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Non-symmetric Lévy processes on the real line

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Niesymetryczne procesy Lévy'ego na prostej rzeczywistej

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Podziękowania

Niniejsza rozprawa jest zwieńczeniem kilkunastu lat pracy i nauki i jest naturalnie wiele osób, które swoim wysiłkiem i wsparciem na różne sposoby przyczyniły się do jej powstania. Na początku chciałbym serdecznie podziękować moim naukowym mentorom. Wyrazy wdzięczności kieruję w pierwszej kolejności do mojego promotora dr. hab. inż. Tomasza Grzywnego za ponad sześcioletną opiekę, za wprowadzenie mnie od podstaw w świat matematyki naukowej, za stworzenie przyjaznej atmosfery do pracy i za ogromną dozę cierpliwości, wsparcia i wyrozumiałości w obliczu słabszych momentów, za wiedzę, którą się ze mną dzielił przez te wszystkie lata oraz za pokładaną we mnie wiarę i nieustanne motywowanie mnie do dalszej pracy. Serdeczne podziękowania kieruję również do mojego promotora pomocniczego dr. inż. Karola Szczypkowskiego za cenne rady i wskazówki podczas pisania tej rozprawy oraz za współtworzenie przyjacielskiej i motywującej atmosfery na Wydziale Matematyki. W tym miejscu chciałbym również serdecznie podziękować wszystkim członkom seminarium „Teoria półgrup Markowa i operatorów Schrödingera" za wiele cennych rad i pouczających dyskusji oraz za przyjazne i wspierające podejście do młodszych kolegów i uczniów. Szczególne podziękowania należą się prof. dr. hab. inż. Krzysztofowi Bogdanowi za bezwarunkową chęć do pomocy w każdym aspekcie, za mnóstwo cennych obserwacji i wskazówek oraz za niezobowiązującego maila z sugestią rozważenia aplikacji na stanowisko studenta stypendysty do grantu Harmonia dwadzieścia godzin przed ostatecznym terminem składania wniosków, a który to mail był pierwszym z wielu punktów zwrotnych prowadzących koniec końców do napisania niniejszej rozprawy. Chciałbym też podziękować tutaj dr. Bartoszowi Trojanowi za cierpliwość, wyrozumiałość, wsparcie i za cenne wskazówki przy pisaniu pracy *Transition densities of subordinators...* oraz za próbę nauczenia mnie eleganckiej redakcji prac naukowych. Wyrazy wdzięczności należą się też dr. hab. inż. Mateuszowi Kwaśnickiemu za pomoc w stawianiu pierwszych kroków w świecie nauki podczas pisania pracy licencjackiej.

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Contents

Streszczenie

Od momentu charakteryzacji procesów Lévy'ego przez Paula Lévy'ego w latach 30. XX wieku doczekały się one ogromnej liczby zastosowań w przeróżnych kontekstach. Między innymi możemy tutaj wymienić liczne aplikacje w świecie finansowym (na przykład do opisu pozornie losowych zjawisk zachodzących na giełdzie), zastosowania do opisu modeli behawioralnych poszukiwania pożywienia przez wiele żywych organizmów, fizyczne modele przepływu cieczy w ośrodkach porowatych i wiele innych. Krótko mówiąc, procesy Lévy'ego przez ostatnie kilkadziesiąt lat nieprzerwanie znajdują się w centrum uwagi matematyków i ich popularność jest efektem synergii dwóch zazębiających się czynników. Po pierwsze, stanowią one szeroką klasę procesów stochastycznych, które z kolei są głównym narzędziem służącym do matematycznego opisu zjawisk losowych. Z drugiej strony, własności stacjonarności i niezależności przyrostów często znacznie upraszczają rachunki i umożliwiają uzyskanie jawnych wyników. W niniejszej dysertacji naszym celem jest dołożenie kolejnej cegiełki do rozległej wiedzy o procesach Lévy'ego w przypadku jednowymiarowym.

Podstawowym obiektem naszych badań jest gęstość przejścia dla procesu Lévy'ego $\mathbf{X} = (X_t \colon t \geq 0)$, która zdefiniowana jest niejawnie poprzez następujący wzór:

$$\mathbb{E}_x f(X_t) = \int_{\mathbb{R}} p(t, x, y) f(y)\, dy, \quad t > 0, x \in \mathbb{R}.$$

$\mathbb{E}_x$ jest tutaj wartością oczekiwaną odpowiadającą prawdopodobieństwu $\mathbb{P}_x$, które z kolei związane jest z procesem startującym z punktu $x \in \mathbb{R}$. Ponadto, z analitycznego punktu widzenia, pod pewnymi założeniami ogólna teoria mówi, że gęstość przejścia rozwiązuje uogólnione równanie ciepła i z tego powodu nazywana jest często *jądrem ciepła*, która to nazwa przyjęła się zarówno w probabilistycznym, jak i analitycznym środowisku. Zagadnienie istnienia i oszacowań jądra ciepła dla różnych procesów Lévy'ego przyciągało uwagę wielu matematyków w ostatnich dekadach, co zaowocowało ogromną liczbą artykułów. Naszym pierwszym celem w niniejszej dysertacji będzie wyprowadzenie oszacowań dla przypadku jednowymiarowych niesymetrycznych procesów Lévy'ego. Nie trzeba tutaj dodawać, że niesymetryczny przypadek jest dużo trudniejszy do analizy od symetrycznego z powodu braku przyjaznej struktury. W niniejszej rozprawie stosujemy dwa podejścia: albo narzucamy jakąś formę kontroli na asymetrię procesu, albo opracowujemy i stosujemy metodę dopasowaną specjalnie do konkretnego analizowanego przypadku. Początkowo stosujemy się do drugiej techniki i skupiamy się na jednowymiarowych procesach Lévy'ego z dobrze zdefiniowaną transformatą Laplace'a. Pierwszą klasą analizowanych procesów są subordynatory, którym poświęcony jest rozdział 3. Na początku koncentrujemy się na asymptotyce gęstości przejścia i ten cel osiągnięty jest w twierdzeniu 3.1.1 przy założeniu dolnego skalowania z wykładnikiem $\alpha - 2$, gdzie $\alpha > 0$, na (minus) drugą pochodną wykładnika Laplace'a ϕ. Dowód opiera się na aproksymacji za pomocą metody punktu siodłowego, a głównym punktem jest uzyskanie odpowiedniej kontroli na holomorficzne rozszerzenie ϕ. Warto tutaj dodać, że oprócz dolnego skalowania nie nakładamy żadnych dodatkowych założeń. W szczególności dopuszczamy miary Lévy'ego, które nie są absolutnie

ciągłe względem miary Lebesgue'a. Na końcu sekcji 3.3 dowodzimy również kolejnego wyniku z bardziej przyjazną formą przestrzenno-czasowego obszaru stosowalności przy założeniu dodatkowego górnego skalowania na ϕ z wykładnikiem $\beta < 1$.

Pozostała część rozdziału 3 poświęcona jest wyprowadzeniu górnych i dolnych oszacowań na gęstość przejścia subordynatora. Górnego oszacowania dowodzimy w twierdzeniu 3.4.4, ale można otrzymać również jego słabszą wersję bez żadnych dodatkowych założeń (zob. twierdzenie 3.4.3). Główne wyniki dotyczące oszacowań dolnych to twierdzenie 3.4.7 i propozycja 3.4.9. Pierwszy z nich jest szczególnie ciekawy z powodu dość zawiłego argumentu dotyczącego nośnika rozkładu granicznej zmiennej losowej, którą uzyskujemy z zastosowania twierdzenia Prochorowa. Podsumowaniem tej sekcji jest ostre dwustronne oszacowanie gęstości przejścia zaprezentowane w twierdzeniu 3.4.13. Tutaj zakładamy już zarówno dolne jak, i górne skalowanie ϕ z wykładnikami $0 < \alpha \leq \beta < 1$ wraz z prawie monotonicznością gęstości miary Lévy'ego $\nu(x)$. Nie powinien tutaj być niczym zaskakującym fakt, że granice obszaru stosowalności oszacowań powiązane są z obszarem zachodzących skalowań. W szczególności, jeśli skalowania są globalne, to oszacowania na gęstość przejścia również są globalne w czasie i przestrzeni i taka wersja twierdzenia 3.4.13 zaprezentowana jest we wstępie rozdziału 3 jako twierdzenie 3.1.2. Ponadto zauważmy, że w celu otrzymania ostrych dwustronnych oszacowań musimy się odseparować od granicznego przypadku $\beta = 1$ lub, mówiąc potocznie, zostać *poniżej* granicznego przypadku. Jest to kolejna manifestacja raczej powszechnego zjawiska odmiennego zachowania się w sytuacjach granicznych w różnych kontekstach. Sekcję zamykamy przykładem zastosowania twierdzenia 3.4.13 dla relatywistycznego α-stabilnego subordynatora. Na końcu rozdziału pokazujemy dwa przykłady zastosowania naszych wyników, które podkreślają istotność naszych wniosków, tj. subordynację poza standardową przestrzenią $\mathbb{R}^d$ oraz ostre oszacowania na funkcję Greena subordynatora (zob. przykłady 3.5.2 i 3.5.3 oraz twierdzenie 3.5.8).

Zauważmy przy okazji, że niektóre ze wstępnych wyników używanych w rozdziale 3. przeniesione są do załącznika A. Głównie są to własności porównywalności i skalowań podstawowych obiektów charakteryzujących subordynator i czytelnik może dostrzec pewne podobieństwo do wyników uzyskanych przez Grzywnego i Szczypkowskiego [43]. Jednakże ale z racji tego, że nasze założenia wyrażane są poprzez (minus) drugą pochodną wykładnika Laplace'a ϕ zamiast, powiedzmy, przez część rzeczywistą wykładnika charakterystycznego, zdecydowaliśmy się dla jasności i kompletności zamieścić wszystkie dowody. Naszą motywacją przy tworzeniu tego załącznika była również chęć odciążenia i bez tego dość obszernego rozdziału 3, jak również zaproponowanie nieco ogólniejszej oprawy, która będzie mogła znaleźć zastosowanie także w nieco innej sytuacji w kolejnym rozdziale.

Niekwestionowaną zaletą podejścia zaprezentowanego w rozdziale 3. jest fakt, że może być ono zastosowane również do innych jednowymiarowych procesów Lévy'ego, jeśli tylko transformata Laplace'a istnieje i jest dobrze zdefiniowana na prawej półpłaszczyźnie zespolonej. W dużej mierze korzystamy z tego w rozdziale 4, gdzie ta sama technika jest zastosowana do jednowymiarowych spektralnie jednostronnych procesów Lévy'ego, ponieważ w tym przypadku istnienie transformaty Laplace'a jest konsekwencją jednostronnych skoków procesu $\mathbf{X}$. Zaznaczamy tutaj, że z racji tego, że metoda rozwinięta w rozdziale 3 jest zastosowana niemal w tej samej formie również w rozdziale 4, jego struktura i treść wykazują wiele podobieństw do wyników i dowodów zaprezentowanych w rozdziale 3. Dla wygody czytelnika pilnujemy dokładnie wszystkich drobniejszych i istotniejszych zmian i komentujemy je przed zaprezentowaniem konkretnego dowodu. Tak więc, asymptotyka gęstości przejścia zaprezentowana w twierdzeniu 4.1.1, jak również jego dowód, idą tym samym torem co dowód twierdzenia 3.1.1. Oszacowania górne i dolne wyrażone przez twierdzenie 4.4.2 oraz lematy 4.4.4 i 4.4.5, jak również ostre dwustronne oszacowania zaprezentowane w twierdzeniu 4.5.3, także mają swoje odpowiedniki. Tak jak w rozdziale 3, zjawisko *separacji* od granicznego przypadku $\alpha = 1$ widoczne jest również w niektórych oszacowaniach w rozdziale 4 z tą różnicą, że tutaj istotne jest, by być *powyżej*, a nie *poniżej* granicznego przypadku. Rozdział zakończony jest przykładem zastosowania twierdzenia 4.5.3 dla spektralnie dodatniego α-stabilnego procesu Lévy'ego.

Przedstawimy teraz problematykę rozdziału 5. Tutaj skupiamy się na analizie czasu pierwszego trafienia w zbiór zwarty i nasze podejście opiera się głównie na teorii potencjału zamiast na technikach analitycznych. W przeciwieństwie do poprzednich rozdziałów, stosujemy tutaj metodę *kontroli asymetrii* poprzez narzucenie globalnego skalowania w większości naszych rozumowań. Obowiązującym w całym rozdziale założeniem jest następujący warunek całkowy:

$$\int_0^\infty \frac{d\xi}{1 + \mathrm{Re}\,\psi(\xi)} < \infty,$$

gdzie ψ jest wykładnikiem charakterystycznym procesu $\mathbf{X}$. U podstaw naszych rozważań leży dość ulotna relacja pomiędzy dwiema różnymi wersjami kompensacji jądra potencjału:

$$S(x) = \int_0^\infty (p(s,0) - p(s,x))\, ds,$$

gdzie $p(t,\,\cdot\,)$ jest zdefiniowane poprzez relację $p(t,x,y) = p(t,y-x)$ dla dowolnych $t > 0$ i $x, y \in \mathbb{R}$, oraz

$$H(x) = \frac{1}{\pi}\int_0^\infty (1 - \cos xs)\,\mathrm{Re}\left[\frac{1}{\psi(s)}\right] ds.$$

Mianowicie okazuje się, że pierwsza z nich jest właściwym obiektem opisującym asymptotyczne zachowanie czasu pierwszego trafienia, lecz jej istotną wadą jest fakt, że nie wiemy *a priori*, czy w ogóle jest dobrze zdefiniowana, podczas gdy druga co prawda jest dobrze zdefiniowana, ale jej zastosowania ograniczają się jedynie do opisania oszacowań czasu trafienia. Problem poprawnego zdefiniowana S jest wysoce nietrywialny i dlatego pokazujemy kilka przykładów jego istnienia w lemacie 5.3.1, wniosku 5.3.2, propozycji 5.3.3 i wniosku 5.3.5. Kwestia asymptotyki czasu pierwszego trafienia w zbiór zwarty rozwiązana jest w twierdzeniu 5.3.7, a ważny przykład, który spełnia jego założenia, jest zapewniony dzięki twierdzeniu 5.2.4, to znaczy gdy część rzeczywista wykładnika charakterystycznego jest funkcją regularnie zmieniającą się w zerze z wykładnikiem $\alpha \in (1,2)$ oraz gdy kontrolujemy asymetrię miary Lévy'ego, to znaczy gdy jest ona postaci

$$\nu(dx) = C_d \mathbb{1}_{x<0}\nu_0(dx) + C_u \mathbb{1}_{x>0}\nu_0(dx),$$

gdzie $\nu_0(dx)$ jest symetryczną miarą Lévy'ego. Zauważmy tutaj, że formę powyżej mają miary Lévy'ego wszystkich procesów syetrycznych, stabilnych, jak również procesów spektralnie jednostronnych.

W celu otrzymania ostrych dwustronnych oszacowań czasu pierwszego trafienia najpierw dowodzimy w twierdzeniu 5.4.4 globalnej, niezmienniczej na skalę nierówności Harnacka. Ten drugi wynik udowodniony jest przy założeniu zerowego pierwszego momentu, to znaczy $\mathbb{E}X_1 = 0$ oraz pod warunkiem globalego skalowania części rzeczywistej wykładnika charakterystycznego z wykładnikiem $\alpha > 1$ i tym samym rozszerza wynik Grzywnego i Ryznara [40], którzy z kolei inspirowali się dowodem Bassa i Levina [3]. Zauważmy, że nie zakładamy tutaj absolutnej ciągłości miary Lévy'ego. Nierówność Harnacka jest następnie zastosowana w celu uzyskania ostrych dwustronnych oszacowań czasu pierwszego trafienia w zbiór zwarty, zob. twierdzenie 5.5.11. Wynik ten otrzymany jest przy założeniu globalnego skalowania $\mathrm{Re}\,\psi$ z wykładnikiem $\alpha > 1$, zerowego pierwszego momentu, tzn. $\mathbb{E}X_1 = 0$ oraz dodatkowego, nietrywialnego założenia, które nie jest *a priori* oczywiste, za to jest trywialnie spełnione dla procesów symetrycznych. Ponadto dzięki niedawnym wynikom Grzywnego [35] jesteśmy w stanie obejść ten problem w przypadku, gdy proces jest spektralnie jednostronny, zob. wniosek 5.5.14. Na zakończenie rozdziału prezentujemy dużą klasę procesów Lévy'ego, które spełniają założenia twierdzenia 5.5.11.

Większość materiału przygotowawczego, włącznie z notacją i wprowadzeniem podstawowych obiektów i ich własności, jest zaprezentowana w rozdziale 2. W szczególności podajemy tam definicję procesu Lévy'ego, wprowadzamy główne narzędzia z teorii fluktuacji i potencjału oraz prezentujemy krótki przegląd własności skalowania i regularnej zmienności. Na koniec chcielibyśmy jeszcze dodać, że każdy kolejny rozdział rozszerza odpowiednią część tego krótkiego streszczenia, zapewnia rys historyczny oraz definiuje właściwy kontekst naszych badań. Zatem w celu znalezienia dalszych informacji i szczegółów na konkretny temat odsyłamy czytelnika do odpowiedniego rozdziału tej dysertacji.

Chapter 1

Introduction

Since the characterisation of Lévy processes by Paul Lévy in the 1930s, they have lived to see myriads of applications in various contexts. One can list here numerous implementations into the financial setting, e.g. to the description of seemingly random phenomena occurring on the stock market, applications to the behaviour model of food tracking of many living organisms, to the physical flow models, for instance in porous media, and many more. To put it in a nutshell, Lévy processes have been the primary class of interest over the last century and their popularity is a synergy of two overlapping factors. First, they form a vast and general class of stochastic processes which are the main tool in mathematical description of random phenomena. On the other hand, the stationarity and independence of increments often significantly simplify calculations and enable explicit results. In this dissertation we aim at contributing to the extensive knowledge about Lévy processes by focusing on the one-dimensional case.

The primary object of our studies is the transition density of the Lévy process $\mathbf{X} = (X_t \colon t \geq 0)$ defined implicitly by the following formula

$$\mathbb{E}_x f(X_t) = \int_{\mathbb{R}} p(t,x,y) f(y) \, dy, \quad t > 0, x \in \mathbb{R}.$$

Here $\mathbb{E}_x$ is the expectation corresponding to the probability $\mathbb{P}_x$ related to the Lévy process starting from $x \in \mathbb{R}$. Moreover, from the analytical point of view, under certain assumptions the general theory states that the transition density solves the generalised heat equation and therefore it has earned the name *heat kernel* which is now commonly used both in the probabilistic and analytical world. The problem of existence and estimates of heat kernels for various Lévy processes has attracted huge attention in the last decades and resulted in abundant number of articles. Our first goal in this dissertation is to establish analogous estimates for the case of one-dimensional non-symmetric Lévy processes. It goes without saying that the non-symmetric case is much harder to handle than the symmetric one due to a lack of a familiar structure. The approach here may be twofold: either to impose some control on the non-symmetry or to invent and apply a tailor-made method specifically to the analysed case. We follow the latter technique and focus on one-dimensional Lévy processes with the well-defined Laplace transform starting with subordinators as our first target in Chapter 3. At the beginning we concentrate on the asymptotic behaviour of the transition density and that goal is achieved in Theorem 3.1.1 under the assumption of the weak lower scaling property with scaling index $\alpha - 2$, where $\alpha > 0$ imposed on the (minus) second derivative of the Laplace exponent ϕ. The method of the proof is based on the saddle point approximation and its crucial component is to establish sufficient control on the holomorphic extension of ϕ. It is worth mentioning that there are no additional assumptions apart from the scaling property; in particular, we allow Lévy measures which are not absolutely continuous with respect to the Lebesgue measure. At the end of Section 3.3 we also give another result with a more accessible form of the space-time validity regime, provided that the additional upper scaling condition with index $\beta < 1$ for ϕ holds true.

