s. Thus, there exists subsequence $(V_{n_k})_{k\in\mathbb{N}}$ such that $V_{n_k} \to +\infty$ $\mathbb{P}_x$-a.s., as $k \to \infty$. By the same argument as in the case $\alpha \in (0,1)$, $\ln S_{n_k} = \ln V_{n_k-1} + \ln T_{n_k} \to +\infty$ in probability. Therefore, there exists a subsequence $(\ln S_{n_{k_l}})_{l\in\mathbb{N}}$ such that $\ln S_{n_{k_l}} \to +\infty$ $\mathbb{P}_x$-a.s., as $l \to \infty$, which implies that $R_\infty = \infty$.

Now let $x < 0$. Then,

$$\begin{aligned}\mathbb{P}_x\big(R_\infty < \infty\big) &= \mathbb{E}_x\big[\mathbb{E}\big(\mathbb{1}_{(0,\infty)}(R_\infty) \mid \mathcal{F}_{R_1}\big)\big] = \mathbb{E}_x\big[\mathbb{E}\big(\mathbb{1}_{(0,\infty)}(R_\infty\circ\theta_{R_1} + R_1) \mid \mathcal{F}_{R_1}\big)\big] \\ &= \mathbb{E}_x\Big[\mathbb{E}_{X_{R_1}}\big(\mathbb{1}_{(0,\infty)}(R_\infty + s)\big)\Big|_{s=R_1}\Big] = \mathbb{E}_x\Big[\mathbb{P}_{X_{R_1}}\big(R_\infty + s < \infty\big)\Big|_{s=R_1}\Big] \\ &= \mathbb{E}_x\big[\mathbb{P}_{X_{R_1}}\big(R_\infty < \infty\big)\big],\end{aligned}$$

since $R_1 < \infty$ a.s. Note that for $\alpha \in (0,1]$, from the first part of the proof, it follows that

$$\mathbb{P}_x\big(R_\infty < \infty\big) = \mathbb{E}_x\big[\mathbb{P}_{X_{R_1}}\big(R_\infty < \infty\big)\big] = \mathbb{E}_x 0 = 0,$$

and for $\alpha \in (1,2)$,

$$\mathbb{P}_x\big(R_\infty < \infty\big) = \mathbb{E}_x\big[\mathbb{P}_{X_{R_1}}\big(R_\infty < \infty\big)\big] = \mathbb{E}_x 1 = 1,$$

which completes the proof. □

4.4 The main theorem

Now we formulate the main theorem of this chapter. Recall that R_∞ denotes the lifetime of the process X.

Theorem 4.11. *The following statements hold $\mathbb{P}_x$-a.s. for every $x \neq 0$.*

1. *If $\alpha \in (0,1)$, then $R_\infty = \infty$ and $\lim_{t\to\infty} |X_t| = \infty$.*
2. *If $\alpha \in (1,2)$, then $0 < R_\infty < \infty$ and $\lim_{t \nearrow R_\infty} X_t = 0$.*
3. *If $\alpha = 1$, then $R_\infty = \infty$ and $\lim_{t\to\infty} X_t$ does not exist. More precisely,*

$$\liminf_{n\to\infty} X_{R_{2n}} = 0, \qquad \limsup_{n\to\infty} X_{R_{2n}} = +\infty.$$

Proof. The case $\alpha = 1$ follows immediately from Theorem 4.6 and Theorem 4.10.

Now assume that $\alpha \neq 1$ and $x \neq 0$. Note that $Y_t := h_{\alpha-1}(X_t) \geq 0$ is a supermartingale with right-continuous trajectories. Indeed, for $s < t$, from Markov property and Theorem 3.20 we get

$$\begin{aligned}\mathbb{E}_x\big[Y_t \mid \mathcal{F}_s\big] &= \mathbb{E}_x\big[h_{\alpha-1}(X_t) \mid \mathcal{F}_s\big] = \mathbb{E}_x\big[h_{\alpha-1}(X_{t-s}\circ\theta_s) \mid \mathcal{F}_s\big] = \mathbb{E}_x\big[h_{\alpha-1}(X_{t-s})\circ\theta_s \mid \mathcal{F}_s\big] \\ &= \mathbb{E}_{X_s}\big[h_{\alpha-1}(X_{t-s})\big] = K_{t-s}h_{\alpha-1}(X_s) \leq h_{\alpha-1}(X_s) = Y_s.\end{aligned}$$

Hence, it is clear that $Z_t := -h_{\alpha-1}(X_t)$ is a submartingale.

Following [53, Chapter II.2], we consider a function $f : \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} \subseteq [0,\infty) \cap \mathbb{Q}$ is a countable set, and define the *number of downcrossings* of the interval $[a,b]$ by the function f as follows. Let $F := \{t_1, t_2, \ldots, t_m\} \subset \mathbb{T}$. For $a, b \in \mathbb{R}$, $a < b$, we define inductively,

$$\begin{aligned} s_1 &= \inf\{t_i :\ t_i \in F,\ f(t_i) > b\}, \\ s_2 &= \inf\{t_i > s_1 :\ t_i \in F,\ f(t_i) < a\}, \\ &\vdots \\ s_{2n+1} &= \inf\{t_i > s_{2n} :\ t_i \in F,\ f(t_i) > b\}, \\ s_{2n+2} &= \inf\{t_i > s_{2n+1} :\ t_i \in F,\ f(t_i) < a\}, \end{aligned}$$

while $\inf(\emptyset) := t_m$. We set

$$D_F(f, [a,b]) := \sup\{n :\ s_{2n} < t_m\}.$$

The *number of downcrossings* of the interval $[a,b]$ by the function $f : \mathbb{T} \to \mathbb{R}$ we define as the number

$$D_\mathbb{T}(f, [a,b]) := \sup\{D_F(f,[a,b]) :\ F \subseteq \mathbb{T},\ F \text{ finite}\}.$$

From the Doob's downcrossing lemma [53, Proposition 2.1, p. 61],

$$\mathbb{E}_x\big[D_{\mathbb{T}}(Z,[a,b])\big] \le \sup_{t\in\mathbb{T}} \frac{\mathbb{E}_x\big[(Z_t-b)^+\big]}{b-a} = \sup_{t\in\mathbb{T}} \frac{\mathbb{E}_x\big[(h_{\alpha-1}(X_t)+b)^-\big]}{b-a} \le \frac{|b|}{b-a} < \infty.$$

Hence, $\mathbb{P}_x\big(D_{\mathbb{T}}(Z,[a,b])=\infty\big)=0$. Let

$$\mathcal{A}_{\mathbb{T}} := \bigcap_{a,b\in\mathbb{Q},\ a<b} \big\{\omega:\ D_{\mathbb{T}}(Z(\omega),[a,b])<\infty\big\}.$$

Then $\mathbb{P}_x(\mathcal{A}_{\mathbb{T}})=1$, because $\mathcal{A}_{\mathbb{T}}$ is an intersection of countable many sets of the measure one.

Let ξ denote the lifetime of the process X, i.e. $\xi=R_\infty$. Moreover, let $\mathbb{T}=[0,\infty)\cap\mathbb{Q}$. If $\omega\in\mathcal{A}_{\mathbb{T}}$, then $D_{\mathbb{T}}(Z(\omega),[a,b])<\infty$ for any rational numbers $a<b$. We claim that the limit $Z(\omega):=\lim_{t\to\xi} Z_t(\omega)$ exists (but it may be infinite). Indeed, assume that this limit does not exist. Then we can find rational numbers $\alpha<\beta$ such that

$$\liminf_{t\to\xi} Z_t(\omega) < \alpha < \beta < \limsup_{t\to\xi} Z_t(\omega).$$

From this fact we conclude that there exists an increasing sequence $(r_n)_{n\in\mathbb{N}}\subset\mathbb{T}$ such that $Z_{r_{2n}}(\omega)\le\alpha<\beta\le Z_{r_{2n-1}}(\omega)$. By taking $\mathbb{I}=\{r_1,r_2,r_3,\ldots\}$ we obtain that

$$D_{\mathbb{T}}(Z(\omega),[\alpha,\beta]) \ge D_{\mathbb{I}}(Z(\omega),[\alpha,\beta]) = \infty,$$

which is a contradiction. Hence, the limit $Z:=-\lim_{t\nearrow\xi} h_{\alpha-1}(X_t)$ exists with probability one.

Assume that $\alpha\in(0,1)$. Then from Theorem 4.10, the lifetime of the process X is infinite a.e. Moreover, from Theorem 4.6, $\lim_{n\to\infty} X_{R_{2n}}=+\infty$ a.e., hence $\lim_{n\to\infty} h_{\alpha-1}(X_{R_{2n}})=0$ a.e. From the uniqueness of the limit we get that $\lim_{t\to\infty} h_{\alpha-1}(X_t)=0$ a.e., hence $\lim_{t\to\infty}|X_t|=+\infty$ a.e.

Now assume that $\alpha\in(1,2)$. Then from Theorem 4.10, the lifetime of the process X is finite a.e. Moreover, from Theorem 4.6, $\lim_{n\to\infty} X_{R_{2n}}=0$ a.e., hence $\lim_{n\to\infty} h_{\alpha-1}(X_{R_{2n}})=0$ a.e. From the uniqueness of the limit we get that $\lim_{t\nearrow\xi} h_{\alpha-1}(X_t)=0$ a.e., hence $\lim_{t\nearrow\xi} X_t=0$ a.e. □

4.5 Feller property

Here we prove that for $\alpha\in(1,2)$, $(K_t)_{t\ge0}$ forms a Feller semigroup on the space $C_0(\mathbb{R}^*)$. For this purpose, we will use the fact that the process X has the finite lifetime for such α.

Corollary 4.12. *For $\alpha\in(1,2)$, $x\neq0$, $t>0$,*

$$\lim_{x\to0} K_t(x,\mathbb{R}) = \lim_{x\to0} \mathbb{P}_x(R_\infty>t) = 0.$$

Proof. From Lemma 3.15 it follows that $K_t(x,\mathbb{R}) = K_{tx^{-\alpha}}(1,\mathbb{R})$ for all $x > 0$. By Theorem 4.10,

$$K_t(x,\mathbb{R}) = \mathbb{P}_1(R_\infty > tx^{-\alpha}) \to 0,$$

as $x \to 0^+$.

Similarly, for all $x < 0$, from Lemma 3.15 it follows that $K_t(x,\mathbb{R}) = K_t(-|x|,\mathbb{R}) = K_{t|x|^{-\alpha}}(-1,\mathbb{R})$. And again by Theorem 4.10,

$$K_t(x,\mathbb{R}) = \mathbb{P}_{-1}(R_\infty > t|x|^{-\alpha}) \to 0,$$

as $x \to 0^-$. □

Lemma 4.13. *For $f \in C_0(\mathbb{R}^*)$ and $x \neq 0$, $K_t f(x) \to f(x)$, as $t \to 0^+$.*

Proof. From Corollary 3.4 and Lemma 3.10 we have

$$\begin{aligned} |K_t f(x) - f(x)| &= \left| \widehat{P}_t f(x) + \int_0^t \widehat{P}_s \widehat{\nu} K_{t-s} f(x)\, \mathrm{d}s - f(x) \right| \\ &\leq \left| \widehat{P}_t f(x) - f(x) \right| + \|f\|_\infty \int_0^t \widehat{P}_s \widehat{\nu} \mathbf{1}(x)\, \mathrm{d}s. \end{aligned}$$

Moreover, from Ikeda–Watanabe formula (2.25), for $x > 0$, we have

$$\int_0^t \widehat{P}_s \widehat{\nu} \mathbf{1}(x)\, \mathrm{d}s = \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_s^D(x,y)\nu(y,z) = \mathbb{P}_x^Y(\tau_D < t) \to 0,$$

as $t \to 0^+$.

Similarly, for $x < 0$,

$$\int_0^t \widehat{P}_s \widehat{\nu} \mathbf{1}(x)\, \mathrm{d}s = \int_0^t \nu(x,D) e^{-\nu(x,D)s}\, \mathrm{d}s = 1 - e^{-\nu(x,D)t} \to 0,$$

as $t \to 0^+$. Hence, from Lemma 3.2 we obtain a desired convergence. □

Theorem 4.14. *For $\alpha \in (1,2)$, $(K_t)_{t\geq 0}$ is a Feller semigroup on $C_0(\mathbb{R}^*)$.*

Proof. From Lemma 3.10 it follows that for any $f \in C_0(\mathbb{R}^*)$, $0 \leq f \leq 1$ we have $0 \leq K_t f \leq 1$, $t \geq 0$. Hence $(K_t)_{t\geq 0}$, is a semigroup of positive contraction operators on $C_0(\mathbb{R}^*)$.

We claim that $K_t C_0(\mathbb{R}^*) \subset C_0(\mathbb{R}^*)$. Let $t > 0$ and $f \in C_0(\mathbb{R}^*)$. Then from Theorem 3.16, $K_t f \in C(\mathbb{R}^*)$. Moreover, from Corollary 4.12,

$$\lim_{x\to 0} \left| K_t f(x) \right| \leq \|f\|_\infty \lim_{x\to 0} K_t \mathbf{1}(x) = \|f\|_\infty \lim_{x\to 0} K_t(x,\mathbb{R}) = 0.$$

From Corollary 3.4 and Lemma 3.10, for $x > 0$,

$$\left| K_t f(x) \right| \leq P_t^D |f|(x) + \|f\|_\infty \int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_s^D(x,y)\nu(y,z).$$

From (2.25), (2.26) and (2.21),

$$\begin{aligned}\big|K_t f(x)\big| &\le P_t^D|f|(x) + \|f\|_\infty \big[1 - \mathbb{P}_x^Y(\tau_D > t)\big] \\ &\approx P_t^D(|f|\mathbb{1}_D)(x) + \|f\|_\infty \Big[1 - \big(1 \wedge |x|^{\alpha/2} t^{-1/2}\big)\Big]. \end{aligned} \tag{4.17}$$

From (4.17) and Theorem 2.2 it follows that $\lim\limits_{x\to+\infty} K_t f(x) = 0$. Similarly, from Corollary 3.4 and Lemma 3.10, for $x < 0$,

$$\begin{aligned}\big|K_t f(x)\big| &\le |f(x)|e^{-\nu(x,D)t} + \|f\|_\infty \int_0^t \nu(x,D)e^{-\nu(x,D)s}\,\mathrm{d}s \\ &= |f(x)|e^{-\nu(x,D)t} + \|f\|_\infty \big[1 - e^{-\nu(x,D)t}\big]. \end{aligned} \tag{4.18}$$

From (4.18) and (2.12) it follows that $\lim\limits_{x\to-\infty} K_t f(x) = 0$. Therefore, we obtained that $K_t f \in C_0(\mathbb{R}^*)$.

Hence, with Lemma 4.13, we finish the proof. □

Combining the results of Theorem 3.16 and Theorem 4.14 we have proved that in fact for $\alpha \in (1, 2)$, $(K_t)_{t\ge0}$ is a doubly Feller semigroup or that process $(X_t)_{t\ge0}$ is doubly Feller.

Chapter 5

Pointwise generator

In this chapter, we are mainly interested in a pointwise generator for the semigroup K. Recall that the classical *infinitesimal generator* (see e.g. [49]) for a Feller semigroup $(T_t)_{t\geq 0}$ on $C_0(\mathscr{X})$ is defined by

$$\mathcal{L}f := \lim_{t\to 0^+} \frac{T_t f - f}{t} \quad \text{in } C_0(\mathscr{X}). \tag{5.1}$$

The domain $\mathcal{D}(\mathcal{L})$ of the generator $\mathcal{L}$ consists of all functions $f \in C_0(\mathscr{X})$ for which the limit (5.1) exists. It is commonly known that dealing with such generators is somehow difficult, especially describing its domains. In our considerations, we want to calculate such limit only for the excessive functions h_β defined in Chapter 3, but as we know such functions do not belong to $C_0(\mathbb{R}^*)$. Thankfully, in our case, it will be sufficient to consider a *pointwise generator*.

Our main goal in this chapter is to derive the *pointwise generator* of the semigroup K on the functions h_β. More precisely, we are interested in the exact form of the limit

$$\lim_{t\to 0^+} \frac{K_t h_\beta(x) - h_\beta(x)}{t},$$

calculated for each $x \neq 0$. The precise statement of this result is given in Theorem 5.15 below.

Before we will proceed with the pointwise generator, in the first two sections we propose quite technical lemmas, which will be helpful in our further considerations.

5.1 Estimation of integrals

Lemma 5.1. *For $\alpha \in (0,2)$, $x > 0$, $y < 0$,*

$$\int_0^{1/2} s^{-1}\Big(1 \wedge \frac{|x|}{s^{1/\alpha}}\Big)^{\alpha/2}\Big(1 \wedge \frac{|y|}{s^{1/\alpha}}\Big)^{-\alpha/2} p_s(x,y)\,\mathrm{d}s \approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big).$$

Proof. Let $A := \int_0^{1/2} s^{-1}\big(1 \wedge \frac{|x|}{s^{1/\alpha}}\big)^{\alpha/2}\big(1 \wedge \frac{|y|}{s^{1/\alpha}}\big)^{-\alpha/2} p_s(x,y)\,\mathrm{d}s$ and consider three cases.

CASE 1. Let $x^\alpha \geq 1/2$. Then of course $|x-y|^\alpha \geq 1/2$ and then from (2.4),

$$\begin{aligned} A &\approx \int_0^{1/2} s^{-1}\Big(1 \wedge \frac{|y|}{s^{1/\alpha}}\Big)^{-\alpha/2} \frac{s}{|x-y|^{\alpha+1}}\,\mathrm{d}s \\ &= |x-y|^{-\alpha-1}\Big[\int_0^{|y|^\alpha \wedge 1/2} \mathrm{d}s + |y|^{-\alpha/2}\int_{|y|^\alpha\wedge 1/2}^{1/2} s^{1/2}\,\mathrm{d}s\Big] \\ &= |x-y|^{-\alpha-1}\Big[\big(|y|^\alpha \wedge 1/2\big) + \tfrac{2}{3}|y|^{-\alpha/2}\Big(\Big(\frac{1}{2}\Big)^{3/2} - \big(|y|^\alpha \wedge 1/2\big)^{3/2}\Big)\Big]. \end{aligned}$$

For $|y|^\alpha > 1/4$,

$$\begin{aligned} A &\approx |x-y|^{-\alpha-1} \\ &\approx |x-y|^{-\alpha-1}\big(1 \vee |y|^{-\alpha/2}\big) \\ &\approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big), \end{aligned}$$

and for $|y|^\alpha \leq 1/4$,

$$\begin{aligned} A &\approx |x-y|^{-\alpha-1}\big[|y|^\alpha + |y|^{-\alpha/2}\big] \\ &= |x-y|^{-\alpha-1}|y|^{-\alpha/2}\big[1 + |y|^{3\alpha/2}\big] \\ &\approx |x-y|^{-\alpha-1}|y|^{-\alpha/2} \\ &\approx |x-y|^{-\alpha-1}\big(1 \vee |y|^{-\alpha/2}\big) \\ &\approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big). \end{aligned}$$

CASE 2. Let $x^\alpha < 1/2$ and $|x-y|^\alpha > 1$. Then $|y|^\alpha = \big(|x-y| - |x|\big)^\alpha > \big(1 - 2^{-1/\alpha}\big)^\alpha$. Moreover,

$$\begin{aligned} A &\approx \int_0^{x^\alpha} |x-y|^{-\alpha-1}\,\mathrm{d}s + \int_{x^\alpha}^{1/2} \frac{x^{\alpha/2}}{\sqrt{s}}|x-y|^{-\alpha-1}\,\mathrm{d}s \\ &= |x-y|^{-\alpha-1}\Big[x^\alpha + 2x^{\alpha/2}\big(2^{-1/2} - x^{\alpha/2}\big)\Big] \\ &= |x-y|^{-\alpha-1}x^{\alpha/2}\big[\sqrt{2} - x^{\alpha/2}\big]. \end{aligned}$$

Note that

$$\frac{1}{\sqrt{2}} = \sqrt{2} - \frac{1}{\sqrt{2}} \leq \sqrt{2} - x^{\alpha/2} \leq \sqrt{2}.$$

Therefore,

$$A \approx |x-y|^{-\alpha-1}x^{\alpha/2} \approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big).$$

CASE 3. Let $x^\alpha < 1/2$ and $|x-y|^\alpha \le 1$. Then $|y|^\alpha = \big(|x-y| - |x|\big)^\alpha \le 1$. Assume that $|x| < |y|$. Then

$$\begin{aligned}
A &\approx \int_0^{x^\alpha} |x-y|^{-\alpha-1}\,\mathrm{d}s + \int_{x^\alpha}^{|y|^\alpha \wedge 1/2} \frac{|x|^{\alpha/2}}{\sqrt{s}} |x-y|^{-\alpha-1}\,\mathrm{d}s \\
&+ \int_{|y|^\alpha \wedge 1/2}^{|x-y|^\alpha \wedge 1/2} |x|^{\alpha/2}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\,\mathrm{d}s + \int_{|x-y|^\alpha \wedge 1/2}^{1/2} |x|^{\alpha/2}|y|^{-\alpha/2} s^{-1/\alpha-1}\,\mathrm{d}s \\
&= |x-y|^{-\alpha-1}|x|^\alpha + 2|x|^{\alpha/2}|x-y|^{-\alpha-1}\Big(\big(|y|^\alpha \wedge 1/2\big)^{1/2} - |x|^{\alpha/2}\Big) \\
&+ |x|^{\alpha/2}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\big[\big(|x-y|^\alpha \wedge 1/2\big) - \big(|y|^\alpha \wedge 1/2\big)\big] \\
&+ \alpha|x|^{\alpha/2}|y|^{-\alpha/2}\big[\big(|x-y|^\alpha \wedge 1/2\big)^{-1/\alpha} - 2^{1/\alpha}\big] \qquad (5.2) \\
&\le \frac{|x|^\alpha}{|x-y|^{\alpha+1}} + \frac{2|x|^{\alpha/2}|y|^{\alpha/2}}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} + \alpha|x|^{\alpha/2}|y|^{-\alpha/2}\big(|x-y|^\alpha \wedge 1/2\big)^{-1/\alpha}.
\end{aligned}$$

Note that $|x|^\alpha = |x|^{\alpha/2}|x|^{\alpha/2} < |x|^{\alpha/2}|y|^{\alpha/2}$. Moreover,

$$|x-y|^\alpha \wedge 1/2 \ge |x-y|^\alpha \wedge \tfrac{1}{2}|x-y|^\alpha = \tfrac{1}{2}|x-y|^\alpha.$$

Therefore,

$$\begin{aligned}
A &\lesssim \frac{|x|^{\alpha/2}|y|^{\alpha/2}}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} = \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[1 + \frac{|y|^\alpha}{|x-y|^\alpha}\right] \lesssim \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} \\
&= \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big).
\end{aligned}$$

For the estimate from below we consider two cases. For $|x| < |y| < (8^{1/\alpha} - 1)|x|$, from (5.2), we have that

$$\begin{aligned}
A &\gtrsim |x-y|^{-\alpha-1}|x|^\alpha = \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} \cdot \left[\frac{|x|\,|y|}{(|x|+|y|)^2}\right]^{\alpha/2} \ge \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} \cdot \left[\frac{|x|}{|x|+|y|}\right]^\alpha \\
&\ge \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} \cdot \left[\frac{|x|}{|x| + (8^{1/\alpha}-1)|x|}\right]^\alpha \approx \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} \\
&= \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{-\alpha/2}\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big).
\end{aligned}$$

For $|y| \ge (8^{1/\alpha} - 1)|x|$, from (5.2), we have that

$$\begin{aligned}
A &\gtrsim 2|x|^{\alpha/2}|x-y|^{-\alpha-1}\Big(\big(|y|^\alpha \wedge 1/2\big)^{1/2} - |x|^{\alpha/2}\Big) \\
&\approx \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left(\frac{|y|}{|x|+|y|}\right)^{\alpha/2}\left[\left(\frac{|y|^\alpha \wedge 1/2}{|x-y|^\alpha}\right)^{1/2} - \left(\frac{|x|}{|x-y|}\right)^{\alpha/2}\right] \\
&\gtrsim \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left(\frac{1}{(8^{1/\alpha}-1)^{-1}+1}\right)^{\alpha/2}\left[\left(\frac{|y| \wedge 2^{-1/\alpha}}{\big((8^{1/\alpha}-1)^{-1}+1\big)|y|}\right)^{\alpha/2} - \left(\frac{1}{8^{1/\alpha}}\right)^{\alpha/2}\right] \\
&\approx \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[\big(8^{1/\alpha}-1)^{-1}+1\big)^{-\alpha/2}\big(1 \wedge 2^{-1/\alpha}|y|^{-1}\big)^{\alpha/2} - 8^{-1/2}\right].
\end{aligned}$$