The remaining part of Chapter 3 is dedicated to derivation of upper and lower estimates of the transition density. The upper estimate is displayed in Theorem 3.4.4, but one can also obtain a weaker version with no additional assumptions (see Theorem 3.4.3). Main results concerning lower estimates are presented in Theorem 3.4.7 and Proposition 3.4.9. The former is particularly interesting due to a rather involved argument about the support of the limit random variable resulting from the application of the Prokhorov theorem. We conclude that part with one cumulative sharp two-sided estimate of the transition density displayed by Theorem 3.4.13. Here we assume both lower and upper scalings on ϕ with $0 < \alpha < \beta < 1$ and the almost monotonicity of the Lévy density $\nu(x)$. It is not surprising that the validity region is closely related to the area where the scaling properties hold true. In particular, if they are global, then the obtained estimate is also global in space and in time and this version is presented in Theorem 3.1.2 in the introduction of Chapter 3. Moreover, note that in order to obtain sharp two-sided estimates we need to separate ourselves from the limit case $\beta = 1$ or, informally speaking, to stay *below* the limit case. This is yet another instance of a rather common phenomenon of different behaviour of limit cases in various settings. We also present one example of an application of Theorem 3.4.13 to the relativistic α-stable subordinator (see Example 3.4.15). The chapter is concluded with two examples of applications of our results which advocate the relevance of our findings, i.e. the subordination beyond the classical $\mathbb{R}^d$ setting and sharp two-sided estimate of the Green function of the subordinator (see Examples 3.5.2 and 3.5.3 and Theorem 3.5.8).

We note in passing that some of the preliminary results used in Chapter 3 are moved to Appendix A. These are mainly comparability and scaling properties of characteristic objects of the subordinator and the reader may observe that they bear some similarity to the ones presented in Grzywny and Szczypkowski [43], but since the assumptions are imposed on the (minus) second derivative of the Laplace exponent ϕ instead of, say, on the real part of the characteristic exponent, we choose to provide all the proofs for the sake of clarity and completeness. Our motivation behind creating the appendix was to make the extensive chapter less overloaded and to provide a more general background which could be applied also in a slightly different setting in the next chapter.

The unquestionable advantage of the approach proposed in Chapter 3 is that it can be applied to other one-dimensional Lévy processes, provided that their Laplace transform exists and is well defined on the right complex half-plane. We largely profit from that in Chapter 4 where the same method is successfully applied to the spectrally one-sided Lévy processes, since in that case the existence of the Laplace transform is a consequence of the restriction on $\mathbf{X}$ to jump only forward. We should comment here that due to the fact that the method developed in Chapter 3 is applied almost in the same form in Chapter 4, its structure and content displays significant similarities to the results and proofs obtained for subordinators. There are however some minor yet crucial differences which mostly preclude the possibility of presentation of both types of processes in one compact form. Therefore, for the sake of clarity and in order to make that part of the dissertation self-contained, we provide a full reasoning instead of only referring to appropriate results in Chapter 3. We do however keep track of this minor changes right before the proofs for the convenience of the reader. So, the asymptotic behaviour of the transition density presented in Theorem 4.1.1 as well as its proof follows the idea behind Theorem 3.1.1. The lower and upper estimates represented by Theorem 4.4.2 and Lemmas 4.4.4 and 4.4.5 as well as sharp two-sided estimate covered by Theorem 4.5.3 also have their counterparts. As in Chapter 3, the phenomenon of *separation* from the limit case $\alpha = 1$ in some of the estimates is also present in Chapter 4 but with the difference that here we will assume to be *over* rather then *below* the limit case. We end that chapter with an example of application of Theorem 4.5.3 for the spectrally positive α-stable Lévy process.

Let us now outline the content of Chapter 5. Here we concentrate on the analysis of the first hitting time of a compact set and our approach is based on the potential theory rather than analytic discussions. Contrary to previous chapters, here we adopt a method of *controlling the non-symmetry* by imposing the global scaling condition in most of our reasonings. The standing assumption in this part is the following integral condition:

$$\int_0^\infty \frac{d\xi}{1+\operatorname{Re}\psi(\xi)} < \infty,$$

where ψ is the characteristic exponent of the process $\mathbf{X}$. On the foundation of our considerations lies a rather elusive interplay between two different versions of compensated potential kernel:

$$S(x) = \int_0^\infty \left(p(s,0) - p(s,x)\right)\, ds,$$

where $p(t,\,\cdot\,)$ is defined for all $t > 0$ and $x, y \in \mathbb{R}$ by the relation $p(t,x,y) = p(t,y-x)$ and

$$H(x) = \frac{1}{\pi}\int_0^\infty (1-\cos xs)\operatorname{Re}\left[\frac{1}{\psi(s)}\right] ds.$$

Namely, it turns out that the first one is appropriate to describe the asymptotic behaviour of the first hitting time but its serious drawback is the fact that we do not a priori know if it exists, while the other is well defined in our setting but its applications are limited to the estimates rather than asymptotics. The question whether S is well defined is highly non-trivial and we provide some examples in Lemma 5.3.1, Corollary 5.3.2, Proposition 5.3.3 and Corollary 5.3.5. The asymptotic behaviour of the first hitting time of a compact set is resolved in Theorem 5.3.7 and an important example which satisfies its assumptions is provided by Theorem 5.2.4, i.e. if the real part of the characteristic exponent is regularly varying at the origin with the scaling index $\alpha \in (1,2)$ and if we control the non-symmetry of the Lévy measure, that is when

$$\nu(dx) = C_d \mathbb{1}_{x<0}\nu_0(dx) + C_u \mathbb{1}_{x>0}\nu_0(dx),$$

where ν_0 is a symmetric Lévy measure. We note that the form above includes all stable processes as well as spectrally one-sided Lévy processes.

In order to obtain sharp estimates of the first hitting time we first prove in Theorem 5.4.4 and then apply the global scale-invariant Harnack inequality. The latter is derived under the assumptions $\mathbb{E}X_1 = 0$ and the global lower scaling property with the index $\alpha > 1$ on the real part of the characteristic exponent and extends the one obtained by Grzywny and Ryznar in [40] with the proof inspired by Bass and Levin [3]. Note that we do not assume the absolute continuity of the Lévy measure. The Harnack inequality is then applied to obtain sharp two-sided estimates of the first hitting time of a compact set, see Theorem 5.5.11. Here the results are obtained under the global scaling on $\operatorname{Re}\psi$ with $\alpha > 1$, $\mathbb{E}X_1 = 0$ and a non-trivial additional assumption which is not a priori obvious but is trivially satisfied for symmetric Lévy processes. Also, thanks to the recent preprint by Grzywny [35], we are able to bypass this problem in the special case of spectrally one-sided Lévy processes in Corollary 5.5.14. We end that chapter with an example of a class of Lévy processes which satisfy the assumptions of Theorem 5.5.11.

Most of the preliminary material, including the notation setting and the introduction of basic objects and its properties, is presented in Chapter 2. In particular, we provide a definition of a

Lévy process, introduce common tools from the potential and fluctuation theory, and provide a short overview of scaling properties and regular variation. In the end we would like to note that each subsequent chapter elaborates a corresponding part of this short introduction, provides a historical background and defines a proper setting of our research. Therefore, for further details on the specific subject we encourage the reader to consult the appropriate chapter of the dissertation.

Chapter 2

Preliminaries

In this chapter we introduce basic objects and definitions that will be in constant use throughout the dissertation. Our goal here is to achieve a sensible compromise between the urge of rigorous and complete introduction, which would result in too extensive volume, and a brief summary, usually presented at the beginning of almost every scientific article. For this reason, we usually refrain ourselves from giving rigorous proofs of these preliminary results but instead we provide complete references and suggestions where the reader may find further information on the specific subject. We start with fixing the notation and then move to an exhaustive discussion of most of the notions appearing in the subsequent chapters.

2.1 Notation

The space of continuous functions on a set D is denoted by $C(D)$. Its subspaces are denoted as follows: functions which are additionally bounded on D are denoted by $C_b(D)$, those from $C_c(D)$ are compactly supported in D. Next, by $C^k(D)$, $k = 1, 2, \ldots$, we denote a space of functions with continuous derivatives up to the order k with the usual convention that $f \in C^\infty(D)$ means that $f \in C^k(D)$ for every $k \in \mathbb{N}$. By $f \in C_0(\mathbb{R}^d)$ we mean that f is continuous and vanishes at infinity. The classes $C_0^k(D)$ and $C_b^k(D)$ are defined analogously with the implicit rule that the respective properties apply to all derivatives up to the order k as well.

By $c, c_1, C, C_1, \ldots$ we denote positive constants. Their values usually are not important and may vary from line to line in the chain of estimates. We do, however, keep track of their dependence on other parameters and to this end we write $c = c(a, b)$ to indicate that the c depends only on a and b. On rare occasions, we will distinguish some constants using a different notation but then its meaning should present no difficulties. For instance, from the context it will be rather clear that C_{H} denotes a constant appearing in the Harnack inequality. We will often use the notation $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

For two functions we write $f \approx g$ on some set D if there is a constant $c > 0$ such that

$$c^{-1} f(x) \le g(x) \le c f(x), \quad x \in D.$$

Accordingly, we write $f \lesssim g$ ($f \gtrsim g$) if the ratio f/g is bounded from above (below) by a positive constant. By $f \sim g$ as $x \to x_0$ we mean that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1.$$

Occasionally we will also write $f = \mathcal{O}(g)$ as $x \to x_0$ if

$$\limsup_{x \to x_0} \frac{f(x)}{g(x)} < \infty.$$

Frequently we will use the generalised inverse function which is defined as follows. For a complex-valued function $f\colon \mathbb{R}^d \mapsto \mathbb{C}$ we set

$$f^{-1}(s) = \sup\{r > 0\colon f^*(r) = s\}, \quad s \geq 0,$$

where f^* is a radial majorant of $\operatorname{Re} f$ i.e.

$$f^*(r) = \sup_{|x| \leq r} \operatorname{Re} f(x), \quad r \geq 0.$$

Note that for continuous functions f we have directly from the definition that

$$f^*(f^{-1}(s)) = s \qquad \text{and} \qquad f^{-1}(f^*(s)) \geq s, \quad s > 0.$$

For a Borel set D and $x \in D$ we let $\delta(D) = \inf\{|x|\colon x \in D\}$ and $\delta_D(x) = \inf\{|x-y|\colon y \in D^c\}$. As usual, the diameter of D is defined as $\operatorname{diam}(D) = \sup\{|x-y|\colon x, y \in D\}$. For $x \in \mathbb{R}$ and $r > 0$ by $B(x,r)$ we denote a ball of radius r centred at x, i.e. $B(x,r) = \{y \in \mathbb{R}\colon |y-x| < r\}$. For simplicity, we write $B_r = B(0,r)$. Finally, for the sake of clarity we declare that all sets, subsets and functions are assumed to be Borel.

2.2 Lévy process and its properties

This section introduces a basic object of our attention, namely a one-dimensional Lévy process. We provide a definition and fundamental properties, discuss its special forms and cases, and develop basic tools from the potential and fluctuation theory. We note that since our setting in this dissertation is the real line, we restrict ourselves to the one-dimensional case for the sake of consistency but it is rather straightforward that the the content presented below may be easily extended to the d-dimensional case. We start with providing the construction of the Lévy process $\mathbf{X}$.

2.2.1 Construction and basic properties

Throughout the dissertation by $\mathbf{X} = \{X_t\colon t \geq 0\}$ we will denote *a Lévy process* on the real line, i.e. a *one-dimensional* stochastic process which starts from 0, has stationary and independent increments, is stochastically continuous, and has *càdlàg* trajectories (fr. *continue à droite, limite à gauche* – right continuous with left limits). Sometimes we will drop the orthodox notation $\mathbf{X}$ in favour of less precise but generally accepted X_t. The distinction whether X_t is a single random variable or a stochastic process should be clear from the context and result in no confusion for the reader. The author is well aware of the scarcity of the description below but the detailed construction of $\mathbf{X}$ is beyond the scope of this work; therefore we will skip frequent technical difficulties in this part and focus on the outline of the construction rather than provide a rigorous reasoning. The general reference here is the book of Sato [93]; the first two chapters of the book are devoted to characterisation and existence of Lévy processes, while the rest provides an immense overview of properties of X_t and will be of our interest later on. We also refer the reader to the first two chapters of Chung and Zhao [25] for an excellent and elementary introduction to the construction and potential theory of Brownian motion which should cause no essential difficulties while extending to more general Lévy processes.

Given two numbers $\sigma \geq 0$, $\gamma \in \mathbb{R}$, and a Borel measure ν satisfying

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (|x|^2 \wedge 1)\, \nu(dx) < \infty,$$

we define *a Lévy-Khintchine exponent* by the following formula:

$$\psi(\xi) = \sigma^2\xi^2 - i\xi\gamma - \int_{\mathbb{R}} (e^{i\xi x} - 1 - i\xi x \mathbb{1}_{|x|<1}(x))\, \nu(dx), \quad \xi \in \mathbb{R}. \tag{2.2.1}$$

The measure ν is called *the Lévy measure* and the triplet (σ, γ, ν) is referred to as *Lévy triplet* or *characteristic triplet* as it can be proved to characterise any Lévy process in a one-to-one correspondence. Indeed, by the Lévy-Khintchine representation for the infinitely divisible measures [93, Theorem 8.1], for every fixed $t > 0$, the function $\varphi\colon \mathbb{R} \mapsto \mathbb{C}$ defined by $\varphi(\xi) = e^{-t\psi(\xi)}$ is a characteristic function of some random variable, which we anticipatively denote by X_t. In other words, for every $t > 0$ there is a measure $p_t(dx)$ such that its Fourier transform satisfies

$$\widehat{p_t}(\xi) = \mathbb{E}e^{i\xi X_t} = \int_{\mathbb{R}} e^{i\xi x}\, p_t(dx) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}.$$

For any $0 \leq t_0 < t_1 < \ldots < t_n$ and $A_0, A_1, \ldots, A_n \subset \mathbb{R}$ we define the finite-dimensional distributions

$$\begin{aligned}\mu_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \int_{\mathbb{R}}\int_{\mathbb{R}} \ldots \int_{\mathbb{R}} & p_{t_0}(dx_0)\mathbb{1}_{A_0}(x_0)\, p_{t_1-t_0}(dx_1)\mathbb{1}_{A_1}(x_0+x_1) \\ & \times \ldots p_{t_n-t_{n-1}}(dx_n)\mathbb{1}_{A_n}(x_0+x_1+\cdots+x_n).\end{aligned} \tag{2.2.2}$$

By the Kolmogorov extension theorem there exists a process $\mathbf{X}$ such that its finite-dimensional distributions are equal to $\mu_{t_0,t_1,\ldots,t_n}$, i.e.

$$\mathbb{P}(X_{t_0} \in A_0, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) = \mu_{t_0,t_1,\ldots,t_n}(A_0 \times A_1 \times \ldots \times A_n).$$

Next, [93, Theorem 10.5] asserts that $\mathbf{X}$ is in fact a Markov process with transition functions given by $p_t(x, A) = p_t(A - x)$. With this notation at hand, the equation (2.2.2) takes much simpler form,

$$\mu_{t_0,\ldots,t_n}(A_0 \times \cdots \times A_n) = \int_{A_0}\int_{A_1} \ldots \int_{A_n} p_{t_0}(dx_0)\, p_{t_1-t_0}(x_0, dx_1) \ldots p_{t_n-t_{n-1}}(x_{n-1}, dx_n),$$

which nicely corresponds to the Chapman-Kolmogorov property of Markov processes. Finally, [93, Theorem 11.5] ensures that there is a modification of $\mathbf{X}$ with càdlàg trajectories, which, with a slight abuse of notation, we will also denote by $\mathbf{X}$, that is the desired Lévy process. Conversely, given a Lévy process, the distribution of an infinitely divisible random variable X_1 (see e.g. [93, Corollary 8.3]) due to [93, Theorem 8.1] corresponds to a unique triplet (σ, γ, ν). The claim is established.

Recall that the triplet $(1, 0, 0)$ generates the one-dimensional Brownian motion. If $\sigma = 0$, then X_t is called *purely non-Gaussian*. The triplet $(0, \gamma, 0)$ corresponds to a deterministic drift $X_t = \gamma t$ and $(0, 0, \nu)$ gives rise to a process which is often labelled as *pure-jump process*. That name is not exactly adequate, as such processes usually have an intrinsic drift (possibly infinite in the case of unbounded variation of sample paths — we will comment on that later on), but, since the community seems to be grown accustomed to this label, we will obey the rule as well.

We also note that if $\sigma = 0$ and $\nu(\mathbb{R}) < \infty$, then $\mathbf{X}$ is a compound Poisson process (possibly with a drift), a case which is excluded from our considerations. Therefore, hereinafter we assume **in the whole dissertation** that either $\sigma > 0$ or $\nu(\mathbb{R}) = \infty$, i.e. $\mathbf{X}$ is not a compound Poisson process. We note in passing that in case of a one-dimensional Brownian motion for every $t > 0$ the transition function $p_t(dx)$ has a density $p_t(x)$ given by (see e.g. [25, Theorem 1.4])

$$p_t(x) = (2\pi t)^{-1/2} \exp\Big(- \frac{|x|^2}{2t} \Big),$$

which solves the classical one-dimensional heat equation and is therefore referred to as *the heat kernel*. The transition density $p_t(\,\cdot\,)$ for general Lévy processes (if it exists) inherits the name, although the issue of solving the corresponding heat equation is much more delicate. We note in passing that, provided that the heat kernel exists, the transition function $p_t(x, \,\cdot\,)$ has a transition density given by $p_t(x, y) = p_t(y - x)$ for every $x, y \in \mathbb{R}$. The sufficient condition for the existence of heat kernel is given by Hartman and Wintner [45]:

$$\lim_{|\xi|\to\infty} \frac{\operatorname{Re}\psi(\xi)}{\ln(1 + |\xi|)} = \infty. \tag{HW}$$

The identity above is referred to as the *Hartman-Wintner condition*. If this is the case, then not only $p_t(\,\cdot\,)$ exists for all $t > 0$, but we also have $p_t \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$. A comprehensive overview of necessary and sufficient conditions for absolute continuity of $p_t(dx)$ is provided by Knopova and Schilling [64]. In view of Bertoin [4, Proposition I.6], the family $\{p_t(\,\cdot\,) \colon t > 0\}$ induces a strongly continuous contraction semigroup on $C_0(\mathbb{R})$ equipped with the supremum norm:

$$P_t f(x) = \int_{\mathbb{R}} f(y)\, p_t(x, dy) = \mathbb{E}_x f(X_t), \quad x \in \mathbb{R}.$$

Alternatively, P_t has *the Feller property* and $\mathbf{X}$ is *a Feller process*. Moreover, [93, Theorem 40.10 and Corollary 40.11] imply that every Lévy process has the *strong Markov property*. Every Feller semigroup admits the *infinitesimal generator* defined as an operator $\mathcal{A}$ with domain $\mathscr{D}(\mathcal{A})$ consisting of functions $f \in C_0(\mathbb{R})$ for which the following limit

$$\mathcal{A} f(x) = \lim_{t\to 0^+} \frac{1}{t}(P_t f(x) - f(x)) = \lim_{t\to 0^+} \frac{\mathbb{E}_x f(X_t) - f(x)}{t}$$

exists in the strong sense, i.e. uniformly on $\mathbb{R}$. For any Lévy process X_t we have $C_0^2(\mathbb{R}) \subset \mathscr{D}(\mathcal{A})$ and

$$\mathcal{A} f(x) = \frac{\sigma^2}{2}\Delta f(x) + \gamma f'(x) + \int_{\mathbb{R}} (f(x+y) - f(x) - y f'(x)\mathbb{1}_{|x|<1}(y))\nu(y)\, dy$$

for any $f \in C_0^2(\mathbb{R})$, where (σ, γ, ν) is the characteristic triplet of $\mathbf{X}$, see e.g. [93, Theorem 31.5].

As above, the probability corresponding to $\mathbf{X}$ will be denoted by $\mathbb{P}$. Note that for any $t > 0$, the image of $\mathbb{P}$ under X_t, that is $\mathbb{P} \circ X_t^{-1}$, is a probability measure on $\mathbb{R}$. This measure is called the *law* or the *distribution* of random variable X_t. Recall that convergence in distribution of random variables is defined precisely as weak convergence of its distributions. Now, let us consider a collection $\mathcal{D}$ of càdlàg functions $\omega \colon [0, \infty) \mapsto \mathbb{R}$ which serves as a space of all admissible sample paths of a Lévy process. From that point of view, $\mu_{\mathbf{X}} = \mathbb{P} \circ \mathbf{X}^{-1}$ is a probability measure on the space of sample paths $\mathcal{D}$. Therefore, we may take $\mathbf{X}$ as a canonical map on $\mathcal{D}$, that is for any $t > 0$,

$$X_t = X_t(\omega) = \omega(t).$$

The detailed description of the space $\mathcal{D}$, as well as the topology it can be endowed with, is beyond the scope of this work; therefore, we will only refer to the books of Gīhman and Skorokhod [31], and Jacod and Shiryaev [53]. Let us also note that usually we will consider a process X_t starting from $x \in \mathbb{R}$; then the corresponding probability distribution and expectation will be denoted by $\mathbb{P}_x$ and $\mathbb{E}_x$, respectively. Accordingly, $\mathbb{P}_x(X_t \in A) = \mathbb{P}(X_t + x \in A)$ for any $t > 0$ and any Borel set $A \subset \mathbb{R}$. As usual, we let $X_{t-} = \lim_{s\to t^-} X_s$ with the convention that $X_{0-} = X_0$. The size of a jump at t is $\Delta X_t = X_t - X_{t-}$.

Motivated by Pruitt [87], we define *the concentration functions* for X_t by setting

$$K(r) = r^{-2}\sigma^2 + r^{-2}\int_{B_r} |x|^2\,\nu(dx), \quad r > 0$$

and

$$h(r) = r^{-2}\sigma^2 + \int_{\mathbb{R}} \left(1 \wedge \frac{|x|^2}{r^2}\right)\nu(dx), \quad r > 0. \tag{2.2.3}$$

Concentration functions are one of the more important characteristics of $\mathbf{X}$. Clearly, $h(r) \geq K(r)$. Moreover, by the Fubini–Tonelli theorem, we get

$$h(r) = 2\int_r^\infty K(s)s^{-1}\,\mathrm{d}s. \tag{2.2.4}$$

Observe that by [87, p. 954] there is a constant c which depends only on the dimension such that

$$\frac{c^{-1}}{h(r)} \leq \mathbb{E}S(r) \leq \frac{c}{h(r)},$$

where $S(r) = \inf\{t > 0\colon |X_t - tb_r| > r\}$ and b_r is the drift compensation defined as

$$b_r = \gamma + \int_{\mathbb{R}} x\big(\mathbb{1}_{|x|<r} - \mathbb{1}_{|x|<1}\big)\,\nu(dx). \tag{2.2.5}$$

Therefore, h may be understood as a measure of expansion of $\mathbf{X}$. Similarly, K describes the local activity and the intensity of small jumps. Recall that $b_r \equiv 0$ if the Lévy process is symmetric, that is when $\gamma = 0$ and the Lévy measure is symmetric. The presence of b_r is due to the fact that the function originally used by Pruitt has also a third component which, roughly speaking, measures the asymmetry of the process $\mathbf{X}$. Moreover, if we define the majorant of the real part of the characteristic exponent

$$\psi^*(r) = \sup_{|z|\leq r} \operatorname{Re}\psi(z),$$

then, by [34, Lemma 4],

$$\frac{1}{24}h(r^{-1}) \leq \psi^*(r) \leq 2h(r^{-1}), \quad r > 0. \tag{2.2.6}$$

The formula above entails that the concentration function will be frequently used throughout the dissertation. Let us provide one more argument by noting that from [87, p. 954] one may also conclude that there is a constant c depending only on the dimension such that

$$\mathbb{P}\left(\sup_{s\leq t}|X_s - sb_r| \geq r\right) \leq cth(r) \qquad \text{and} \qquad \mathbb{P}\left(\sup_{s\leq t}|X_s - sb_r| \leq r\right) \leq \frac{c}{th(r)}.$$

As previously, the formulae above become much simpler if $\mathbf{X}$ is symmetric. These estimates are so routinely in use that they earned its own name *Pruitt's estimates* and so they will be referred to as such from this moment on with no further comment.

In general, Lévy processes exhibit two different types of sample paths' behaviour for large time. Namely, we say that X_t is *recurrent* if

$$\liminf_{t\to\infty} |X_t| = 0 \quad \text{a.s.}$$

and *transient* if

$$\lim_{t\to\infty} |X_t| = \infty \quad \text{a.s.}$$

Any Lévy process $\mathbf{X}$ is either recurrent or transient, or in other words, the transience/recurrence dichotomy holds true (see [93, Theorem 35.4]). One can also distinguish a subclass of *point recurrent* processes which is characterised by the evidently stronger condition

$$\limsup_{t\to\infty} \mathbb{1}_{\{0\}}(X_t) = 1 \quad \text{a.s.}$$

[93, Theorem 35.4] states that transience and recurrence are closely related to the existence of potential measures which will be elaborated in Subsection 2.2.2, see remarks under (2.2.14). For now let us only mention that these objects are fundamental in the development of potential theory and its ill-definiteness usually coerces a different type of analysis. The reader may also consult [93, Theorem 37.5, Corollary 37.6 and Remark 37.7] for criteria of Chung-Fuchs type. For instance, an easy application of these results leads to the conclusion that every non-degenerate Lévy process in dimension $d \geq 3$ is transient ([93, Theorem 37.8]).

Example 2.2.1. Suppose $\mathbf{X}$ is a symmetric, d-dimensional rotationally-invariant α-stable process with $\alpha \in (0, 2]$. Then [93, Theorems 37.16 and 37.18, Remark 37.21 and Example 43.7] assert that $\mathbf{X}$ is transient if $\alpha < d$, recurrent if $\alpha \geq d$, and point recurrent if $\alpha > d$.

Let us now distinguish two different kinds of processes which will be intensively studied later on in the dissertation. It is well known (see e.g. Revuz and Yor [89, Corollary 2.5]) that sample paths of Brownian motion are of unbounded variation. Let us therefore assume that $\sigma = 0$ and suppose that

$$\int_{\mathbb{R}} (|x| \wedge 1)\, \nu(dx) < \infty. \tag{2.2.7}$$

Then we may rewrite (2.2.1) as follows:

$$\psi(\xi) = -i\xi b - \int_{\mathbb{R}} (e^{i\xi x} - 1)\, \nu(dx), \tag{2.2.8}$$

where $b = \gamma - \int_{B_1} x\, \nu(dx)$ is sometimes called *the drift* of X_t. The strict justification of that name is quite technical and we refer to [93, Chapter 4] for a comprehensive study, but the informal explanation is rather simple. If (2.2.7) holds true, then almost surely the sample path of X_t is of bounded variation (see [93, Theorem 21.9]) and we may write

$$X_t = bt + \sum_{s \leq t} \Delta X_s, \quad t \geq 0. \tag{2.2.9}$$

Here, thanks to the property of bounded variation, the infinite sum of jumps of X_t is almost surely convergent and X_t may be represented as an independent sum of a drift process and a pure-jump part, see also [93, Theorem 19.3]. If we additionally assume that $b \geq 0$ and that the Lévy measure is supported on the positive half-line, then [93, Theorem 21.5] implies that the underlying process almost surely has a non-decreasing sample path. This kind of behaviour is nicely depicted by (2.2.9). We note in passing that this result is in line with our intuitions but remains valid only under the assumption of bounded variation. In fact, even if there is no Gaussian part and the Lévy measure is concentrated on the positive half-line, but the condition (2.2.7) is not satisfied, then the process is oscillating and the support of the distribution of X_t is equal to the whole real line for all $t > 0$. Intuitively, this is due to the fact that if (2.2.7) is not satisfied, then the sum of jumps of $\mathbf{X}$ in (2.2.9) is almost surely divergent and cannot be compensated by any finite b. Therefore, there must be some compensation so that the process could exist at all. One can also recall the Lévy-Khintchine formula (2.2.1) and observe that the formal splitting of the non-local part as in (2.2.8) would result in two divergent integrals, the first one corresponding to the divergent sum of jumps and the second — to the infinite negative drift. The rigorous formulation of the considerations above is provided by [93, Theorem 24.10], see also a comment under [93, Theorem 21.5].

We denote such one-dimensional, non-decreasing process by $\mathbf{T} = (T_t : t \geq 0)$ and call it a *subordinator* to underline the connection with the abstract notion of subordination introduced in the 1950s by Bochner [8] and Phillips [81]. We postpone a more detailed discussion to Chapter 3, where subordinators will be the main focus of our attention. For now let us note that in such case it is convenient to consider the Laplace transform of $\mathbf{T}$ instead of usually preferred Fourier transform. Namely, there is a function $\phi\colon [0,\infty) \mapsto [0,\infty)$, called the *Laplace exponent* of T_t such that

$$\mathbb{E}e^{-\lambda T_t} = \int_{[0,\infty)} e^{-\lambda x}\, p_t(dx) = e^{-t\phi(\lambda)}, \quad \lambda \geq 0.$$

The Laplace exponent ϕ is in fact a Bernstein function and its numerous properties shall be studied and exploited in Chapter 3. A general reference concerning Bernstein functions is the book [94]. We also refer to [93, Chapter 6] and [4, Chapter III] for a complete introduction to subordination theory. Let us also note that directly from the definition it follows that every non-trivial subordinator is transient.

Subordinators are a special case of *spectrally one-sided Lévy processes*, i.e. one-dimensional Lévy processes with the Lévy measure supported on one of the half-lines. It can be deduced from the Lévy-Khintchine formula (2.2.1) that every one-dimensional Lévy process may be represented as a difference of independent spectrally-positive Lévy processes. In view of [93, Theorem 21.9], any one-dimensional Lévy process of bounded variation is a difference of independent subordinators. Let us consider a different case and assume that (2.2.7) does not hold. Since a negative of a spectrally-positive Lévy process is spectrally-negative, we may and do restrict ourselves to the spectrally-positive case, that is when $\operatorname{supp}\nu \subset [0,\infty)$. Such processes, due to their specific structure, find natural applications in finance modelling but are also interesting from the theoretical point of view. Namely, the absence of negative jumps entails the existence of the Laplace transform (see [4, Chapter VII]), that is there is a function $\varphi\colon [0,\infty) \mapsto \mathbb{R}$ such that

$$\mathbb{E}e^{-\lambda X_t} = e^{t\varphi(\lambda)}, \quad \lambda \geq 0. \tag{2.2.10}$$

By holomorphic extension one can see that

$$\varphi(\lambda) = \sigma^2\lambda^2 - \gamma\lambda + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{x<1}\right) \nu(dx), \quad \lambda \geq 0. \tag{2.2.11}$$

Since $\mathbb{P}(X_1 < -1) > 0$ by [93, Theorem 24.10], we have $\varphi(\lambda) \to \infty$ as $\lambda \to \infty$. We emphasize two significant differences between Laplace exponents of subordinators and spectrally positive Lévy processes. First, by differentiating (2.2.11) twice, we deduce that φ is convex, contrary to the subordinator case. Moreover, φ is not necessarily positive. Indeed, by differentiating

(2.2.10) with respect to λ, setting $t = 1$ and taking the limit $\lambda \to 0^+$ yields

$$\mathbb{E}X_1 = -\varphi'(0^+) = \gamma + \int_{[1,\infty)} x\,\nu(dx). \tag{2.2.12}$$

We see that $\mathbb{E}X_1 \in (-\infty, +\infty]$. In particular, if $\mathbb{E}X_1 > 0$, then $\varphi < 0$ in the neighbourhood of the origin. The second property will cause some technical difficulties in Chapter 4.

2.2.2 Potential theory

In this subsection we introduce basic objects of potential theory which will be in use throughout the dissertation. In what follows we assume only that the transition density of $\mathbf{X}$ exists for all $t > 0$; any additional assumptions shall be explicitly stated. In particular, we do not assume the absolute continuity of the Lévy measure $\nu(dx)$. We allow that kind of facilitation due to the fact that the existence of heat kernel in every setting considered in this dissertation is a more or less immediate consequence of the assumptions imposed. Nevertheless, one can obtain without essential difficulties a bigger generality.

First, for any given Lévy process $\mathbf{X}$, by $\widehat{\mathbf{X}}$ we denote its *dual process*, i.e. $\widehat{\mathbf{X}} = -\mathbf{X}$. Any objects corresponding to $\widehat{\mathbf{X}}$ will be also denoted by the symbol „$\widehat{\ }$". The term *dual* is motivated by two identities provided by [4, Proposition II.1 and Lemma II.2]. The first one asserts an analytical interpretation, to wit: for any non-negative measurable functions f, g and for any $t \geq 0$,

$$\int_{\mathbb{R}} P_t f(x) g(x)\, dz = \int_{\mathbb{R}} f(x) \widehat{P}_t g(x)\, dx,$$

whereas the latter yields a probabilistic motivation. Namely, for fixed $t > 0$ the reversed process $(X_{(t-s)-} - X_t \colon 0 \leq s \leq t)$ is equal in distribution to the dual process $(\widehat{X}_s \colon 0 \leq s \leq t)$. The concept of duality plays a vital role in the development of potential theory, see e.g. Chapter VI of Blumenthal and Getoor [7], but here its part will be rather marginal — we will use some objects (and their properties) related to the dual process in order to describe the behaviour of hitting times in Chapter 5, but the theory itself will not be developed. The proper introduction of these objects is postponed to the moment when basic elements of fluctuation theory are established.

By τ_D we denote *the first exit time* of $\mathbf{X}$ from an open set D, i.e.

$$\tau_D = \inf\{t > 0 \colon X_t \notin D\}.$$

[93, Theorem 40.13] ensures that τ_D is a stopping time. For a closed set F we define *the first hitting time* of F as the first exit time from its complement F^c, that is $T_F = \tau_{F^c}$. If $F = \{a\}$ is a singleton, then slightly abusing the notation we write $T_{\{a\}} = T_a$.

Let us consider a process $\mathbf{X}^D$ defined as $\mathbf{X}$ *killed* after exiting the open set D. Then its heat kernel is given by *Hunt's formula* (also called *the sweeping formula*):

$$p_t^D(x,y) = p_t(x,y) - \mathbb{E}_x[\tau_D < t; p_{t-\tau_D}(X_{\tau_D}, y)], \quad t > 0, x, y \in \mathbb{R}. \tag{2.2.13}$$

and is due to Hunt [49], who originally proved (2.2.13) for $\mathbf{X}$ being a Brownian motion. The reader may also consult [25, Theorem 2.4] for a neat presentation of the proof and for the conclusion that the result holds also for general Lévy processes. We observe that (2.2.13) has a simple probabilistic interpretation: we remove sample paths which exit D before time t so that they have no impact on the transition density of X_t^D. From (2.2.13) we clearly have that $0 \leq p_t^B \leq p_t^D \leq p_t$ for any $B \subset D \subset \mathbb{R}^d$. For completeness we note that although one may expect the Dirichlet heat kernel to vanish on the complement of D, this is not a priori obvious for general processes and domains. For instance, if $\mathbf{X}$ is isotropic and D is sufficiently regular, say, Lipschitz, then by the Blumenthal 0-1 law, $\mathbb{P}_x(\tau_D = 0) = 1$ for $x \in D^c$ and consequently, $p_t^D(x, y) = 0$ for $x \in D^c$ or $y \in D^c$. In general, however, this is a delicate matter and requires a special attention. Since this problem will not be relevant in the present dissertation, let us only refer to [11, Remark 1.9] for a neat elaboration on the subject.

Next, for any $\lambda > 0$, we let U^λ be *the λ-potential kernel*, that is the Laplace transform of the heat kernel:

$$U^\lambda(x, y) = \int_0^\infty e^{-\lambda x} p_t(x, y)\, dt. \tag{2.2.14}$$

Since $p_t(x, y) = p_t(y - x)$ for $x, y \in \mathbb{R}$, we also have $U^\lambda(x, y) = U^\lambda(y - x)$ for $x, y \in \mathbb{R}$. If U^λ exists for $\lambda = 0$, then we set $U^0 = U$ and call U *the potential kernel*. In view of [93, Theorem 35.4], the existence of the potential kernel implies that the underlying process is transient. If this is not the case, then various notions of compensation are considered in order to make the integral defining U^λ convergent, see e.g. Port and Stone [86], Bogdan and Żak [15] or Grzywny and Ryznar [40]. We also refer to Chapter 5 for elaboration and further discussion.

By analogy, one may also define the λ-potential kernel for the killed process $\mathbf{X}^D$ by setting

$$G_D^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p^D(t; x, y)\, dt, \quad x, y \in D.$$

The function G_D^λ is called the *λ-Green function*. The 0-Green function (if it exists) is simply called *the Green function* and we denote it by G_D. Accordingly,

$$G_D(x, y) = \int_0^\infty p_t^D(x, y)\, dt, \quad x, y \in \mathbb{R}.$$

The Green function gives rise to the *Green operator*

$$G_D[f](x) = \int_{\mathbb{R}^d} G_D(x, y) f(y)\, dy, \quad x \in \mathbb{R},$$

for suitable functions f. Observe that if f is non-negative, then the Tonelli theorem implies

$$\int_{\mathbb{R}} G_D(x, y) f(y) dy = \int_{\mathbb{R}} \int_0^\infty p^D(t, x, y) f(y)\, dt\, dy = \mathbb{E}_x \int_0^{\tau_D} f(X_t)\, dt,$$

which suggests that G_D may be interpreted as the occupation time density of X_t up to the first exit time τ_D. In particular, for $f \equiv 1$ we obtain $G_D[\mathbf{1}](x) = \mathbb{E}_x \tau_D$. The question whether G_D is well defined (that is: finite) is a non-trivial task itself and, in general, it depends on both the process $\mathbf{X}$ and the domain D. It is clear, however, that the sufficient conditions are transience of $\mathbf{X}$ or boundedness of D — in such case, the claim follows from Pruitt's estimates.

Recall that by the strong Markov property, for any $t > 0$, $x, y \in \mathbb{R}$ and $B \subsetneq D$,

$$p_t^D(x, y) = p_t^B(x, y) + \mathbb{E}_x\big[\tau_B < t; p_{t-\tau_B}^D(X_{\tau_B}, y)\big].$$

Thus, integrating the above equality against dt yields

$$G_D(x, y) = G_B(x, y) + \mathbb{E}_x G_D(X_{\tau_B}, y), \quad x, y \in \mathbb{R}. \tag{2.2.15}$$

Moreover, by integration of the identity (2.2.13) with respect to time, for any $\lambda > 0$ the following Hunt formula (or the sweeping formula) for the λ-Green function holds true:

$$G_D^\lambda(x, y) = U^\lambda(x, y) - \mathbb{E}_x\left[e^{-\lambda \tau_D} U^\lambda(X_{\tau_D}, y)\right], \quad x, y \in \mathbb{R}. \tag{2.2.16}$$

If U is well defined (that is: finite) ,then the equality above is valid also for $\lambda = 0$. If this is not the case, then in some cases the aforementioned *compensated potential kernel* may be inserted in place of infinite U, the example here being Theorem A.4 from Grzywny, Kassmann and the author [36].

We say that a function f is *regular harmonic* (with respect to X_t) in an open set D if it has *the mean value property* in D, i.e.

$$f(x) = \mathbb{E}_x f(X_{\tau_D}), \quad x \in D. \tag{2.2.17}$$

Here we assume that the integral on the right-hand side is absolutely convergent. If (2.2.17) holds for every bounded open set B such that $\overline{B} \subset D$, then f is *harmonic* in D. For instance, (2.2.15) implies that for every $y \in D$ the function $x \mapsto G_D(x, y)$ is harmonic in $D \setminus \{y\}$ and regular harmonic in every open set B such that $\overline{B} \subset D \setminus \{y\}$. By the strong Markov property every regular harmonic function is harmonic.

We note that such notion of harmonicity is in line with the classical formulation of the mean value property for the Laplace operator, cf. [25, Theorems 1.9 and 1.24] in the multi-dimensional case. Recall that the process $\mathbf{X}$ corresponding to Δ is in fact a Brownian motion. Thus, if we take $B = B(x, r)$, then, due to continuity of sample paths and rotational invariance, we conclude that the distribution of X_{τ_B} under $\mathbb{P}_x$ is uniform on a sphere $\partial B(x, r)$. This property stays in stark contrast with any $\mathbf{X}$ with a non-zero non-local part, where, due the presence of jumps, the support of X_{τ_D} is in general the whole D^c. The distribution of X_{τ_D} is called the *harmonic measure* and is denoted by $P_D(x, \cdot\,)$, i.e. for any Borel $A \subset \mathbb{R}^d$,

$$P_D(x, A) = \mathbb{P}_x(X_{\tau_D} \in A).$$

The distribution of the first exit time is a delicate matter and we comment on it right now. To this end, let us evoke the celebrated Ikeda-Watanabe formula for a joint distribution of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$ restricted to the event $\{\tau_D < \infty, X_{\tau_D -} \neq X_{\tau_D}\}$, i.e.

$$\mathbb{P}_x(\tau_D \in I, X_{\tau_D -} \in A, X_{\tau_D} \in B) = \int_I \int_{B-y} \int_A p^D(u, x, dy)\, \nu(dz)\, du. \tag{2.2.18}$$

The result is due to Ikeda and Watanabe [50]; see also Bogdan, Rosiński, Serafin and Wojciechowski [13, Section 4.2 and equation (4.13)] for a neat derivation of (2.2.18) via the so-called Lévy system formula. In particular, if we restrict ourselves to the case when the Lévy measure is absolutely continuous with respect to the Lebesgue measure, then taking $I = (0, \infty)$ and $A = D$ yields

$$\mathbb{P}_x(\tau_D < \infty, X_{\tau_D -} \neq X_{\tau_D}, X_{\tau_D} \in B) = \int_B \int_D G_D(x, y) \nu(z - y)\, dy\, dz.$$

Then it follows that the function

$$P_D(x, z) = \int_D G_D(x, y) \nu(y - z)\, dy, \quad x \in D, z \in D^c, \tag{2.2.19}$$

is the density of the distribution of X_{τ_D} restricted to the event that $\mathbf{X}$ exits D by a jump. $P_D(x, \cdot\,)$ is called the *Poisson kernel*, again in analogy to the classical case. The identities (2.2.18) and (2.2.19) are both referred to as the Ikeda-Watanabe formula and although it should be clear from the setting which one is applied at the moment, for the sake of clarity we will provide both the name and the number. To summarise, in general, the harmonic measure may contain a singular part on ∂D and has a density $P_D(x, z)$ on D^c which corresponds to a discontinuous exit from D. A natural question arises in which situations the identity $\mathbb{P}_x(X_{\tau_D} \in \partial D) = 0$

holds true for all $x \in D$. This is, however, beyond the scope of this dissertation; therefore, let us only note one vital example of Lipschitz sets which was stated by Millar [72] and clarified by Sztonyk [99]. A more general geometric condition is given by means of the so-called *volume density condition* by Wu [104] or in the Appendix of Bogdan, Grzywny, Pietruska-Pałuba and Rutkowski [9].