Recall that $|y| \le 1$ and then

$$\begin{aligned}
A &\gtrsim \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[\big((8^{1/\alpha}-1)^{-1}+1\big)^{-\alpha/2}\big(1\wedge 2^{-1/\alpha}\big)^{\alpha/2} - 8^{-1/2}\right]\\
&= \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[\frac{(8^{1/\alpha}-1)^{\alpha/2}}{4} - \frac{1}{2\sqrt{2}}\right]\\
&\approx \big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{-\alpha/2}\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big).
\end{aligned}$$

Now assume that $|x| \ge |y|$. Then, similarly,

$$\begin{aligned}
A &\approx \int_0^{|y|^\alpha} |x-y|^{-\alpha-1}\,\mathrm{d}s + \int_{|y|^\alpha}^{x^\alpha} s^{1/2}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\,\mathrm{d}s\\
&+ \int_{x^\alpha}^{|x-y|^\alpha\wedge 1/2} |x|^{\alpha/2}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\,\mathrm{d}s + \int_{|x-y|^\alpha\wedge 1/2}^{1/2} s^{-1/\alpha-1}|x|^{\alpha/2}|y|^{-\alpha/2}\,\mathrm{d}s\\
&= |x-y|^{-\alpha-1}|y|^\alpha + \tfrac{2}{3}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\big[|x|^{3\alpha/2}-|y|^{3\alpha/2}\big]\\
&+ |x|^{\alpha/2}|y|^{-\alpha/2}|x-y|^{-\alpha-1}\big[\big(|x-y|^\alpha\wedge 1/2\big)-|x|^\alpha\big]\\
&+ \alpha|x|^{\alpha/2}|y|^{-\alpha/2}\big[\big(|x-y|^\alpha\wedge 1/2\big)^{-1/\alpha}-2^{1/\alpha}\big]\\
&\lesssim \frac{|y|^\alpha}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} + \frac{\alpha|x|^{\alpha/2}|y|^{-\alpha/2}}{\big(|x-y|^\alpha\wedge 1/2\big)^{1/\alpha}}\\
&\lesssim \frac{|y|^\alpha}{|x-y|^{\alpha+1}} + \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}. \qquad (5.3)
\end{aligned}$$

Note that from the assumption, we have that

$$\frac{|y|^\alpha}{|x-y|^{\alpha+1}} = \frac{|y|^{-\alpha/2}}{|x-y|}\cdot\left[\frac{|y|}{|x-y|}\right]^\alpha |y|^{\alpha/2} \le \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}.$$

Hence,

$$A \lesssim \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|} = \big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{-\alpha/2}\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big).$$

For the estimate from below, we consider two cases. For $|x| \ge |y| \ge 2^{2/3-2/\alpha}|x|$, from (5.3),

$$\begin{aligned}
A &\gtrsim \frac{|y|^\alpha}{|x-y|^{\alpha+1}} = \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[\frac{|y|^{3/2}|x|^{-1/2}}{|x-y|}\right]^\alpha \ge \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left[\frac{|y|^{3/2}|x|^{-1/2}}{2|x|}\right]^\alpha\\
&= \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left(\frac{|y|}{2^{2/3}|x|}\right)^{3\alpha/2} \ge \frac{1}{8}\frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\\
&\approx \big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{-\alpha/2}\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big).
\end{aligned}$$

Similarly, for $|y| < 2^{2/3-2/\alpha}|x|$, from (5.3),

$$\begin{aligned} A &\gtrsim |y|^{-\alpha/2}|x-y|^{-\alpha-1}\big[|x|^{3\alpha/2} - |y|^{3\alpha/2}\big] \geq |y|^{-\alpha/2}|x-y|^{-\alpha-1}|x|^{3\alpha/2}\big[1-2^{\alpha-3}\big] \\ &\approx \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left(\frac{|x|}{|x-y|}\right)^{\alpha} \geq \frac{|x|^{\alpha/2}|y|^{-\alpha/2}}{|x-y|}\left(\frac{|x|}{2|x|}\right)^{\alpha} \\ &\approx \big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{-\alpha/2}\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big), \end{aligned}$$

which completes the proof. □

Lemma 5.2. *For $\alpha \in (0,2)$, $t > 0$, $x > 0$ and $y < 0$,*

$$\int_0^\infty \Big(1 \wedge \frac{z^{\alpha/2}}{\sqrt{t}}\Big) p_t(x,z)\nu(z,y)\,\mathrm{d}z \approx t^{-1}\Big(1\wedge\frac{|y|}{t^{1/\alpha}}\Big)^{-\alpha/2} p_t(x,y).$$

Proof. Let $I_t(x,y) := \int_0^\infty \big(1\wedge\frac{z^{\alpha/2}}{\sqrt{t}}\big)p_t(x,z)\nu(z,y)\,\mathrm{d}z$. By substitution $z = t^{1/\alpha}w$, from (2.3), we get

$$I_t(t^{1/\alpha}x, t^{1/\alpha}y) = t^{-1/\alpha-1}\int_0^\infty \big(1\wedge w^{\alpha/2}\big)p_1(x,w)\nu(w,y)\,\mathrm{d}w = t^{-1/\alpha-1}I_1(x,y), \tag{5.4}$$

hence it suffices to find an estimation of $I_1(x,y)$.

First, assume that $x < 1/2$. From (2.4) we then have

$$\begin{aligned} I_1(x,y) &\approx \int_0^\infty \big(1\wedge w^{\alpha/2}\big)\big(1\wedge|x-w|^{-\alpha-1}\big)|w-y|^{-\alpha-1}\,\mathrm{d}w \\ &\approx \int_0^1 w^{\alpha/2}|w-y|^{-\alpha-1}\,\mathrm{d}w + \int_1^\infty |x-w|^{-\alpha-1}|w-y|^{-\alpha-1}\,\mathrm{d}w =: A(y) + B(x,y). \end{aligned}$$

Using the substitution $w = |y|u$ we obtain that

$$A(y) = |y|^{-\alpha/2}\int_0^{1/|y|} \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u. \tag{5.5}$$

Furthermore, for $|y| \leq 1$,

$$\int_0^1 \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u \leq \int_0^{1/|y|} \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u \leq \int_0^\infty \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u < \infty,$$

hence $\int_0^{1/|y|} \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u \approx 1$. Moreover, for $|y| > 1$,

$$\int_0^{1/|y|} \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u \approx \int_0^{1/|y|} u^{\alpha/2}\,\mathrm{d}u \approx |y|^{-\alpha/2-1}.$$

Therefore,

$$\int_0^{1/|y|} \frac{u^{\alpha/2}}{(u+1)^{\alpha+1}}\,\mathrm{d}u \approx 1\wedge|y|^{-\alpha/2-1}.$$

From (5.5),

$$A(y) \approx |y|^{-\alpha-1} \wedge |y|^{-\alpha/2}. \tag{5.6}$$

For the integral $B(x,y)$ we proceed as follows,

$$B(x,y) \lesssim \int_1^\infty |w-y|^{-\alpha-1}\,\mathrm{d}w \le \int_1^\infty w^{-\alpha-1}\,\mathrm{d}w \approx 1,$$

and

$$B(x,y) \le |y|^{-\alpha-1} \int_1^\infty |w-1/2|^{-\alpha-1}\,\mathrm{d}w \approx |y|^{-\alpha-1}.$$

Hence,

$$B(x,y) \lesssim 1 \wedge |y|^{-\alpha-1}. \tag{5.7}$$

Furthermore, note that from (5.6) and (5.7),

$$I_1(x,y) \approx A(y) + B(x,y) \lesssim \big(|y|^{-\alpha-1} \wedge |y|^{-\alpha/2}\big) + \big(1 \wedge |y|^{-\alpha-1}\big) \lesssim |y|^{-\alpha-1} \wedge |y|^{-\alpha/2}.$$

Indeed, for $|y| \le 1$ it is obvious, because $|y|^{-\alpha/2} \ge 1$. On the other hand, for $|y| > 1$, $1 \wedge |y|^{-\alpha-1} = |y|^{-\alpha-1} = |y|^{-\alpha-1} \wedge |y|^{-\alpha/2}$. Further, it is obvious that

$$I_1(x,y) \gtrsim A(y) \approx |y|^{-\alpha-1} \wedge |y|^{-\alpha/2},$$

hence

$$I_1(x,y) \approx |y|^{-\alpha-1} \wedge |y|^{-\alpha/2}. \tag{5.8}$$

Moreover, for $|y| < 1$ we have $|x-y| \lesssim 1$, hence from (5.8), we get

$$I_1(x,y) \approx |y|^{-\alpha/2} \approx \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big).$$

Similarly, for $|y| \ge 1$, we have $|x-y| = x + |y| \ge x + 1 > x + 2x = 3x$ and $|x-y| \ge |y| = |x-y| - |x| > |x-y| - \frac{1}{3}|x-y| = \frac{2}{3}|x-y|$ and from (5.8),

$$I_1(x,y) \approx |y|^{-\alpha-1} \approx |x-y|^{-\alpha-1} = \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big).$$

To sum up, we obtain so far that for $x \in (0, 1/2)$,

$$I_1(x,y) \approx \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big). \tag{5.9}$$

Now, assume that $x \geq 1/2$. From (2.4) we have

$$
\begin{aligned}
I_1(x,y) &\approx \int_0^\infty \big(1 \wedge w^{\alpha/2}\big)\big(1 \wedge |x-w|^{-\alpha-1}\big)|w-y|^{-\alpha-1}\,\mathrm{d}w \\
&\approx \int_0^{1/4} w^{\alpha/2}|x-w|^{-\alpha-1}|w-y|^{-\alpha-1}\mathrm{d}w \\
&+ \int_{|x-w|<1/4} |w-y|^{-\alpha-1}\mathrm{d}w \\
&+ \int_{\substack{|x-w|\geq 1/4 \\ w\geq 1/4}} |x-w|^{-\alpha-1}|w-y|^{-\alpha-1}\mathrm{d}w =: C(x,y)+D(x,y)+E(x,y).
\end{aligned}
$$

From (5.6),

$$
\begin{aligned}
C(x,y) &\approx |x|^{-\alpha-1}\int_0^{1/4} w^{\alpha/2}|w-y|^{-\alpha-1}\,\mathrm{d}w \approx |x|^{-\alpha-1}A(4y) \\
&\approx |x|^{-\alpha-1}\big(|4y|^{-\alpha-1} \wedge |4y|^{-\alpha/2}\big) \approx |x|^{-\alpha-1}\big(|y|^{-\alpha-1} \wedge |y|^{-\alpha/2}\big).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
D(x,y) &= \int_{|x-w|<1/4} |w-y|^{-\alpha-1}\mathrm{d}w \\
&= \alpha^{-1}(x+|y|)^{-\alpha}\left[\Big(1-\frac{1/4}{x+|y|}\Big)^{-\alpha} - \Big(1+\frac{1/4}{x+|y|}\Big)^{-\alpha}\right] \\
&\approx \alpha^{-1}(x+|y|)^{-\alpha}\frac{1/4}{x+|y|} \\
&\approx |x-y|^{-\alpha-1}.
\end{aligned}
$$

From the fact that $a \vee b \approx a+b$, $a,b \geq 0$, we also get that

$$
\begin{aligned}
E(x,y) &= \int_{\substack{|x-w|\geq 1/4 \\ w\geq 1/4}} \big(|x-w| \wedge |w-y|\big)^{-\alpha-1}\big(|x-w| \vee |w-y|\big)^{-\alpha-1}\mathrm{d}w \\
&\approx \int_{\substack{|x-w|\geq 1/4 \\ w\geq 1/4}} \big(|x-w| \wedge |w-y|\big)^{-\alpha-1}\big(|x-w| + |w-y|\big)^{-\alpha-1}\mathrm{d}w \\
&\leq |x-y|^{-\alpha-1} \int_{\substack{|x-w|\geq 1/4 \\ w\geq 1/4}} \big(|x-w| \wedge |w-y|\big)^{-\alpha-1}\,\mathrm{d}w \\
&\approx |x-y|^{-\alpha-1}.
\end{aligned}
$$

Hence, in case $x \geq 1/2$ we obtain that

$$
I_1(x,y) \approx |x|^{-\alpha-1}\big(|y|^{-\alpha-1} \wedge |y|^{-\alpha/2}\big) + |x-y|^{-\alpha-1}. \tag{5.10}
$$

Let $|y| \geq 1$. It is obvious that $|x| \cdot |y| \approx \big(|x| \wedge |y|\big)\big(|x| + |y|\big) = \big(|x| \wedge |y|\big)|x-y|$ and then from (5.10),

$$\begin{aligned} I_1(x,y) &\approx \big(|x| \cdot |y|\big)^{-\alpha-1} + |x-y|^{-\alpha-1} \approx |x-y|^{-\alpha-1}\big[1 + \big(|x| \wedge |y|\big)^{-\alpha-1}\big] \\ &\approx |x-y|^{-\alpha-1} = \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big). \end{aligned}$$

Now assume that $|y| < 1$. Then $|x-y| = x + |y| \geq \frac{1}{2}|y| + |y| = \frac{3}{2}|y|$ and $|x| = |x-y| - |y| \geq |x-y| - \frac{2}{3}|x-y| = \frac{1}{3}|x-y|$. Therefore, from (5.10),

$$\begin{aligned} I_1(x,y) &\approx |x|^{-\alpha-1}|y|^{-\alpha/2} + |x-y|^{-\alpha-1} \approx |x-y|^{-\alpha-1}\big(1 + |y|^{-\alpha/2}\big) \approx |x-y|^{-\alpha-1}|y|^{-\alpha/2} \\ &= |x-y|^{-\alpha-1}\big(1 \wedge |y|\big)^{-\alpha/2} \approx \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big). \end{aligned}$$

Hence, for $x \geq 1/2$,

$$I_1(x,y) \approx \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big). \tag{5.11}$$

From (5.9), (5.11) and (2.4) it follows that

$$I_1(x,y) \approx \big(1 \wedge |y|\big)^{-\alpha/2} p_1(x,y), \qquad x > 0,\ y < 0,$$

and then from (5.4) and (2.3), for $t > 0$,

$$\begin{aligned} I_t(x,y) &= t^{-1/\alpha-1} I_1\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big) \\ &\approx t^{-1}\Big(1 \wedge \frac{|y|}{t^{1/\alpha}}\Big)^{-\alpha/2} t^{-1/\alpha} p_1\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big) \\ &= t^{-1}\Big(1 \wedge \frac{|y|}{t^{1/\alpha}}\Big)^{-\alpha/2} p_t(x,y), \end{aligned}$$

which completes the proof. □

Corollary 5.3. *For $\alpha \in (0,2)$, $t > 0$, $x > 0$ and $y < 0$,*

$$\mathcal{J}(t,x,y) := \int_D p_t^D(x,z)\nu(z,y)\,\mathrm{d}z \approx t^{-1}\Big(1 \wedge \frac{|x|}{t^{1/\alpha}}\Big)^{\alpha/2}\Big(1 \wedge \frac{|y|}{t^{1/\alpha}}\Big)^{-\alpha/2} p_t(x,y).$$

Proof. It follows directly from (2.20), (2.21) and from Lemma 5.2. □

Corollary 5.4. *For $\alpha \in (0,2)$, $t > 0$ and $x > 0$,*

$$\int_D \mathrm{d}z \int_{D^c} \mathrm{d}y\, p_t^D(x,z)\nu(z,y) \approx t^{-1}\left[\Big(\frac{|x|}{t^{1/\alpha}}\Big)^{-\alpha} \wedge \Big(\frac{|x|}{t^{1/\alpha}}\Big)^{\alpha/2}\right].$$

Proof. From the Tonelli's theorem and from Corollary 5.3,

$$\begin{aligned} I(t,x) &:= \int_D \mathrm{d}z \int_{D^c} \mathrm{d}y\, p_t^D(x,z)\nu(z,y) \\ &\approx t^{-1}\Big(1 \wedge \frac{|x|}{t^{1/\alpha}}\Big)^{\alpha/2} \int_{D^c} \Big(1 \wedge \frac{|y|}{t^{1/\alpha}}\Big)^{-\alpha/2} p_t(x,y)\,\mathrm{d}y. \end{aligned} \tag{5.12}$$

Hence, it suffices to find an estimation of $A(t,x) := \int_{-\infty}^{0} \big(1 \wedge \frac{|y|}{t^{1/\alpha}}\big)^{-\alpha/2} p_t(x,y)\,\mathrm{d}y$. Note that by substitution $y = t^{1/\alpha}w$,

$$A(t, t^{1/\alpha}x) = \int_{-\infty}^{0} \big(1 \wedge |w|\big)^{-\alpha/2} p_1(x,w)\,\mathrm{d}w = A(1,x). \tag{5.13}$$

Moreover, from (2.4),

$$\begin{aligned} A(1,x) &\approx \int_0^{\infty} \big(1 \wedge |y|\big)^{-\alpha/2}\big(1 \wedge |x+y|^{-\alpha-1}\big)\,\mathrm{d}y \\ &= \int_0^1 |y|^{-\alpha/2}\big(1 \wedge |x+y|^{-\alpha-1}\big)\,\mathrm{d}y + \int_1^{\infty} \big(1 \wedge |x+y|^{-\alpha-1}\big)\,\mathrm{d}y. \end{aligned}$$

For $|x| \le 1$,

$$A(1,x) \approx \int_0^1 |y|^{-\alpha/2}\,\mathrm{d}y + \int_1^{\infty} |x+y|^{-\alpha-1}\,\mathrm{d}y \approx 1,$$

and for $|x| > 1$,

$$\begin{aligned} A(1,x) &\approx \int_0^1 |y|^{-\alpha/2}|x+y|^{-\alpha-1}\,\mathrm{d}y + \int_1^{\infty} |x+y|^{-\alpha-1}\,\mathrm{d}y \\ &= |x|^{-3\alpha/2} \int_0^{1/x} w^{-\alpha/2}(1+w)^{-\alpha-1}\,\mathrm{d}w + |x|^{-\alpha} \int_{1/x}^{\infty} \frac{\mathrm{d}w}{(w+1)^{\alpha+1}} \\ &\approx |x|^{-3\alpha/2} \int_0^{1/x} w^{-\alpha/2}\,\mathrm{d}w + |x|^{-\alpha} \\ &\approx |x|^{-\alpha-1} + |x|^{-\alpha} \approx |x|^{-\alpha}. \end{aligned}$$

Hence,

$$A(1,x) \approx 1 \wedge |x|^{-\alpha}, \tag{5.14}$$

and from (5.13),

$$A(t,x) = A\big(1, t^{-1/\alpha}x\big) \approx 1 \wedge t|x|^{-\alpha}.$$

Combining this result with (5.12) we get

$$I(t,x) \approx t^{-1}\Big(1 \wedge \frac{|x|^{\alpha}}{t}\Big)^{1/2}\Big(1 \wedge \frac{t}{|x|^{\alpha}}\Big) = t^{-1}\big(t|x|^{-\alpha} \wedge t^{-1/2}|x|^{\alpha/2}\big),$$

which is our claim. □

Lemma 5.5. *For $\alpha \in (0,2)$, $t > 0$, $x > 0$, $y < 0$,*

$$\int_0^t s^{-1}\Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{s}}\Big)\Big(1 \wedge \frac{|y|}{s^{1/\alpha}}\Big)^{-\alpha/2} p_s(x,y) e^{-\nu(y,D)(t-s)}\,\mathrm{d}s \approx \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{|y|^{\alpha/2}}{\sqrt{t}}\Big) p_t(x,y).$$

Proof. Let $I_t(x,y) := \int_0^t s^{-1}\big(1\wedge\frac{|x|^{\alpha/2}}{\sqrt{s}}\big)\big(1\wedge\frac{|y|}{s^{1/\alpha}}\big)^{-\alpha/2} p_s(x,y)e^{-\nu(y,D)(t-s)}\,\mathrm{d}s$. By substitution $s = tu$ and from (2.3) and (2.11) we get

$$\begin{aligned} &I_t(t^{1/\alpha}x, t^{1/\alpha}y) &(5.15)\\ &= \int_0^t s^{-1}\Big(1\wedge\frac{\sqrt{t}|x|^{\alpha/2}}{\sqrt{s}}\Big)\Big(1\wedge\frac{t^{1/\alpha}|y|}{s^{1/\alpha}}\Big)^{-\alpha/2} p_s(t^{1/\alpha}x, t^{1/\alpha}y)e^{-\nu(t^{1/\alpha}y,D)(t-s)}\,\mathrm{d}s \\ &= t^{-1/\alpha}\int_0^1 u^{-1}\Big(1\wedge\frac{|x|^{\alpha/2}}{\sqrt{u}}\Big)\Big(1\wedge\frac{|y|}{u^{1/\alpha}}\Big)^{-\alpha/2} p_u(x,y)e^{-\nu(y,D)(1-u)}\,\mathrm{d}u \\ &= t^{-1/\alpha}I_1(x,y). &(5.16)\end{aligned}$$

Hence it suffices to find an estimation of $I_1(x,y)$.

Note that

$$I_1(x,y) = \int_0^{1/2}\ldots\,\mathrm{d}u + \int_{1/2}^1\ldots\,\mathrm{d}u := A(x,y) + B(x,y). \tag{5.17}$$

For $u\in[1/2,1)$ we have,

$$u^{-1}\Big(1\wedge\frac{|x|^{\alpha/2}}{\sqrt{u}}\Big)\Big(1\wedge\frac{|y|}{u^{1/\alpha}}\Big)^{-\alpha/2} p_u(x,y) \approx \big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{-\alpha/2}p_1(x,y).$$

Furthermore, note that $1-e^{-\eta/2}\approx 1\wedge\eta$, $\eta\geq 0$. Indeed, for $\eta\geq 1$, $1-e^{-\eta/2}\approx 1 = 1\wedge\eta$. For $0\leq\eta<1$ there exists a constant $c\in(0,\eta)$ such that $1-e^{-\eta/2} = \frac{\eta}{2}\big(1-\frac{1}{4}e^{-c}\eta\big)$. Then, $1-e^{-\eta/2}\approx\frac{\eta}{2}\approx 1\wedge\eta$, which follows from the inequality $1\geq 1-\frac{1}{4}e^{-c}\eta\geq 3/4$. From this observation, from (2.12), it follows that

$$\int_{1/2}^1 e^{-\nu(y,D)(1-u)}\,\mathrm{d}u = \frac{1}{\nu(y,D)}\Big[1-e^{-\nu(y,D)/2}\Big] \approx \frac{1}{\nu(y,D)}\big(1\wedge\nu(y,D)\big) \approx \big(1\wedge|y|\big)^{\alpha}.$$

Therefore,

$$B(x,y)\approx\big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{\alpha/2}p_1(x,y). \tag{5.18}$$

For $u\in(0,1/2)$ there exists a constant $C>0$ such that

$$e^{-\nu(y,D)(1-u)}\leq e^{-C|y|^{-\alpha}}\lesssim\big(1\wedge|y|\big)^{\alpha+1}. \tag{5.19}$$

From Lemma 5.1 and (5.19),

$$A(x,y)\lesssim\big(1\wedge|x|\big)^{\alpha/2}\big(1\wedge|y|\big)^{\alpha/2}\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big)\big(1\wedge|y|\big).$$

Note that for $|x-y|<1$ obviously we have $|y|<1$ and

$$\big(|x-y|^{-1}\wedge|x-y|^{-\alpha-1}\big)\big(1\wedge|y|\big) = \frac{|y|}{|x-y|} < 1 = \big(1\wedge|x-y|^{-\alpha-1}\big).$$

Similarly, for $|x-y| \geq 1$,

$$\big(|x-y|^{-1} \wedge |x-y|^{-\alpha-1}\big)\big(1 \wedge |y|\big) = |x-y|^{-\alpha-1}\big(1 \wedge |y|\big) \leq |x-y|^{-\alpha-1} = \big(1 \wedge |x-y|^{-\alpha-1}\big).$$

Hence,

$$A(x,y) \lesssim \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{\alpha/2}\big(1 \wedge |x-y|^{-\alpha-1}\big) \approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{\alpha/2} p_1(x,y). \tag{5.20}$$

From (5.17), (5.18) and (5.20) we get

$$I_1(x,y) \approx \big(1 \wedge |x|\big)^{\alpha/2}\big(1 \wedge |y|\big)^{\alpha/2} p_1(x,y). \tag{5.21}$$

and then from (5.15) and (5.21) and (2.3), for $t > 0$,

$$I_t(x,y) = t^{-1/\alpha} I_1\big(t^{-1/\alpha}x, t^{-1/\alpha}y\big) \approx \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{|y|^{\alpha/2}}{\sqrt{t}}\Big) p_t(x,y),$$

which completes the proof. □

From Lemma 5.2 and Lemma 5.5 and (2.22) the following conclusion follows immediately.