Let us now introduce one more powerful tool in the potential theory. First, we say that the *Harnack inequality* holds for X_t if for any $0 < r < R$ there is a constant $C_H = C_H(d, r, R)$ such that for every function f which is non-negative in $\mathbb{R}^d$ and harmonic in B_R,

$$\sup_{x \in B_r} f(x) \leq C_{\mathrm{H}} \inf_{x \in B_r} f(x). \tag{2.2.20}$$

The Harnack inequality is said to be *scale invariant* if the dependence on r is in fact expressed by the ratio r/R; in such case, the usual choice is $r = R/2$. If further C_{H} does not depend on R, then the Harnack inequality is *global scale invariant*.

The nature of the inequality (2.2.20) is local in the sense that it describes the behaviour of harmonic functions on subsets of B_R which are separated from the boundary by a positive number. In particular, as $r \to R^-$, one should expect the constant C_H to explode. The history of Harnack inequalities dates back to the end of XIX century to the work of Harnack [44] and since then they have been intensively studied and exploited in various applications. We refer here to the excellent survey article by Kassmann [57] which covers the historical summary as well as a review of classical results. Note here one crucial difference with the non-local case: here the harmonic function f, due to the non-locality of the underlying operator, is required to be non-negative on the whole $\mathbb{R}^d$ instead of only on B_R. For the visualisation of contrast between local and non-local setting, one may consult Bass and Levin [3]. Since the appearance of that paper numerous generalisations have been obtained, see e.g. Grzywny and Ryznar [40]. Some contribution to this subject is given in Chapter 5. Among its many consequences one should list a classical application to local Hölder regularity of harmonic function, c.f. [3] and Mimica [74], or an easy corollary about the estimates of the derivative of the renewal function for the ladder height process (defined later in the chapter), which in turn entail estimates of the density of the distribution of the supremum process, see Chaumont and Małecki [18]. Another application is provided in Chapter 5 where (2.2.20) is applied to deduce certain boundary behaviour of harmonic function, which eventually implies estimates of the first hitting time of points and intervals.

For the sake of completeness, we note that the phenomena that occur near the boundary of D are captured by the so-called *boundary Harnack inequality*. In short, it states that harmonic functions decay at the same rate or, in other words, that their ratio has bounded oscillation close to the boundary. Perhaps the most general setting of the boundary Harnack inequality for jump processes is provided by Bogdan, Kumagai and Kwaśnicki [12], and we refer to this article for further development.

2.2.3 Elements of fluctuation theory

In this subsection we impose the following integral condition:

$$\int_0^\infty \frac{d\xi}{1 + \operatorname{Re}\psi(\xi)} < \infty. \tag{2.2.21}$$

Such condition implies that $\operatorname{Re}\psi$ is unbounded, hence it must not be a characteristic function of a compound Poisson process and consequently, $\operatorname{Re}\psi(\xi) > 0$ for $\xi \neq 0$. First, observe that

(2.2.21) implies that $\mathbf{X}$ is of unbounded variation. Indeed, suppose the converse; then $\mathbf{X}$ can be depicted as a difference of two independent subordinators. It follows that $\operatorname{Re}\psi$ has at most linear growth and hence,

$$\int_0^\infty \frac{d\xi}{1+\operatorname{Re}\psi(\xi)} = \infty,$$

which is a contradiction.

Next, given a closed set $B \subset \mathbb{R}^d$, we say that a point $x \in \mathbb{R}^d$ is *regular for* B if $\mathbb{P}_x(T_B = 0) = 1$. If $B = \{x\}$, then we say that x is regular for itself, which is equivalent to saying that 0 is regular for itself. Now, (2.2.21) together with Bretagnolle [16, Theoreme 7 and 8] imply that 0 is regular for itself, i.e. $\mathbb{P}^0(T_0 = 0) = 1$. This, in turn, combined with Kesten [58, Theorem 1], yields that $\mathbb{P}_x(T_0 < \infty) = 1$, that is our process hits 0 almost surely in finite time. Furthermore, by (2.2.21) together with Rogozin [91], we have that

$$\limsup_{t\to 0^+} \frac{X_t}{t} = \infty \quad \text{a.s.} \qquad \text{and} \qquad \liminf_{t\to 0^+} \frac{X_t}{t} = -\infty \quad \text{a.s.} \tag{2.2.22}$$

In particular, it follows that 0 is regular for half-lines $(-\infty, 0)$ and $(0, \infty)$. Also, (2.2.22) immediately implies that $\mathbb{P}(\inf\{t > 0\colon X_t \neq 0\} = 0) = 1$, that is 0 is *an instantaneous point*.

With these preliminary results at hand, let us consider a reflected process $\mathbf{S} - \mathbf{X}$, where $S_t = \sup_{s\le t} X_s$. Observe that the zero set of the reflected process coincides with the set of points of new maxima of $\mathbf{X}$. By $\mathbf{L} = (L_t\colon t \geq 0)$ we denote *the local time at* 0 for the process $\mathbf{S} - \mathbf{X}$, that is an increasing and continuous process which increases precisely on the closure of the zero set of $\mathbf{S} - \mathbf{X}$ (or equivalently, increases only on the closure of the set of new maxima of $\mathbf{X}$). The problem of defining and constructing local times is highly non-trivial and is beyond the scope of this short introduction. They will not be, however, objects of our analysis and we introduce them only to formally define other objects of fluctuation theory. Therefore, we restrict ourselves here to referring to [4, Chapter IV] and Kyprianou [68, Chapter 6] for a comprehensive study. Let us only provide one example where the local time at 0 can be easily defined as a functional of $\mathbf{X}$. Namely, if X_t is spectrally-negative, then, due to the absence of positive jumps, we conclude that every new maximum is attained in a continuous way. Thus, we may simply take $L_t = S_t$.

Next, by $\mathbf{L}^{-1}$ we denote the right-continuous inverse of $\mathbf{L}$, i.e. $L_t^{-1} = \inf\{s \geq 0\colon L(s) > t\}$. It follows directly from the definition that $\mathbf{L}^{-1}$ recovers times when new maxima occur and is therefore called *the ascending time ladder process*. Now define $\mathbf{H} = (H_t\colon t \geq 0)$ by setting $H_t = S_{L_t^{-1}} = X_{L_t^{-1}}$ whenever $L_t^{-1} < \infty$ and $H_t = \infty$ otherwise. Consequently, $\mathbf{H}$ recovers values of maxima and is by analogy called *the ascending ladder height process*. The pair $(\mathbf{L}^{-1}, \mathbf{H})$ is referred to as *the ascending ladder process* and its properties are intensively studied in [4, Chapter VI].

We may now introduce *the renewal function* V as a potential measure of the interval $[0, x]$ for the ascending ladder height process H_t, that is

$$V(x) = \int_0^\infty \mathbb{P}(H_s \leq x)\, ds, \quad x \geq 0,$$

with the convention that $V(x) = 0$ for $x < 0$. Similarly, we let

$$\widehat{V}(x) = \int_0^\infty \mathbb{P}(\widehat{H}_s \leq x)\, ds, \quad x \geq 0,$$

and $\widehat{V}(x) = 0$ for $x < 0$. Clearly, if $\mathbf{X}$ is symmetric, then $\widehat{\mathbf{X}} = \mathbf{X}$ and consequently $\widehat{V} = V$. Directly from definitions of V and $\widehat{V}$ we conclude that both are sub-additive. Moreover, by

Silverstein [95, Theorem 1 and 2], $\widehat{V}$ and $\widehat{V}'$ are harmonic on $(0,\infty)$ and V and V' are coharmonic on $(0,\infty)$. In fact, also from [95], we have that both V' and $\widehat{V}'$ are positive on $(0,\infty)$, hence both V and $\widehat{V}$ are actually strictly increasing on $(0,\infty)$. We also note that using monotonicity and sub-additivity of V and $\widehat{V}$, for any $\lambda \geq 1$ and any $r>0$,

$$V(\lambda x) \leq 2\lambda V(x) \qquad \text{and} \qquad \widehat{V}(\lambda x) \leq 2\lambda \widehat{V}(x).$$

Eventually let us mention that there is a neat connection provided by [4, Theorem VI.20] between renewal functions and Green function for the half line. Namely,

$$G_{(0,\infty)}(x,y) = \int_0^x \widehat{V}'(u)V'(y-x+u)\,du, \quad y>x>0. \tag{2.2.23}$$

2.3 Regular variation and weak scaling properties

This section is devoted to notions of regular variation and scaling properties as well as their consequences. Especially the latter is nowadays a standard assumption in the literature imposed either on the Lévy density or on (the real part of) the characteristic exponent. For this reason, we are content here with merely a proper introduction and postpone putting it in context to appropriate chapters. For the sake of completeness let us also remark that the concept of regular variation in all its glory is presented in Bingham, Goldie and Teugels [5], and we will refer to it whenever possible.

We say that a function $f\colon [0,\infty) \mapsto [0,\infty)$ is *regularly varying at infinity* with index $\alpha \in \mathbb{R}$ if for every $\lambda \geq 1$,

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha}.$$

In this case, we write shortly $f \in \mathcal{R}^{\infty}_{\alpha}$. Analogously, f is *regularly varying at the origin* with index $\alpha \in \mathbb{R}$ if for every $\lambda \geq 1$,

$$\lim_{x\to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^{\alpha},$$

and then we write $f \in \mathcal{R}^{0}_{\alpha}$. If $\alpha = 0$, then f is said to be *slowly varying* at infinity (at the origin). Out of many useful properties of regularly varying functions, we distinguish [5, Theorem 1.5.6] which shall hereinafter be referred to as *Potter bounds*. Namely, if f is regularly varying at infinity with index α, then for any $C>1$ and $\varepsilon>0$ there is $x_0 = x_0(C,\varepsilon)$ such that for all $x, y \geq x_0$,

$$\frac{f(y)}{f(x)} \leq C\left(\left(\frac{y}{x}\right)^{\alpha+\varepsilon} \vee \left(\frac{y}{x}\right)^{\alpha-\varepsilon}\right).$$

For other remarkable results on regularly varying functions we refer the reader to [5] — see e.g. [5, Proposition 1.5.8 and Theorem 1.6.1] for famous Karamata theorem, [5, Theorem 1.7.1] for the Karamata tauberian theorem, or [5, Theorem 1.7.2] for the monotone density theorem.

Next, we introduce evidently weaker conditions of lower and upper scaling which will be frequently used in the dissertation. Namely, we say that $f\colon [0,\infty) \mapsto [0,\infty)$ has the *weak lower scaling property* at infinity if there are $\alpha \in \mathbb{R}$, $c \in (0,1]$ and $x_0 \geq 0$ such that for all $\lambda \geq 1$ and $x > x_0$,

$$f(\lambda x) \geq c\lambda^{\alpha} f(x).$$

We denote it briefly as $f \in \mathrm{WLSC}(\alpha, c, x_0)$. The triple (α, c, x_0) will be usually referred to as *scalings*. Observe that, if $\alpha > \alpha'$, then $\mathrm{WLSC}(\alpha, c, x_0) \subsetneq \mathrm{WLSC}(\alpha', c, x_0)$. If $x_0 = 0$, then we simply write $f \in \mathrm{WLSC}(\alpha, c)$. Similarly, f has the *weak upper scaling property* if there are $\beta \in \mathbb{R}$, $C \geq 1$ and $x_0 \geq 0$ such that for all $\lambda \geq 1$ and $x > x_0$,

$$f(\lambda x) \leq C\lambda^{\beta} f(x).$$

In this case we write $f \in \mathrm{WUSC}(\beta, c, x_0)$. If $x_0 = 0$, then we denote it as $f \in \mathrm{WUSC}(\beta, C)$.

We say that a function $f\colon [0, \infty) \mapsto [0, \infty)$ has a *doubling property* on (x_0, ∞) for some $x_0 \geq 0$ if there is $C \geq 1$ such that for all $x > x_0$,

$$C^{-1} f(x) \leq f(2x) \leq C f(x).$$

Notice that a non-increasing function with the weak lower scaling property has the doubling property. The same is true for a non-decreasing function with the weak upper scaling property.

Finally, we say that a function $f\colon [0, \infty) \mapsto [0, \infty)$ is *almost increasing* on (x_0, ∞) for some $x_0 \geq 0$ if there is $c \in (0, 1]$ such that for all $y \geq x > x_0$,

$$cf(x) \leq f(y).$$

It is almost decreasing on (x_0, ∞) if there is $C \geq 1$ such that for all $y \geq x > x_0$,

$$Cf(x) \geq f(y).$$

In view of [10, Lemma 11], $f \in \mathrm{WLSC}(\alpha, c, x_0)$ if and only if the function

$$(x_0, \infty) \ni x \mapsto x^{-\alpha} f(x)$$

is almost increasing. Similarly, $f \in \mathrm{WUSC}(\beta, C, x_0)$ if any only if the function

$$(x_0, \infty) \ni x \mapsto x^{-\beta} f(x)$$

is almost decreasing.

Chapter 3

Transition densities of subordinators of positive order

3.1 Introduction

From this moment on we assume that $d = 1$. The material of this chapter is taken from the preprint by Grzywny, the author and Trojan [38].

Let $\mathbf{T} = (T_t : t \geq 0)$ be a subordinator with the Laplace exponent ϕ. Let us first provide the historical background and define a proper context of our research. As aforementioned in Chapter 2, the abstract introduction of the subordination dates back to 1950s and is due to Bochner [8] and Philips [81]. In the language of the semigroup theory, for a Bernstein function ϕ and a bounded C_0-semigroup $(e^{-t\mathcal{A}} : t \geq 0)$ with $-\mathcal{A}$ being its generator on some Banach space $\mathscr{X}$, via Bochner integral one can define an operator $\mathcal{B} = \phi(\mathcal{A})$ such that $-\mathcal{B}$ also generates a bounded C_0-semigroup $(e^{-t\mathcal{B}} : t \geq 0)$ on $\mathscr{X}$. The semigroup $(e^{-t\mathcal{B}} : t \geq 0)$ is then said to be *subordinated* to $(e^{-t\mathcal{A}} : t \geq 0)$, and although it may be very different from the original one, its properties clearly follow from properties of both the *parent* semigroup and the involved Bernstein function. See for example Gomilko and Tomilov [32] and the references therein.

From probabilistic point of view, due to positivity and monotonicity, subordinators naturally appear as a random time change functions of Lévy processes, or more generally, Markov processes. Namely, if $(X_t : t \geq 0)$ is a Markov process and $(T_t : t \geq 0)$ is an independent subordinator, then $Y_t = X_{T_t}$ is again a Markov process with a transition function given by

$$\mathbb{P}_x(Y_t \in A) = \int_{[0,\infty)} \mathbb{P}_x(X_s \in A)\,\mathbb{P}(T_t \in ds).$$

The procedure just described is called a subordination of a Markov process and can be interpreted as a probabilistic form of the equality $\mathcal{B} = \phi(\mathcal{A})$. Here $\mathcal{A}$ and $\mathcal{B}$ are (minus) generators of semigroups associated to processes X_t and Y_t, respectively. From analytical point of view, the transition density of Y_t (the integral kernel of $e^{-t\mathcal{B}}$) can be obtained as a time average of transition density of X_t with respect to the distribution of T_t. In particular, by taking $\mathcal{A} = -\Delta$ and changing the time of (that is: subordinating) a Brownian motion, one can obtain a large class of subordinated Brownian motions. A principal example here is an α-stable subordinator with the Laplace exponent $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$, which gives rise to the symmetric, rotation-invariant α-stable process and corresponds to the special case of fractional powers of semigroup $(e^{-t\mathcal{A}^\alpha} : t \geq 0)$. For this reason, distributional properties of subordinators were often studied with reference to heat kernel estimates of subordinated Brownian motions, see e.g. Kim and Mimica [60] and the references therein.

Our intention in this chapter is to concentrate on the subordinator itself rather than on its time-change applications and to obtain asymptotic behaviour as well as upper and lower estimates of its transition density in a possibly wide generality. For the stable case we refer the reader to Hawkes [46], where the author investigates the growth of sample paths of a stable subordinator and obtains the asymptotic behaviour of its distribution function as well as its density. The former result on the growth of T_t was later extended by Fristedt and Pruitt [30]. The latter, i.e. tail probability estimates for subordinators and non-decreasing random walks, were considered by Jain and Pruitt in [54]. In a more general setting some related results were obtained e.g. by Iksanov, Kabluchko and Marynych [51], Picard [82], or Vaudeva and Divanji [102]. Some new examples of families of subordinators with explicit transition densities were given by Burridge, Kuznetsov and Kwaśnicki [17]. A result more related to our contribution is the work of Fahrenwaldt [29], where the author derives explicit approximate expressions for the transition density. However, the assumptions imposed are rather restrictive in the sense that subordinators which are admissible in [29] are approximately stable.

The generality level we propose is expressed by scaling conditions. Our **standing** assumption in this chapter is the weak lower scaling property imposed on the (minus) second derivative of the Laplace exponent $-\phi''$. The first result provides the asymptotic behaviour of the transition density. Note that in view of Proposition A.1.8 we may conclude that the Hartman-Wintner condition (HW) is satisfied and consequently, the probability distribution of T_t has a density $p(t,\,\cdot\,)$.

Theorem 3.1.1. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ*. Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$*,* $x_0 \geq 0$*, and* $\alpha > 0$*. Then for each* $\varepsilon > 0$ *there is* $M_0 > 0$ *such that*

$$\left| p\big(t, t\phi'(w)\big)\sqrt{2\pi t(-\phi''(w))}\exp\Big\{t\big(\phi(w)-w\phi'(w)\big)\Big\} - 1\right| \leq \varepsilon,$$

provided that $w > x_0$ *and* $tw^2(-\phi''(w)) > M_0$*.*

We note that due to the scaling property of $-\phi''$ we have that the function $w \mapsto w^2\,(-\phi''(w))$ is almost increasing and tends to infinity as $w \to \infty$; therefore, for fixed $t > 0$ the admissible set of w is in fact a certain right half-line.

The result above yields a following approximation of the transition density which is valid on certain appropriate region:

$$p(t,x) \approx \frac{1}{\sqrt{t(-\phi''(w))}}\exp\Big\{-t\big(\phi(w)-w\phi'(w)\big)\Big\},$$

where $w = (\phi')^{-1}(x/t)$. The identity above is displayed by Corollaries 3.3.4 and 3.3.6. In particular, we see the estimate above is genuinely sharp, i.e. the constants appearing in the exponential factors are the same on both sides of the estimate.

The proof of the result above is provided in Section 3.3. Theorem 3.1.1 is essential for the whole chapter because it provides the asymptotic behaviour of the transition density which is later used in derivation of upper and lower estimates. The key argument in the proof is the lower estimate of the holomorphic extension of the Laplace exponent ϕ (see Lemma 3.3.1) which justifies the inversion of the Laplace transform and allows us to perform the saddle point type approximation. We note that the only assumption is the weak lower scaling property on the second derivative of the Laplace exponent. In particular, we do not assume the absolute continuity of the Lévy measure ν. Observe also that the asymptotics are valid in some region described by means of both space and time variable. By freezing one of them, we obtain as corollaries the results similar to [29], see e.g. Corollary 3.3.5.

Next, in Section 3.4, we apply Theorem 3.1.1 in order to obtain various upper and lower estimates of the transition density. A perhaps striking remark here is that we do not state our assumptions and results by means of the Laplace exponent ϕ but by means of its second derivative and a related function $\Phi(x) = x^2(-\phi''(x))$ (see Theorems 3.1.1, 3.4.3 and 3.4.4). It may seem surprising at first sight, but it is in fact in line with estimates of general Lévy processes. Namely, usually the transition density of a Lévy process is described by the generalized inverse of the real part of the characteristic exponent ψ^{-1} (see e.g. Grzywny and Szczypkowski [43] or Knopova and Kulik [63]). It turns out that in our setting one can show (see Proposition A.1.5) that the lower scaling property implies that $\Phi^{-1} \approx \psi^{-1}$ for x sufficiently large. We note also that in some cases Φ may be significantly different from the Laplace exponent ϕ, the example here being ϕ regularly varying at infinity with the scaling index $\alpha = 1$. However, if one assumes additional upper scaling condition with scaling parameter $\beta - 2$ for β strictly between 0 and 1, then by virtue of Proposition 3.4.2 these two objects are comparable. We should also note here that although our starting point and the main object to work with is the Laplace exponent ϕ, in many cases the primary object is either the real part of the characteristic exponent or the Lévy measure ν; in such case results are often presented by means of or require its tail decay. Therefore, it would be convenient to have a connection between these objects and this problem is addressed in Appendix A through Propositions A.1.8 and A.1.9, where we prove that scaling property of $\operatorname{Re}\psi$ or the tail of the Lévy measure ν indeed implies the scaling property of $-\phi''$.

Main results concerning upper and lower estimates are covered by Theorems 3.4.4, 3.4.7 and 3.4.10. They are afterwards merged together with Theorem 3.1.1 in Theorem 3.4.13, which provides sharp two-sided estimate of the transition density under both lower and upper scaling conditions with $0 < \alpha \leq \beta < 1$. Below we present its special case where the imposed scalings are assumed to be global. We note that a similar result to Theorem 3.1.2 under more restrictive assumptions appeared in Chen, Kim, Kumagai and Wang [23]. Their result, however, is not sharp, i.e. different constants appear in the exponent on both sides of the estimate, while in the following theorem both constants are equal to 1.

Theorem 3.1.2. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ. *Suppose that* $\phi \in \mathrm{WLSC}(\alpha, c) \cap \mathrm{WUSC}(\beta, C)$ *for some* $c \in (0,1]$, $C \geq 1$ *and* $0 < \alpha \leq \beta < 1$. *We also assume that the Lévy measure* ν *has an almost decreasing density* $\nu(x)$. *Then for all* $t \in (0,\infty)$ *and* $x > 0$,

$$p(t,x) \approx \begin{cases} \big(t\big(-\phi''(w)\big)\big)^{-1/2} \exp\left\{-t(\phi(w) - w\phi'(w))\right\}, & \text{if } 0 < x\phi^{-1}(1/t) \leq 1, \\ tx^{-1}\phi(1/x), & \text{if } 1 < x\phi^{-1}(1/t), \end{cases}$$

where $w = (\phi')^{-1}(x/t)$.

Two possible applications of main results of this chapter are presented in Section 3.5. The first one addresses the problem of subordination beyond the classical $\mathbb{R}^d$ setting through Examples 3.5.2 and 3.5.3. In the second we concentrate on global two-sided estimate of the Green function of $\mathbf{T}$. Namely, under the assumption of Theorem 3.1.2, we prove that for all $x > 0$,

$$U(x) \approx \frac{1}{x\phi(1/x)}.$$

We refer the reader to Theorem 3.5.8 for a precise formulation. We note in passing that this result refines estimates obtained by Kim, Song and Vondraček [62] and extends the class of admissible subordinators.

Finally, we also remark that the main results of this chapter hold true when $-\phi''$ is a function regularly varying at infinity with regularity index $\alpha-2$, where $\alpha>0$. This follows easily by Potter bounds for regularly varying functions, which immediately imply both lower and upper scaling properties. Moreover, if additionally $\alpha<1$, then, by Karamata's theorem and monotone density theorem, regular variation of $-\phi''$ with index $\alpha-2$ is equivalent to regular variation of ϕ with index α. However, the situation is a bit different in the case $\alpha=1$. Namely, by [5, Proposition 1.5.9b], the regular variation of $-\phi''$ still yields the slow variation of $-\phi'$ and the application of Karamata's theorem implies regular variation of ϕ. However, the other direction in general does not hold, since then the monotone density theorem yields a void conclusion.

3.2 Bernstein functions

In this section we recall some basic facts about Bernstein functions and derive a number of preliminary results concerning the Laplace exponent ϕ. Roughly speaking, our goal here is to prove that scaling properties are preserved by differentiation and integration of ϕ and its derivatives. These properties often will be assumed later in this chapter and such approach seems to be natural, at least in the sense that they are rather standard in the theory of the subordinated Brownian motion, cf. for instance Kim and Mimica [59, 60]. For an immense overview of Bernstein functions and their applications, we refer the reader to the book of Schilling, Song and Vondraček [94].

Let us first recall a definition. A function $f\colon(0,\infty)\mapsto[0,\infty)$ is *completely monotone* if it is smooth and

$$(-1)^n f^{(n)}\geq 0$$

for all $n\in\mathbb{N}_0$. Next, a function ϕ is called a *Bernstein* function if it is a non-negative smooth function such that ϕ' is completely monotone.

Let ϕ be a Bernstein function. By [94, Theorem 3.2], there are two non-negative numbers a and b, and a Radon measure μ on $(0,\infty)$ satisfying

$$\int_{(0,\infty)}(1\wedge s)\,\mu(ds)<\infty,$$

and such that

$$\phi(\lambda)=a+b\lambda+\int_{(0,\infty)}(1-e^{-\lambda s})\,\mu(ds),\quad \lambda\geq 0. \tag{3.2.1}$$

A Bernstein function ϕ is called *complete* if the measure μ has a completely monotone density with respect to Lebesgue measure. Note that from the representation above we immediately obtain that

$$\phi'(\lambda)=b+\int_{(0,\infty)}se^{-\lambda s}\,\mu(ds),\quad \lambda\geq 0,$$

and

$$-\phi''(\lambda)=\int_{(0,\infty)}s^2e^{-\lambda s}\,\mu(ds),\quad \lambda\geq 0.$$

In particular, we see that ϕ' and $-\phi''$ are strictly positive on $(0,\infty)$, provided that the measure μ is non-zero or in other words, ϕ is not linear. In view of [52, Lemma 3.9.34], for all $n\in\mathbb{N}$ we have

$$\frac{(-1)^{n+1}}{n!}\lambda^n\phi^{(n)}(\lambda)\leq\phi(\lambda),\quad \lambda>0. \tag{3.2.2}$$

Since ϕ is concave, for each $\lambda \geq 1$ and $x > 0$ we have

$$\phi(\lambda x) \leq \phi'(x)(\lambda - 1)x + \phi(x),$$

thus, by (3.2.2),

$$\phi(\lambda x) \leq \lambda\phi(x). \tag{3.2.3}$$

We start with a preliminary result concerning completely monotone functions.

Proposition 3.2.1. *Let f be a completely monotone function. Suppose that f has a doubling property on (x_0, ∞) for some $x_0 \geq 0$. Then there is $C > 0$ such that for all $x > x_0$,*

$$f(x) \geq Cx|f'(x)|.$$

Proof. Without loss of generality we may assume that $f \not\equiv 0$. Clearly,

$$f(x) - f(x/2) = \int_{x/2}^{x} f'(s)\, ds \leq \tfrac{1}{2}xf'(x).$$

Since f is completely monotone, it is a positive function and

$$f(x/2) \geq \tfrac{1}{2}x|f'(x)|,$$

which together with the doubling property, gives

$$f(x) \geq Cx|f'(x)|$$

for $x > 2x_0$. Hence, we obtain our assertion in the case $x_0 = 0$. If $x_0 > 0$, then we observe that the function

$$[x_0, 2x_0] \ni x \mapsto \frac{x|f'(x)|}{f(x)}$$

is continuous and positive, thus bounded. This completes the proof. □

Proposition 3.2.2. *Let f be a completely monotone function. Suppose that $-f' \in \mathrm{WLSC}(\tau, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\tau \leq -1$. Then $f \in \mathrm{WLSC}(1 + \tau, c, x_0)$. Analogously, if $-f' \in \mathrm{WUSC}(\tau, C, x_0)$ for some $C \geq 1$, $x_0 \geq 0$ and $\tau \leq -1$, then $(f - f(\infty)) \in \mathrm{WUSC}(\tau, C, x_0)$.*

Proof. Let $\lambda > 1$. For $y > x > x_0$, we have

$$f(\lambda x) - f(\lambda y) = -\int_{\lambda x}^{\lambda y} f'(s)\, ds = -\lambda \int_{x}^{y} f'(\lambda s)\, ds \geq -c\lambda^{1+\tau} \int_{x}^{y} f'(s)\, ds = c\lambda^{1+\tau}(f(x) - f(y)),$$

thus

$$f(\lambda x) \geq c\lambda^{1+\tau} f(x) + f(\lambda y) - c\lambda^{1+\tau} f(y).$$

Since f is non-negative and non-increasing, we can take y approaching infinity to get

$$\begin{aligned} f(\lambda x) &\geq c\lambda^{1+\tau} f(x) + \left(1 - c\lambda^{1+\tau}\right) \lim_{y\to\infty} f(y) \\ &\geq c\lambda^{1+\tau} f(x), \end{aligned}$$

where in the last inequality we have also used that $1 \geq c\lambda^{1+\tau}$. The second part of the proposition can be proved in much the same way. □

Proposition 3.2.3. *Let ϕ be a Bernstein function with $\phi(0) = 0$. Then $\phi \in \mathrm{WLSC}(\alpha, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$, and $\alpha > 0$ if and only if $\phi' \in \mathrm{WLSC}(\alpha - 1, c', x_0)$ for some $c' \in (0, 1]$. Furthermore, if $\phi \in \mathrm{WLSC}(\alpha, c, x_0)$, then there is $C \geq 1$ such that for all $x > x_0$,*

$$x\phi'(x) \leq \phi(x) \leq Cx\phi'(x). \tag{3.2.4}$$

Proof. Suppose first that $\phi' \in \mathrm{WLSC}(\alpha - 1, c, x_0)$. Without loss of generality we may assume that $\phi' \not\equiv 0$. We claim that (3.2.4) holds true. In view of (3.2.2), it is enough to show that there is $C \geq 1$ such that for all $x > x_0$,

$$\phi(x) \leq Cx\phi'(x).$$

First, let us observe that, by the weak lower scaling property of ϕ',

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} \phi'(s)\, ds \leq c^{-1}\phi'(x) \int_{x_0}^{x} (s/x)^{-1+\alpha}\, ds \leq \frac{1}{c\alpha} x\phi'(x). \tag{3.2.5}$$

Thus we get the assertion in the case $x_0 = 0$. If $x_0 > 0$, then it is enough to show that there is $C > 0$ such that for all $x > x_0$,

$$x\phi'(x) \geq C. \tag{3.2.6}$$

Since $\phi' \in \mathrm{WLSC}(\alpha - 1, c, x_0)$, the function

$$(x_0, \infty) \ni x \mapsto x\phi'(x)$$

is almost increasing. Hence, for $x \geq 2x_0$ we have

$$x\phi'(x) \geq c2x_0\phi'(2x_0).$$

To conclude (3.2.6), we notice that $\phi'(x)$ is positive and continuous in $[x_0, 2x_0]$. Now, by (3.2.6), we get

$$x\phi'(x) \geq C\phi(x_0)$$

for all $x > x_0$, which, together with (3.2.5), implies (3.2.4) and the scaling property of ϕ follows.

Now assume that $\phi \in \mathrm{WLSC}(\alpha, c, x_0)$. By monotonicity of ϕ', for $0 < s < t$,

$$\frac{\phi(tx) - \phi(sx)}{\phi(x)} \leq \frac{x(t - s)\phi'(sx)}{\phi(x)}.$$

For $s = 1$, by the lower scaling, we get

$$\frac{x(t - 1)\phi'(x)}{\phi(x)} \geq \frac{\phi(tx)}{\phi(x)} - 1 \geq ct^{\alpha} - 1$$

for all $x > x_0$. Thus, for $t = 2^{1/\alpha}c^{-1/\alpha}$, we obtain that $x\phi'(x) \gtrsim \phi(x)$ for all $x > x_0$. Invoking (3.2.2), we conclude (3.2.4). In particular, ϕ' has the weak lower scaling property. This completes the proof. □

The next proposition provides the control of ϕ' by its derivative $-\phi''$. Observe that the reverse inequality under the assumption of the weak lower scaling property of $-\phi''$ follows by Propositions 3.2.2 and 3.2.3 together with (3.2.2).

Proposition 3.2.4. *Let ϕ be a Bernstein function. Suppose that $-\phi'' \in \mathrm{WUSC}(\beta - 2, C, x_0)$ for some $C \geq 1$, $x_0 \geq 0$, and $\beta < 1$. Then for all $x > x_0$,*

$$\phi'(x) \leq \frac{C}{1 - \beta} x(-\phi''(x)) + b.$$

Proof. Without loss of generality we may assume that $\phi'' \not\equiv 0$. By the scaling property, for $x > x_0$ we have

$$\frac{\phi'(x) - b}{x(-\phi''(x))} = \int_x^\infty \frac{t}{x} \frac{(-\phi''(t))}{(-\phi''(x))} \frac{dt}{t} \le C \int_x^\infty \left(\frac{t}{x}\right)^{-1+\beta} \frac{dt}{t} = C\frac{1}{1-\beta},$$

which concludes the proof. □

Remark 3.2.5. Let ϕ be a Bernstein function such that $\phi(0) = 0$. Suppose that $-\phi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \ge 0$ and $\alpha \in (0, 1]$. Since ϕ' is completely monotone, by Proposition 3.2.2, $\phi' \in \mathrm{WLSC}(\alpha - 1, c, x_0)$. Therefore, by Proposition 3.2.3, we conclude that $\phi \in \mathrm{WLSC}(\alpha, c_1, x_0)$ for some $c_1 \in (0, 1]$.

Remark 3.2.6. As stated in the introduction, our standing assumption in this chapter is the weak lower scaling property imposed on $-\phi''$ with the scaling index $\alpha - 2$, where $\alpha > 0$. In formulations of our result we do not put any upper bound on α, but let us note here that $-\phi''$ is non-increasing and integrable at infinity; thus, we in fact always have $\alpha < 1$. This simple fact will be useful in the proof of Theorem 3.1.1.

Proposition 3.2.7. *Let f be a completely monotone function. Suppose that*

$$f \in \mathrm{WLSC}(\alpha - 1, c, x_0) \cap \mathrm{WUSC}(\beta - 1, C, x_0)$$

for some $c \in (0, 1]$, $C \ge 1$, $x_0 \ge 0$ and $0 < \alpha \le \beta < 1$. Then

$$-f' \in \mathrm{WLSC}(\alpha - 2, c', x_0) \cap \mathrm{WUSC}(\beta - 2, C', x_0)$$

for some $c' \in (0, 1]$ and $C' \ge 1$.

Proof. By monotonicity of $-f'$, for $0 < s < t$,

$$\frac{-x(t-s)f'(tx)}{f(x)} \le \frac{f(sx) - f(tx)}{f(x)} \le \frac{-x(t-s)f'(sx)}{f(x)}. \tag{3.2.7}$$

Taking $s = 1$ in the second inequality, the weak upper scaling property yields

$$\frac{-x(t-1)f'(x)}{f(x)} \ge 1 - \frac{f(tx)}{f(x)} \ge 1 - ct^{\beta-1}$$

for all $x > x_0$. By selecting $t > 1$ such that $ct^{\beta-1} \le \frac{1}{2}$, we obtain that $x\big(-f'(x)\big) \gtrsim f(x)$ for $x > x_0$. Similarly, taking $t = 1$ in the first inequality in (3.2.7), by the weak lower scaling property we get

$$\frac{-x(1-s)f'(x)}{f(x)} \le \frac{f(sx)}{f(x)} - 1 \le c^{-1}s^{\alpha-1} - 1$$

for all $x > x_0/s$. By selecting $0 < s < 1$ such that $s^{\alpha-1} \ge 2c$, we obtain $x\big(-f'(x)\big) \lesssim f(x)$ for $x > x_0/s$. Hence,

$$f(x) \approx x\big(-f'(x)\big) \tag{3.2.8}$$

for all $x > x_0/s$. Therefore, lower and upper scaling properties follow from (3.2.8) and the scaling properties of f. This finishes the proof for the case $x_0 = 0$. If $x_0 > 0$, we notice that both f and $-f'$ are positive and continuous, thus at the possible expense of the constants we get (3.2.8) for all $x > x_0$. □

Now, by combining Propositions 3.2.3 and 3.2.7, we immediately get the following corollary.

Corollary 3.2.8. *Let ϕ be a Bernstein function such that $\phi(0)=0$. Suppose that*

$$\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$$

for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$ and $0 < \alpha \leq \beta < 1$. Then

$$-\phi'' \in \mathrm{WLSC}(\alpha-2, c', x_0) \cap \mathrm{WUSC}(\beta-2, C', x_0)$$

for some $c' \in (0,1]$ and $C' \geq 1$.

The last result of this section states that there is a complete Bernstein function $\widetilde{\phi}$ which is similar to ϕ in a sense that both the functions $\widetilde{\phi}$ and ϕ as well as their (minus) second derivatives are comparable. This feature shall be handy in deriving Green function estimates in Section 3.5 (see the proof of Claim 3.5.5), where Lemma 3.2.9 will be used to prove a tricky estimate by shifting the problem onto $\widetilde{\phi}$ and then applying some results which are already known for complete Bernstein functions.

Lemma 3.2.9. *Let ϕ be a Bernstein function. Suppose that $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha > 0$. Then there exists a complete Bernstein function $\widetilde{\phi}$ such that $\widetilde{\phi} \approx \phi$ for $x > 0$, and $-\widetilde{\phi}'' \approx -\phi''$ for $x > x_0$.*

Proof. Let us define

$$\widetilde{\phi}(\lambda) = a + b\lambda + \int_{(0,\infty)} \frac{\lambda s}{\lambda s + 1}\, \mu(ds).$$

By [94, Theorem 6.2 (vi)] the function $\widetilde{\phi}$ is a complete Bernstein function. Since, for $y > 0$,

$$\frac{y}{y+1} \approx (1 - e^{-y}),$$

we get $\widetilde{\phi}(\lambda) \approx \phi(\lambda)$. Moreover,

$$\widetilde{\phi}''(\lambda) = -2 \int_{(0,\infty)} \frac{s^2}{(\lambda s + 1)^3}\, \mu(ds).$$

Now, observe that, by Corollary A.1.2,

$$\int_{(0,1/\lambda)} \frac{s^2}{(\lambda s+1)^3}\, \mu(ds) \approx \lambda^{-2} K(1/\lambda) \approx -\phi''(\lambda), \quad \lambda > x_0,$$

with the implied constant independent of λ. Moreover, by Proposition A.1.3, we clearly have

$$\int_{(1/\lambda,\infty)} \frac{s^2}{(\lambda s+1)^3}\, \mu(ds) \lesssim \lambda^{-2} \mu((1/\lambda,\infty)) \lesssim \lambda^{-2} h(1/\lambda) \lesssim \lambda^{-2} K(1/\lambda), \quad \lambda > x_0,$$

with the implied constant independent of λ. Hence, due to Corollary A.1.2, we obtain $-\widetilde{\phi}''(\lambda) \approx -\phi''(\lambda)$ for $\lambda > x_0$. □

3.3 Asymptotic behaviour of densities

As stated in the introduction of this chapter, we let $\mathbf{T} = (T_t \colon t \geq 0)$ be a subordinator with the Laplace exponent ϕ with $a = 0$, cf. (3.2.1). We note in passing that $a > 0$ corresponds to the subordinator killed at an independent exponential time with parameter a, cf. [94, Chapter 5]. Recall that we adopt here a different form of the Lévy-Khintchine representation of $\mathbf{T}$, cf. (2.2.1) and (2.2.8):

$$\begin{aligned} \psi(\xi) &= -i\xi b - \int_{(0,\infty)} (e^{i\xi x} - 1)\, \nu(dx) \\ &= -i\xi\gamma - \int_{(0,\infty)} (e^{i\xi x} - 1 - i\xi x \mathbb{1}_{x<1})\, \nu(dx) \end{aligned} \tag{3.3.1}$$

with $b = \gamma - \int_{B_1} x\, \nu(dx)$. Since ϕ is a Bernstein function, it also admits the integral representation (3.2.1). As it may be easily checked (see e.g. [94, Proposition 3.6]), we have $\mu = \nu$, $a = 0$, and $\psi(\xi) = \phi(-i\xi)$. In particular, $\phi(0) = 0$.