Corollary 5.6. *For $\alpha \in (0,2)$, $t > 0$, $x > 0$ and $y < 0$,*

$$\mathcal{K}(t,x,y) := \int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,y)e^{-\nu(y,D)(t-s)} \approx \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{|y|^{\alpha/2}}{\sqrt{t}}\Big) p_t(x,y).$$

Proof. From (2.20) and (2.21),

$$\begin{aligned}\mathcal{K}(t,x,y) &\approx \int_0^t \mathrm{d}s \int_D \mathrm{d}z \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{s}}\Big)\Big(1 \wedge \frac{|z|^{\alpha/2}}{\sqrt{s}}\Big) p_s(x,z)\nu(z,y)e^{-\nu(y,D)(t-s)} \\ &= \int_0^t \mathrm{d}s \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{s}}\Big) e^{-\nu(y,D)(t-s)} \int_D \Big(1 \wedge \frac{|z|^{\alpha/2}}{\sqrt{s}}\Big) p_s(x,z)\nu(z,y)\,\mathrm{d}z.\end{aligned}$$

From Lemma 5.2,

$$\mathcal{K}(t,x,y) \approx \int_0^t s^{-1}\Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{s}}\Big)\Big(1 \wedge \frac{|y|}{s^{1/\alpha}}\Big)^{-\alpha/2} p_s(x,y)e^{-\nu(y,D)(t-s)}\,\mathrm{d}s,$$

and further, from Lemma 5.5,

$$\mathcal{K}(t,x,y) \approx \Big(1 \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)\Big(1 \wedge \frac{|y|^{\alpha/2}}{\sqrt{t}}\Big) p_t(x,y),$$

which is our claim. □

Corollary 5.7. *For $\alpha \in (0,2)$, $t > 0$ and $x > 0$,*

$$\mathcal{L}(t,x) := \int_0^t \mathrm{d}s \int_D \mathrm{d}z \int_{D^c} \mathrm{d}y\, p_s^D(x,z)\nu(z,y)e^{-\nu(y,D)(t-s)} \approx \Big(\frac{|x|^{\alpha/2}}{\sqrt{t}}\Big)^{-2} \wedge \frac{|x|^{\alpha/2}}{\sqrt{t}}.$$

Proof. By substitution $s = tu$, $y = t^{1/\alpha}a$ and $z = t^{1/\alpha}b$ we get

$$\mathcal{L}(t, t^{1/\alpha}x) = \int_0^1 \mathrm{d}u \int_D \mathrm{d}b \int_{D^c} \mathrm{d}a \, t^{1+2/\alpha} p_{tu}^D(t^{1/\alpha}x, t^{1/\alpha}b)\nu(t^{1/\alpha}b, t^{1/\alpha}a)e^{-t\nu(t^{1/\alpha}a,D)(1-u)}.$$

From (2.15) and (2.11),

$$\mathcal{L}(t, t^{1/\alpha}x) = \int_0^1 \mathrm{d}u \int_D \mathrm{d}b \int_{D^c} \mathrm{d}a \, p_u^D(x, b)\nu(b, a)e^{-\nu(a,D)(1-u)} = \mathcal{L}(1, x). \tag{5.22}$$

Hence, it suffices to find an estimation of $\mathcal{L}(1, x)$.

Note that from the Tonelli's theorem and from Corollary 5.6,

$$\mathcal{L}(1, x) = \int_{-\infty}^0 \mathcal{K}(1, x, y) \, \mathrm{d}y \approx \big(1 \wedge |x|^{\alpha/2}\big) \int_{-\infty}^0 \big(1 \wedge |y|^{\alpha/2}\big)\big(1 \wedge |x - y|^{-\alpha-1}\big) \, \mathrm{d}y.$$

Moreover,

$$\begin{aligned} A &:= \int_{-\infty}^0 \big(1 \wedge |y|^{\alpha/2}\big)\big(1 \wedge |x - y|^{-\alpha-1}\big) \, \mathrm{d}y \\ &\approx \int_{-1}^0 |y|^{\alpha/2}\big(1 \wedge |x - y|^{-\alpha-1}\big) \, \mathrm{d}y + \int_{-\infty}^{-1} \big(1 \wedge |x - y|^{-\alpha-1}\big) \, \mathrm{d}y. \end{aligned}$$

For $|x| \le 1$,

$$A \approx \int_{-1}^0 |y|^{\alpha/2} \, \mathrm{d}y + \int_{-\infty}^{-1} |x - y|^{-\alpha-1} \, \mathrm{d}y \approx 1 = |x|^{-\alpha}\big(1 \wedge |x|\big)^\alpha,$$

and for $|x| > 1$,

$$\begin{aligned} A &\approx \int_{-1}^0 |y|^{\alpha/2}|x - y|^{-\alpha-1} \, \mathrm{d}y + \int_{-\infty}^{-1} |x - y|^{-\alpha-1} \, \mathrm{d}y \lesssim \int_{-\infty}^0 |x - y|^{-\alpha-1} \, \mathrm{d}y \approx |x|^{-\alpha} \\ &= |x|^{-\alpha}\big(1 \wedge |x|\big)^\alpha, \end{aligned}$$

and

$$A \gtrsim \int_{-\infty}^{-1} |x - y|^{-\alpha-1} \, \mathrm{d}y = |x|^{-\alpha} \int_{1/x}^\infty \frac{\mathrm{d}z}{(z + 1)^{\alpha+1}} \gtrsim |x|^{-\alpha} = |x|^{-\alpha}\big(1 \wedge |x|\big)^\alpha.$$

Hence, $A \approx |x|^{-\alpha}\big(1 \wedge |x|\big)^\alpha$.

Therefore, we obtain that

$$\mathcal{L}(1, x) \approx \big(1 \wedge |x|\big)^{\alpha/2}|x|^{-\alpha}\big(1 \wedge |x|\big)^\alpha = |x|^{-\alpha} \wedge |x|^{\alpha/2}. \tag{5.23}$$

Hence, from (5.22) and (5.23),

$$\mathcal{L}(t, x) = \mathcal{L}(1, t^{-1/\alpha}x) \approx t|x|^{-\alpha} \wedge t^{-1/2}|x|^{\alpha/2},$$

which is the desired conclusion. □

5.2 Convergence of integrals

Lemma 5.8. *For* $x, y > 0$*,*

$$\lim_{t \to 0^+} \frac{p_t^D(x,y)}{t} = \nu(x,y).$$

Proof. By the Hunt's formula (2.14) and the Ikeda–Watanabe formula (2.25),

$$\frac{p_t^D(x,y)}{t} = \frac{p_t(x,y)}{t} - \frac{1}{t}\mathbb{E}_x^Y\left[\tau_D < t;\ p_{t-\tau_D}(Y_{\tau_D}, y)\right]. \tag{5.24}$$

From Pòlya [52] (see also Cygan et al. [26]), $\lim_{t\to 0^+} p_t(x,y)/t = \nu(x,y)$. Moreover, it is obvious that $Y_{\tau_D} < 0$, hence $|Y_{\tau_D} - y| > y$ and from (2.4) we get

$$\frac{1}{t}\mathbb{E}_x^Y\left[\tau_D < t;\ p_{t-\tau_D}(Y_{\tau_D}, y)\right] \lesssim \frac{1}{t}\mathbb{E}_x^Y\left[\tau_D < t; \frac{t-\tau_D}{|Y_{\tau_D} - y|^{\alpha+1}}\right] \leq |y|^{-1-\alpha}\,\mathbb{P}_x^Y(\tau_D < t) \to 0, \tag{5.25}$$

as $t \to 0^+$. ☐

Lemma 5.9. *The function*

$$(0,\infty) \times D \times \overline{D}^c \ni (t,x,y) \mapsto \mathcal{J}(t,x,y) = \int_D p_t^D(x,z)\nu(z,y)\,\mathrm{d}z$$

is continuous.

Proof. Let $g(t,x,y,z) := p_t^D(x,z)\nu(z,y)$. The function $D \ni z \mapsto g(t,x,y,z)$ is integrable for $t > 0$, $x \in D$ and $y \in \overline{D}^c$ (see e.g. (2.25)). Hence, the function $\mathcal{J}(t,x,y)$ is well-defined.

Let $\varepsilon > 0$ and define $T = [\varepsilon, \infty)$, $K = [\varepsilon, \infty)$, $L = (-\infty, -\varepsilon]$. We will show that for all $(t,x,y) \in T \times K \times L$ and for any sequence $(t_n, x_n, y_n) \subset T \times K \times L$ such that $\lim_{n\to\infty}(t_n, x_n, y_n) = (t,x,y)$, we have $\lim_{n\to\infty} \mathcal{J}(t_n, x_n, y_n) = \mathcal{J}(t,x,y)$.

Note that $T \times K \times L \ni (t,x,y) \mapsto p_t^D(x,y)$ is continuous. Therefore, the function $(t,x,y) \mapsto g(t,x,y,z)$, $z \in D$, is continuous on $T \times K \times L$, hence $g(t_n, x_n, y_n, z) \to g(t,x,y,z)$ as $n \to \infty$. Furthermore, from (2.22) it follows that for $(t,x,y) \in T \times K \times L$ and $z \in D$ we have

$$|g(t,x,y,z)| \lesssim t^{-1/\alpha}\nu(z,y) \lesssim \varepsilon^{-1/\alpha}|z+\varepsilon|^{-1-\alpha}.$$

Moreover, $\int_D |z+\varepsilon|^{-1-\alpha}\,\mathrm{d}z < \infty$, hence from the dominated convergence theorem we then obtain continuity of the function $\mathcal{J}$ on $T \times K \times L$.

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that the function $\mathcal{J}$ is continuous on $(0,\infty) \times D \times \overline{D}^c$. ☐

Lemma 5.10. *For $x > 0$ and $y < 0$ we have*

$$\mathcal{J}(t,x,y) = \int_D p_t^D(x,z)\nu(z,y)\,\mathrm{d}z \to \nu(x,y),$$

as $t \to 0^+$.

Proof. Let $\varepsilon > 0$ and define $K = [\varepsilon, \infty)$, $L = (-\infty, -\varepsilon]$. Let $U, V \subset \mathbb{R}$, $U \neq V$, be the open sets such that $x \in K \subset U \subset V \subset D$. Let $\{\alpha_1, \alpha_2\}$ be a partition of unity for the sets U^c and V, i.e. for $i = 1, 2$, the functions $\alpha_i : D \to [0, 1]$ satisfy the following conditions: $\alpha_i \in C^\infty(D)$, $\operatorname{supp}\alpha_1 \subseteq V$, $\operatorname{supp}\alpha_2 \subseteq U^c \cap D$, $\alpha_1 = 1$ on U, $\alpha_2 = 1$ on $V^c \cap D$ and $\alpha_1(x) + \alpha_2(x) = 1$ for $x \in D$.

For $z \in D$ and $y \in L$ let $\varphi_1(z,y) := \alpha_1(z)\nu(z,y)$, $\varphi_2(z,y) := \alpha_2(z)\nu(z,y)$. Then $\operatorname{supp}\varphi_1 \subseteq V$, $\operatorname{supp}\varphi_2 \subseteq U^c \cap D$ and

$$\mathcal{J}(t,x,y) = \int_D p_t^D(x,z)\varphi_1(z,y)\,\mathrm{d}z + \int_D p_t^D(x,z)\varphi_2(z,y)\,\mathrm{d}z =: A(t,x,y) + B(t,x,y).$$

From the construction it follows that $\varphi_1(\cdot, y) \in C_0(D)$, hence from Theorem 2.3,

$$A(t,x,y) = \int_D p_t^D(x,z)\varphi_1(z,y)\,\mathrm{d}z \to \varphi_1(x,y) = \nu(x,y),$$

as $t \to 0^+$.

We will show that $B(t,x,y) \to 0$, as $t \to 0^+$. From (2.22) we have

$$\begin{aligned} B(t,x,y) &= \int_{\operatorname{supp}\varphi_2(\cdot,y)} p_t^D(x,z)\varphi_2(z,y)\,\mathrm{d}z \\ &\leq \int_{U^c\cap D} p_t^D(x,z)\nu(z,y)\,\mathrm{d}z \\ &\lesssim t \int_{U^c\cap D} |x-z|^{-\alpha-1}|z-y|^{-\alpha-1}\,\mathrm{d}z. \end{aligned}$$

Let $\rho := \operatorname{dist}(K, U^c \cap D) > 0$ and $\eta := \operatorname{dist}(L, U^c \cap D)$. Then,

$$B(t,x,y) \lesssim t\rho^{-\alpha-1}\eta^{-\alpha-1}|U^c \cap D| \to 0,$$

as $t \to 0^+$. □

Lemma 5.11. *For $x > 0$, $y < 0$ and $\mu \in \{0, 1\}$ we have*

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,y)e^{-\mu\nu(y,D)(t-s)} = \nu(x,y).$$

Proof. Let $\varepsilon > 0$ and $K := [\varepsilon, \infty)$, $L := (-\infty, -\varepsilon]$. Without loss of generality, we may assume that $t \leq 1$. Let $x \in K$ and $y \in L$. From Lemma 5.9 and Lemma 5.10, by putting

$\mathcal{J}(0,x,y) := \nu(x,y)$, we obtain a continuous function $t \mapsto \mathcal{J}(t,x,y)$ on the interval $[0,1]$. Hence, by the Mean Value Theorem it follows that there exists $c = c(t) \in [0,t]$ such that

$$\begin{aligned}\frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,y)e^{-\mu\nu(y,D)(t-s)} &= \frac{1}{t}\int_0^t \mathcal{J}(s,x,y)e^{-\mu\nu(y,D)(t-s)}\,\mathrm{d}s \\ &= \mathcal{J}(c(t),x,y)e^{-\mu\nu(y,D)(t-c(t))}.\end{aligned}$$

Therefore, from Lemma 5.10,

$$\begin{aligned}\lim_{t\to 0^+}\frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,y)e^{-\mu\nu(y,D)(t-s)} &= \lim_{t\to 0^+}\mathcal{J}(c(t),x,y)e^{-\mu\nu(y,D)(t-c(t))} \\ &= \nu(x,y),\end{aligned}$$

which completes the proof. □

Lemma 5.12. *For $x > 0$ we have*

$$\lim_{t\to 0^+}\frac{1}{t}\mathbb{P}_x^Y(\tau_D < t) = \lim_{t\to 0^+}\frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,D^c) = \nu(x,D^c).$$

Proof. The first equality follows from (2.25). We will prove the latter equality. Without loss of generality, we may assume that $t \le 1$. Note that from the Tonelli's theorem,

$$A(t,x) := \frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,D^c) = \int_{D^c}\left[\frac{1}{t}\int_0^t \mathcal{J}(s,x,y)\,\mathrm{d}s\right]\mathrm{d}y.$$

From a similar argument as in the proof of Lemma 5.11 we conclude that there exists $c = c(t) \in [0,t]$, such that

$$A(t,x) = \int_{D^c}\mathcal{J}(c(t),x,y)\,\mathrm{d}y.$$

From Corollary 5.3 and (2.4),

$$\begin{aligned}\mathcal{J}(c(t),x,y) &\lesssim (c(t))^{-1}\Big(1 \wedge \frac{|y|}{(c(t))^{1/\alpha}}\Big)^{-\alpha/2} p_{c(t)}(x,y) \\ &\le (c(t))^{-1}\big(1 \wedge |y|\big)^{-\alpha/2} p_{c(t)}(x,y) \\ &\lesssim \big(1 \wedge |y|\big)^{-\alpha/2}|x-y|^{-\alpha-1}.\end{aligned}$$

Moreover,

$$\begin{aligned}\int_{D^c}\big(1 \wedge |y|\big)^{-\alpha/2}|x-y|^{-\alpha-1}\,\mathrm{d}y &= \int_D \big(1 \wedge y\big)^{-\alpha/2}(x+y)^{-\alpha-1}\,\mathrm{d}y \\ &= \int_0^1 \frac{\mathrm{d}y}{y^{\alpha/2}(x+y)^{\alpha+1}} + \int_1^\infty \frac{\mathrm{d}y}{(x+y)^{\alpha+1}} < \infty.\end{aligned}$$

Hence, we use the dominated convergence theorem to obtain that

$$\lim_{t\to 0^+} A(t,x) = \lim_{t\to 0^+}\int_{D^c}\mathcal{J}(c(t),x,y)\,\mathrm{d}y = \int_{D^c}\lim_{t\to 0^+}\mathcal{J}(c(t),x,y)\,\mathrm{d}y = \nu(x,D^c),$$

where the latter convergence follows from Lemma 5.10. □

Lemma 5.13. *Let $\alpha \in (0,2)$. Assume that $f:\mathbb{R}^* \to [0,\infty)$ is a function such that for some $x \neq 0$ we have $\widehat{\nu} f(x) < \infty$. Then*

$$\lim_{t\to 0^+} \frac{1}{t} K_{t,1} f(x) = \widehat{\nu} f(x).$$

Proof. Assume that $x > 0$. Note that from the Tonelli's theorem,

$$\begin{aligned}
\frac{1}{t} K_{t,1} f(x) &= \frac{1}{t} \int_0^t \mathrm{d}s \int_D \mathrm{d}z \int_{D^c} \mathrm{d}y\, p_s^D(x,z)\nu(z,y) e^{-\nu(y,D)(t-s)} f(y) \\
&= \int_{D^c} \left[\frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}z\, p_s^D(x,z)\nu(z,y)e^{-\nu(y,D)(t-s)}\right] f(y)\,\mathrm{d}y \\
&= \int_{D^c} t^{-1}\mathcal{K}(t,x,y) f(y)\,\mathrm{d}y.
\end{aligned}$$

From Corollary 5.6 and (2.4),

$$t^{-1}\mathcal{K}(t,x,y) \lesssim t^{-1} p_t(x,y) \lesssim \nu(x,y).$$

Moreover,

$$\int_{D^c} \nu(x,y) f(y)\,\mathrm{d}y = \widehat{\nu} f(x) < \infty.$$

Therefore, we can use the dominated convergence theorem to obtain that from Lemma 5.11 with $\mu = 1$ we get

$$\lim_{t\to 0^+} \frac{1}{t} K_{t,1} f(x) = \int_{D^c} \lim_{t\to 0^+} t^{-1}\mathcal{K}(t,x,y) f(y)\,\mathrm{d}y = \int_{D^c} \nu(x,y) f(y)\,\mathrm{d}y = \widehat{\nu} f(x).$$

The case $x < 0$ we prove in the same way, but we propose it for the convenience of the reader. Assume that $x < 0$. From the Tonelli's theorem,

$$\begin{aligned}
\frac{1}{t} K_{t,1} f(x) &= \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_D \mathrm{d}z\, e^{-\nu(x,D)r} \nu(x,y) p_{t-r}^D(y,z) f(z) \\
&= \int_D \left[\frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y\, p_{t-r}^D(z,y)\nu(y,x)e^{-\nu(x,D)r}\right] f(z)\,\mathrm{d}z \\
&= \int_D \left[\frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}y\, p_s^D(z,y)\nu(y,x)e^{-\nu(x,D)(t-s)}\right] f(z)\,\mathrm{d}z \\
&= \int_D t^{-1}\mathcal{K}(t,z,x) f(z)\,\mathrm{d}z.
\end{aligned}$$

From Corollary 5.6 and (2.4),

$$t^{-1}\mathcal{K}(t,z,x) \lesssim t^{-1} p_t(z,x) \lesssim \nu(z,x).$$

Moreover,

$$\int_D \nu(z,x) f(z)\,\mathrm{d}z = \widehat{\nu} f(x) < \infty.$$

Therefore, we can use the dominated convergence theorem to obtain that from Lemma 5.11 with $\mu = 1$,

$$\lim_{t\to 0^+} \frac{1}{t} K_{t,1} f(x) = \int_D \lim_{t\to 0^+} t^{-1}\mathcal{K}(t,z,x) f(z)\,\mathrm{d}z = \int_D \nu(z,x) f(z)\,\mathrm{d}z = \widehat{\nu} f(x),$$

which is our conclusion. □

Corollary 5.14. *For $\alpha \in (0,2)$ and $\beta \in (-1,\alpha)$, we have*

$$\lim_{t\to 0^+} \frac{1}{t} K_{t,1} h_\beta(x) = \widehat{\nu} h_\beta(x), \qquad x \neq 0, \tag{5.26}$$

Proof. Note that for $x > 0$ we have

$$\widehat{\nu} h_\beta(x) \approx \int_{D^c} \frac{|y|^\beta}{|x-y|^{\alpha+1}}\,\mathrm{d}y = x^{\beta-\alpha} \int_0^\infty \frac{w^\beta}{(1+w)^{\alpha+1}}\,\mathrm{d}w = x^{\beta-\alpha}\mathfrak{B}(\beta+1,\alpha-\beta) < \infty.$$

Similarly, for $x < 0$, $\widehat{\nu} h_\beta(x) < \infty$. Therefore, we use Lemma 5.13 to get (5.26). □

5.3 The main theorem

Now we introduce the main result of this chapter.

Theorem 5.15. *Assume that $x \neq 0$, $\alpha \in (0,2)$, $\beta(\alpha-\beta-1) \geq 0$ and let*

$$\mathcal{C}(\alpha,\beta,x) := \begin{cases} \alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta) - \gamma(\alpha,\beta), & x > 0, \\ \alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta), & x < 0, \end{cases}$$

where

$$\gamma(\alpha,\beta) := \int_0^1 \frac{(t^\beta - 1)(1-t^{\alpha-\beta-1})}{(1-t)^{\alpha+1}}\,\mathrm{d}t \leq 0.$$

Then $\mathcal{C}(\alpha,\beta,x) \geq 0$ and

$$\lim_{t\to 0^+} \frac{h_\beta(x) - K_t h_\beta(x)}{t} = \mathcal{A}_{1,\alpha}\mathcal{C}(\alpha,\beta,x) h_\beta(x)|x|^{-\alpha}.$$

Moreover, $\mathcal{C}(\alpha,\beta,x) = 0$ if and only if $\beta = 0$ or $\beta = \alpha - 1$.

To prove this theorem, we first have to find the analogous result for the α-stable killed Lévy process Y. Below, we prove it for a wider class of functions.

Lemma 5.16. *Let $x > 0$ and $\alpha \in (0,2)$. Assume that $f \in C^2(D)$ is a function such that*

$$\int_D \frac{|f(y)|}{(1+y)^{\alpha+1}}\,\mathrm{d}y < \infty.$$

Then

$$\lim_{t\to 0^+} \frac{f(x) - \widehat{P}_t f(x)}{t} = \text{p.v.} \int_D (f(x) - f(y))\nu(x,y)\,\mathrm{d}y + f(x)\nu(x,D^c).$$

Proof. Let $t > 0$. Note that

$$\frac{f(x) - \widehat{P}_t f(x)}{t} = \frac{1}{t}\int_D (f(x) - f(y)) p_t^D(x, y)\,\mathrm{d}y + f(x)\frac{1 - p_t^D(x, D)}{t}. \tag{5.27}$$

From (2.25) we have that

$$\frac{1 - p_t^D(x, D)}{t} = \frac{1}{t}\mathbb{P}_x^Y(\tau_D < t) = \frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}y\, p_s^D(x, y)\nu(y, D^c).$$

Hence, from Lemma 5.12,

$$\lim_{t\to 0^+} \frac{1 - p_t^D(x, D)}{t} = \nu(x, D^c). \tag{5.28}$$

Combining (5.27) and (5.28) we obtain the following equality

$$\lim_{t\to 0^+} \frac{f(x) - \widehat{P}_t f(x)}{t} = \lim_{t\to 0^+} \int_D (f(x) - f(y))\frac{p_t^D(x, y)}{t}\,\mathrm{d}y + f(x)\nu(x, D^c),$$

with the assumption that the latter limit exists — we will show it in the further part of the proof. We will find this limit in a few steps.

Let $\varepsilon \in (0, x/2]$ be an arbitrary constant. Then we write that

$$\begin{aligned} &\int_D (f(x) - f(y))\frac{p_t^D(x, y)}{t}\,\mathrm{d}y \\ &= \int_{D\cap\{|x-y|<\varepsilon\}} (f(x) - f(y))\frac{p_t^D(x, y)}{t}\,\mathrm{d}y + \int_{D\cap\{|x-y|\ge\varepsilon\}} (f(x) - f(y))\frac{p_t^D(x, y)}{t}\,\mathrm{d}y \\ &=: I_t + II_t. \end{aligned} \tag{5.29}$$

Note that from (2.22) it follows that

$$|f(x) - f(y)|\frac{p_t^D(x, y)}{t} \lesssim |f(x) - f(y)| \cdot |x - y|^{-\alpha-1},$$

and

$$\begin{aligned} &\int_{D\cap\{|x-y|\ge\varepsilon\}} |f(x) - f(y)| \cdot |x - y|^{-\alpha-1}\,\mathrm{d}y \\ &\le |f(x)| \int_{D\cap\{|x-y|\ge\varepsilon\}} |x - y|^{-\alpha-1}\,\mathrm{d}y + \int_{D\cap\{|x-y|\ge\varepsilon\}} \frac{|f(y)|}{|x - y|^{\alpha+1}}\,\mathrm{d}y. \end{aligned} \tag{5.30}$$

Moreover, for $|x - y| \ge \varepsilon$ there exists a constant $\widehat{C} = \widehat{C}(x, \varepsilon) > 0$, such that $\widehat{C}(y + 1) \le |x - y|$. To prove it we consider two cases. Let $y \le x - \varepsilon$. Then for $C_1 := \frac{\varepsilon}{x+1}$ we have $C_1 < \frac{\varepsilon}{x+1-\varepsilon} \le \frac{\varepsilon}{y+1} \le \frac{x-y}{y+1}$, which gives us the inequality $C_1(y + 1) < |x - y|$. In case $y \ge x + \varepsilon$ we set $C_2 := \frac{\varepsilon}{1+3x/2}$ and then we observe that $C_2 < \frac{\varepsilon}{1+x+\varepsilon} = \frac{1-\frac{x}{x+\varepsilon}}{1+\frac{1}{x+\varepsilon}} \le \frac{1-\frac{x}{y}}{1+\frac{1}{y}} = \frac{y-x}{y+1}$. This gives us $C_2(y + 1) < |x - y|$. Thus, the claim follows by taking $\widehat{C} := C_1 \wedge C_2$.