In this section we study the asymptotic behaviour of the probability density of T_t. In the whole section we assume that $\phi'' \not\equiv 0$, otherwise $T_t = bt$ is deterministic. The main result of this section is Theorem 3.1.1 formulated in the introduction of this chapter and proven at the end of the present section. Let us start by showing an estimate of the real part of the holomorphic extension of ϕ.

Lemma 3.3.1. *Suppose that $-\phi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$, and $\alpha > 0$. Then there exists $C > 0$ such that for all $w > x_0$ and $\lambda \in \mathbb{R}$,*

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) \geq C\lambda^2\big(- \phi''(|\lambda| \vee w)\big).$$

Proof. By the integral representation (3.2.1), for $\lambda \in \mathbb{R}$ we have

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) = \int_{(0,\infty)} (1 - \cos(\lambda s))e^{-ws}\, \nu(ds).$$

In particular,

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) = \mathrm{Re}\,(\phi(w - i\lambda) - \phi(w)).$$

Thus, it is sufficient to consider $\lambda > 0$. We can estimate

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) \geq \int_{(0,1/\lambda)} (1 - \cos(\lambda s))e^{-ws}\, \nu(ds) \gtrsim \lambda^2 \int_{(0,1/\lambda)} s^2 e^{-ws}\, \nu(ds). \tag{3.3.2}$$

Due to Lemma A.1.1 we obtain, for $\lambda \geq w$,

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) \gtrsim \lambda^2 \int_{(0,1/\lambda)} s^2\, \nu(ds) \gtrsim \lambda^2(-\phi''(\lambda)).$$

If $w > \lambda > 0$, then by (3.3.2) we have

$$\mathrm{Re}\,(\phi(w + i\lambda) - \phi(w)) \gtrsim \lambda^2 \int_{(0,1/w)} s^2 e^{-ws} \mu(ds) \geq e^{-1}\lambda^2 \int_{(0,1/w)} s^2 \mu(ds),$$

which, by Lemma A.1.1, completes the proof. □

Proof of Theorem 3.1.1. Let $x = t\phi'(w)$ and $M > 0$. We first show that

$$p(t, x) = \frac{1}{2\pi} \cdot \frac{e^{-t\Theta(x/t,0)}}{\sqrt{t(-\phi''(w))}} \int_{\mathbb{R}} \exp\bigg\{ - t\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t(-\phi''(w))}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\bigg)\bigg\}\, du, \tag{3.3.3}$$

provided that $w > x_0$ and $tw^2(-\phi''(w)) > M$, where for $\lambda \in \mathbb{R}$ we have set

$$\Theta(x/t,\lambda) = \phi(w+i\lambda) - \frac{x}{t}(w+i\lambda). \tag{3.3.4}$$

To do so, let us recall that

$$\mathbb{E}(e^{-\lambda T_t}) = e^{-t\phi(\lambda)}, \quad \lambda \geq 0.$$

Thus, by the Mellin's inversion formula (see e.g. [27, Chapter 3]), if the limit

$$\lim_{L\to\infty} \frac{1}{2\pi i}\int_{w-iL}^{w+iL} e^{-t\phi(\lambda)+\lambda x}\,d\lambda \quad \text{exists,} \tag{3.3.5}$$

then the probability density $p(t,\,\cdot\,)$ of T_t satisfies

$$p(t,x) = \lim_{L\to\infty} \frac{1}{2\pi i}\int_{w-iL}^{w+iL} e^{-t\phi(\lambda)+\lambda x}\,d\lambda.$$

Therefore, our task is to justify the statement (3.3.5). For $L > 0$ we write

$$\frac{1}{2\pi i}\int_{w-iL}^{w+iL} e^{-t\phi(\lambda)+\lambda x}\,d\lambda = \frac{1}{2\pi}\int_{-L}^{L} e^{-t\Theta(x/t,\lambda)}\,d\lambda.$$

By the change of variables

$$\lambda = \frac{u}{\sqrt{t(-\phi''(w))}},$$

we obtain

$$\begin{aligned}\int_{-L}^{L} e^{-t\Theta(x/t,\lambda)}\,d\lambda &= e^{-t\Theta(x/t,0)}\int_{-L}^{L}\exp\Big\{-t\Big(\Theta(x/t,\lambda)-\Theta(x/t,0)\Big)\Big\}\,d\lambda \\ &= \frac{e^{-t\Theta(x/t,0)}}{\sqrt{t(-\phi''(w))}}\times \\ &\int_{-L\sqrt{t(-\phi''(w))}}^{L\sqrt{t(-\phi''(w))}}\exp\left\{-t\left(\Theta\Big(\frac{x}{t},\frac{u}{\sqrt{t(-\phi''(w))}}\Big)-\Theta\Big(\frac{x}{t},0\Big)\right)\right\}du.\end{aligned}$$

We claim that there is $C > 0$ not depending on M, such that for all $u \in \mathbb{R}$,

$$t\,\mathrm{Re}\left(\Theta\Big(\frac{x}{t},\frac{u}{\sqrt{t(-\phi''(w))}}\Big)-\Theta\Big(\frac{x}{t},0\Big)\right) \geq C(u^2\wedge(|u|^\alpha M^{1-\alpha/2})), \tag{3.3.6}$$

provided that $w > x_0$ and $tw^2(-\phi''(w)) > M$. Recall that in view of Remark 3.2.6 we have $\alpha < 1$. Indeed, by (3.3.4) and Lemma 3.3.1, for $w > x_0$ we get

$$t\,\mathrm{Re}\left(\Theta\Big(\frac{x}{t},\frac{u}{\sqrt{t(-\phi''(w))}}\Big)-\Theta\Big(\frac{x}{t},0\Big)\right) \gtrsim \frac{|u|^2}{\phi''(w)}\phi''\Big(\frac{|u|}{\sqrt{t(-\phi''(w))}}\vee w\Big). \tag{3.3.7}$$

We next estimate the right-hand side of (3.3.7). If $|u| \leq w\sqrt{t(-\phi''(w))}$, then

$$\frac{|u|^2}{\phi''(w)}\phi''\Big(\frac{|u|}{\sqrt{t(-\phi''(w))}}\vee w\Big) = |u|^2.$$

Otherwise, since $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$, we obtain

$$\begin{aligned}\frac{|u|^2}{\phi''(w)}\phi''\Big(\frac{|u|}{\sqrt{t(-\phi''(w))}} \vee w\Big) &\geq c|u|^2\Big(\frac{|u|}{\sqrt{tw^2(-\phi''(w))}}\Big)^{-2+\alpha} \\ &= c|u|^\alpha\big(tw^2(-\phi''(w))\big)^{1-\alpha/2} \\ &\geq cM^{1-\alpha/2}|u|^\alpha.\end{aligned}$$

Hence, we deduce (3.3.6). To finish the proof of (3.3.5), we invoke the dominated convergence theorem. Consequently, by Mellin's inversion formula we obtain (3.3.3).

Our next task is to show that for each $\varepsilon > 0$ there is $M_0 > 0$ such that

$$\left|\int_{\mathbb{R}} \exp\bigg\{-t\Big(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t(-\phi''(w))}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\Big)\bigg\}\, du - \int_{\mathbb{R}} e^{-\frac{1}{2}u^2}\, du\right| \leq \varepsilon, \tag{3.3.8}$$

provided that $w > x_0$ and $tw^2(-\phi''(w)) > M_0$. In view of (3.3.6), by taking $M_0 > 1$ sufficiently large, we get

$$\left|\int_{|u|\geq M_0^{1/4}} \exp\bigg\{-t\Big(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t(-\phi''(w))}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\Big)\bigg\}\, du\right| \leq \int_{|u|\geq M_0^{1/4}} e^{-C|u|^\alpha}\, du \leq \varepsilon, \tag{3.3.9}$$

and

$$\int_{|u|\geq M_0^{1/4}} e^{-\frac{1}{2}u^2}\, du \leq \varepsilon. \tag{3.3.10}$$

Next, we claim that there is $C > 0$ such that

$$\left|t\Big(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t(-\phi''(w))}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\Big) - \frac{1}{2}|u|^2\right| \leq C|u|^3 M_0^{-\frac{1}{2}}. \tag{3.3.11}$$

Indeed, since

$$\partial_\lambda \Theta\Big(\frac{x}{t}, 0\Big) = 0,$$

by the Taylor formula, we get

$$\begin{aligned}\left|t\Big(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t(-\phi''(w))}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\Big) - \frac{1}{2}|u|^2\right| &= \left|\frac{1}{2}\partial_\lambda^2\Theta\Big(\frac{x}{t}, \xi\Big)\frac{|u|^2}{-\phi''(w)} - \frac{1}{2}|u|^2\right| \\ &= \frac{|u|^2}{2|\phi''(w)|}\big|\phi''(w+i\xi) - \phi''(w)\big|,\end{aligned} \tag{3.3.12}$$

where ξ is some number satisfying

$$|\xi| \leq \frac{|u|}{\sqrt{t(-\phi''(w))}}. \tag{3.3.13}$$

Observe that

$$|\phi''(w+i\xi) - \phi''(w)| \leq \int_{(0,\infty)} s^2 e^{-ws}|e^{-i\xi s} - 1|\,\nu(ds) \leq 2|\xi|\int_{(0,\infty)} s^3 e^{-ws}\,\nu(ds) = 2|\xi|\phi'''(w).$$

Note that $-\phi''$ is a non-increasing function with the weak lower scaling property, hence it is doubling. Since it is also completely monotone, by Proposition 3.2.1, for $w > x_0$,

$$-\phi''(w) \gtrsim w\phi'''(w),$$

which together with (3.3.13) give

$$|\phi''(w+i\xi)-\phi''(w)| \leq C\frac{|u|}{\sqrt{t(-\phi''(w))}}\cdot\frac{-\phi''(w)}{w} \leq CM_0^{-\frac{1}{2}}|u|(-\phi''(w)), \tag{3.3.14}$$

whenever $tw^2(-\phi''(w)) > M_0$. Now, (3.3.11) easily follows by (3.3.14) and (3.3.12).

Finally, since for any $z \in \mathbb{C}$,

$$\left|e^z - 1\right| \leq |z|e^{|z|},$$

by (3.3.11), we obtain

$$\left|\int_{|u|\leq M_0^{1/4}} \exp\left\{-t\left(\Theta\left(\frac{x}{t},\frac{u}{\sqrt{t(-\phi''(w))}}\right)-\Theta\left(\frac{x}{t},0\right)\right)\right\} - e^{-\frac{1}{2}|u|^2}\,du\right|$$
$$\leq CM_0^{-\frac{1}{2}}\int_{|u|\leq M_0^{1/4}} \exp\left\{-\tfrac{1}{2}|u|^2 + CM_0^{-\frac{1}{2}}|u|^3\right\}|u|^3\,du \leq \varepsilon,$$

provided that M_0 is sufficiently large, which, together with (3.3.9) and (3.3.10), completes the proof of (3.3.8) and the theorem follows. □

Remark 3.3.2. If $x_0 = 0$, then the constant M_0 in Theorem 3.1.1 depends only on α and c. If $x_0 > 0$, it also depends on

$$\sup_{x\in[x_0,2x_0]} \frac{x\phi'''(x)}{-\phi''(x)}.$$

Remark 3.3.3. Let us note here one crucial observation which will be vital for our development later on. After a close inspection, one may conclude that the only tools needed in the proof of Theorem 3.1.1 are: the properties of the Laplace exponent ϕ (including in particular the weak lower scaling property of $-\phi''$ but also the integral representation (3.2.1)), Lemma 3.3.1 and Proposition 3.2.1. Lemma 3.3.1 is also a consequence of scaling property through Lemma A.1.1, while the application of Proposition 3.2.1 follows by the properties of the Laplace exponent ϕ. This feature suggests that the same method may be applied to other types of one-dimensional Lévy processes, provided that the Laplace transform exists and its exponent enjoys some regularity properties. A natural candidate would be a class of spectrally one-sided Lévy processes, since in this case the existence of the Laplace transform follows by an easy argument, see e.g. Chapter VII of Bertoin [4]. The same remark may be applied for other results of this chapter as well and will be our main motivation in Chapter 4.

By Theorem 3.1.1, we immediately get the following corollaries. Note that since ϕ' is non-increasing and $\lim_{\lambda\to\infty}\phi'(\lambda) = b$, an additional lower bound on x appears. This observation agrees with the fact that the support of the distribution of T_t is precisely $[tb,\infty)$.

Corollary 3.3.4. *Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2,c,x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$, *and* $\alpha > 0$. *Then there is* $M_0 > 0$ *such that*

$$p(t,x) \approx \frac{1}{\sqrt{t(-\phi''(w))}}\exp\Big\{-t(\phi(w)-w\phi'(w))\Big\},$$

uniformly on the set

$$\Big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}\colon tb < x < t\phi'(x_0^+) \text{ and } tw^2(-\phi''(w)) > M_0\Big\}$$

where $w = (\phi')^{-1}(x/t)$.

Corollary 3.3.5. *Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$*,* $x_0 \geq 0$*, and* $\alpha > 0$*. Assume also that* $b = 0$*. Then for any* $x > 0$*,*

$$\lim_{t\to\infty} p(t,x)\sqrt{t(-\phi''(w))}\exp\Big\{t\big(\phi(w)-w\phi'(w)\big)\Big\} = (2\pi)^{-1/2},$$

where $w = (\phi')^{-1}(x/t)$*.*

By imposing on $-\phi''$ an additional condition of the weak upper scaling, we can further simplify the description of the set where the sharp estimates of $p(t,x)$ hold.

Corollary 3.3.6. *Suppose that* $\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ *for some* $c \in (0,1]$*,* $C \geq 1$*,* $x_0 \geq 0$*, and* $0 < \alpha \leq \beta < 1$*. Assume also that* $b = 0$*. Then there is* $\delta > 0$ *such that*

$$p(t,x) \approx \frac{1}{\sqrt{t(-\phi''(w))}} \exp\Big\{-t\big(\phi(w)-w\phi'(w)\big)\Big\},$$

uniformly on the set

$$\Big\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}\colon 0 < x\phi^{-1}(1/t) < \delta, \text{ and } 0 \leq t\phi(x_0) \leq 1\Big\} \tag{3.3.15}$$

where $w = (\phi')^{-1}(x/t)$*.*

Proof. By Proposition 3.2.3, there is $C_1 \geq 1$ such that for all $u > x_0$,

$$\phi(u) \leq C_1 u\phi'(u),$$

thus, for (t,x) belonging to the set (3.3.15),

$$\frac{x}{t} < \delta\frac{1}{t\phi^{-1}(1/t)} = \delta\frac{\phi(\phi^{-1}(1/t))}{\phi^{-1}(1/t)} \leq C_1\delta\phi'(\phi^{-1}(1/t)). \tag{3.3.16}$$

By Proposition 3.2.3, $\phi' \in \mathrm{WLSC}(-1+\alpha, c, x_0)$, hence for all $D \geq 1$,

$$\phi'(D\phi^{-1}(1/t)) \geq cD^{-1+\alpha}\phi'(\phi^{-1}(1/t)).$$

By taking δ sufficiently small, we get

$$D = \left(\frac{c}{C_1\delta}\right)^{\frac{1}{1-\alpha}} \geq 1,$$

thus, by (3.3.16), we obtain

$$\frac{x}{t} < \phi'(D\phi^{-1}(1/t)),$$

which implies that

$$w = (\phi')^{-1}(x/t) > D\phi^{-1}(1/t). \tag{3.3.17}$$

In particular, $w > x_0$. On the other hand, by Propositions 3.2.3 and 3.2.4, there is $c_1 \in (0,1]$ such that

$$tw^2(-\phi''(w)) \geq c_1 t\phi(w).$$

By Remark 3.2.5, $\phi \in \mathrm{WLSC}(\alpha, c_2, x_0)$ for some $c_2 \in (0,1]$. Therefore,

$$t\phi(w) = \frac{\phi(w)}{\phi(\phi^{-1}(1/t))} \geq c_2\left(\frac{w}{\phi^{-1}(1/t)}\right)^{\alpha},$$

which together with (3.3.17), gives

$$tw^2(-\phi''(w)) \gtrsim \delta^{-\frac{\alpha}{1-\alpha}} > M_0$$

for δ sufficiently small. Hence, by Corollary 3.3.4, we conclude the proof. □

3.4 Estimates of the density

The aim of this section is to provide upper and lower bounds on the density of T_t as well as discuss the situation when these two coincide. We provide some historical and contemporary references, but it goes without saying that the literature on the subject is gargantuan and the list below is far from complete. As usual, the first results in the non-local setting were obtained for isotropic α-stable process in $\mathbb{R}^d$ by Pòlya [84], and Blumenthal and Getoor [6], and provided basis for studies of more complicated processes, e.g. subordinated Brownian motions (e.g. Mimica [75] and Song [97]), isotropic unimodal Lévy processes (for instance Bogdan, Grzywny and Ryznar [10], Cygan, Grzywny and Trojan [26], or Grzywny, Ryznar and Trojan [41]), and even more general symmetric Markov processes (see Chen, Kim and Kumagai [21] or Chen and Kumagai [24]). One may, among others, list the articles on heat kernel estimates for jump processes of finite range by Chen, Kim and Kumagai [20] or with lower intensity of higher jumps by Mimica [73] and Sztonyk [100]. While a great many of articles with explicit results is devoted to symmetric processes or those which are, in a certain appropriate sense, similar to the symmetric ones, the non-symmetric case is in general harder to handle due to lack of familiar structure. This problem was approached in many different ways, see e.g. Bogdan, Sztonyk and Knopova [14], Knopova and Kulik [63] or Picard [82]. For more specific class of stable processes we refer the reader to Hiraba [47], Pruitt and Taylor [88], and Watanabe [103]. Overall, one has to impose some control on the non-symmetry in order to obtain estimates in an easy-to-handle form. This idea was applied in the recent preprint by Grzywny and Szczypkowski [42], where the authors considered the case of the Lévy measure being comparable to some unimodal Lévy measure. The methods developed in [42] and in the article of the same authors [43] were a inspiration for our results of this section.

Just for the record, we note that we always assume that $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha \in (0,1]$. In particular, by Proposition A.1.8 and the Hartman-Wintner condition (HW), the probability distribution of T_t has a density $p(t,\,\cdot\,)$. To express the majorant on $p(t,\,\cdot\,)$, it is convenient to set

$$\Phi(x) = x^2(-\phi''(x)), \quad x > 0.$$

Obviously, $\Phi \in \mathrm{WLSC}(\alpha, c, x_0)$. Recall that the generalized inverse function Φ^{-1} is defined as

$$\Phi^{-1}(x) = \sup\{r > 0 \colon \Phi^*(r) = x\}$$

where

$$\Phi^*(r) = \sup_{0<x\leq r} \Phi(x).$$

We note that a number of useful preliminary results is shifted to Appendix A due to their versatility and applicability also in Chapter 4. Let us begin with the following observation on the exponent appearing in Theorem 3.1.1. The claim is rather elementary, but it will be useful in the proof of the lower bound in Theorem 3.4.7.

Proposition 3.4.1. *Suppose that $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha > 0$. Then there is $C > 0$ such that for all $x > x_0$,*

$$(\phi(x) - x\phi'(x)) \leq C\Phi(x). \tag{3.4.1}$$

Proof. We have

$$(\phi(x) - x\phi'(x)) - (\phi(x_0) - x_0\phi'(x_0^+)) = \int_{x_0}^{x} \Phi(u)\frac{du}{u} = \int_{x_0/x}^{1} \Phi(xu)\frac{du}{u}$$

where

$$\phi'(x_0^+) = \lim_{x \to x_0^+} \phi'(x).$$

By the weak lower scaling property of Φ, for any $x_0/x < u \leq 1$ we have

$$\Phi(x) \geq c u^{-\alpha} \Phi(xu),$$

thus,

$$(\phi(x) - x\phi'(x)) - (\phi(x_0) - x_0\phi'(x_0^+)) \lesssim \Phi(x) \int_0^1 u^{\alpha-1}\, du,$$

which proves (3.4.1) if $x_0 = 0$. For $x_0 > 0$ we denote $c_1 = \phi(x_0) - x_0\phi'(x_0)$ and using the scaling property of Φ we conclude that

$$c_1 = \frac{c_1}{\Phi(x_0)} \cdot \Phi(x_0) \leq \frac{c_1}{\Phi(x_0)} c^{-1} \left(\frac{x_0}{x}\right)^{\alpha} \Phi(x) \leq \frac{c_1 c^{-1}}{\Phi(x_0)} \cdot \Phi(x),$$

provided that $x > x_0$ and the claim follows. □

The next result together with Proposition A.1.5 explain, at least to some extent, the usage of Φ instead of ϕ in estimating the transition density. As already mentioned, by virtue of Proposition A.1.5 and results by Grzywny and Szczypkowski [42, 43], we see that the function Φ^{-1} specifies not only the magnitude of the supremum of the transition density but also its localisation. We will prove the analogue for our setting in Theorem 3.4.7, but for now let us stress once again that, in general, the expected comparability $\phi \approx \Phi^*$ need not hold and one has to impose additional upper scaling to obtain such conclusion.