Furthermore, note that from the above observations we get that

$$\int_{D\cap\{|x-y|\geq\varepsilon\}}\frac{|f(y)|}{|x-y|^{\alpha+1}}\,\mathrm{d}y\leq\widehat{C}^{-1}\int_{D\cap\{|x-y|\geq\varepsilon\}}\frac{|f(y)|}{(1+y)^{\alpha+1}}\,\mathrm{d}y<\infty.$$

Hence, the integral of the left-hand side of (5.30) is finite. Therefore, from the dominated convergence theorem and by Lemma 5.8,

$$\lim_{t\to0^+}II_t=\int_{D\cap\{|x-y|\geq\varepsilon\}}(f(x)-f(y))\nu(x,y)\,\mathrm{d}y. \tag{5.31}$$

For now, we consider the integral I_t. From the Hunt's formula (2.14) we get

$$\begin{aligned}I_t&=\int_{D\cap\{|x-y|<\varepsilon\}}(f(x)-f(y))\frac{p_t(x,y)}{t}\,\mathrm{d}y-\int_{D\cap\{|x-y|<\varepsilon\}}(f(x)-f(y))\mathcal{H}(t,x,y)\,\mathrm{d}y\\&=:A_t-B_t,\end{aligned} \tag{5.32}$$

where $\mathcal{H}(t,x,y):=t^{-1}\mathbb{E}_x^Y\big[\tau_D<t;\ p_{t-\tau_D}(Y_{\tau_D},y)\big]$. From (5.24) and (5.25) it follows that

$$|f(x)-f(y)|\mathcal{H}(t,x,y)\lesssim|f(x)-f(y)|\cdot|y|^{-\alpha-1}.$$

Moreover,

$$\begin{aligned}&\int_{D\cap\{|x-y|<\varepsilon\}}\frac{|f(x)-f(y)|}{|y|^{\alpha+1}}\,\mathrm{d}y\\&\leq|f(x)|\int_{D\cap\{|x-y|<\varepsilon\}}\frac{\mathrm{d}y}{|y|^{\alpha+1}}+\int_{D\cap\{|x-y|<\varepsilon\}}\frac{|f(y)|}{|y|^{\alpha+1}}\,\mathrm{d}y.\end{aligned} \tag{5.33}$$

Note that for $|x-y|<\varepsilon$ there exists a constant $\widetilde{C}=\widetilde{C}(x)>0$ such that $y\approx1+y$. Indeed, let $\widetilde{C}:=\frac{2}{x}+4$. Then $\frac{3x}{2}<\frac{\widetilde{C}x}{2}-1$. Recall that $\varepsilon\leq\frac{x}{2}$. Then, from the assumption, it is obvious that $\frac{x}{2}<y<\frac{3x}{2}$ and then $\widetilde{C}^{-1}(1+y)<\widetilde{C}^{-1}\big(1+\frac{3x}{2}\big)<\frac{x}{2}<y<3\cdot\frac{x}{2}<3(1+y)$, which is our claim.

Furthermore, from the above result we obtain that

$$\int_{D\cap\{|x-y|<\varepsilon\}}\frac{|f(y)|}{|y|^{\alpha+1}}\,\mathrm{d}y\approx\int_{D\cap\{|x-y|<\varepsilon\}}\frac{|f(y)|}{(1+y)^{\alpha+1}}\,\mathrm{d}y<\infty.$$

Hence, the integral of the left-hand side of (5.33) is finite. Hence, from the dominated convergence theorem and from (5.25),

$$\lim_{t\to0^+}B_t=\int_{D\cap\{|x-y|<\varepsilon\}}(f(x)-f(y))\lim_{t\to0^+}\mathcal{H}(t,x,y)\,\mathrm{d}y=0. \tag{5.34}$$

Now we consider the integral A_t. Let $t>0$ be small enough that $t^{1/\alpha}<\varepsilon$. Note that by the symmetry of p_t,

$$\begin{aligned}\int_{D\cap\{|x-y|<\varepsilon\}}(y-x)p_t(x,y)\,\mathrm{d}y&=\int_x^{x+\varepsilon}(y-x)p_t(x,y)\,\mathrm{d}y+\int_{x-\varepsilon}^{x}(y-x)p_t(x,y)\,\mathrm{d}y\\&=\int_x^{x+\varepsilon}(y-x)p_t(x,y)\,\mathrm{d}y+\int_x^{x+\varepsilon}(x-w)p_t(x,2x-w)\,\mathrm{d}w\\&=\int_x^{x+\varepsilon}(y-x)p_t(x,y)\,\mathrm{d}y-\int_x^{x+\varepsilon}(w-x)p_t(x,w)\,\mathrm{d}w=0.\end{aligned}$$

Hence,

$$\begin{aligned}|A_t| &= \left|\int_{D\cap\{|x-y|<\varepsilon\}} (f(x)-f(y))\frac{p_t(x,y)}{t}\,\mathrm{d}y\right| \\ &= \left|\int_{D\cap\{|x-y|<\varepsilon\}} \big(f(y)-f(x)-(y-x)f'(x)\big)\frac{p_t(x,y)}{t}\,\mathrm{d}y\right| \\ &\le \int_{D\cap\{|x-y|<\varepsilon\}} \big|f(y)-f(x)-(y-x)f'(x)\big|\frac{p_t(x,y)}{t}\,\mathrm{d}y.\end{aligned}$$

From the Taylor's theorem,

$$f(y)-f(x)-(y-x)f'(x) = \frac{f''(c)}{2}(y-x)^2,$$

where c lies strictly between x and y. From the fact that $\varepsilon \le x/2$, it follows that $x-\varepsilon \ge x/2$. Hence, for $|x-y|<\varepsilon$,

$$\begin{aligned}\big|f(y)-f(x)-(y-x)f'(x)\big| &\le \sup_{c\in(x-\varepsilon,x+\varepsilon)} |f''(c)||y-x|^2 \\ &\le \sup_{c\in(x/2,3x/2)} |f''(c)||y-x|^2 =: C(x)|y-x|^2.\end{aligned}$$

Hence,

$$|A_t| \le C(x)\int_{D\cap\{|x-y|<\varepsilon\}} |y-x|^2\frac{p_t(x,y)}{t}\,\mathrm{d}y.$$

From (2.4), there exists a constant $C_0>0$ such that

$$\begin{aligned}|A_t| &\le C(x)C_0\int_{D\cap\{|x-y|<\varepsilon\}} |y-x|^2\frac{1}{t}\Big(t^{-1/\alpha}\wedge\frac{t}{|x-y|^{\alpha+1}}\Big)\,\mathrm{d}y \\ &= C(x)C_0t^{-1/\alpha-1}\int_{D\cap\{|x-y|<t^{1/\alpha}\}} |y-x|^2\,\mathrm{d}y + C(x)C_0\int_{D\cap\{t^{1/\alpha}\le|x-y|<\varepsilon\}} |y-x|^{1-\alpha}\,\mathrm{d}y \\ &\le \tfrac{2}{3}C(x)C_0t^{-1/\alpha-1}t^{3/\alpha} + C(x)C_0\int_{D\cap\{|x-y|<\varepsilon\}} |y-x|^{1-\alpha}\,\mathrm{d}y \\ &= C(x)C_0\big[\tfrac{2}{3}t^{2/\alpha-1}+\tfrac{2}{2-\alpha}\varepsilon^{2-\alpha}\big].\end{aligned}$$

Using the above observations, we conclude that

$$\limsup_{t\to0^+}|A_t| \le \tfrac{2}{2-\alpha}C(x)C_0\,\varepsilon^{2-\alpha} =: \widehat{C}(x,\alpha)\varepsilon^{2-\alpha}. \tag{5.35}$$

Combining (5.32), (5.34) and (5.35) we get

$$\liminf_{t\to0^+} I_t \ge \liminf_{t\to0^+} A_t - \limsup_{t\to0^+} B_t = \liminf_{t\to0^+} A_t \ge -\widehat{C}(x,\alpha)\varepsilon^{2-\alpha}, \tag{5.36}$$

and

$$\limsup_{t\to0^+} I_t \le \limsup_{t\to0^+} A_t - \liminf_{t\to0^+} B_t = \limsup_{t\to0^+} A_t \le \widehat{C}(x,\alpha)\varepsilon^{2-\alpha}. \tag{5.37}$$

Hence, from (5.29), (5.31) and (5.36),

$$\liminf_{t\to 0^+}\int_D (f(x)-f(y))\frac{p_t^D(x,y)}{t}\,\mathrm{d}y \\ \geq -\widehat{C}(x,\alpha)\varepsilon^{2-\alpha}+\int_{D\cap\{|x-y|\geq\varepsilon\}}(f(x)-f(y))\nu(x,y)\,\mathrm{d}y. \tag{5.38}$$

We will show that there exists the limit

$$\text{p.v.}\int_D (f(x)-f(y))\nu(x,y)\,\mathrm{d}y := \lim_{\varepsilon\to 0^+}\int_{D\cap\{|x-y|\geq\varepsilon\}}(f(x)-f(y))\nu(x,y)\,\mathrm{d}y.$$

The existence follows from the dominated convergence theorem. Indeed, by the same argument as in the previous part of the proof we know that

$$\int_{D\cap\{|x-y|\geq\varepsilon\}}(f(x)-f(y))\nu(x,y)\,\mathrm{d}y \\ = \int_D \mathbb{1}_{\{|x-y|\geq\varepsilon\}}\big(f(x)-f(y)-\mathbb{1}_{B(x,x/2)}(y)(y-x)f'(x)\big)\nu(x,y)\,\mathrm{d}y.$$

Furthermore, from the Taylor's theorem we get

$$\mathbb{1}_{\{|x-y|\geq\varepsilon\}}\left|f(x)-f(y)-\mathbb{1}_{B(x,x/2)}(y)(y-x)f'(x)\right| \\ \leq \mathbb{1}_{B(x,x/2)^c}(y)\big(|f(x)|+|f(y)|\big)+\mathbb{1}_{B(x,x/2)}(y)\sup_{y\in B(x,x/2)}|f''(y)||y-x|^2 \\ =: \mathbb{1}_{B(x,x/2)^c}(y)\big(|f(x)|+|f(y)|\big)+\mathbb{1}_{B(x,x/2)}(y)C(x)|y-x|^2,$$

and then

$$\int_{D\cap B(x,x/2)^c}\frac{|f(x)|+|f(y)|}{|x-y|^{\alpha+1}}\,\mathrm{d}y + C(x)\int_{B(x,x/2)}|y-x|^{1-\alpha}\,\mathrm{d}y<\infty.$$

Note that the finiteness of the above expression follows from the previous part of the proof. Then from the dominated convergence theorem we obtain that

$$\lim_{\varepsilon\to 0^+}\int_{D\cap\{|x-y|\geq\varepsilon\}}(f(x)-f(y))\nu(x,y)\,\mathrm{d}y \\ = \int_D \big(f(x)-f(y)-\mathbb{1}_{B(x,x/2)}(y)(y-x)f'(x)\big)\nu(x,y)\,\mathrm{d}y,$$

and in particular the above limit exists. Hence, by taking $\varepsilon\to 0^+$ in (5.38), we get

$$\liminf_{t\to 0^+}\int_D (f(x)-f(y))\frac{p_t^D(x,y)}{t}\,\mathrm{d}y \geq \text{p.v.}\int_D (f(x)-f(y))\nu(x,y)\,\mathrm{d}y.$$

Similarly, from (5.29), (5.31) and (5.37), we obtain that

$$\limsup_{t\to 0^+}\int_D (f(x)-f(y))\frac{p_t^D(x,y)}{t}\,\mathrm{d}y \leq \text{p.v.}\int_D (f(x)-f(y))\nu(x,y)\,\mathrm{d}y.$$

Therefore,

$$\lim_{t\to 0^+}\int_D (f(x)-f(y))\frac{p_t^D(x,y)}{t}\,\mathrm{d}y = \text{p.v.}\int_D (f(x)-f(y))\nu(x,y)\,\mathrm{d}y,$$

which ends the proof. □

The following corollary is a direct use of the previous lemma for $h_\beta(x)$.

Corollary 5.17. *For $x > 0$, $\alpha \in (0, 2)$ and $\beta \in (-1, \alpha)$ we have*

$$\lim_{t\to 0^+} \frac{h_\beta(x) - \widehat{P}_t h_\beta(x)}{t} = \text{p.v.} \int_D (h_\beta(x) - h_\beta(y))\nu(x, y)\,\mathrm{d}y + h_\beta(x)\nu(x, D^c). \tag{5.39}$$

Lemma 5.18. *For $\alpha \in (0, 2)$ and $\beta(\alpha - \beta - 1) \geq 0$,*

$$\lim_{t\to 0^+} \frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(x) = 0, \qquad x \neq 0.$$

Proof. First, assume that $x > 0$. Then from Corollary 3.4 it follows that

$$K_t = \widehat{P}_t + \int_0^t \widehat{P}_r \widehat{\nu} \widehat{P}_{t-r}\,\mathrm{d}r + \int_0^t \int_r^t \widehat{P}_r \widehat{\nu} \widehat{P}_{s-r} \widehat{\nu} K_{t-s}\,\mathrm{d}s\,\mathrm{d}r, \tag{5.40}$$

and from Corollary 3.21 we have that

$$\begin{aligned}
&\frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(x) \\
&= \frac{1}{t} \int_0^t \int_r^t \widehat{P}_r \widehat{\nu} \widehat{P}_{s-r} \widehat{\nu} K_{t-s} h_\beta(x)\,\mathrm{d}s\,\mathrm{d}r \\
&= \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \int_r^t \mathrm{d}s \int_D \mathrm{d}v\, p_r^D(x, y)\nu(y, z) e^{-\nu(z,D)(s-r)} \nu(z, v) K_{t-s} h_\beta(v) \\
&\leq \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \int_r^t \mathrm{d}s \int_D \mathrm{d}v\, p_r^D(x, y)\nu(y, z) e^{-\nu(z,D)(s-r)} \nu(z, v) h_\beta(v).
\end{aligned}$$

Direct calculations show that for $z < 0$,

$$\int_0^\infty \nu(z, v) h_\beta(v)\,\mathrm{d}v \approx \int_0^\infty \frac{v^\beta}{|z - v|^{\alpha+1}}\,\mathrm{d}v = |z|^{\beta-\alpha} \mathfrak{B}(\beta + 1, \alpha - \beta) \approx |z|^{\beta-\alpha}.$$

Therefore, from (2.12),

$$\begin{aligned}
\frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(x) &\lesssim \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \int_r^t \mathrm{d}s\, p_r^D(x, y)\nu(y, z) e^{-\nu(z,D)(s-r)} |z|^{\beta-\alpha} \\
&\approx \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x, y)\nu(y, z)|z|^\beta \big[1 - e^{-\nu(z,D)(t-r)}\big] \\
&\lesssim \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x, y)\nu(y, z)|z|^\beta \big(1 \wedge t|z|^{-\alpha}\big).
\end{aligned}$$

Using the substitution $r = tu$ we get

$$\frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(x) \lesssim \int_0^1 \mathrm{d}u \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_{tu}^D(x, y)\nu(y, z)|z|^\beta \big(1 \wedge t|z|^{-\alpha}\big),$$

and by (2.22),

$$\begin{aligned}
&\frac{1}{t}\sum_{n=2}^{\infty} K_{t,n}h_\beta(x)\\
&\lesssim \int_0^1 \mathrm{d}u \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z \left(1\wedge \frac{y^{\alpha/2}}{\sqrt{tu}}\right)\left((tu)^{-1/\alpha}\wedge \frac{tu}{|x-y|^{\alpha+1}}\right)\nu(y,z)|z|^\beta\left(1\wedge t|z|^{-\alpha}\right)\\
&\lesssim \int_0^1 \mathrm{d}u \int_{D\cap\{|x-y|<(tu)^{1/\alpha}\}} \mathrm{d}y \int_{D^c} \mathrm{d}z\, (tu)^{-1/\alpha}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\left(1\wedge t|z|^{-\alpha}\right)\\
&+\int_0^1 \mathrm{d}u \int_{D\cap\{|x-y|\geq(tu)^{1/\alpha}\}} \mathrm{d}y \int_{D^c} \mathrm{d}z \left(1\wedge \frac{y^{\alpha/2}}{\sqrt{tu}}\right)\frac{tu}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\left(1\wedge t|z|^{-\alpha}\right)\\
&=: I_t + II_t.
\end{aligned}$$

If t is sufficiently small compared with x in I_t, then $|z-x|\lesssim|z-y|$ and

$$\begin{aligned}
I_t &\lesssim \int_0^1 \mathrm{d}u \int_{D\cap\{|x-y|<(tu)^{1/\alpha}\}} \mathrm{d}y \int_{D^c} \mathrm{d}z\, (tu)^{-1/\alpha}\frac{|z|^\beta}{|x-z|^{\alpha+1}}\left(1\wedge t|z|^{-\alpha}\right)\\
&\lesssim \int_0^1 \mathrm{d}u \int_{D^c} \mathrm{d}z\, \frac{|z|^\beta}{|x-z|^{\alpha+1}}\left(1\wedge t|z|^{-\alpha}\right)\\
&= \int_{D^c} \frac{|z|^\beta}{|x-z|^{\alpha+1}}\left(1\wedge t|z|^{-\alpha}\right)\mathrm{d}z,
\end{aligned}$$

and the latter integral converges to 0 as $t\to 0^+$ from the dominated convergence theorem. Indeed, $\left(1\wedge t|z|^{-\alpha}\right)\leq 1$ and

$$\int_{-\infty}^0 \frac{|z|^\beta}{|x-z|^{\alpha+1}}\,\mathrm{d}z = x^{\beta-\alpha}\int_0^\infty \frac{w^\beta}{(w+1)^{\alpha+1}}\,\mathrm{d}w = x^{\beta-\alpha}\,\mathfrak{B}(\beta+1,\alpha-\beta)<\infty.$$

It remains to show that $II_t\to 0$ as $t\to 0^+$. We will show that in a few steps.

The integrand in II_t is non-negative, so from the Tonelli's theorem, we can change the order of integrals. Hence, for now, we consider only integral over u. We have

$$\int_0^1 \mathbb{1}_{\{|x-y|\geq(tu)^{1/\alpha}\}}\left(1\wedge\frac{y^{\alpha/2}}{\sqrt{tu}}\right)u\,\mathrm{d}u = \int_0^{1\wedge(|x-y|^\alpha/t)}\left(1\wedge\frac{y^{\alpha/2}}{\sqrt{tu}}\right)u\,\mathrm{d}u.$$

Here we consider two cases. If $y\geq x/2$ then $y\geq|x-y|$ and hence $\frac{y^\alpha}{t}\geq 1\wedge\frac{|x-y|^\alpha}{t}\geq u$. This means that $\frac{y^{\alpha/2}}{\sqrt{tu}}\geq 1$, so

$$\int_0^{1\wedge(|x-y|^\alpha/t)}\left(1\wedge\frac{y^{\alpha/2}}{\sqrt{tu}}\right)u\,\mathrm{d}u = \int_0^{1\wedge(|x-y|^\alpha/t)} u\,\mathrm{d}u = \frac{1}{2}\left(1\wedge\frac{|x-y|^\alpha}{t}\right)^2.$$

If $y \in (0, x/2)$ then similarly we have that $\frac{y^\alpha}{t} < \frac{|x-y|^\alpha}{t}$. Hence, for sufficiently small t,

$$\begin{aligned}
\int_0^{1\wedge(|x-y|^\alpha/t)} \Big(1\wedge \frac{y^{\alpha/2}}{\sqrt{tu}}\Big) u \,\mathrm{d}u &\le \int_0^{1\wedge(y^\alpha/t)} u\,\mathrm{d}u + \int_{1\wedge(y^\alpha/t)}^{1\wedge(|x-y|^\alpha/t)} \frac{y^{\alpha/2}}{\sqrt{t}}\sqrt{u}\,\mathrm{d}u \\
&= \frac{1}{2}\Big(1\wedge\frac{y^\alpha}{t}\Big)^2 + \frac{2}{3}\frac{y^{\alpha/2}}{\sqrt{t}}\Big[\Big(1\wedge\frac{|x-y|^\alpha}{t}\Big)^{3/2} - \Big(1\wedge\frac{y^\alpha}{t}\Big)^{3/2}\Big] \\
&\lesssim \Big(1\wedge\frac{y^\alpha}{t}\Big)^2 + \Big(\frac{y^\alpha}{t}\Big)^{1/2} \\
&\lesssim \frac{y^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

Now we consider again the integral II_t. Using previous observations, we get

$$\begin{aligned}
II_t &= \int_0^1 \mathrm{d}u \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\; \mathbb{1}_{\{|x-y|\ge (tu)^{1/\alpha}\}} \Big(1\wedge\frac{y^{\alpha/2}}{\sqrt{tu}}\Big)\frac{tu}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\big(1\wedge t|z|^{-\alpha}\big) \\
&\lesssim \int_0^{x/2}\mathrm{d}y\int_{D^c}\mathrm{d}z\,\frac{y^{\alpha/2}}{\sqrt{t}}\frac{t}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\big(1\wedge t|z|^{-\alpha}\big) \\
&+ \int_{x/2}^\infty \mathrm{d}y\int_{D^c}\mathrm{d}z\,\Big(1\wedge\frac{|x-y|^\alpha}{t}\Big)^2\frac{t}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\big(1\wedge t|z|^{-\alpha}\big) \\
&=: A_t + B_t.
\end{aligned}$$

For $y \in (0, x/2)$ it is obvious that $|x-y| \approx x$ and then

$$\begin{aligned}
A_t &\le \sqrt{t}\int_0^{x/2}\mathrm{d}y\int_{D^c}\mathrm{d}z\,\frac{y^{\alpha/2}}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}} \\
&\approx x^{-\alpha-1}\sqrt{t}\int_0^{x/2}\mathrm{d}y\int_{D^c}\mathrm{d}z\,y^{\alpha/2}\frac{|z|^\beta}{|y-z|^{\alpha+1}} \\
&\approx x^{-\alpha-1}\sqrt{t}\int_0^{x/2} y^{\beta-\alpha/2}\,\mathrm{d}y \\
&\lesssim x^{\beta-3\alpha/2}\sqrt{t}\to 0,
\end{aligned}$$

as $t \to 0^+$.

Since, for $y > x/2$ and $z < 0$, we have $|y-z| > |x/2 - z| = \frac{1}{2}|x-2z|$, then by substitution $w = 2z$ we get

$$\begin{aligned}
B_t &= \int_{x/2}^\infty \mathrm{d}y\int_{D^c}\mathrm{d}z\,\Big(1\wedge\frac{|x-y|^\alpha}{t}\Big)^2\frac{t}{|x-y|^{\alpha+1}}\frac{|z|^\beta}{|y-z|^{\alpha+1}}\big(1\wedge t|z|^{-\alpha}\big) \\
&\lesssim \int_{|x-y|\le t^{1/\alpha}}\mathrm{d}y\int_{D^c}\mathrm{d}w\,\frac{|x-y|^{\alpha-1}}{t}\frac{|w|^\beta}{|x-w|^{\alpha+1}}\big(1\wedge t|w|^{-\alpha}\big) \\
&+ \int_{\{y\ge x/2\}\cap\{|x-y|>t^{1/\alpha}\}}\mathrm{d}y\int_{D^c}\mathrm{d}w\frac{t}{|x-y|^{\alpha+1}}\frac{|w|^\beta}{|x-w|^{\alpha+1}}\big(1\wedge t|w|^{-\alpha}\big) \\
&=: C_t + D_t.
\end{aligned}$$

Note that from Tonelli's theorem we get

$$C_t \lesssim \int_{D^c}\frac{|w|^\beta}{|x-w|^{\alpha+1}}\big(1\wedge t|w|^{-\alpha}\big)\,\mathrm{d}w \to 0,$$

as $t \to 0^+$, which follows in the same way as in the case of I_t. For the integral D_t, again from the Tonelli's theorem, we have

$$\begin{aligned}
D_t &= \int_{\{y \ge x/2\} \cap \{|x-y| > t^{1/\alpha}\}} \mathrm{d}y \int_{D^c} \mathrm{d}w \, \frac{t}{|x-y|^{\alpha+1}} \frac{|w|^\beta}{|x-w|^{\alpha+1}} \left(1 \wedge t|w|^{-\alpha}\right) \\
&\le \int_{D^c} \frac{t|w|^\beta}{|x-w|^{\alpha+1}} \left(1 \wedge t|w|^{-\alpha}\right) \int_{\{|x-y| > t^{1/\alpha}\}} \frac{\mathrm{d}y}{|x-y|^{\alpha+1}} \, \mathrm{d}w \\
&= \int_{D^c} \frac{t|w|^\beta}{|x-w|^{\alpha+1}} \left(1 \wedge t|w|^{-\alpha}\right) \int_{\{|u| > t^{1/\alpha}\}} \frac{\mathrm{d}u}{|u|^{\alpha+1}} \, \mathrm{d}w \\
&= \frac{2}{\alpha} \int_{D^c} \frac{|w|^\beta}{|x-w|^{\alpha+1}} \left(1 \wedge t|w|^{-\alpha}\right) \mathrm{d}w \to 0,
\end{aligned}$$

as $t \to 0^+$. The latter convergence follows from the dominated convergence theorem.