Proposition 3.4.2. *Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0) \cap \mathrm{WUSC}(\beta - 2, C, x_0)$ *for some* $c \in (0, 1]$, $C \geq 1$, $x_0 \geq 0$, *and* $0 < \alpha \leq \beta < 1$. *Assume also that* $b = 0$. *Then for all* $x > x_0$,

$$\Phi^*(x) \approx \phi(x), \tag{3.4.2}$$

and for all $r > \Phi(x_0)$,

$$\Phi^{-1}(r) \approx \phi^{-1}(r).$$

Furthermore, there is $c' \in (0, 1]$ *such that for all* $\lambda \geq 1$ *and* $r > 1/\Phi(x_0)$,

$$\Phi^{-1}(\lambda r) \geq c' \lambda^{1/\beta} \Phi^{-1}(r). \tag{3.4.3}$$

Proof. Let us observe that, by (3.2.2), Proposition 3.2.3 and Proposition 3.2.4, there is $c_1 \in (0, 1]$ such that for all $x > x_0$,

$$2\phi(x) \geq \Phi(x) \geq c_1 \phi(x).$$

Now the proof of the lemma is similar to the proof of Proposition A.1.5; therefore, it is omitted. □

3.4.1 Estimates from above

In this section we show the upper estimates of $p(t, \cdot)$. Before embarking on the proof let us recall that

$$b_r = b + \int_{(0,r)} s\, \nu(ds), \quad r > 0.$$

We observe that in view of (3.3.1) the definition above is in line with the usual one given in Chapter 2 by (2.2.5). Let us define $\zeta\colon [0,\infty) \mapsto [0,\infty]$,

$$\zeta(s) = \begin{cases} \infty & \text{if } s = 0, \\ \Phi^*(1/s) & \text{if } 0 < s \leq x_0^{-1}, \\ A\phi(1/s) & \text{if } x_0^{-1} < s, \end{cases}$$

where $A = \Phi^*(x_0)/\phi(x_0) \in (0,2]$ by virtue of (3.2.2). We observe that ζ is positive and A is taken so that it is also non-increasing.

The next result is a version of upper estimate of the transition density. Its distinctive feature is the fact that we do not impose any additional assumptions on the Lévy measure, so, in particular, ν is allowed to be singular. Clearly, putting some restrictions on ν results in sharper estimates (see Theorem 3.4.4), but it is interesting that the scaling property sole is enough to get some information.

Theorem 3.4.3. *Let* $\mathbf{T}$ *be a subordinator with the Lévy–Khintchine exponent* ψ *and the Laplace exponent* ϕ. *Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$ *and* $\alpha > 0$. *Then there is* $C > 0$ *such that for all* $t \in (0, 1/\Phi(x_0))$ *and* $x \in \mathbb{R}$,

$$p\Big(t, x + tb_{1/\psi^{-1}(1/t)}\Big) \leq C\Phi^{-1}(1/t) \cdot \min\left\{1, t\zeta(|x|)\right\}. \tag{3.4.4}$$

In particular, for all $t \in (0, 1/\Phi(x_0))$ *and* $x \geq 2et\phi'(\psi^{-1}(1/t))$,

$$p(t, x + tb) \leq C\Phi^{-1}(1/t) \cdot \min\left\{1, t\zeta(x)\right\}. \tag{3.4.5}$$

Proof. Without loss of generality we can assume $b = 0$. Indeed, otherwise it is enough to consider a shifted process $\widetilde{T}_t = T_t - tb$. Next, let us observe that for any Borel set $B \subset \mathbb{R}$, we have

$$\nu(B) \lesssim \int_{(\delta(B),\infty)} \left(1 - e^{-s/\delta(B)}\right) \nu(ds) \leq \phi(1/\delta(B)). \tag{3.4.6}$$

Furthermore, for $\delta(B) < 1/x_0$, by Proposition A.1.3 and Corollary A.1.2,

$$\nu(B) \leq h(\delta(B)) \lesssim \Phi^*(1/\delta(B)).$$

Thus, $\nu(B) \lesssim \zeta(\delta(B))$. We claim that ζ has doubling property on $(0,\infty)$. Indeed, since $-\phi''$ is non-increasing function with the weak lower scaling property, it has doubling property on (x_0,∞); thus, for $0 < s < x_0^{-1}$,

$$\zeta(\tfrac{1}{2}s) \approx 4s^{-2}(-\phi''(2/s)) \lesssim s^{-2}(-\phi''(1/s)) \lesssim \zeta(s).$$

This completes the argument in the case $x_0 = 0$. If $x_0 > 0$, then by (3.2.3), for $s > 2x_0^{-1}$, we have

$$\zeta(\tfrac{1}{2}s) = A\phi(2/s) \leq 2A\phi(1/s) \leq 2\zeta(s).$$

Lastly, the function

$$[\tfrac{1}{2}x_0, x_0] \ni x \mapsto \frac{\Phi^*(2x)}{\phi(x)}$$

is continuous, thus it is bounded.

Next, for $s > 0$ and $x \in \mathbb{R}$,

$$s \vee |x| - \tfrac{1}{2}|x| \geq \tfrac{1}{2}s,$$

thus, by monotonicity and doubling property of ζ, we get

$$\zeta\big(s \vee |x| - \tfrac{1}{2}|x|\big) \lesssim \zeta(s).$$

Hence, by (2.2.3) and (A.1.6), for $r > 0$,

$$\int_{(r,\infty)} \zeta\big(s \vee x - \tfrac{1}{2}x\big)\, \nu(dx) \lesssim \zeta(s) h(r) \lesssim \zeta(s)\psi^*(1/r). \tag{3.4.7}$$

Since ψ^* has the weak lower scaling property and satisfies (A.1.7), by [43, Proposition 3.4] together with Proposition A.1.4, there are $C > 0$ and $t_1 \in (0, \infty]$ such that for all $t \in (0, t_1)$,

$$\int_{\mathbb{R}} e^{-t \operatorname{Re}\psi(\xi)} |\xi| \, d\xi \leq C\big(\psi^{-1}(1/t)\big)^2. \tag{3.4.8}$$

If $x_0 = 0$, then $t_1 = \infty$. If $t_1 < 1/\Phi(x_0)$, we can expand the above estimate for $t_1 \leq t < 1/\Phi(x_0)$ using positivity of the right hand side and monotonicity of the left hand side.

In view of (3.4.6), (3.4.7) and (3.4.8), by [56, Theorem 1] with $\gamma = 0$, there are $C_1, C_2, C_3 > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in \mathbb{R}$,

$$\begin{aligned} & p\Big(t, x + t b_{1/\psi^{-1}(1/t)}\Big) \\ & \quad \leq C_1 \psi^{-1}(1/t) \cdot \min\Big\{1, t\zeta\big(\tfrac{1}{4}|x|\big) + \exp\Big\{-C_2 |x| \psi^{-1}(1/t) \log\big(1 + C_3 |x| \psi^{-1}(1/t)\big)\Big\}\Big\}. \end{aligned}$$

Let us consider $x > 0$ and $t \in (0, 1/\Phi(x_0))$ such that $t\zeta(x) \leq 1$. We claim that

$$\exp\Big\{-C_2 x \psi^{-1}(1/t) \log\big(1 + C_3 x \psi^{-1}(1/t)\big)\Big\} \lesssim t\zeta(x). \tag{3.4.9}$$

First, suppose that $x > x_0^{-1}$. Let us observe that the function

$$[0, \infty) \ni u \mapsto u \exp\Big\{-C_2 u \log\big(1 + C_3 u\big)\Big\}$$

is bounded. Therefore,

$$\exp\Big\{-C_2 x \psi^{-1}(1/t) \log\big(1 + C_3 x \psi^{-1}(1/t)\big)\Big\} \lesssim \frac{1}{x\psi^{-1}(1/t)}. \tag{3.4.10}$$

Since $x\phi^{-1}(1/t) \geq 1$, by (3.2.3), we have

$$t\phi(1/x) = \frac{\phi(1/x)}{\phi(x\phi^{-1}(1/t) \cdot 1/x)} \geq \frac{1}{x\phi^{-1}(1/t)}. \tag{3.4.11}$$

Next, in light of (3.2.2), for all $y > 0$,

$$\tfrac{1}{2}\Phi^*(y) \leq \phi(y),$$

hence, by the monotonicity of ϕ^{-1},

$$\begin{aligned} \phi^{-1}(1/t) &= \phi^{-1}\big(\tfrac{1}{2}\Phi^*(\Phi^{-1}(2/t))\big) \\ &\leq \phi^{-1}\big(\phi(\Phi^{-1}(2/t))\big) \\ &= \Phi^{-1}(2/t) \\ &\leq C\psi^{-1}(1/t) \end{aligned} \tag{3.4.12}$$

where in the last step we have used Proposition A.1.5. Putting (3.4.10), (3.4.11), and (3.4.12) together, we obtain (3.4.9) as claimed.

Now let $0 < x \leq x_0^{-1}$. Observe that the function

$$[0,\infty) \ni u \mapsto u^2 \exp\Big\{-C_2 u \log\left(1 + C_3 u\right)\Big\}$$

is also bounded. Hence,

$$\exp\Big\{-C_2 x\psi^{-1}(1/t)\log\left(1 + C_3 x\psi^{-1}(1/t)\right)\Big\} \lesssim \frac{1}{\left(x\psi^{-1}(1/t)\right)^2}. \tag{3.4.13}$$

Since $x\Phi^{-1}(1/t) \geq 1$, using (A.1.3) we get

$$t\Phi^*(1/x) = \frac{\Phi^*(1/x)}{\Phi^*\left(x\Phi^{-1}(1/t)\cdot 1/x\right)} \geq \frac{1}{\left(x\Phi^{-1}(1/t)\right)^2}. \tag{3.4.14}$$

Hence, putting together (3.4.13) and (3.4.14), and invoking Proposition A.1.4, we again obtain (3.4.9).

Finally, using doubling property of ζ, we get

$$\zeta(\tfrac{1}{4}x) \lesssim \zeta(x),$$

thus, an another application of Proposition A.1.5 leads to (3.4.4).

For the proof of (3.4.5), we observe that

$$\phi'(\lambda) = \int_{(0,\infty)} x e^{-\lambda x}\,\nu(dx) \geq e^{-1}\int_{(0,1/\lambda)} x\,\nu(dx).$$

Thus,

$$b_{1/\psi^{-1}(1/t)} = \int_{(0,1/\psi^{-1}(1/t))} x\,\nu(dx) \leq e\phi'(\psi^{-1}(1/t)).$$

Hence, by monotonicity and doubling property of ζ, for $x > 2et\phi'(\psi^{-1}(1/t))$, we obtain

$$\zeta\Big(x - tb_{1/\psi^{-1}(1/t)}\Big) \leq \zeta\left(\frac{x}{2}\right) \lesssim \zeta(x),$$

and the theorem follows. □

Now we define $\eta\colon [0,\infty) \mapsto [0,\infty]$,

$$\eta(s) = s^{-1}\zeta(s) = \begin{cases} \infty & \text{if } s = 0, \\ s^{-1}\Phi^*(1/s) & \text{if } 0 < s \leq x_0^{-1}, \\ As^{-1}\phi(1/s) & \text{if } x_0^{-1} < s, \end{cases}$$

where $A = \Phi^*(x_0)/\phi(x_0) \in (0,2]$. Notice that, by (3.2.2), if $2t\zeta(|x|) \leq 1$, then $t\Phi^*(1/|x|) \leq 1$, and so

$$\eta(|x|) = |x|^{-1}\zeta(|x|) \leq \Phi^{-1}(1/t)\zeta(|x|).$$

Therefore,

$$\min\left\{\Phi^{-1}(1/t), t\eta(|x|)\right\} \leq 4\Phi^{-1}(1/t)\cdot\min\left\{1, t\zeta(|x|)\right\}.$$

Thus, η is a better majorant of $p(t,\cdot)$ than ζ, but one has to pay a price of additional (but still not very restrictive) assumption on the Lévy measure. The following result refines Theorem 3.4.3.

Theorem 3.4.4. *Let $\mathbf{T}$ be a subordinator with the Lévy–Khintchine exponent ψ and the Laplace exponent ϕ. Suppose that $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha > 0$. We also assume that the Lévy measure ν has an almost decreasing density $\nu(x)$. Then there is $C > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in \mathbb{R}$,*

$$p\Big(t, x + tb_{1/\psi^{-1}(1/t)}\Big) \leq C \min\left\{\Phi^{-1}(1/t), t\eta(|x|)\right\}. \tag{3.4.15}$$

In particular, for all $t \in (0, 1/\Phi(x_0))$ and $x \geq 2et\phi'(\psi^{-1}(1/t))$,

$$p(t, x + tb) \leq C \min\left\{\Phi^{-1}(1/t), t\eta(x)\right\}. \tag{3.4.16}$$

Proof. Without loss of generality we can assume $b = 0$. Let us observe that for any $\lambda > 0$,

$$\phi(\lambda) \geq \int_0^{1/\lambda} (1 - e^{-\lambda s})\nu(s)\, ds \gtrsim \nu(1/\lambda)\lambda^{-1},$$

and

$$-\phi''(\lambda) \geq \int_0^{1/\lambda} s^2 e^{-\lambda s}\nu(s)\, ds \gtrsim \nu(1/\lambda)\lambda^{-3}.$$

Hence,

$$\nu(x) \lesssim \eta(x) \qquad \text{for all } x > 0. \tag{3.4.17}$$

Since η is non-increasing, for any Borel subset $B \subset \mathbb{R}$,

$$\nu(B) \lesssim \int_{B \cap (0,\infty)} \eta(x)\, dx \lesssim \eta(\delta(B)) \operatorname{diam}(B). \tag{3.4.18}$$

Arguing as in the proof of Theorem 3.4.3, we conclude that η has doubling property on $(0,\infty)$. Using that and monotonicity of η, for $s > 0$ and $x \in \mathbb{R}$,

$$\eta(s \vee x - \tfrac{1}{2}x) \leq \eta(\tfrac{1}{2}s) \lesssim \eta(s).$$

Therefore, by (A.1.6), for $r > 0$,

$$\int_r^\infty \eta(s \vee x - \tfrac{1}{2}x)\nu(x)\, dx \lesssim \eta(s)\psi^*(1/r). \tag{3.4.19}$$

Since ψ^* has the weak lower scaling property and satisfies (A.1.7), by [43, Theorem 3.1] and Proposition A.1.4, there are $C > 0$ and $t_1 \in (0, \infty]$ such that for all $t \in (0, t_1)$,

$$\int_{\mathbb{R}} e^{-t \operatorname{Re}\psi(\xi)}\, d\xi \leq C\psi^{-1}(1/t). \tag{3.4.20}$$

If $x_0 = 0$, then $t_1 = \infty$. If $t_1 < 48/\Phi(x_0)$, we can expand the above estimate for $t_1 \leq t < 48/\Phi(x_0)$ using positivity of the right hand side and monotonicity of the left hand side.

In view of (3.4.18), (3.4.19), and (3.4.20), by [42, Theorem 5.2], there is $C > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in \mathbb{R}$,

$$p\Big(t, x + tb_{1/\psi^{-1}(1/t)}\Big) \leq C\psi^{-1}(1/t) \cdot \min\Big\{1, t\big(\psi^{-1}(1/t)\big)^{-1}\eta(|x|) + \big(1 + |x|\psi^{-1}(1/t)\big)^{-3}\Big\}.$$

We claim that

$$\frac{\psi^{-1}(1/t)}{\big(1 + |x|\psi^{-1}(1/t)\big)^3} \lesssim t\eta(|x|) \tag{3.4.21}$$

whenever $t\eta(|x|) \leq \frac{A}{2}\Phi^{-1}(1/t)$.

First, let us show that for any $\varepsilon \in (0,1]$, the condition $t\eta(|x|) \leq \frac{A\varepsilon}{2}\Phi^{-1}(1/t)$ implies that

$$t\Phi^*\left(\frac{1}{|x|}\right) \leq \varepsilon|x|\Phi^{-1}\left(\frac{1}{t}\right). \tag{3.4.22}$$

Indeed, by (3.2.2), we have $|x|\eta(|x|) \geq \frac{A}{2}\Phi^*(1/|x|)$, thus

$$\varepsilon|x|\Phi^{-1}\left(\frac{1}{t}\right) \geq \frac{2}{A}t|x|\eta(|x|) \geq t\Phi^*\left(\frac{1}{|x|}\right).$$

Notice also that $\varepsilon^{1/3}|x|\Phi^{-1}(1/t) \geq 1$, since otherwise, by (A.1.3),

$$1 < t\Phi^*\left(\frac{1}{\varepsilon^{1/3}|x|}\right) < \frac{1}{\varepsilon^{2/3}}t\Phi^*\left(\frac{1}{|x|}\right),$$

which entails that $\varepsilon^{2/3} < t\Phi^*(1/|x|)$, i.e. $\varepsilon^{1/3}|x|\Phi^{-1}(1/t) < \varepsilon^{-2/3}t\Phi^*(1/|x|)$ contrary to (3.4.22).

To show (3.4.21), let us suppose that $t\eta(|x|) \leq \frac{A}{2}\Phi^{-1}(1/t)$, thus $|x|\Phi^{-1}(1/t) \geq 1$. By (A.1.3), we have

$$t\Phi^*(1/|x|) = \frac{\Phi^*(1/|x|)}{\Phi^*(|x|\Phi^{-1}(1/t) \cdot 1/|x|)} \geq \frac{1}{(|x|\Phi^{-1}(1/t))^2},$$

which, by Proposition A.1.5, gives

$$t|x|\eta(|x|) \geq \tfrac{A}{2}t\Phi^*(1/|x|) \gtrsim \frac{|x|\psi^{-1}(1/t)}{(1+|x|\psi^{-1}(1/t))^3},$$

proving (3.4.21), and (3.4.15) follows. The inequality (3.4.16) holds by the same argument as in the proof of Theorem 3.4.3. □

Remark 3.4.5. In statements of Theorems 3.4.3 and 3.4.4, we can replace $b_{1/\psi^{-1}(1/t)}$ with $b_{1/\Phi^{-1}(1/t)}$. Indeed, let us observe that if $0 < r_1 \leq r_2 < 1/x_0$, then

$$|b_{r_1} - b_{r_2}| \leq \int_{(r_1,r_2]} s\,\nu(ds) \leq r_1^{-1}r_2^2 h(r_2) \lesssim r_1^{-1}r_2^2\psi^*(1/r_2), \tag{3.4.23}$$

where in the last estimate we have used (A.1.6). Hence, by (A.1.11), we get

$$|b_{r_1} - b_{r_2}| \lesssim r_1^{-1}r_2^2\Phi^*(1/r_2). \tag{3.4.24}$$

Therefore, by (3.4.23), (3.4.24), and Proposition A.1.5, there is $C \geq 1$ such that

$$\left|b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)}\right| \leq C\frac{1}{t\Phi^{-1}(1/t)}, \tag{3.4.25}$$

provided that $0 < t < 1/\Phi(x_0)$. Now, let us suppose that $8C^2t\zeta(|x|) \leq 1$. Then, by (3.2.3) and (A.1.3),

$$\frac{1}{t} \geq 8C^2\zeta(|x|) \geq 4C^2\Phi^*\left(\frac{1}{|x|}\right) \geq \Phi^*\left(\frac{2C}{|x|}\right),$$

that is

$$|x| \geq \frac{2C}{\Phi^{-1}(1/t)}. \tag{3.4.26}$$

Hence, by (3.4.25),

$$\Big|x + t\Big(b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)}\Big)\Big| \geq |x| - \frac{C}{\Phi^{-1}(1/t)} \geq \frac{|x|}{2},$$

which together with monotonicity and the doubling property of ζ, gives

$$\zeta\Big(\big|x + t(b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)})\big|\Big) \lesssim \zeta(|x|).$$

Similarly, if $t\eta(|x|) \leq \frac{A\varepsilon}{2}\Phi^{-1}(1/t)$, then

$$|x|\Phi^{-1}(1/t) \geq \varepsilon^{-1/3},$$

thus, by taking $\varepsilon = (2C)^{-3}$, we obtain (3.4.26). Hence, by monotonicity and the doubling property of η, we again obtain

$$\eta\bigg(\Big|x + t\Big(b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)}\Big)\Big|\bigg) \lesssim \eta(|x|).$$

3.4.2 Estimates from below

In this part we develop estimates from below on the transition density $p(t, \cdot)$. The main result is Theorem 3.4.7 with the proof inspired by the ideas from Picard [82] and the proof of [43, Theorem 5.3]. We note that the significant step in the proof in which we obtain the lower bound on the density of one part of the decomposed subordinator $\mathbf{T}$ is justified by the sharp estimate provided by Theorem 3.1.1. Because of that we are able to generalize results obtained in [82] to the case when $-\phi''$ satisfies the weak lower scaling of index $\alpha - 2$ for $\alpha > 0$ together with a certain additional condition. Theorem 3.4.7 is then used to prove the lower estimate of $p(t, \cdot)$ on the right side of the supremum, which, together with Theorems 3.1.1 and 3.4.4, completes the picture and provokes a question on sharpness of our results, which is exclusively answered in the next subsection with the ultimate summary given by Theorem 3.4.13. We hint, however, that the additional upper scaling condition will come into play in this case.