Now let $x < 0$. From Lemma 3.3 we have the following equality:

$$\begin{aligned}
\frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(x) &= \frac{1}{t} \sum_{n=1}^{\infty} \int_0^t \widehat{P}_r \widehat{\nu} K_{t-r,n} h_\beta(x) \, \mathrm{d}r \\
&= \frac{1}{t} \int_0^t \mathrm{d}r \int_D \mathrm{d}z \, e^{-\nu(x,D)r} \nu(x,z) \sum_{n=1}^{\infty} K_{t-r,n} h_\beta(z) \\
&= \int_0^1 \mathrm{d}r \int_D \mathrm{d}z \, e^{-\nu(x,D)tr} \nu(x,z) \sum_{n=1}^{\infty} K_{t(1-r),n} h_\beta(z) \\
&\le \int_0^1 \mathrm{d}r \int_D \mathrm{d}z \, \nu(x,z) \sum_{n=1}^{\infty} K_{t(1-r),n} h_\beta(z).
\end{aligned}$$

From the first part of the proof, we know that $\lim_{t\to 0^+} \frac{1}{t} \sum_{n=2}^{\infty} K_{t,n} h_\beta(z) = 0$, where $z > 0$. Hence, of course, $\lim_{t\to 0^+} \sum_{n=2}^{\infty} K_{t,n} h_\beta(z) = 0$. Moreover, from Corollary 5.14 it follows that $\lim_{t\to 0^+} K_{t,1} h_\beta(z) = 0$. Combining this results, for $z > 0$, we get $\lim_{t\to 0^+} \sum_{n=1}^{\infty} K_{t,n} h_\beta(z) = 0$. From Corollary 3.21 it follows that $\sum_{n=1}^{\infty} K_{t(1-r),n} h_\beta(z) \le K_{t(1-r)} h_\beta(z) \le h_\beta(z)$ and we observe that

$$\int_0^1 \mathrm{d}r \int_D \mathrm{d}z \, \nu(x,z) h_\beta(z) \approx \int_0^\infty \frac{z^\beta}{|x-z|^{\alpha+1}} \, \mathrm{d}z = x^{\beta-\alpha} \mathfrak{B}(\beta+1, \alpha-\beta) < \infty.$$

Therefore, by the dominated convergence theorem,

$$\int_0^1 \mathrm{d}r \int_D \mathrm{d}z \, \nu(x,z) \sum_{n=1}^{\infty} K_{t(1-r),n} h_\beta(z) \to 0,$$

as $t \to 0^+$. □

Proof of Theorem 5.15. Let $\mathcal{G}(t,x) := (h_\beta(x) - K_t h_\beta(x))/t$ and assume that $x > 0$. Then from Corollary 5.17,

$$\begin{aligned}
\lim_{t\to 0^+} \mathcal{G}(t,x) &= \lim_{t\to 0^+} \frac{h_\beta(x) - \widehat{P}_t h_\beta(x)}{t} - \lim_{t\to 0^+} \frac{1}{t} \sum_{n=1}^{\infty} K_{t,n} h_\beta(x) \\
&= \text{p.v.} \int_D (h_\beta(x) - h_\beta(y)) \nu(x,y) \, \mathrm{d}y + h_\beta(x) \nu(x, D^c) - \lim_{t\to 0^+} \frac{1}{t} \sum_{n=1}^{\infty} K_{t,n} h_\beta(x).
\end{aligned}$$

From Corollary 5.14 and from Lemma 5.18 we have

$$\lim_{t\to 0^+} \frac{1}{t}\sum_{n=1}^{\infty} K_{t,n}h_\beta(x) = \int_{D^c} h_\beta(y)\nu(x,y)\,\mathrm{d}y.$$

Therefore,

$$\begin{aligned}\lim_{t\to 0^+} \mathcal{G}(t,x) &= \text{p.v.}\int_D (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y + \int_{D^c}(h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y \quad (5.41)\\ &= \text{p.v.}\int_{\mathbb{R}} (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y.\end{aligned}$$

Now we will calculate the exact values of the above integrals. Using the substitution $y = |x|z$ we obtain,

$$\text{p.v.}\int_D (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y = \mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\,\text{p.v.}\int_0^\infty \frac{1-z^\beta}{|1-z|^{\alpha+1}}\,\mathrm{d}z.$$

From Section 5 of [9] we have

$$\text{p.v.}\int_0^\infty \frac{1-z^\beta}{|1-z|^{\alpha+1}}\,\mathrm{d}z = -\gamma(\alpha,\beta) \geq 0.$$

Hence,

$$\text{p.v.}\int_D (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y = -\mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\gamma(\alpha,\beta). \quad (5.42)$$

Moreover, from (2.13),

$$\begin{aligned}\int_{D^c} (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y &= h_\beta(x)\nu(x,D^c) - \mathcal{A}_{1,\alpha}\int_{-\infty}^0 \frac{|y|^\beta}{|x-y|^{\alpha+1}}\,\mathrm{d}y\\ &= \mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta)\big]. \quad (5.43)\end{aligned}$$

From (5.41), (5.42) and (5.43) we get

$$\lim_{t\to 0^+}\mathcal{G}(t,x) = \mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta) - \gamma(\alpha,\beta)\big].$$

Now consider the case $x<0$. Then,

$$\mathcal{G}(t,x) = h_\beta(x)\frac{1-e^{-\nu(x,D)t}}{t} - \frac{1}{t}K_{t,1}h_\beta(x) - \frac{1}{t}\sum_{n=2}^{\infty} K_{t,n}h_\beta(x).$$

From Corollary 5.14 and Lemma 5.18 we obtain that

$$\lim_{t\to 0^+}\mathcal{G}(t,x) = \int_D (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y. \quad (5.44)$$

Moreover, from (2.12),

$$\begin{aligned}\int_D (h_\beta(x)-h_\beta(y))\nu(x,y)\,\mathrm{d}y &= h_\beta(x)\nu(x,D) - \mathcal{A}_{1,\alpha}\int_0^\infty \frac{y^\beta}{|x-y|^{\alpha+1}}\,\mathrm{d}y\\ &= \mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta)\big]. \quad (5.45)\end{aligned}$$

Hence,

$$\lim_{t\to 0^+} \mathcal{G}(t,x) = \mathcal{A}_{1,\alpha}|x|^{\beta-\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta)\big].$$

Now, it remains to show that $\mathcal{C}(\alpha,\beta,x) > 0$ for $\beta(\alpha-\beta-1) > 0$. From [9, (5.3)] it follows that $\gamma(\alpha,\beta) < 0$. Hence, it suffices to show that for $\beta(\alpha-\beta-1) > 0$, $\mathcal{M}(\alpha,\beta) := \alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta) > 0$. From the properties of the Beta function we obtain that

$$\mathcal{M}(\alpha,\beta) = \frac{1}{\Gamma(\alpha+1)}\big[\Gamma(\alpha) - \Gamma(\beta+1)\Gamma(\alpha-\beta)\big].$$

We will consider two cases.

Assume that $1 < \alpha < 2$ and $0 < \beta < \alpha - 1$ and define a function $(0,\alpha-1) \ni \beta \mapsto F(\beta) := \Gamma(\alpha) - \Gamma(\beta+1)\Gamma(\alpha-\beta)$. Then

$$F'(\beta) = \Gamma(\beta+1)\Gamma(\alpha-\beta)\big[\psi(\alpha-\beta) - \psi(\beta+1)\big],$$

where ψ denotes the digamma function. From the fact that the digamma function is increasing on $(0,\infty)$, we obtain that for $\beta \in (0,(\alpha-1)/2)$, $F'(\beta) > 0$ and for $\beta \in ((\alpha-1)/2, \alpha-1)$, $F'(\beta) < 0$. Hence, for $\beta = (\alpha-1)/2$ the function F has the global maximum on the interval $(0,\alpha-1)$. Moreover, for $\beta \in (0,\alpha-1)$, $F(\beta) > F(0) = F(\alpha-1) = 0$. Thus, $\mathcal{M}(\alpha,\beta) > 0$.

Now, assume that $0 < \alpha < 1$ and $\alpha - 1 < \beta < 0$. Using the same notation as in the previous case we get that for $\beta \in (\alpha-1,(\alpha-1)/2)$, $F'(\beta) > 0$ and for $\beta \in ((\alpha-1)/2, 0)$, $F'(\beta) < 0$. And then we conclude that again for $\beta = (\alpha-1)/2$ the function F has the global maximum on the interval $(\alpha-1,0)$ and moreover, $F(\beta) > 0$ for $\beta \in (\alpha-1,0)$. Hence, $\mathcal{M}(\alpha,\beta) > 0$. $\square$

Chapter 6

Function spaces and the Dirichlet form

In this chapter, we prove the Hardy inequality for K in case $\alpha \neq 1$ and investigate the Dirichlet form $\mathcal{E}$ corresponding to the process X. We also prove various characterizations of the domain of the form $\mathcal{E}$.

6.1 Dirichlet form

From Section 3.2 we know that $(K_t)_{t\geq 0}$ is symmetric and strongly continuous contraction semigroup on $L^2(\mathbb{R})$. Following [19, p. 3] or [35, p. 23], for $t > 0$,

$$\mathcal{E}^{(t)}(u,v) := \frac{1}{t}\langle u - K_t u, v\rangle_{L^2(\mathbb{R})}, \qquad u,v \in L^2(\mathbb{R}),$$

is a symmetric form on $L^2(\mathbb{R})$. From general theory (see e.g. [19], [35]) it turns out that for $u \in L^2(\mathbb{R})$, $\mathcal{E}^{(t)}(u,u)$ is non-negative. Moreover, if $t > 0$ decreases, then $\mathcal{E}^{(t)}(u,u)$ is increasing (it follows from the spectral representation of $(K_t)_{t\geq 0}$). Therefore, the limit $\lim_{t\to 0^+} \mathcal{E}^{(t)}(u,u) = \sup_{t>0} \mathcal{E}^{(t)}(u,u)$ exists, and we may then set

$$\mathcal{F} := \{u \in L^2(\mathbb{R}) : \lim_{t\to 0^+} \mathcal{E}^{(t)}(u,u) < \infty\},$$

$$\mathcal{E}(u,v) := \lim_{t\to 0^+} \mathcal{E}^{(t)}(u,v), \qquad u,v \in \mathcal{F}. \tag{6.1}$$

Then $\mathcal{E}$ becomes a closed symmetric form on $L^2(\mathbb{R})$ corresponding to the semigroup $(K_t)_{t\geq 0}$ (see [19, Chapter 1, p. 3]). Moreover, from Lemma 3.10 it follows that for each $t \geq 0$ the operator K_t is *Markovian*, i.e. for $u \in L^2(\mathbb{R})$, $0 \leq u \leq 1$ we have $0 \leq K_t u \leq 1$. Hence, from [35, Theorem 1.4.1], the form $\mathcal{E}$ is *Markovian* (for the definition of Markovian property of symmetric forms see e.g. [35, p. 4]). Therefore, the form $(\mathcal{E},\mathcal{F})$ corresponding to the semigroup $(K_t)_{t\geq 0}$ is in fact a *Dirichlet form*.

In what follows we will use the following abbreviation: $\mathcal{E}[u] := \mathcal{E}(u,u)$, $u \in \mathcal{F}$. We want to extend this definition to the whole space $L^2(\mathbb{R})$, by setting $\mathcal{E}[v] = \infty$ if $v \notin \mathcal{F}$. This convention will be useful in the formulation of the Hardy inequality for the form $\mathcal{E}$.

From the general properties of the Dirichlet forms we can prove the following (see proof of the Theorem 1.4.2 in [35]): if $u_1, u_2 \in \mathcal{F}$, $w \in L^2(\mathbb{R})$ satisfy

$$|w(x) - w(y)| \leq |u_1(x) - u_1(y)| + |u_2(x) - u_2(y)|, \qquad x, y \in \mathbb{R},$$

$$|w(x)| \leq |u_1(x)| + |u_2(x)|, \qquad x \in \mathbb{R},$$

then $w \in \mathcal{F}$ and

$$\sqrt{\mathcal{E}[w]} \leq \sqrt{\mathcal{E}[u_1]} + \sqrt{\mathcal{E}[u_2]}. \tag{6.2}$$

By taking $u_1, u_2 \in \mathcal{F}$ and $w := u_1 + u_2 \in L^2(\mathbb{R})$ from (6.2) we obtain the following triangle inequality for $\mathcal{E}^{1/2}$:

$$\sqrt{\mathcal{E}[u_1 + u_2]} \leq \sqrt{\mathcal{E}[u_1]} + \sqrt{\mathcal{E}[u_2]}, \qquad u_1, u_2 \in \mathcal{F}. \tag{6.3}$$

Hence, from the definition of the form $\mathcal{E}$ and from the property (6.3), it follows that in fact $\sqrt{\mathcal{E}[\cdot]}$ is a seminorm on $\mathcal{F}$. From (6.3) it follows also that $\mathcal{F}$ is in fact a linear subspace of $L^2(\mathbb{R})$.

On the space $\mathcal{F}$ we define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ and a norm $\|\cdot\|_{\mathcal{F}}$ by the following expressions:

$$\langle u, v \rangle_{\mathcal{F}} := \langle u, v \rangle_{L^2(\mathbb{R})} + \mathcal{E}(u,v), \qquad u, v \in \mathcal{F},$$

$$\|u\|^2_{\mathcal{F}} := \|u\|^2_{L^2(\mathbb{R})} + \mathcal{E}[u], \qquad u \in \mathcal{F}. \tag{6.4}$$

The fact that $\langle \cdot, \cdot \rangle_{\mathcal{F}}$ is an inner product is an elementary exercise, and we will not present its details. Moreover, let us observe that the norm $\|\cdot\|_{\mathcal{F}}$ is norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}}$. Then from the fact that $\mathcal{E}$ is a Dirichlet form, $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ becomes a Hilbert space.

6.2 Hardy inequality

In this short section we prove the *Hardy inequality* for the Dirichlet form $\mathcal{E}$ on $L^2(\mathbb{R})$. To prove it, we use the methods proposed in Bogdan, Dyda, Kim [10].

Theorem 6.1 (Hardy inequality)**.** *For* $u \in L^2(\mathbb{R})$ *and* $\alpha \in (0,1) \cup (1,2)$,

$$\mathcal{E}[u] \geq (\mathcal{C}_\alpha + \mathcal{D}_\alpha) \int_D u^2(x)|x|^{-\alpha}\, \mathrm{d}x + \mathcal{C}_\alpha \int_{D^c} u^2(x)|x|^{-\alpha}\, \mathrm{d}x, \tag{6.5}$$

where

$$\mathcal{C}_\alpha = \mathcal{A}_{1,\alpha}\Big[\alpha^{-1} - \frac{\big(\Gamma(\frac{\alpha+1}{2})\big)^2}{\Gamma(\alpha+1)}\Big] > 0, \qquad \mathcal{D}_\alpha = \mathcal{A}_{1,\alpha} \int_0^1 \frac{(1 - t^{(\alpha-1)/2})^2}{(1-t)^{\alpha+1}}\, \mathrm{d}t > 0.$$

Proof. Let $h_\beta(x) = |x|^\beta$ for $\beta(\alpha-\beta-1) > 0$ and $v := u/h_\beta$, with the convention that for $h_\beta(x) = 0$ or ∞, we have $v(x) = 0$. Of course $vh_\beta \in L^2(\mathbb{R})$, because $|vh_\beta| \le |u|$. Moreover, from Corollary 3.21 we know that $vK_th_\beta \in L^2(\mathbb{R})$. Therefore, for $t > 0$ we have

$$\mathcal{E}^{(t)}[vh_\beta] = \frac{1}{t}\langle v(h_\beta - K_th_\beta), vh_\beta\rangle_{L^2(\mathbb{R})} + \frac{1}{t}\langle vK_th_\beta - K_t(vh_\beta), vh_\beta\rangle_{L^2(\mathbb{R})} =: A_t + B_t.$$

We observe that from the inequality $ab \le \frac{1}{2}(a^2+b^2)$, $a, b \in \mathbb{R}$, and Lemma 3.5,

$$\begin{aligned}
&\frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v(x)h_\beta(x)v(y)h_\beta(y) \\
&\le \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(x)h_\beta^2(x) + \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(y)h_\beta^2(y) \\
&= \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(x)h_\beta^2(x) + \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(x)h_\beta^2(x) \\
&= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(x)h_\beta^2(x) \\
&\le \frac{1}{t}\int_{\mathbb{R}} v^2(x)h_\beta^2(x)\,\mathrm{d}x < \infty.
\end{aligned}$$

Hence, also from Lemma 3.5,

$$\begin{aligned}
B_t &= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v(x)h_\beta(x)h_\beta(y)(v(x)-v(y)) \\
&= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(x)h_\beta(x)h_\beta(y) - \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v(x)h_\beta(x)v(y)h_\beta(y) \\
&= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v^2(y)h_\beta(y)h_\beta(x) - \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v(x)h_\beta(x)v(y)h_\beta(y) \\
&= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, v(y)h_\beta(y)h_\beta(x)(v(y)-v(x)).
\end{aligned}$$

Moreover, from the above equalities, we can write that

$$B_t = \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\, h_\beta(x)h_\beta(y)(v(x)-v(y))^2 \ge 0.$$

From that fact, we get the following inequality

$$\mathcal{E}^{(t)}[vh_\beta] \ge A_t = \int_{\mathbb{R}} v^2(x)h_\beta(x)\frac{h_\beta(x) - K_th_\beta(x)}{t}\,\mathrm{d}x.$$

From Corollary 3.21, the integrand is non-negative and then from Fatou's lemma and Theorem 5.15 we have

$$\begin{aligned}
\mathcal{E}[vh_\beta] &\ge \int_{\mathbb{R}} v^2(x)h_\beta(x)\liminf_{t\to 0^+}\frac{h_\beta(x) - K_th_\beta(x)}{t}\,\mathrm{d}x \\
&= \mathcal{A}_{1,\alpha}\int_{\mathbb{R}} v^2(x)h_\beta^2(x)\mathcal{C}(\alpha,\beta,x)|x|^{-\alpha}\,\mathrm{d}x.
\end{aligned}$$

Thus,

$$\mathcal{E}[u] \ge (\mathcal{C}_{\alpha,\beta} + \mathcal{D}_{\alpha,\beta})\int_D u^2(x)|x|^{-\alpha}\,\mathrm{d}x + \mathcal{C}_{\alpha,\beta}\int_{D^c} u^2(x)|x|^{-\alpha}\,\mathrm{d}x, \tag{6.6}$$

where

$$\mathcal{C}_{\alpha,\beta} = \mathcal{A}_{1,\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta)\big], \qquad \mathcal{D}_{\alpha,\beta} = -\mathcal{A}_{1,\alpha}\gamma(\alpha,\beta).$$

The fact that $\mathcal{C}_{\alpha,\beta} > 0$ and $\mathcal{D}_{\alpha,\beta} > 0$ follows also from Theorem 5.15.

Now, we will show that the constants $\mathcal{C}_{\alpha,\beta}$ and $\mathcal{D}_{\alpha,\beta}$ have the biggest values for $\beta = (\alpha-1)/2$. Recall that

$$\mathcal{D}_{\alpha,\beta} = -\mathcal{A}_{1,\alpha}\int_0^1 \frac{(t^\beta - 1)(1-t^{\alpha-\beta-1})}{(1-t)^{\alpha+1}}\,\mathrm{d}t,$$

and define the function

$$u(\beta,t) = \frac{(t^\beta - 1)(1-t^{\alpha-\beta-1})}{(1-t)^{\alpha+1}},$$

where $t\in(0,1)$ and $\beta(\alpha-\beta-1)>0$. We will calculate the derivative

$$\frac{\partial}{\partial\beta}\mathcal{D}_{\alpha,\beta} = -\mathcal{A}_{1,\alpha}\frac{\partial}{\partial\beta}\int_0^1 u(\beta,t)\,\mathrm{d}t.$$

We want to use [57, Theorem 11.5] to change the order of derivative and integral. Obviously, the function $(0,1)\ni t\mapsto u(\beta,t)$ is integrable for each considered β and the function $\beta\mapsto u(\beta,t)$ is differentiable for each $t\in(0,1)$. Moreover, note that for $t\in(0,1)$,

$$\left|\frac{\partial}{\partial\beta}u(\beta,t)\right| = \frac{|\ln t|}{(1-t)^{\alpha+1}}\big|t^\beta - t^{\alpha-\beta-1}\big|. \tag{6.7}$$

We will show that, for $t\in(0,1)$ and $\beta(\alpha-\beta-1)>0$,

$$\big|t^\beta - t^{\alpha-\beta-1}\big| \le \big|1-t^{\alpha-1}\big|. \tag{6.8}$$

We consider two cases. First, assume that $1<\alpha<2$ and $0<\beta<\alpha-1$. Then for $\beta\in(0,\frac{\alpha-1}{2})$, $t^{\alpha-1}\le t^{\alpha-1-2\beta}$ and

$$\big|1-t^{\alpha-1}\big| = 1-t^{\alpha-1} \ge 1-t^{\alpha-1-2\beta} \ge t^\beta\big(1-t^{\alpha-1-2\beta}\big) = t^\beta - t^{\alpha-\beta-1} = \big|t^\beta - t^{\alpha-\beta-1}\big|.$$

Similarly, for $\beta\in[\frac{\alpha-1}{2},\alpha-1)$, $t^{\alpha-1}\le t^{2\beta-\alpha+1}$ and

$$\big|1-t^{\alpha-1}\big| = 1-t^{\alpha-1} \ge 1-t^{2\beta-\alpha+1} \ge t^{\alpha-\beta-1}\big(1-t^{2\beta-\alpha+1}\big) = t^{\alpha-\beta-1} - t^\beta = \big|t^\beta - t^{\alpha-\beta-1}\big|.$$

Now assume that $0<\alpha<1$ and $\alpha-1<\beta<0$. Note that (6.8) is equivalent to the inequality

$$\big|t^{-\beta} - t^{(2-\alpha)+\beta-1}\big| \le \big|1-t^{(2-\alpha)-1}\big|. \tag{6.9}$$

Let $\gamma := 2-\alpha\in(1,2)$ and $\delta := -\beta\in(0,\gamma-1)$ and from the first case we get the following inequality

$$\big|t^\delta - t^{\gamma-\delta-1}\big| \le \big|1-t^{\gamma-1}\big|,$$

which is equivalent to the inequality (6.9). Thus, we have proved (6.8) in the second case.

From (6.7) and (6.8) we have

$$\left|\frac{\partial}{\partial\beta}u(\beta,t)\right| \leq \frac{|\ln t|}{(1-t)^{\alpha+1}}|1-t^{\alpha-1}|. \tag{6.10}$$

Moreover,

$$I := \int_0^1 \frac{|\ln t|}{(1-t)^{\alpha+1}}|1-t^{\alpha-1}|\,\mathrm{d}t < \infty. \tag{6.11}$$

We will show that again in two cases.

Assume that $\alpha \in (0,1)$. Note that for $t \in (1/2,1)$ we have $|\ln t| \approx 1-t$ and $t^{\alpha-1}-1 \leq t^{-1}-1 = (1-t)/t$. Then

$$\begin{aligned} I &= \int_0^{1/2} \frac{|\ln t|}{(1-t)^{\alpha+1}}(t^{\alpha-1}-1)\,\mathrm{d}t + \int_{1/2}^1 \frac{|\ln t|}{(1-t)^{\alpha+1}}(t^{\alpha-1}-1)\,\mathrm{d}t \\ &\lesssim \int_0^{1/2} |\ln t|\, t^{\alpha-1}\,\mathrm{d}t + \int_{1/2}^1 \frac{\mathrm{d}t}{t(1-t)^{\alpha-1}} \\ &\lesssim \int_0^{1/2} |\ln t|\, t^{\alpha-1}\,\mathrm{d}t + \int_{1/2}^1 \frac{\mathrm{d}t}{(1-t)^{\alpha-1}} < \infty. \end{aligned}$$

Now let $\alpha \in (1,2)$. Then for $t \in (0,1)$, $1-t^{\alpha-1} \leq 1-t$. Hence,

$$\begin{aligned} I &= \int_0^{1/2} \frac{|\ln t|}{(1-t)^{\alpha+1}}(1-t^{\alpha-1})\,\mathrm{d}t + \int_{1/2}^1 \frac{|\ln t|}{(1-t)^{\alpha+1}}(1-t^{\alpha-1})\,\mathrm{d}t \\ &\lesssim \int_0^{1/2} |\ln t|\,\mathrm{d}t + \int_{1/2}^1 \frac{|\ln t|}{(1-t)^{\alpha}}\,\mathrm{d}t \\ &\approx \int_0^{1/2} |\ln t|\,\mathrm{d}t + \int_{1/2}^1 \frac{\mathrm{d}t}{(1-t)^{\alpha-1}} < \infty. \end{aligned}$$

Thus we have proved (6.11).

Therefore, from [57, Theorem 11.5] we obtain that

$$\frac{\partial}{\partial\beta}\mathcal{D}_{\alpha,\beta} = \mathcal{A}_{1,\alpha}\int_0^1 \frac{\ln t}{(1-t)^{\alpha+1}}\left[t^{\alpha-\beta-1}-t^{\beta}\right]\mathrm{d}t.$$

Note that for $\beta < (\alpha-1)/2$, $\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{D}_{\alpha,\beta} > 0$ and for $\beta > (\alpha-1)/2$, $\frac{\mathrm{d}}{\mathrm{d}\beta}\mathcal{D}_{\alpha,\beta} < 0$. Hence, $\mathcal{D}_{\alpha,\beta}$ has the biggest value for $\beta = (\alpha-1)/2$. Moreover, from the proof of Theorem 5.15, it follows that the constant $\mathcal{C}_{\alpha,\beta}$ has the biggest value also for $\beta = (\alpha-1)/2$.

Thus, by taking the maximum over $\beta \in (0,\alpha-1)$ in (6.6) we get the desired inequality:

$$\mathcal{E}[u] \geq (\mathcal{C}_{\alpha,(\alpha-1)/2} + \mathcal{D}_{\alpha,(\alpha-1)/2})\int_D u^2(x)|x|^{-\alpha}\,\mathrm{d}x + \mathcal{C}_{\alpha,(\alpha-1)/2}\int_{D^c} u^2(x)|x|^{-\alpha}\,\mathrm{d}x. \quad \square$$

Above we have proved the Hardy inequality for $\alpha \in (0,1)\cup(1,2)$. In case $\alpha = 1$ the above proof is also valid, but with this method we will obtain only trivial inequality of the form $\mathcal{E}[u] \geq 0$, $u \in L^2(\mathbb{R})$.

From Theorem 6.1 we have obvious corollary.

Corollary 6.2. *For $u \in L^2(\mathbb{R})$ and $\alpha \in (0,1) \cup (1,2)$,*

$$\int_{\mathbb{R}} u^2(x)|x|^{-\alpha}\,\mathrm{d}x \lesssim \mathcal{E}[u].$$

6.3 The characterization of a domain of the Dirichlet form

Recall that $D = (0,\infty)$ and define the form $\mathcal{E}_D$ on $L^2(\mathbb{R})$ with its natural domain $\mathcal{D}(\mathcal{E}_D)$ as follows:

$$\mathcal{E}_D(u,v) := \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} (u(x)-u(y))(v(x)-v(y))\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y, \qquad u,v \in L^2(\mathbb{R}),$$

and

$$\mathcal{D}(\mathcal{E}_D) := \{u \in L^2(\mathbb{R}) :\ \mathcal{E}_D(u,u) < \infty\}.$$

Similarly, we write $\mathcal{E}_D[u] := \mathcal{E}_D(u,u)$. Note that the set $\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c$ in the definition of $\mathcal{E}_D$ is equal to the sum $(D\times D)\cup(D\times D^c)\cup(D^c\times D)$.

We want to define a norm on the space $\mathcal{D}(\mathcal{E}_D)$. For this purpose we prove the following lemma.

Lemma 6.3. *A function $p : \mathcal{D}(\mathcal{E}_D) \to \mathbb{R}$ defined by $p(u) := \sqrt{\mathcal{E}_D[u]}$ is a seminorm.*

Proof. The absolute homogeneity follows immediately from the definition of the form $\mathcal{E}_D$. We will prove the triangle inequality for p, i.e. the inequality $p(u+v) \le p(u)+p(v)$, $u,v \in \mathcal{D}(\mathcal{E}_D)$. We will use the analogous methods as in the proof of the Minkowski inequality (see Stein and Shakarchi [64, Theorem 1.2]).

Let $u,v \in \mathcal{D}(\mathcal{E}_D)$. From the inequality $(a-b)^2 \le 2a^2+2b^2$, $a,b\in\mathbb{R}$, we get

$$\begin{aligned}2p^2(u+v) &= \iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} (u(x)+v(x)-u(y)-v(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\le 4\mathcal{E}_D[u]+4\mathcal{E}_D[v] < \infty,\end{aligned}$$

which means that $p(u+v)<\infty$. Moreover, we may assume that $p(u+v)>0$, because if $p(u+v)=0$, then the triangle inequality for p is obvious.

From the inequality $|x+y| \le |x|+|y|$, $x,y\in\mathbb{R}$, and from the Hölder inequality we get

$$\begin{aligned}2p^2(u+v) &= \iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} (u(x)+v(x)-u(y)-v(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\le \iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} |u(x)-u(y)||u(x)+v(x)-u(y)-v(y)|\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &+ \iint_{\mathbb{R}\times\mathbb{R}\setminus D^c\times D^c} |v(x)-v(y)||u(x)+v(x)-u(y)-v(y)|\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\ &\le 2p(u)p(u+v)+2p(v)p(u+v).\end{aligned}$$

Dividing both sides of the above inequality by $2p(u+v)$ we get a desired triangle inequality for p. $\square$

In what follows we will also consider the inner product $\langle\cdot,\cdot\rangle_{\mathcal{E}_D}$ and the norm $\|\cdot\|_{\mathcal{E}_D}$ corresponding to the form $\mathcal{E}_D$ defined by the following expressions:

$$\langle u,v\rangle_{\mathcal{E}_D} := \langle u,v\rangle_{L^2(\mathbb{R})} + \mathcal{E}_D(u,v), \qquad u,v\in\mathcal{D}(\mathcal{E}_D),$$

$$\|u\|^2_{\mathcal{E}_D} := \|u\|^2_{L^2(\mathbb{R})} + \mathcal{E}_D[u], \qquad u\in\mathcal{D}(\mathcal{E}_D).$$

In this section we propose the connection between the Dirichlet form $(\mathcal{E},\mathcal{F})$ corresponding to the semigroup $(K_t)_{t\geq 0}$ and the form $(\mathcal{E}_D,\mathcal{F}^*)$, where

$$\mathcal{F}^* := \Big\{u\in\mathcal{D}(\mathcal{E}_D):\ \int_{\mathbb{R}} u^2(x)|x|^{-\alpha}\,\mathrm{d}x<\infty\Big\}.$$

The first connection is established in the following proposition, which describes the explicit form of the Dirichlet form $\mathcal{E}$ on the smooth functions with compact support.

Theorem 6.4. *Let $u,v\in C_c^\infty(\mathbb{R}^*)$. Then $u,v\in\mathcal{D}(\mathcal{E}_D)\cap\mathcal{F}$ and*

$$\mathcal{E}(u,v)=\mathcal{E}_D(u,v). \tag{6.12}$$

Proof. We will show that $C_c^\infty(\mathbb{R}^*)\subset\mathcal{D}(\mathcal{E}_D)$. Let $u\in C_c^\infty(\mathbb{R}^*)$ and observe that it is obvious that there exist $f\in C_c^\infty(D)$ and $g\in C_c^\infty\big(\overline{D}^c\big)$ such that $u=f+g$. From the inequality $(a+b)^2\leq 2a^2+2b^2$, $a,b\in\mathbb{R}$, it suffices to show that $f,g\in\mathcal{D}(\mathcal{E}_D)$.

It is obvious that $f,g\in L^2(\mathbb{R})$. Note that from Tonelli's theorem,

$$\mathcal{E}_D[f]=\frac{1}{2}\iint_{D\times D}(f(x)-f(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y+\int_D f^2(x)\nu(x,D^c)\,\mathrm{d}x. \tag{6.13}$$

From (2.13) it follows that

$$\int_D f^2(x)\nu(x,D^c)\,\mathrm{d}x\lesssim\|f\|^2_\infty\int_{\mathrm{supp}(f)}|x|^{-\alpha}\,\mathrm{d}x<\infty,$$

hence it suffices to show the finiteness of the first integral in (6.13). From the fact that $f\in C_c^\infty(D)$ and by the symmetry it is obvious that

$$\begin{aligned}
&\iint_{D\times D}(f(x)-f(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\leq\iint_{(\mathrm{supp}\,f\times D)\cup(D\times\mathrm{supp}\,f)}(f(x)-f(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&=2\int_{\mathrm{supp}\,f}\mathrm{d}x\int_D\mathrm{d}y\,(f(x)-f(y))^2\nu(x,y)\\
&\approx\int_{\mathrm{supp}\,f}\mathrm{d}x\int_{D\cap\{|y-x|<1\}}\mathrm{d}y\,(f(x)-f(y))^2\nu(x,y)\\
&+\int_{\mathrm{supp}\,f}\mathrm{d}x\int_{D\cap\{|y-x|\geq 1\}}\mathrm{d}y\,(f(x)-f(y))^2\nu(x,y)=:A+B.
\end{aligned}$$

In case of the integral A we note that from Lagrange mean value theorem $|f(x) - f(y)| = |f'(c)||x-y|$, where c lies between x and y. Moreover, for $|y-x| < 1$, $|f'(c)| \le M$ for some constant $M > 0$. Then,

$$A \lesssim \int_{\operatorname{supp} f} \mathrm{d}x \int_{\{|y-x|<1\}} \mathrm{d}y\, |x-y|^2 \nu(x,y) \approx \int_{\operatorname{supp} f} \mathrm{d}x \int_{\{|w|<1\}} |w|^{1-\alpha}\, \mathrm{d}w < \infty.$$

Moreover,

$$B \lesssim 2\, \|f\|_\infty^2 \int_{\operatorname{supp} f} \mathrm{d}x \int_{\{|y-x|\ge 1\}} \mathrm{d}y\, |x-y|^{-\alpha-1} \lesssim \int_{\operatorname{supp} f} \mathrm{d}x \int_{\{|w|\ge 1\}} \mathrm{d}y\, |w|^{-\alpha-1} < \infty.$$

Thus we have proved that $\mathcal{E}_D[f] < \infty$.

Similarly,

$$\mathcal{E}_D[g] = \int_{D^c} g^2(x)\nu(x,D)\, \mathrm{d}x \lesssim \|g\|_\infty^2 \int_{\operatorname{supp}(g)} |x|^{-\alpha}\, \mathrm{d}x < \infty.$$

Hence, $f, g \in \mathcal{D}(\mathcal{E}_D)$.

Let $u, v \in C_c^\infty(\mathbb{R}^*)$ and recall that

$$\begin{aligned}\mathcal{E}(u,v) &= \lim_{t\to 0^+} \frac{1}{t} \int_{\mathbb{R}} \big[u(x) - K_t u(x)\big] v(x)\, \mathrm{d}x \\ &= \lim_{t\to 0^+} \frac{1}{t} \int_{D} \big[u(x) - K_t u(x)\big] v(x)\, \mathrm{d}x + \lim_{t\to 0^+} \frac{1}{t} \int_{D^c} \big[u(x) - K_t u(x)\big] v(x)\, \mathrm{d}x \\ &=: I + II. \end{aligned} \tag{6.14}$$

Therefore, without loss of generality, in what follows, we may assume that $t \in (0,1)$.

Let $x > 0$. From Corollary 3.4,

$$\begin{aligned}&\frac{1}{t}\big[u(x) - K_t u(x)\big] \\ &= \frac{1}{t}\big[u(x) - P_t^D u(x)\big] - \frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x,y)\nu(y,z) K_{t-r} u(z). \end{aligned} \tag{6.15}$$

We will show that

$$\begin{aligned}\mathcal{C}^D(u,v) &:= \lim_{t\to 0^+} \frac{1}{t} \int_D \big[u(x) - P_t^D u(x)\big] v(x)\, \mathrm{d}x \\ &= \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))(v(x)-v(y))\nu(x,y) + \int_D u(x)v(x)\nu(x,D^c)\, \mathrm{d}x. \end{aligned} \tag{6.16}$$

The exact form of the form $\mathcal{C}^D$ for the process Y is commonly known (see e.g. Fukushima et al. [35, Theorem 4.5.2, p. 185]), but we propose its proof for completion of the argument.

Note that $u(x) = u(x)p_t^D(x,D) + u(x)\big(1 - p_t^D(x,D)\big)$. Thus,

$$\begin{aligned}
&\frac{1}{t}\int_D \big[u(x) - P_t^D u(x)\big] v(x)\,\mathrm{d}x \\
&= \int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))v(x)\frac{p_t^D(x,y)}{t} + \int_D u(x)v(x)\frac{1-p_t^D(x,D)}{t}\,\mathrm{d}x \\
&=: A_t + B_t.
\end{aligned}$$

From (2.22) it follows that $p_t^D(x,y)/t \lesssim \nu(x,y)$. Moreover, from the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, $a, b \in \mathbb{R}$, we have

$$\begin{aligned}
&\int_D \mathrm{d}x \int_D \mathrm{d}y\,|u(x)-u(y)||v(x)|\frac{p_t^D(x,y)}{t} \\
&\le \|v\|_\infty\, t^{-1} \int_{D\cap\operatorname{supp}(v)} \mathrm{d}x \int_D \mathrm{d}y\,(|u(x)| + |u(y)|)p_t^D(x,y) \\
&\le \|u\|_\infty \|v\|_\infty\, t^{-1}\left[\int_{D\cap\operatorname{supp}(v)} p_t^D(x,D)\,\mathrm{d}x + \int_{D\cap\operatorname{supp}(v)} \mathrm{d}x \int_{D\cap\operatorname{supp}(u)} p_t^D(x,y)\,\mathrm{d}y\right] \\
&\le \|u\|_\infty \|v\|_\infty\, t^{-1}\big[|\operatorname{supp}(v)| + t^{-1/\alpha}|\operatorname{supp}(v)|\cdot|\operatorname{supp}(u)|\big] < \infty.
\end{aligned}$$

From the Fubini's theorem and from the symmetry of p^D,

$$\begin{aligned}
A_t &= \int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))v(x)\frac{p_t^D(x,y)}{t} \\
&= -\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))v(y)\frac{p_t^D(x,y)}{t}.
\end{aligned}$$

Hence,

$$A_t = \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))(v(x)-v(y))\frac{p_t^D(x,y)}{t}.$$

Note that from the inequalities $p_t^D(x,y)/t \lesssim \nu(x,y)$ and $ab \le \frac{1}{2}(a^2+b^2)$, $a, b \in \mathbb{R}$, and from the fact that

$$\begin{aligned}
&\int_D \mathrm{d}x \int_D \mathrm{d}y\,|u(x)-u(y)||v(x)-v(y)|\nu(x,y) \\
&\le \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))^2\nu(x,y) + \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(v(x)-v(y))^2\nu(x,y) \\
&\le \mathcal{E}_D[u] + \mathcal{E}_D[v] < \infty,
\end{aligned}$$

it follows that we can use the dominated convergence theorem to obtain that

$$\begin{aligned}
\lim_{t\to 0^+} A_t &= \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))(v(x)-v(y)) \lim_{t\to 0^+}\frac{p_t^D(x,y)}{t} \\
&= \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))(v(x)-v(y))\nu(x,y).
\end{aligned}$$

The latter convergence follows from Lemma 5.8.

From (2.25), (2.26) and from Corollary 5.4 it follows that

$$\frac{1-p_t^D(x,D)}{t} = \frac{1}{t}\int_0^t \mathrm{d}s \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_s^D(x,y)\nu(y,z) \lesssim |x|^{-\alpha}, \tag{6.17}$$

and

$$\int_D u(x)v(x)|x|^{-\alpha}\,\mathrm{d}x \le \|u\|_\infty \|v\|_\infty \int_{D\cap \operatorname{supp}(u)} |x|^{-\alpha}\,\mathrm{d}x < \infty.$$

Therefore, again from the dominated convergence theorem, (6.17) and Lemma 5.12 it follows that

$$\lim_{t\to 0^+} B_t = \int_D u(x)v(x) \lim_{t\to 0^+} \frac{1-p_t^D(x,D)}{t}\,\mathrm{d}x = \int_D u(x)v(x)\nu(x,D^c)\,\mathrm{d}x.$$

Thus, we have proved (6.16).

Further, from the fact that $|K_t u(z)| \le \|u\|_\infty$, $t>0$, $z\ne 0$, and from Fubini's theorem,

$$\begin{aligned} C(t,x) :&= \frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x,y)\nu(y,z)K_{t-r}u(z) \\ &= \int_0^1 \mathrm{d}r \int_{D^c} \mathrm{d}z \Big[\int_D p_{tr}^D(x,y)\nu(y,z)\,\mathrm{d}y\Big] K_{t(1-r)}u(z) \\ &= \int_0^1 \mathrm{d}r \int_{D^c} \mathcal{J}(tr,x,z)K_{t(1-r)}u(z)\,\mathrm{d}z. \end{aligned}$$

From Corollary 5.3 and from (2.4),

$$\begin{aligned} \mathcal{J}(tr,x,z)|K_{t(1-r)}u(z)| &\lesssim \|u\|_\infty (tr)^{-1}\Big(1\wedge \frac{|z|}{(tr)^{1/\alpha}}\Big)^{-\alpha/2} \frac{tr}{|x-z|^{\alpha+1}} \\ &\lesssim \|u\|_\infty \Big(1\wedge \frac{|z|}{r^{1/\alpha}}\Big)^{-\alpha/2} |x-z|^{-\alpha-1}, \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \mathrm{d}r \int_{D^c} \Big(1\wedge \frac{|z|}{r^{1/\alpha}}\Big)^{-\alpha/2} |x-z|^{-\alpha-1}\,\mathrm{d}z \\ &= \int_0^1 \mathrm{d}r \int_{-\infty}^{-r^{1/\alpha}} |x-z|^{-\alpha-1}\,\mathrm{d}z + \int_0^1 \mathrm{d}r \int_{-r^{1/\alpha}}^0 \sqrt{r}|z|^{-\alpha/2}|x-z|^{-\alpha-1}\,\mathrm{d}z \\ &\le \int_{-\infty}^0 |x-z|^{-\alpha-1}\,\mathrm{d}z + \int_0^1 \sqrt{r}\mathrm{d}r \int_{-\infty}^0 |z|^{-\alpha/2}|x-z|^{-\alpha-1}\,\mathrm{d}z < \infty. \end{aligned}$$

Therefore, from the dominated convergence theorem, Lemma 5.10 and Lemma 4.13 it follows that

$$\lim_{t\to 0^+} C(t,x) = \int_0^1 \mathrm{d}r \int_{D^c} \lim_{t\to 0^+} \mathcal{J}(tr,x,z)K_{t(1-r)}u(z)\,\mathrm{d}z = \int_{D^c} \nu(x,z)u(z)\,\mathrm{d}z. \tag{6.18}$$

From Corollary 5.4 we have

$$\frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y \int_{D^c} \mathrm{d}z\, p_r^D(x,y)\nu(y,z)|K_{t-r}u(z)| \lesssim \|u\|_\infty |x|^{-\alpha},$$

and

$$\int_D v(x)|x|^{-\alpha}\,\mathrm{d}x \le \|v\|_\infty \int_{D\,\cap\,\mathrm{supp}\,(v)} |x|^{-\alpha}\,\mathrm{d}x < \infty.$$

Hence, from the dominated convergence theorem and from (6.18),

$$\lim_{t\to 0^+}\int_D v(x)C(t,x)\,\mathrm{d}x = \int_D v(x)\lim_{t\to 0^+} C(t,x)\,\mathrm{d}x = \int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, u(y)v(x)\nu(x,y). \tag{6.19}$$

Combining (6.15), (6.16) and (6.19) we get

$$\begin{aligned} I &= \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))(v(x)-v(y))\nu(x,y) \\ &\quad + \int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\,(u(x)-u(y))v(x)\nu(x,y). \end{aligned} \tag{6.20}$$

Now assume that $x<0$. From Corollary 3.4 we have

$$\frac{1}{t}\big[u(x)-K_t u(x)\big] = u(x)\frac{1-e^{-\nu(x,D)t}}{t} - \frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y\, e^{-\nu(x,D)r}\nu(x,y)K_{t-r}u(y). \tag{6.21}$$

Moreover, from the inequality $1-e^{-x}\le x$, $x\ge 0$, and from the fact that

$$\int_{D^c} |u(x)v(x)|\nu(x,D)\,\mathrm{d}x \lesssim \|u\|_\infty \|v\|_\infty \int_{D^c\,\cap\,\mathrm{supp}\,(u)} |x|^{-\alpha}\,\mathrm{d}x < \infty,$$

we can use the dominated convergence theorem to obtain that

$$\begin{aligned} \lim_{t\to 0^+}\int_{D^c} u(x)v(x)\frac{1-e^{-\nu(x,D)t}}{t}\,\mathrm{d}x &= \int_{D^c} u(x)v(x)\nu(x,D)\,\mathrm{d}x \\ &= \int_{D^c} dx \int_D \mathrm{d}y\, u(x)v(x)\nu(x,y). \end{aligned} \tag{6.22}$$

Furthermore,

$$e^{-\nu(x,D)tr}\int_D \nu(x,y)|K_{t(1-r)}u(y)|\,\mathrm{d}y \le \|u\|_\infty \nu(x,D),$$

and

$$\int_{D^c} \mathrm{d}x \int_0^1 \mathrm{d}r\,|v(x)|\nu(x,D) \lesssim \|v\|_\infty \int_{D^c\,\cap\,\mathrm{supp}\,(v)} |x|^{-\alpha} < \infty.$$

Then, from the dominated convergence theorem and from Lemma 4.13,

$$\begin{aligned} &\lim_{t\to 0^+}\int_{D^c} v(x)\left[\frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}y\, e^{-\nu(x,D)r}\nu(x,y)K_{t-r}u(y)\right]\mathrm{d}x \\ &= \lim_{t\to 0^+}\int_{D^c}\mathrm{d}x \int_0^1 \mathrm{d}r\, v(x)e^{-\nu(x,D)tr}\int_D \nu(x,y)K_{t(1-r)}u(y)\,\mathrm{d}y \\ &= \int_{D^c}\mathrm{d}x \int_0^1 \mathrm{d}r\, v(x)\lim_{t\to 0^+}\int_D \nu(x,y)K_{t(1-r)}u(y)\,\mathrm{d}y \\ &= \int_{D^c}\mathrm{d}x \int_D \mathrm{d}y\, u(y)v(x)\nu(x,y). \end{aligned} \tag{6.23}$$

Combining (6.21), (6.22) and (6.23) we get

$$II = \int_{D^c} dx \int_D \mathrm{d}y\, (u(x) - u(y))v(x)\nu(x,y). \tag{6.24}$$

From (6.14), (6.20) and (6.24),

$$\begin{aligned}\mathcal{E}(u,v) &= \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))(v(x)-v(y))\nu(x,y)\\ &+ \int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y)\\ &+ \int_{D^c} dx \int_D \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y).\end{aligned} \tag{6.25}$$

Note that from the inequality $ab \le \frac{1}{2}(a^2+b^2)$, $a, b \in \mathbb{R}$, it follows that

$$\begin{aligned}&\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, |u(x)-u(y)||v(x)|\nu(x,y)\\ &\le \frac{1}{2}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))^2\nu(x,y) + \frac{1}{2}\int_D v^2(x)\nu(x,D^c)\,\mathrm{d}x\\ &\lesssim \mathcal{E}_D[u] + \frac{1}{2}\|v\|_\infty^2 \int_{D\cap \operatorname{supp}(v)} |x|^{-\alpha} < \infty.\end{aligned}$$

Thus, from the Fubini's theorem and the symmetry of ν we get

$$\begin{aligned}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y) &= \int_{D^c} \mathrm{d}y \int_D \mathrm{d}x\, (u(x)-u(y))v(x)\nu(x,y)\\ &= -\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))v(y)\nu(x,y).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y)\\ &= \frac{1}{2}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y) - \frac{1}{2}\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))v(y)\nu(x,y).\end{aligned} \tag{6.26}$$

Similarly, we prove that

$$\begin{aligned}&\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y)\\ &= \frac{1}{2}\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\, (u(x)-u(y))v(x)\nu(x,y) - \frac{1}{2}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\, (u(x)-u(y))v(y)\nu(x,y).\end{aligned} \tag{6.27}$$

Combining (6.25), (6.26) and (6.27) we get (6.12), which completes the proof. □

The above result gives us explicit form of the Dirichlet form $\mathcal{E}$ only for the smooth functions with compact support. To extend this result for a larger set of functions, we will study equivalent definitions of the domain $\mathcal{F}$.

Lemma 6.5. *For $u \in L^2(\mathbb{R})$, $\mathcal{E}_D[u] \le \mathcal{E}[u]$.*

Proof. From Lemma 3.10 we have

$$\begin{aligned}\mathcal{E}^{(t)}[u] &= \frac{1}{t}\int_{\mathbb{R}} \big[u(x)K_t(x,\mathbb{R}) - K_t u(x) + u(x)\big(1-K_t(x,\mathbb{R})\big)\big]u(x)\,\mathrm{d}x \\ &= \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))u(x) + \frac{1}{t}\int_{\mathbb{R}} u^2(x)\big(1-K_t(x,\mathbb{R})\big)\,\mathrm{d}x \\ &\ge \frac{1}{t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))u(x). \end{aligned} \tag{6.28}$$

Moreover, $\int_{\mathbb{R}} u^2(x)K_t\mathbf{1}(x)\,\mathrm{d}x < \infty$ and then from Lemma 3.5,

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))u(x) &= \int_{\mathbb{R}} u^2(x)K_t\mathbf{1}(x)\,\mathrm{d}x - \int_{\mathbb{R}} u(x)K_t u(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} K_t u^2(x)\,\mathrm{d}x - \int_{\mathbb{R}} u(x)K_t u(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(y)-u(x))u(y). \end{aligned}$$

Hence,

$$\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))u(x) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))^2. \tag{6.29}$$

Recall that $K_{t,1}(x,D) = 0$ for $x > 0$ and $K_{t,1}(x,D^c) = 0$ for $x < 0$. Combining (6.28) and (6.29) we obtain that

$$\begin{aligned}\mathcal{E}^{(t)}[u] &\ge \frac{1}{2t}\int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} K_t(x,\mathrm{d}y)\,(u(x)-u(y))^2 \\ &\ge \frac{1}{2t}\int_D \mathrm{d}x \int_D \mathrm{d}y\, p_t^D(x,y)(u(x)-u(y))^2 + \frac{1}{2t}\int_D \mathrm{d}x \int_{D^c} K_{t,1}(x,\mathrm{d}y)(u(x)-u(y))^2 \\ &+ \frac{1}{2t}\int_{D^c} \mathrm{d}x \int_D K_{t,1}(x,\mathrm{d}y)(u(x)-u(y))^2 \\ &= \frac{1}{2t}\int_D \mathrm{d}x \int_D \mathrm{d}y\, p_t^D(x,y)(u(x)-u(y))^2 \\ &+ \frac{1}{2}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\,(u(x)-u(y))^2 \left[\frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}a\, p_r^D(x,a)\nu(a,y)e^{-\nu(y,D)(t-r)}\right] \\ &+ \frac{1}{2}\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))^2 \Big[\frac{1}{t}\int_0^t \mathrm{d}r \int_D \mathrm{d}b\, e^{-\nu(x,D)r}\nu(x,b)p_{t-r}^D(b,y)\Big]. \end{aligned}$$

From the Fatou's lemma, Lemma 5.8 and Lemma 5.11,

$$\begin{aligned}\mathcal{E}[u] &\ge \frac{1}{2}\int_D \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))^2\nu(x,y) \\ &+ \frac{1}{2}\int_D \mathrm{d}x \int_{D^c} \mathrm{d}y\,(u(x)-u(y))^2\nu(x,y) \\ &+ \frac{1}{2}\int_{D^c} \mathrm{d}x \int_D \mathrm{d}y\,(u(x)-u(y))^2\nu(y,x) \\ &= \mathcal{E}_D[u], \end{aligned}$$

which is our claim. □

For $u \in L^2(\mathbb{R})$ and $A \subset \mathbb{R}$ we set

$$u_A(x) := u(x)\,\mathbb{1}_A(x) = \begin{cases} u(x), & x \in A, \\ 0, & x \notin A. \end{cases}$$

Of course $u_A \in L^2(\mathbb{R})$ and $u = u_D + u_{\overline{D}^c}$ a.e. With such definition, we have the following lemma.

Lemma 6.6. *Let $u \in \mathcal{F}^*$. Then each of the functions u_D and $u_{\overline{D}^c}$ belongs to $\mathcal{D}(\mathcal{E}_D)$.*

Proof. From (2.13) we have

$$\begin{aligned} \mathcal{E}_D[u_D] &= \frac{1}{2}\int_D\int_D (u(x)-u(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_D u^2(x)\nu(x,D^c)\,\mathrm{d}x \\ &\lesssim \mathcal{E}_D[u] + \int_D u^2(x)|x|^{-\alpha}\,\mathrm{d}x < \infty. \end{aligned}$$

Then similarly,

$$\mathcal{E}_D[u_{\overline{D}^c}] = \int_{D^c} u^2(x)\nu(x,D)\,\mathrm{d}x \approx \int_{D^c} u^2(x)|x|^{-\alpha}\,\mathrm{d}x < \infty.$$ □

Lemma 6.7. *Let $u \in L^2(\mathbb{R})$. Assume that there exists a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^*)$ such that $\|u-u_n\|_{\mathcal{F}} \to 0$ as $n \to \infty$, then $\|u-u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$. Conversely, if there exists $(u_n) \subset C_c^\infty(\mathbb{R}^*)$ such that $\|u-u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$, then $\|u-u_n\|_{\mathcal{F}} \to 0$ as $n \to \infty$. Hence,*

$$\overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{F}}} = \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{E}_D}}.$$

Proof. Assume that there exists $(u_n) \subset C_c^\infty(\mathbb{R}^*)$ such that $\|u-u_n\|_{\mathcal{F}} \to 0$. Then (u_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_{\mathcal{F}}$, i.e. $\|u_n-u_m\|_{\mathcal{F}}^2 = \|u_n-u_m\|_{L^2(\mathbb{R})}^2 + \mathcal{E}[u_n-u_m] \to 0$ as $n,m \to \infty$. From Theorem 6.4 we get that $\|u_n-u_m\|_{L^2(\mathbb{R})}^2 + \mathcal{E}_D[u_n-u_m] \to 0$ as $n,m \to \infty$. Thus $\|u_n-u_m\|_{\mathcal{E}_D} \to 0$, $n,m \to \infty$. From the fact that $\mathcal{E}_D$ is closed and symmetric form (for the proof of the closedness see Voight [65, Lemma 2.19]), it follows that there exists $\widehat{u} \in \mathcal{D}(\mathcal{E}_D)$ such that $\|\widehat{u}-u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$ (see [35, p. 4]). We have obtained so far that in particular $\|u-u_n\|_{L^2(\mathbb{R})} \to 0$ and $\|\widehat{u}-u_n\|_{L^2(\mathbb{R})} \to 0$, $n \to \infty$. Hence, from the uniqueness of the limit in $L^2(\mathbb{R})$ it follows that $\widehat{u} = u$ a.e. Therefore, $\|u-u_n\|_{\mathcal{E}_D} = \|\widehat{u}-u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$.

We know that $\mathcal{E}$ is a Dirichlet form, hence it is closed. Therefore, the second implication follows in the same way. □

Lemma 6.8. *If $u \in \mathcal{D}(\mathcal{E}_D)$ and $u = 0$ a.e. on D then there exists a sequence $(u_n) \subset C_c^\infty(\overline{D}^c)$ such that $\|u-u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$.*

Proof. Note that

$$\|u\|_{\mathcal{E}_D}^2 = \int_{D^c} |u(x)|^2\big(1+\nu(x,D)\big)\,\mathrm{d}x = \int_{\overline{D}^c} \Big[u(x)\sqrt{1+\nu(x,D)}\Big]^2\,\mathrm{d}x. \tag{6.30}$$

Let $f(x) := u(x)\sqrt{1+\nu(x,D)}$ for $x \in \overline{D}^c$. Then from (6.30) it follows that $f \in L^2(\overline{D}^c)$. From Lemma 3.1 in [63, p. 222] it follows that there exists a sequence $(f_n) \subset C_c^\infty\big(\overline{D}^c\big)$ such that $\|f - f_n\|_{L^2(\overline{D}^c)} \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}\|f - f_n\|_{L^2(\overline{D}^c)}^2 &= \int_{D^c} |f(x) - f_n(x)|^2\,\mathrm{d}x = \int_{D^c} \Big[u(x)\sqrt{1+\nu(x,D)} - f_n(x)\Big]^2\,\mathrm{d}x \\ &= \int_{D^c} \Big[u(x) - u_n(x)\Big]^2 \big(1+\nu(x,D)\big)\,\mathrm{d}x \to 0, \end{aligned} \tag{6.31}$$

where $u_n(x) := \frac{f_n(x)}{\sqrt{1+\nu(x,D)}}$ for $x \in \overline{D}^c$. It is an easy exercise to show that $u_n \in C_c^\infty(\overline{D}^c)$. From (6.30) and (6.31) we get

$$\|u - u_n\|_{\mathcal{E}_D}^2 = \int_{D^c} \big[u(x) - u_n(x)\big]^2\big(1+\nu(x,D)\big)\,\mathrm{d}x \to 0,$$

as $n \to \infty$. □

Theorem 6.9. *For $\alpha \in (0,1) \cup (1,2)$ we have the following equalities between the domains:*

$$\mathcal{F} = \mathcal{F}^* = \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{E}_D}} = \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{F}}}.$$

Proof. Note that the inclusion $\overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{F}}} \subset \mathcal{F}$ is obvious by the definition. Therefore, from Lemma 6.7 it suffices to show the following inclusions: $\mathcal{F} \subset \mathcal{F}^* \subset \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{E}_D}}$.

Assume that $u \in \mathcal{F}$, i.e. $u \in L^2(\mathbb{R})$ and $\mathcal{E}[u] < \infty$. Then from Lemma 6.5, $\mathcal{E}_D[u] < \infty$. Moreover, from Corollary 6.2, $\int_{\mathbb{R}} u^2(x)|x|^{-\alpha}\,\mathrm{d}x \lesssim \mathcal{E}[u] < \infty$. Hence, $u \in \mathcal{F}^*$.

Assume that $u \in \mathcal{F}^*$. From Lemma 6.6 it follows that u_D and $u_{\overline{D}^c}$ belongs to $\mathcal{D}(\mathcal{E}_D)$. Note that

$$\mathcal{E}_D[u_D] = \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} (u(x) - u(y))^2\nu(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

and by Fiscella et al. [34, Theorem 6] it follows that there exists a sequence $(f_n) \subset C_c^\infty(D)$ such that $\|u_D - f_n\|_{L^2(\mathbb{R})} + \mathcal{E}_D^{1/2}[u_D - f_n] \to 0$ as $n \to \infty$. Therefore, $\|u_D - f_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$. Moreover, from Lemma 6.8 for $u_{\overline{D}^c}$ there exists a sequence $(g_n) \subset C_c^\infty(\overline{D}^c)$ such that $\|u_{\overline{D}^c} - g_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$ let $u_n := f_n + g_n$ and note that $u_n \in C_c^\infty(\mathbb{R}^*)$. Furthermore,

$$\|u - u_n\|_{\mathcal{E}_D} = \|(u_D - f_n) + (u_{\overline{D}^c} - g_n)\|_{\mathcal{E}_D} \le \|u_D - f_n\|_{\mathcal{E}_D} + \|u_{\overline{D}^c} - g_n\|_{\mathcal{E}_D} \to 0,$$

as $n \to \infty$. Thus $u \in \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{E}_D}}$. □

For an analogous result to this one given in Theorem 6.9 see [14, Proposition 6.3].

In terms of Dirichlet forms, we may then say that the form $(\mathcal{E}, \mathcal{F})$ is in fact regular. Indeed, by the general theory of Dirichlet forms (see [35, p. 6]) it follows that the symmetric form $\mathcal{E}$ is called *regular* if there exists a subset $\mathcal{C}$ of $\mathcal{F} \cap C_c(\mathbb{R}^*)$ such that $\mathcal{C}$ is dense in $\mathcal{F}$ with norm $\|\cdot\|_{\mathcal{F}}$ and dense in $C_c(\mathbb{R}^*)$ with uniform norm. The set $\mathcal{C}$ is then called a *core* of the form $\mathcal{E}$. From Theorem 6.9 it follows immediately that $\mathcal{C} = C_c^\infty(\mathbb{R}^*)$ is a core for $(\mathcal{E}, \mathcal{F})$. According to this, we get the following corollary.

Corollary 6.10. *For $\alpha \in (0,1) \cup (1,2)$ the form $(\mathcal{E}, \mathcal{F})$ is regular.*

The next proposition describes the explicit form of the form $\mathcal{E}$ on the larger class of functions than in the Theorem 6.4.

Theorem 6.11. *For $\alpha \in (0,1) \cup (1,2)$ and $u \in \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{F}}}$,*

$$\mathcal{E}[u] = \mathcal{E}_D[u].$$

Proof. Let $u \in \overline{C_c^\infty(\mathbb{R}^*)}^{\|\cdot\|_{\mathcal{F}}}$. Then there exists a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^*)$ such that $\|u - u_n\|_{\mathcal{F}} \to 0$, $n \to \infty$. From (6.3), (6.4) and from Theorem 6.4 it follows that

$$\sqrt{\mathcal{E}[u]} = \sqrt{\mathcal{E}[(u-u_n)+u_n]} \leq \sqrt{\mathcal{E}[u-u_n]} + \sqrt{\mathcal{E}[u_n]} \leq \|u-u_n\|_{\mathcal{F}} + \sqrt{\mathcal{E}_D[u_n]}.$$

By taking $n \to \infty$, we get

$$\sqrt{\mathcal{E}[u]} \leq \lim_{n\to\infty} \sqrt{\mathcal{E}_D[u_n]}. \tag{6.32}$$

We will show that $\mathcal{E}_D[u_n] \to \mathcal{E}_D[u]$, $n \to \infty$. Indeed, from Lemma 6.7 it follows that $\|u - u_n\|_{\mathcal{E}_D} \to 0$ as $n \to \infty$ and then from Lemma 6.3, by the inverse triangle inequality for $\sqrt{\mathcal{E}_D[\cdot]}$, we have

$$\left|\sqrt{\mathcal{E}_D[u]} - \sqrt{\mathcal{E}_D[u_n]}\right| \leq \sqrt{\mathcal{E}_D[u-u_n]} \leq \|u-u_n\|_{\mathcal{E}_D} \to 0, \tag{6.33}$$

as $n \to \infty$. Thus, from (6.32) and (6.33) we get the inequality $\mathcal{E}[u] \leq \mathcal{E}_D[u]$. Which together with Lemma 6.5 ends the proof. ◻

Chapter 7

Boundary problems

This chapter is devoted to finding the direct solutions of two nonlocal boundary problems. For this purpose, we use the *Dynkin characteristic operator*, which is defined in Section 7.1. The first problem involves the λ-*potentials* (or the *resolvents*), see. Corollary 7.2 below. It is a well-known formula for the Feller generators (see e.g. Dynkin [31, Theorem 1.1, p. 24]) and we prove that it holds in our case also for the Dynkin characteristic operator. The second problem is the main result of this chapter — we prove that with certain assumptions on the function f, the solution of the Neumann boundary problem

$$\begin{cases} (-\Delta)^{\alpha/2} u &= f, \quad \text{in } D, \\ \mathcal{N}_{\alpha/2} u &= f, \quad \text{in } \overline{D}^c. \end{cases}$$

is given by the Green operator G of K.

7.1 Dynkin characteristic operator

The *Dynkin characteristic* operator for the process $X = (X_t)_{t\geq 0}$ is defined by the following expression

$$\mathcal{D}f(x) = \lim_{r\to 0^+} \frac{\mathbb{E}_x f(X_{\tau_{B(x,r)}}) - f(x)}{\mathbb{E}_x \tau_{B(x,r)}}, \quad x \neq 0. \tag{7.1}$$

Here $x \in \mathbb{R}^*$ and a function f are such that the limit exists.

Assume that $x > 0$ and a function f are such that the limit (7.1) exists. Then, from the construction of the process X, from Lemma 3.8 and from the Ikeda–Watanabe formula (2.25) it follows that $\mathbb{E}_x f(X_{\tau_{B(x,r)}}) = \mathbb{E}^Y_x f(Y_{\tau_{B(x,r)}})$ and $\mathbb{E}_x \tau_{B(x,r)} = \mathbb{E}^Y_x \tau_{B(x,r)}$. Hence,

$$\mathcal{D}f(x) = \lim_{r\to 0^+} \frac{\mathbb{E}^Y_x f(Y_{\tau_{B(x,r)}}) - f(x)}{\mathbb{E}^Y_x \tau_{B(x,r)}}, \qquad x > 0.$$

Moreover, from Kwaśnicki [47, Lemma 3.3] it follows that

$$\mathcal{D}f(x) = -(-\Delta)^{\alpha/2}f(x) := \text{p.v.}\int_{\mathbb{R}}(f(y) - f(x))\nu(x,y)\,\mathrm{d}y, \qquad x > 0, \tag{7.2}$$

i.e. the operator $\mathcal{D}$, for $x > 0$, is the fractional Laplacian on $\mathbb{R}$.

Now, assume that f is bounded and let $x < 0$ and $r \in (0, |x|)$. Then, from the construction of the process X, it is obvious that

$$\mathbb{E}_x f(X_{\tau_{B(x,r)}}) = \int_D f(y)k(x, \mathrm{d}y) = \int_D f(y)\frac{\nu(x,y)}{\nu(x,D)}\,\mathrm{d}y,$$

and

$$\mathbb{E}_x\tau_{B(x,r)} = \frac{1}{\nu(x,D)},$$

which follows from the fact that for $x < 0$, $\tau_{B(x,r)}$ is the random variable from the exponential distribution with mean $1/\nu(x,D)$. Hence,

$$\mathcal{D}f(x) = -\mathcal{N}_{\alpha/2}f(x) = -\int_D (f(x) - f(y))\nu(x,y)\,\mathrm{d}y, \qquad x < 0, \tag{7.3}$$

i.e. the operator $\mathcal{D}$, for $x < 0$, is the *nonlocal normal derivative* (see e.g. Dipierro et al. [29]). Further, note that from (7.3), we also have the following equality:

$$\mathcal{D}f(x) = \widehat{\nu}f(x) - f(x)\nu(x,D). \tag{7.4}$$

7.2 The λ-potentials

Here assume that $f \in C_b(\mathbb{R}^*)$ and define the *λ-potential*, $\lambda > 0$, of the semigroup $K = (K_t)_{t\geq 0}$ by

$$U_\lambda f(x) := \mathbb{E}_x\int_0^\infty e^{-\lambda t}f(X_t)\,\mathrm{d}t = \int_0^\infty e^{-\lambda t}K_t f(x)\,\mathrm{d}t, \qquad x \neq 0.$$

Then of course $U_\lambda|f| < \infty$ for all $\lambda > 0$. Indeed, for $x \neq 0$, from Lemma 3.10,

$$\begin{aligned} U_\lambda|f|(x) &= \int_0^\infty e^{-\lambda t}K_t|f|(x)\,\mathrm{d}t \leq \|f\|_\infty \int_0^\infty e^{-\lambda t}K_t\mathbf{1}(x)\,\mathrm{d}t \\ &\leq \|f\|_\infty \int_0^\infty e^{-\lambda t}\,\mathrm{d}t = \lambda^{-1}\|f\|_\infty < \infty. \end{aligned}$$

Proposition 7.1. *Let $\alpha \in (0,2)$, $\lambda > 0$ and $f \in C_b(\mathbb{R}^*)$. Then, $u = U_\lambda f$ is the solution of the equation $(\mathcal{D} - \lambda \mathrm{I})u = -f$ in $\mathbb{R}^*$.*

Proof. For $\lambda > 0$ and $f \in C_b(\mathbb{R}^*)$, $u = U_\lambda f$ is well-defined. From [31, Theorem 5.1] it follows that

$$\mathbb{E}_x\big(e^{-\lambda\tau_{B(x,r)}}u(X_{\tau_{B(x,r)}})\big) = u(x) - \mathbb{E}_x\int_0^{\tau_{B(x,r)}} e^{-\lambda t}f(X_t)\,\mathrm{d}t.$$

Recall that we may use cited theorem, because from Chapter 3.3 in Dynkin [31] it follows that each right-continuous process is strongly measurable.

Hence, for $x \neq 0$,

$$\mathcal{D}u(x) = \lim_{r\to 0^+} \frac{\mathbb{E}_x\big[(1-e^{-\lambda\tau_{B(x,r)}})u(X_{\tau_{B(x,r)}})\big]}{\mathbb{E}_x\tau_{B(x,r)}} - \lim_{r\to 0^+}\frac{1}{\mathbb{E}_x\tau_{B(x,r)}}\mathbb{E}_x\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(X_t)\,\mathrm{d}t$$
$$=: I - II. \tag{7.5}$$

Assume that $x > 0$. Since $\tau_{B(x,r)} \leq \zeta^{(1)}$, we have $X_t = Y_t$ for $t < \tau_{B(x,r)}$. Therefore, from Fubini's theorem,

$$\begin{aligned}\mathbb{E}_x\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(X_t)\,\mathrm{d}t &= \mathbb{E}_x^Y\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(Y_t)\,\mathrm{d}t\\ &= \mathbb{E}_x^Y\int_0^{\infty} e^{-\lambda t} f(Y_t)\,\mathbb{1}_{[t,\infty)}\left(\tau_{B(x,r)}\right)\mathrm{d}t\\ &= \int_0^{\infty}\mathbb{E}_x^Y\left[e^{-\lambda t} f(Y_t)\,\mathbb{1}_{[t,\infty)}\left(\tau_{B(x,r)}\right)\right]\mathrm{d}t\\ &= \int_0^{\infty}\mathbb{E}_0^Y\left[e^{-\lambda t} f(x+Y_t)\,\mathbb{1}_{[t,\infty)}\left(\tau_{B(0,r)}\right)\right]\mathrm{d}t.\end{aligned}$$

From the scaling property of $(Y_t, \tau_{B(x,r)})$ with respect to $\mathbb{P}_0^Y$ (see (2.6)) and again from the Fubini's theorem,

$$\begin{aligned}\mathbb{E}_x^Y\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(Y_t)\,\mathrm{d}t &= \int_0^{\infty}\mathbb{E}_0^Y\left[e^{-\lambda t} f(x+rY_{r^{-\alpha}t})\,\mathbb{1}_{[t,\infty)}\left(r^{\alpha}\tau_{B(0,1)}\right)\right]\mathrm{d}t\\ &= \mathbb{E}_0^Y\int_0^{r^{\alpha}\tau_{B(0,1)}} e^{-\lambda t} f(x+rY_{r^{-\alpha}t})\,\mathrm{d}t.\end{aligned}$$

Using the substitution $s = r^{-\alpha}t$ we get

$$\mathbb{E}_x^Y\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(Y_t)\,\mathrm{d}t = r^{\alpha}\,\mathbb{E}_0^Y\int_0^{\tau_{B(0,1)}} e^{-\lambda s r^{\alpha}} f(x+rY_s)\,\mathrm{d}s.$$

Moreover, from the shift invariance of the process Y, $\mathbb{E}_x^Y\tau_{B(x,r)} = \mathbb{E}_0^Y\tau_{B(0,r)} = r^{\alpha}\mathbb{E}_0^Y\tau_{B(0,1)}$. Hence, from the dominated convergence theorem,

$$\begin{aligned}II &= \lim_{r\to 0^+}\frac{1}{\mathbb{E}_x^Y\tau_{B(x,r)}}\mathbb{E}_x^Y\int_0^{\tau_{B(x,r)}} e^{-\lambda t} f(Y_t)\,\mathrm{d}t\\ &= \lim_{r\to 0^+}\frac{1}{\mathbb{E}_0^Y\tau_{B(0,1)}}\mathbb{E}_0^Y\int_0^{\tau_{B(0,1)}} e^{-\lambda s r^{\alpha}} f(x+rY_s)\,\mathrm{d}s = f(x).\end{aligned}\tag{7.6}$$

In case of the limit I we proceed similarly: from (2.7),

$$\begin{aligned}I &= \lim_{r\to 0^+}\frac{\mathbb{E}_x^Y\left[(1-e^{-\lambda\tau_{B(x,r)}})u(Y_{\tau_{B(x,r)}})\right]}{\mathbb{E}_x^Y\tau_{B(x,r)}}\\ &= \lim_{r\to 0^+}\frac{\mathbb{E}_0^Y\left[(1-e^{-\lambda r^{\alpha}\tau_{B(0,1)}})u(x+rY_{\tau_{B(0,1)}})\right]}{r^{\alpha}\,\mathbb{E}_0^Y\tau_{B(0,1)}}\\ &= \frac{1}{\mathbb{E}_0^Y\tau_{B(0,1)}}\lim_{r\to 0^+}\mathbb{E}_0^Y\left[\frac{1-e^{-\lambda r^{\alpha}\tau_{B(0,1)}}}{r^{\alpha}}u(x+rY_{\tau_{B(0,1)}})\right].\end{aligned}$$

Note that $u(x) = \int_0^\infty e^{-\lambda t} K_t f(x)\,\mathrm{d}t$ is bounded, which follows from the fact that f is bounded. Moreover, from Theorem 3.16, it follows that the function $\mathbb{R}^* \ni x \mapsto u(x)$ is continuous. Hence, from the inequality $1 - e^{-z} \le z$ for $z > 0$, it follows that we can use the dominated convergence theorem to obtain that

$$I = \frac{1}{\mathbb{E}_0^Y \tau_{B(0,1)}} \mathbb{E}_0^Y \left[\lim_{r \to 0+} \frac{1 - e^{-\lambda r^\alpha \tau_{B(0,1)}}}{r^\alpha} u(x + rY_{\tau_{B(0,1)}}) \right] = \lambda u(x). \tag{7.7}$$

Combining (7.5), (7.6) and (7.7) we obtain the equality $\mathcal{D}u(x) = \lambda u(x) - f(x)$, $x > 0$.

Now assume that $x < 0$. From the construction of the process X and from Lemma 3.8 it follows that

$$\begin{aligned} I &= \frac{\mathbb{E}_x\big[(1 - e^{-\lambda R_1}) u(X_{R_1})\big]}{\mathbb{E}_x R_1} \\ &= \nu(x, D)\, \mathbb{E}_x\big[(1 - e^{-\lambda R_1}) u(X_{R_1})\big] \\ &= \nu(x, D) \int_0^\infty \mathrm{d}s \int_D \mathrm{d}y\, e^{-\nu(x,D)s} \nu(x, y)(1 - e^{-\lambda s}) u(y) \\ &= \int_0^\infty \nu(x, D) e^{-\nu(x,D)s} (1 - e^{-\lambda s})\,\mathrm{d}s \int_D \nu(x, y) u(y)\,\mathrm{d}y \\ &= \frac{\lambda}{\nu(x, D) + \lambda} \widehat{\nu} u(x). \end{aligned} \tag{7.8}$$

Similarly,

$$\begin{aligned} II &= f(x) \frac{1}{\mathbb{E}_x R_1} \mathbb{E}_x \int_0^{R_1} e^{-\lambda t}\,\mathrm{d}t \\ &= f(x) \nu(x, D) \lambda^{-1} \mathbb{E}_x\big[1 - e^{-\lambda R_1}\big] \\ &= f(x) \nu(x, D) \lambda^{-1} \int_0^\infty \nu(x, D) e^{-\nu(x,D)s} \big(1 - e^{-\lambda s}\big)\,\mathrm{d}s \\ &= f(x) \frac{\nu(x, D)}{\nu(x, D) + \lambda}. \end{aligned} \tag{7.9}$$

Combining (7.5), (7.8) and (7.9) we obtain the equality

$$\mathcal{D}u(x) = \frac{\lambda}{\nu(x, D) + \lambda} \widehat{\nu} u(x) - f(x) \frac{\nu(x, D)}{\nu(x, D) + \lambda}, \qquad x < 0. \tag{7.10}$$

From (7.4) the equation (7.10) takes the form

$$\mathcal{D}u(x) = \frac{\lambda}{\nu(x, D) + \lambda} \big[\mathcal{D}u(x) + u(x)\nu(x, D)\big] - f(x) \frac{\nu(x, D)}{\nu(x, D) + \lambda}, \tag{7.11}$$

which is equivalent to the equation $\mathcal{D}u(x) = \lambda u(x) - f(x)$, $x < 0$. □

From Proposition 7.1, (7.2) and (7.3) we have the following corollary.

Corollary 7.2. *Let $\alpha \in (0, 2)$, $\lambda > 0$ and $f \in C_b(\mathbb{R}^*)$. Then, $u = U_\lambda f$ is the solution of the following Neumann problem for the fractional Laplacian*

$$\begin{cases} \big[(-\Delta)^{\alpha/2} + \lambda \mathrm{I}\big] u &= f, \quad \text{in } D, \\ \big[\mathcal{N}_{\alpha/2} + \lambda \mathrm{I}\big] u &= f, \quad \text{in } \overline{D}^c. \end{cases}$$

7.3 The 0-potential

In this section, we assume that $f \in C_c(\mathbb{R}^*)$, i.e. f is a bounded continuous function with compact support. We define the 0-*potential* or the *Green operator* of the semigroup $K = (K_t)_{t\geq 0}$ by

$$Gf(x) := \mathbb{E}_x \int_0^\infty f(X_t)\,\mathrm{d}t = \int_0^\infty K_t f(x)\,\mathrm{d}t, \qquad x \neq 0. \tag{7.12}$$

Lemma 7.3. *For* $x \neq 0$, $\alpha \in (0,2)$, $\beta(\alpha-\beta-1) > 0$, $G|x|^{\beta-\alpha} < \infty$.

Proof. Let $x \neq 0$. Recall that $h_\beta(x) = |x|^\beta$. From Theorem 5.15,

$$\lim_{t\to 0^+} \frac{h_\beta(y) - K_t h_\beta(y)}{t} = \mathcal{A}_{1,\alpha}\mathcal{C}(\alpha,\beta,y)|y|^{\beta-\alpha} \geq \mathcal{C}_1(\alpha,\beta)|y|^{\beta-\alpha}, \qquad y \neq 0, \tag{7.13}$$

where $\mathcal{C}_1(\alpha,\beta) := \mathcal{A}_{1,\alpha}\big[\alpha^{-1} - \mathfrak{B}(\beta+1,\alpha-\beta)\big]$. For $s,t>0$, from Corollary 3.21, $h_\beta - K_t h_\beta \geq 0$ and $K_s\big[h_\beta - K_t h_\beta\big](x) \geq 0$. Hence, from (7.13) and from Fatou's lemma,

$$\begin{aligned}
G|x|^{\beta-\alpha} &= \int_0^\infty \int_{\mathbb{R}} K_s(x,\mathrm{d}y)|y|^{\beta-\alpha}\,\mathrm{d}s \\
&\leq \big(\mathcal{C}_1(\alpha,\beta)\big)^{-1} \int_0^\infty \int_{\mathbb{R}} K_s(x,\mathrm{d}y) \lim_{t\to 0^+} \frac{h_\beta(y) - K_t h_\beta(y)}{t}\,\mathrm{d}s \\
&\leq \big(\mathcal{C}_1(\alpha,\beta)\big)^{-1} \liminf_{t\to 0^+} \int_0^\infty K_s\left[\frac{h_\beta - K_t h_\beta}{t}\right](x)\,\mathrm{d}s \\
&\approx \liminf_{t\to 0^+} \frac{1}{t}\int_0^\infty \big[K_s h_\beta(x) - K_{s+t}h_\beta(x)\big]\,\mathrm{d}s \\
&= \liminf_{t\to 0^+} \frac{1}{t}\sum_{i=1}^\infty \int_{(i-1)t}^{it} \big[K_s h_\beta(x) - K_{s+t}h_\beta(x)\big]\,\mathrm{d}s.
\end{aligned} \tag{7.14}$$

Again, from Corollary 3.21, for $s,t>0$, $K_{t+s}h_\beta(x) = K_s\big(K_t h_\beta\big)(x) \leq K_s h_\beta(x)$, hence the function $s \mapsto K_s h_\beta(x)$ is non-increasing. Therefore, from (7.14),

$$\begin{aligned}
G|x|^{\beta-\alpha} &\lesssim \liminf_{t\to 0^+} \frac{1}{t}\sum_{i=1}^\infty \int_{(i-1)t}^{it} \big[K_{(i-1)t}h_\beta(x) - K_{(i+1)t}h_\beta(x)\big]\,\mathrm{d}s \\
&= \liminf_{t\to 0^+} \sum_{i=1}^\infty \big[K_{(i-1)t}h_\beta(x) - K_{(i+1)t}h_\beta(x)\big].
\end{aligned} \tag{7.15}$$

Note that for every $n \in \mathbb{N}$,

$$h_\beta(x) + K_t h_\beta(x) = \sum_{i=1}^n \big[K_{(i-1)t}h_\beta(x) - K_{(i+1)t}h_\beta(x)\big] + K_{nt}h_\beta(x) + K_{(n+1)t}h_\beta(x),$$

so

$$h_\beta(x) + K_t h_\beta(x) \geq \sum_{i=1}^\infty \big[K_{(i-1)t}h_\beta(x) - K_{(i+1)t}h_\beta(x)\big]. \tag{7.16}$$

From (7.15), (7.16) and from Theorem 5.15,

$$G|x|^{\beta-\alpha} \lesssim \liminf_{t\to 0^+} \big[h_\beta(x) + K_t h_\beta(x)\big] = 2h_\beta(x) < \infty. \qquad \square$$

The following corollary is an immediate consequence of the previous lemma.

Corollary 7.4. *Let $x \neq 0$, $\alpha \in (0,2)$ and $\beta(\alpha-\beta-1) > 0$. If f is a function such that $|f(x)| \lesssim |x|^{\beta-\alpha}$, then $G|f|(x) < \infty$. In particular,*

- *$G(1+|x|)^{-1-\kappa} < \infty$ for every $\kappa > 0$,*
- *if f is a bounded and compactly supported function on $\mathbb{R}^*$, then $G|f|(x) < \infty$.*

Proof. We will prove that $G(1+|x|)^{-1-\kappa} < \infty$ for every $\kappa > 0$. The other statements of the Corollary are obvious.

We observe that for every $\alpha \in (0,1)\cup(1,2)$ and $\kappa > 0$ there exists β such that $\beta(\alpha-\beta-1) > 0$ and $\beta \geq \alpha - 1 - \kappa$. For such β we have

$$(1+|x|)^{-1-\kappa} \leq (1+|x|)^{\beta-\alpha} \leq |x|^{\beta-\alpha}, \qquad x \neq 0,$$

and the Corollary follows from Lemma 7.3. □

Proposition 7.5. *Let $\alpha \in (0,1)\cup(1,2)$, $f \in C_c(\mathbb{R}^*)$. Then, $u = Gf$ is the solution of the equation $\mathcal{D}u = -f$ in $\mathbb{R}^*$.*

Proof. From Corollary 7.4, $u = Gf$ is well-defined. From [31, Theorem 5.1] it follows that

$$\mathbb{E}_x u(X_{\tau_{B(x,r)}}) = u(x) - \mathbb{E}_x \int_0^{\tau_{B(x,r)}} f(X_t)\,\mathrm{d}t. \tag{7.17}$$

Hence, for $x > 0$,

$$\begin{aligned}
\mathcal{D}u(x) &= -\lim_{r\to 0^+} \frac{1}{\mathbb{E}_x\tau_{B(x,r)}}\mathbb{E}_x\int_0^{\tau_{B(x,r)}} f(X_t)\,\mathrm{d}t \\
&= -\lim_{r\to 0^+} \frac{1}{\mathbb{E}_x\tau_{B(x,r)}}\left[\mathbb{E}_x\int_0^{\tau_{B(x,r)}} \big(f(X_t)-f(x)\big)\,\mathrm{d}t + f(x)\mathbb{E}_x\tau_{B(x,r)}\right] \\
&= -f(x) - \lim_{r\to 0^+} \frac{1}{\mathbb{E}_x\tau_{B(x,r)}}\mathbb{E}_x\int_0^{\tau_{B(x,r)}} \big(f(X_t)-f(x)\big)\,\mathrm{d}t \\
&= -f(x) - \lim_{r\to 0^+} \frac{1}{\mathbb{E}_x^Y\tau_{B(x,r)}}\mathbb{E}_x^Y\int_0^{\tau_{B(x,r)}} \big(f(Y_t)-f(x)\big)\,\mathrm{d}t.
\end{aligned} \tag{7.18}$$

Note that from Fubini's theorem,

$$\begin{aligned}
\mathcal{D}u(x) &= -f(x) - \lim_{r\to 0^+} \frac{1}{\mathbb{E}_x^Y\tau_{B(x,r)}}\int_0^\infty \mathbb{E}_x^Y\left[\big(f(Y_t)-f(x)\big)\,\mathbb{1}_{[t,\infty)}\left(\tau_{B(x,r)}\right)\right]\mathrm{d}t \\
&= -f(x) - \lim_{r\to 0^+} \frac{1}{\mathbb{E}_0^Y\tau_{B(0,r)}}\int_0^\infty \mathbb{E}_0^Y\left[\big(f(x+Y_t)-f(x)\big)\,\mathbb{1}_{[t,\infty)}\left(\tau_{B(0,r)}\right)\right]\mathrm{d}t.
\end{aligned}$$

From the scaling property of $(Y_t, \tau_{B(x,r)})$ with respect to $\mathbb{P}_0^Y$ (see (2.6)), Fubini's theorem and by the substitution $s = r^{-\alpha}t$ we get

$$\begin{aligned}\mathcal{D}u(x) &= -f(x) - \lim_{r\to 0^+} \frac{1}{r^\alpha \mathbb{E}_0^Y \tau_{B(0,1)}} \int_0^\infty \mathbb{E}_0^Y\left[\left(f(x+rY_{r^{-\alpha}t}) - f(x)\right) \mathbb{1}_{[t,\infty)}\left(r^\alpha \tau_{B(0,1)}\right)\right] \mathrm{d}t \\ &= -f(x) - \lim_{r\to 0^+} \frac{1}{r^\alpha \mathbb{E}_0^Y \tau_{B(0,1)}} \mathbb{E}_0^Y \int_0^{r^\alpha \tau_{B(0,1)}} \left(f(x+rY_{r^{-\alpha}t}) - f(x)\right) \mathrm{d}t \\ &= -f(x) - \lim_{r\to 0^+} \frac{1}{\mathbb{E}_0^Y \tau_{B(0,1)}} \mathbb{E}_0^Y \int_0^{\tau_{B(0,1)}} \left(f(x+rY_s) - f(x)\right) \mathrm{d}s.\end{aligned}$$

Hence, from the dominated convergence theorem,

$$\mathcal{D}u(x) = -f(x) - \frac{1}{\mathbb{E}_0^Y \tau_{B(0,1)}} \mathbb{E}_0^Y \int_0^{\tau_{B(0,1)}} \lim_{r\to 0^+} \left(f(x+rY_s) - f(x)\right) \mathrm{d}s = -f(x).$$

Now, consider $x < 0$. From (7.17) and from the construction of the process X, it is obvious that

$$\mathcal{D}u(x) = -\lim_{r\to 0^+} \frac{1}{\mathbb{E}_x \tau_{B(x,r)}} \mathbb{E}_x \int_0^{\tau_{B(x,r)}} f(X_t)\,\mathrm{d}t = -f(x). \qquad \square$$

From Proposition 7.5, (7.2) and (7.3) we obtain the following theorem, which is our main result of the thesis.

Theorem 7.6. *Let $\alpha \in (0,1) \cup (1,2)$ and $f \in C_c(\mathbb{R}^*)$. Then, for $x \neq 0$ we have $G|f|(x) < \infty$ and $u = Gf$ is the solution of the following Neumann problem for the fractional Laplacian*

$$\begin{cases} (-\Delta)^{\alpha/2} u &= f, \quad \text{in } D, \\ \mathcal{N}_{\alpha/2} u &= f, \quad \text{in } \overline{D}^c. \end{cases}$$

Bibliography

[1] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[2] R. Bañuelos and K. Bogdan. Symmetric stable processes in cones. *Potential Anal.*, 21(3):263–288, 2004.

[3] O. Barndorff-Nielsen, T. Mikosch, and S. Resnick. *Lévy Processes: Theory and Applications*. Birkhäuser Boston, 2001.

[4] H. Bateman, A. Erdélyi, et al. *Higher transcendental functions vol. 1*. McGraw-Hill, 1953.

[5] S. Blomberg, S. Rathnayake, and C. Moreau. Beyond Brownian motion and the Ornstein–Uhlenbeck process: Stochastic diffusion models for the evolution of quantitative characters. *Am. Nat.*, 195(2):145–165, 2020.

[6] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.

[7] R. M. Blumenthal and R. K. Getoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York-London, 1968.

[8] K. Bogdan. Representation of α-harmonic functions in Lipschitz domains. *Hiroshima Math. J.*, 29(2):227–243, 1999.

[9] K. Bogdan, K. Burdzy, and Z.-Q. Chen. Censored stable processes. *Probab. Theory Related Fields*, 127(1):89–152, 2003.

[10] K. Bogdan, B. Dyda, and P. Kim. Hardy inequalities and non-explosion results for semigroups. *Potential Anal.*, 44(2):229–247, 2016.

[11] K. Bogdan and T. Grzywny. Heat kernel of fractional Laplacian in cones. *Colloq. Math.*, 118(2):365–377, 2010.

[12] K. Bogdan, T. Grzywny, K. Pietruska-Pałuba, and A. Rutkowski. Extension and trace for nonlocal operators. *J. Math. Pures Appl. (9)*, 137:33–69, 2020.

[13] K. Bogdan, T. Grzywny, and M. Ryznar. Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.*, 38(5):1901–1923, 2010.

[14] K. Bogdan, T. Komorowski, and L. Marino. Anomalous diffusion limit for a kinetic equation with a thermostatted interface. *Probability Theory and Related Fields*, pages 1–49, 12 2023.

[15] K. Bogdan and M. Kunze. The fractional laplacian with reflections. *Potential Analysis*, pages 1–29, 11 2023.

[16] K. Bogdan, J. Rosiński, G. Serafin, and Ł. Wojciechowski. Lévy systems and moment formulas for mixed Poisson integrals. In *Stochastic analysis and related topics*, volume 72 of *Progr. Probab.*, pages 139–164. Birkhäuser/Springer, Cham, 2017.

[17] K. Bogdan and S. Sydor. On nonlocal perturbations of integral kernels. In *Semigroups of operators — theory and applications*, volume 113 of *Springer Proc. Math. Stat.*, pages 27–42. Springer, Cham, 2015.

[18] C. Bucur and E. Valdinoci. *Nonlocal Diffusion and Applications*. Lecture Notes of the Unione Matematica Italiana. Springer International Publishing, 2016.

[19] Z.-Q. Chen and M. Fukushima. *Symmetric Markov processes, time change, and boundary theory*, volume 35 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2012.

[20] Z.-Q. Chen, P. Kimy, and R. Song. Heat kernel estimates for Dirichlet fractional Laplacian. *Journal of the European Mathematical Society*, 12(5):1307–1329, 01 2010.

[21] Z.-Q. Chen and K. Kuwae. On doubly Feller property. *Osaka J. Math.*, 46(4):909–930, 2009.

[22] K. Chung and J. Walsh. *Markov Processes, Brownian Motion, and Time Symmetry*. Grundlehren der mathematischen Wissenschaften. Springer New York, 2005.

[23] K. L. Chung. *Lectures from Markov processes to Brownian motion*, volume 249 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York-Berlin, 1982.

[24] K. L. Chung and Z. X. Zhao. *From Brownian motion to Schrödinger's equation*, volume 312 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1995.

[25] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[26] W. Cygan, T. Grzywny, and B. Trojan. Asymptotic behavior of densities of unimodal convolution semigroups. *Trans. Amer. Math. Soc.*, 369(8):5623–5644, 2017.

[27] A. de Acosta. A new proof of the Hartman–Wintner law of the iterated logarithm. *Ann. Probab.*, 11(2):270–276, 1983.

[28] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[29] S. Dipierro, X. Ros-Oton, and E. Valdinoci. Nonlocal problems with Neumann boundary conditions. *Rev. Mat. Iberoam.*, 33(2):377–416, 2017.

[30] B. Dyda. A fractional order Hardy inequality. *Illinois J. Math.*, 48(2):575–588, 2004.

[31] E. B. Dynkin. *Markov Processes*, volume 1. New York Acadmic Press Inc., 1965.

[32] K. Engel and R. Nagel. *A Short Course on Operator Semigroups*. Universitext (Berlin. Print). Springer, 2006.

[33] M. Felsinger, M. Kassmann, and P. Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.

[34] A. Fiscella, R. Servadei, and E. Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.

[35] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

[36] R. K. Getoor. *Markov Processes: Ray Processes and Right Processes*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1975.

[37] H. Gjessing, O. Aalen, and N. Hjort. Frailty models based on Lévy processes. *Advances in Applied Probability*, 35:532–550, 06 2003.

[38] F. Grube and T. Hensiek. Robust nonlocal trace spaces and Neumann problems. *Nonlinear Analysis*, 241:113481, 2024.

[39] P. Hartman and A. Wintner. On the Law of the Iterated Logarithm. *American Journal of Mathematics*, 63(1):169–176, 1941.

[40] N. Ikeda, M. Nagasawa, and S. Watanabe. A construction of Markov processes by piecing out. *Proceedings of the Japan Academy*, 42(4):370 – 375, 1966.

[41] N. Ikeda and S. Watanabe. On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. *J. Math. Kyoto Univ.*, 2:79–95, 1962.

[42] T. Jakubowski and P. Maciocha. Ground-state representation for the fractional Laplacian on the half-line. *Probab. Math. Statist.*, 43(1):83–108, 2023.

[43] K. Kaleta and P. Sztonyk. Estimates of transition densities and their derivatives for jump Lévy processes. *J. Math. Anal. Appl.*, 431(1):260–282, 2015.

[44] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[45] P. Kim, R. Song, and Z. Vondraček. Positive self-similar Markov processes obtained by resurrection. *Stochastic Process. Appl.*, 156:379–420, 2023.

[46] P. Kim, R. Song, and Z. Vondraček. Potential theory of Dirichlet forms degenerate at the boundary: the case of no Killing potential. *Math. Ann.*, 388(1):511–542, 2024.

[47] M. Kwaśnicki. Ten equivalent definitions of the fractional Laplace operator. *Fract. Calc. Appl. Anal.*, 20(1):7–51, 2017.

[48] M. Landis, J. Schraiber, and M. Liang. Phylogenetic analysis using Lévy processes: Finding jumps in the evolution of continuous traits. *Syst. Biol.*, 62((2)):193–204, 2013.

[49] Z.-M. Ma and M. Röckner. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer Berlin, Heidelberg, 1992.

[50] P.-A. Meyer. Renaissance, recollements, mélanges, ralentissement de processus de markov. *Annales de l'institut Fourier*, 25(3-4):465–497, 1975.

[51] V. Millot, Y. Sire, and K. Wang. Asymptotics for the fractional Allen-Cahn equation and stationary nonlocal minimal surfaces. *Arch. Ration. Mech. Anal.*, 231(2):1129–1216, 2019.

[52] G. Pòlya. On the zeros of an integral function represented by Fourier's integral. *Messenger of Math.*, 52:185–188, 1923.

[53] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*. Number 293 in Grundlehren der mathematischen Wissenschaften. Springer, Berlin, 3rd edition, 1999.

[54] X. Ros-Oton. Nonlocal elliptic equations in bounded domains: a survey. *Publ. Mat.*, 60(1):3–26, 2016.

[55] A. Rutkowski. The Dirichlet problem for nonlocal Lévy-type operators. *Publicacions Matemàtiques*, 62(1):213 – 251, 2018.

[56] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[57] R. L. Schilling. *Measures, Integrals and Martingales*. Cambridge University Press, 2005.

[58] W. Schoutens. *Levy Processes in Finance: Pricing Financial Derivatives*. Wiley Series in Probability and Statistics. Wiley, 2003.

[59] R. Servadei and E. Valdinoci. Mountain pass solutions for non-local elliptic operators. *J. Math. Anal. Appl.*, 389(2):887–898, 2012.

[60] R. Servadei and E. Valdinoci. Variational methods for non-local operators of elliptic type. *Discrete Contin. Dyn. Syst.*, 33(5):2105–2137, 2013.

[61] M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[62] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Communications on Pure and Applied Mathematics*, 60:67 – 112, 01 2007.

[63] E. M. Stein and R. Shakarchi. *Real analysis: Measure theory, integration, and Hilbert spaces*, volume 3 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2005.

[64] E. M. Stein and R. Shakarchi. *Functional Analysis: Introduction to Further Topics in Analysis*. Princeton University Press, 2011.

[65] P. Voight. Nonlocal operators on domains. *PhD thesis, Bielefeld University*, 2017.

[66] Z. Vondraček. A probabilistic approach to a non-local quadratic form and its connection to the Neumann boundary condition problem. *Math. Nachr.*, 294(1):177–194, 2021.

[67] F. Werner. Concatenation and pasting of right processes. *Electron. J. Probab.*, 26:Paper No. 50, 21, 2021.