First, let us prove the following variant of the celebrated Pruitt's estimates [87, Section 3] adapted to subordinators.

Proposition 3.4.6. *Let* $\mathbf{T}$ *be a subordinator with the Lévy–Khintchine exponent*

$$\psi(\xi) = -i\xi b - \int_{(0,\infty)} (e^{i\xi x} - 1)\,\nu(dx).$$

Then there is an absolute constant $c > 0$ *such that for all* $\lambda > 0$ *and* $t > 0$*,*

$$\mathbb{P}\Big(\sup_{0\leq s\leq t} |T_s - sb_\lambda| \geq \lambda\Big) \leq cth(\lambda).$$

Proof. We are going to apply the estimates [87, (3.2)]. To do so, we need to express the Lévy–Khintchine exponent of $T_s - sb_\lambda$ in the form used in [87, Section 3], namely

$$\begin{aligned}\widetilde{\psi}(\xi) &= \psi(\xi) + i\xi b_\lambda \\ &= -i\xi\Big(b - b_\lambda + \int_{(0,\infty)} \frac{y}{1+|y|^2}\,\nu(dy)\Big) - \int_{(0,\infty)} \Big(e^{i\xi y} - 1 - \frac{iy\xi}{1+|y|^2}\Big)\,\nu(dy).\end{aligned}$$

Since

$$\int_{(0,\lambda]} \frac{y|y|^2}{1+|y|^2}\,\nu(dy) - \int_{(\lambda,\infty)} \frac{y}{1+|y|^2}\,\nu(dy) = \int_{(0,\lambda]} y\,\nu(dy) - \int_{(0,\infty)} \frac{y}{1+|y|^2}\,\nu(dy),$$

we have

$$\begin{aligned} M(\lambda) &= \frac{1}{\lambda}\left|b - b_\lambda + \int_{(0,\infty)} \frac{y}{1+|y|^2}\,\nu(dy) + \int_{(0,\lambda]} \frac{y|y|^2}{1+|y|^2}\,\nu(dy) - \int_{(\lambda,\infty)} \frac{y}{1+|y|^2}\,\nu(dy)\right| \\ &= \frac{1}{\lambda}\left|b - b_\lambda + \int_{(0,\lambda]} y\,\nu(dy)\right| = 0. \end{aligned}$$

Hence, by [87, (3.2)],

$$\mathbb{P}\Big(\sup_{0\le s\le t} |T_s - sb_\lambda| \ge \lambda\Big) \le csh(\lambda),$$

as desired. □

Theorem 3.4.7. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ. *Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \ge 0$, *and* $\alpha > 0$, *and assume that one of the following conditions holds true:*

1. $-\phi'' \in \mathrm{WUSC}(\beta-2, C, x_0)$ *for some* $C \ge 1$ *and* $\alpha \le \beta < 1$, *or*

2. $-\phi''$ *is a function regularly varying at infinity with index* -1. *If* $x_0 = 0$, *we also assume that* $-\phi''$ *is regularly varying at zero with index* -1.

Then there is $M_0 > 0$ *such that for each* $M \ge M_0$ *there exist* $C > 0$ *and* $\rho_0 > 0$, *so that for all* $t \in (0, 1/\Phi(x_0))$, $0 < \rho_1 < \rho_0$, $\rho_2 > 0$ *and all* $x > 0$ *satisfying*

$$-\frac{\rho_1}{\Phi^{-1}(1/t)} \le x - t\phi'(\Phi^{-1}(M/t)) \le \frac{\rho_2}{\Phi^{-1}(1/t)},$$

we have

$$p\,(t,x) \ge C\Phi^{-1}(1/t). \tag{3.4.27}$$

Remark 3.4.8. From the proof of Theorem 3.4.7 it stems that if $x_0 = 0$, then one can obtain the same statement under a mixture of conditions 1. and 2. Namely, it holds if $-\phi''$ is (-1)-regular at infinity and satisfies upper scaling at the origin with the scaling index $\beta - 2$, where $\alpha \le \beta < 1$. Alternatively, one can assume that $-\phi''$ satisfies upper scaling at infinity with index $\beta - 2$, where $\alpha \le \beta < 1$, and varies regularly at zero with index -1. The same remark applies to Proposition 3.4.9.

Proof. First, let us observe that it is enough to prove that (3.4.27) holds true for all $t \in (0, 1/\Phi(x_0))$ and all $x > 0$ satisfying

$$-\frac{\rho_1}{\Phi^{-1}(M/t)} \le x - t\phi'(\Phi^{-1}(M/t)) \le \frac{\rho_2}{\Phi^{-1}(M/t)}.$$

Indeed, since Φ^{-1} is non-decreasing and has upper scaling property (see (A.1.13) in Proposition A.1.5), it has a doubling property. Hence, the theorem will follow immediately with possibly modified ρ_0.

Without loss of generality we can assume that $b = 0$. Let $\lambda > 0$, whose value will be specified later. We decompose the Lévy measure $\nu(dx)$ as follows: let $\nu_1(dx)$ be the restriction of $\frac{1}{2}\nu(dx)$ to the interval $(0, \lambda]$, and

$$\nu_2(dx) = \nu(dx) - \nu_1(dx).$$

We set

$$\phi_1(u) = \int_{(0,\infty)} \left(1 - e^{-us}\right) \nu_1(ds), \qquad \phi_2(u) = \int_{(0,\infty)} \left(1 - e^{-us}\right) \nu_2(ds).$$

Let us denote by $\mathbf{T}^{(j)}$ the subordinator having the Laplace exponent ϕ_j, for $j \in \{1, 2\}$. Let $\psi_j(\xi) = \phi_j(-i\xi)$. Notice that $\frac{1}{2}\nu \leq \nu_2 \leq \nu$; thus,

$$\tfrac{1}{2}\phi \leq \phi_2 \leq \phi,$$

and for every $n \in \mathbb{N}$,

$$\tfrac{1}{2}(-1)^{n+1}\phi^{(n)} \leq (-1)^{n+1}\phi_2^{(n)} \leq (-1)^{n+1}\phi^{(n)}. \tag{3.4.28}$$

Therefore, for all $u > 0$,

$$\tfrac{1}{2}\Phi(u) \leq \Phi_2(u) \leq \Phi(u). \tag{3.4.29}$$

Next, by Theorem 3.1.1, the random variables $T_t^{(2)}$ and T_t are absolutely continuous. Let us denote by $p^{(2)}(t, \cdot\,)$ and $p(t, \cdot\,)$ the densities of $T_t^{(2)}$ and T_t, respectively.

Let $M \geq 2M_0 + 1$, where M_0 is determined in Corollary 3.3.4 for the process $\mathbf{T}^{(2)}$. For $0 < t < 1/\Phi(x_0)$, we set

$$x_t = t\phi_2'(\Phi^{-1}(M/t)).$$

Since $\Phi^{-1}(M/t) > x_0$, we have

$$\frac{x_t}{t} = \phi_2'(\Phi^{-1}(M/t)) \leq \phi_2'(x_0).$$

Let

$$w_2 = (\phi_2')^{-1}(x_t/t) = \Phi^{-1}(M/t).$$

Then, by (3.4.29), we get

$$\Phi_2(w_2) \geq \frac{1}{2}\Phi(\Phi^{-1}(M/t)) = \frac{M}{2t} \geq \frac{M_0}{t}.$$

Moreover, by Proposition 3.4.1 together with (3.4.29), we get

$$t(\phi_2(w_2) - w_2\phi_2'(w_2)) \lesssim t\Phi_2(w_2) \lesssim 1.$$

Hence, by Corollary 3.3.4,

$$p^{(2)}(t, x_t) \gtrsim \frac{1}{\sqrt{t(-\phi_2'')(w_2)}}. \tag{3.4.30}$$

Notice that, by (3.4.28) and Remark 3.3.2, the implied constant in (3.4.30) is independent of t and λ. Since

$$(-\phi_2'')(w_2) \leq (-\phi'')(\Phi^{-1}(M/t)) = \frac{M}{t\big(\Phi^{-1}(M/t)\big)^2},$$

by (3.4.30) and monotonicity of Φ^{-1}, we get

$$p^{(2)}(t, x_t) \geq C_1\Phi^{-1}(1/t) \tag{3.4.31}$$

for some constant $C_1 > 0$.

Next, by the Fourier inversion formula

$$\sup_{x\in\mathbb{R}} \left|\partial_x p^{(2)}(t,x)\right| \lesssim \int_{\mathbb{R}} e^{-t\operatorname{Re}\psi_2(\xi)}|\xi|\,d\xi \lesssim \int_{\mathbb{R}} e^{-\frac{t}{2}\operatorname{Re}\psi(\xi)}|\xi|\,d\xi,$$

thus, by [43, Proposition 3.4], and Propositions A.1.4 and A.1.5, we see that there is $C_2 > 0$ such that for all $t \in (0, 1/\Phi(x_0))$,

$$\sup_{x\in\mathbb{R}} \left|\partial_x p^{(2)}(t,x)\right| \leq C_2\big(\Phi^{-1}(1/t)\big)^2.$$

By the mean value theorem, for $y \in \mathbb{R}$, we get

$$\left|p^{(2)}(t,y+x_t) - p^{(2)}(t,x_t)\right| \leq C_2|y|\big(\Phi^{-1}(1/t)\big)^2.$$

Hence, for $y \in \mathbb{R}$ satisfying

$$|y| \leq \frac{C_1}{2C_2\Phi^{-1}(1/t)},$$

by (3.4.31), we get

$$p^{(2)}(t,y+x_t) \geq p^{(2)}(t,x_t) - C_2|y|\big(\Phi^{-1}(1/t)\big)^2 \geq \frac{C_1}{2}\Phi^{-1}(1/t).$$

Therefore,

$$\begin{aligned}
p(t,x) &= \int_{\mathbb{R}} p^{(2)}(t,x-y)\mathbb{P}(T_t^{(1)} \in dy) \\
&\geq \frac{C_1}{2}\Phi^{-1}(1/t)\cdot\mathbb{P}\Big(\big|x - x_t - T_t^{(1)}\big| \leq \frac{C_0}{\Phi^{-1}(1/t)}\Big) \\
&\geq \frac{C_1}{2}\Phi^{-1}(1/t)\cdot\mathbb{P}\Big(\big|x - x_t - T_t^{(1)}\big| \leq \frac{C_0}{\Phi^{-1}(M/t)}\Big) \\
&= \frac{C_1}{2}\Phi^{-1}(1/t)\cdot\mathbb{P}\Big(\Big|x - \widetilde{x}_t - \big(\tfrac{1}{2}tb_\lambda - (\widetilde{x}_t - x_t)\big) - \big(T_t^{(1)} - \tfrac{1}{2}tb_\lambda\big)\Big| \leq \frac{C_0}{\Phi^{-1}(M/t)}\Big),
\end{aligned}$$

where we have set $C_0 = C_1(2C_2)^{-1}$ and

$$\widetilde{x}_t = t\phi'\big(\Phi^{-1}(M/t)\big).$$

Let $\rho_0 = \frac{1}{2}C_0$ and

$$\lambda = \frac{1}{\Phi^{-1}(M/t)}. \tag{3.4.32}$$

We then have

$$\frac{1}{2}tb_\lambda - (\widetilde{x}_t - x_t) = \frac{1}{2}tb_\lambda - t\phi_1'(1/\lambda) = \frac{t}{2}\int_{(0,\lambda]} s\big(1 - e^{-s/\lambda}\big)\,\nu(ds).$$

Thus, $\frac{1}{2}tb_\lambda - (\widetilde{x}_t - x_t)$ is non-negative and in view of (A.1.6) and (3.4.32),

$$\tfrac{1}{2}tb_\lambda - (\widetilde{x}_t - x_t) \leq C_3 t\lambda\Phi(1/\lambda) = \frac{C_3 M}{\Phi^{-1}(M/t)}, \tag{3.4.33}$$

for some constant $C_3 > 0$. Next, setting

$$\rho(t) = \lambda^{-1}\Big(\tfrac{1}{2}tb_\lambda - (\widetilde{x}_t - x_t)\Big),$$

we get

$$\inf_{t\in(0,1/\Phi(x_0))}\Big\{\mathbb{P}\Big(\Big|x - \widetilde{x}_t - \lambda\rho(t) - (T_t^{(1)} - \tfrac{1}{2}tb_\lambda)\Big| \leq C_0\lambda\Big) : x \geq 0, -\rho_1\lambda \leq x - \widetilde{x}_t \leq \rho_2\lambda\Big\}$$
$$\geq \inf_{t\in(0,1/\Phi(x_0))}\Big\{\mathbb{P}\Big(\Big|y - \lambda^{-1}(T_t^{(1)} - \tfrac{1}{2}tb_\lambda)\Big| \leq C_0\Big) :\ -\rho_1 - \rho(t) \leq y \leq \rho_2\Big\}. \tag{3.4.34}$$

Hence, the problem is reduced to showing that the infimum above is positive. Let us consider a collection $\{Y_t : t \in (0, 1/\Phi(x_0))\}$ of infinitely divisible non-negative random variables $Y_t = \lambda^{-1}(T_t^{(1)} - \frac{1}{2}tb_\lambda)$. The Lévy measure corresponding to Y_t is

$$\mu_t(B) = t\nu_1(\lambda B) \tag{3.4.35}$$

for any Borel subset $B \subset \mathbb{R}$. Since for each $R > 1$,

$$b_{R\lambda}^{(1)} = \int_{(0,R\lambda]} y\,\nu_1(dy) = \frac{1}{2}\int_{(0,\lambda]} y\,\nu(dy) = \frac{1}{2}b_\lambda,$$

by Proposition 3.4.6,

$$\mathbb{P}(|Y_t| \geq R) = \mathbb{P}\Big(\Big|T_t^{(1)} - \tfrac{1}{2}tb_\lambda\Big| \geq R\lambda\Big) \lesssim t\int_{(0,\infty)} \min\{1, R^{-2}\lambda^{-2}s^2\}\,\nu_1(ds),$$

thus,

$$\mathbb{P}(|Y_t| \geq R) \lesssim t\lambda^{-2}R^{-2}\int_{(0,\lambda]} s^2\,\nu(ds) \lesssim tR^{-2}h(\lambda) \lesssim tR^{-2}\Phi(1/\lambda)$$

where in the last estimate we have used (A.1.6). Therefore, recalling (3.4.32), we conclude that the collection is tight. Next, let $((Y_{t_n}, y_n) : n \in \mathbb{N})$ be a sequence realizing the infimum in (3.4.34). By the Prokhorov theorem, we can assume that $(Y_{t_n} : n \in \mathbb{N})$ is weakly convergent to the random variable Y_0. We note that Y_{t_n} has the probability distribution supported in $\big[-\frac{1}{2}t_n\lambda_n^{-1}b_{\lambda_n}, \infty\big)$ where λ_n is defined as λ corresponding to t_n.

Suppose that $(t_n : n \in \mathbb{N})$ contains a subsequence convergent to $t_0 > 0$. Then $Y_0 = Y_{t_0}$ and the support of its probability distribution equals $\big[-\frac{1}{2}t_0\lambda_0^{-1}b_{\lambda_0}, \infty\big)$. Since $\rho(t_0) \leq \frac{1}{2}t_0\lambda_0^{-1}b_{\lambda_0}$, we easily conclude that the infimum in (3.4.34) is positive.

Hence, it remains to investigate the case when $(t_n : n \in \mathbb{N})$ has no positive accumulation points. If zero is the only accumulation point, then $(\lambda_n : n \in \mathbb{N})$ has a subsequence convergent to zero. Otherwise (t_n) diverges to infinity, thus $x_0 = 0$ and (λ_n) contains a subsequence diverging to infinity. In view of (3.4.33), $\rho(t)$ is uniformly bounded in t. Thus, after taking a subsequence, we may and do assume that there exists a limit

$$\widetilde{\rho} = \lim_{n\to\infty} \rho(t_n).$$

By compactness we can also assume that $(y_n : n \in \mathbb{N})$ converges to $y_0 \in [-\rho_1 - \widetilde{\rho}, \rho_2]$. Consequently, to prove that the infimum in (3.4.34) is positive, it is sufficient to show that

$$\mathbb{P}(|y_0 - Y_0| \leq \tfrac{1}{2}C_0) > 0. \tag{3.4.36}$$

Observe that (3.4.36) is trivially satisfied if the support of the probability distribution of Y_0 is the whole real line. Therefore, we can assume that Y_0 is purely non-Gaussian. In view of [93, Theorem 8.7], it is also infinitely divisible.

Given a continuous function $w\colon \mathbb{R} \mapsto \mathbb{R}$ satisfying

$$\big|w(x) - 1\big| \leq C'|x|, \qquad \text{and} \qquad \big|w(x)\big| \leq C'|x|^{-1}, \tag{3.4.37}$$

we write the Lévy–Khintchine exponent of Y_{t_n} in the form

$$\psi_n(\xi) = -i\xi\gamma_n - \int_{(0,\infty)} \big(e^{i\xi s} - 1 - i\xi s w(s)\big)\, \mu_{t_n}(ds)$$

where

$$\gamma_n = \int_{(0,\infty)} s w(s)\, \mu_{t_n}(ds) - \tfrac{1}{2}\lambda_n^{-1} t_n b_{\lambda_n}.$$

Since $(Y_{t_n} : n \in \mathbb{N})$ converges weakly to Y_0, there are $\gamma_0 \in \mathbb{R}$ and σ-finite measure μ_0 on $(0,\infty)$ satisfying

$$\int_{(0,\infty)} \min\left\{1, s^2\right\} \mu_0(ds) < \infty,$$

such that the Lévy–Khintchine exponent of Y_0 is

$$\psi_0(\xi) = -i\xi\gamma_0 - \int_{(0,\infty)} \big(e^{i\xi s} - 1 - i\xi s w(s)\big)\, \mu_0(ds)$$

where

$$\gamma_0 = \lim_{n\to\infty} \gamma_n.$$

Moreover, for any bounded continuous function $f\colon \mathbb{R} \mapsto \mathbb{R}$ vanishing in a neighbourhood of zero, we have

$$\lim_{n\to\infty} \int_{(0,\infty)} f(s)\, \mu_{t_n}(ds) = \int_{(0,\infty)} f(s)\, \mu_0(ds). \tag{3.4.38}$$

Next, let us fix w satisfying (3.4.37) which equals 1 on $[0,1]$. In view of (3.4.35) and the definition of ν_1, the support of μ_{t_n} is contained in $[0,1]$. Hence, $\gamma_n = 0$ for every $n \in \mathbb{N}$ and consequently, $\gamma_0 = 0$. We also conclude that $\operatorname{supp} \mu_0 \subset [0,1]$.

At this stage, we consider the cases (i) and (ii) separately. In (ii) we need to distinguish two possibilities: if (t_n) tends to zero, then also (λ_n) approaches to zero, and we impose that $-\phi''$ is a function regularly varying at infinity with index -1; otherwise, (t_n) tends to infinity as well as (λ_n); thus, $x_0 = 0$, and we additionally assume that $-\phi''$ is a function regularly varying at zero with index -1. For the sake of clarity of presentation, we restrict attention to the first possibility only. In the second one the reasoning is analogous. We show that the support of the probability distribution of Y_0 is the whole real line. By [93, Theorem 24.10], the latter can be deduced from

$$\int_{(0,\infty)} \min\{1, s\}\, \mu_0(ds) = \infty. \tag{3.4.39}$$

Since $\operatorname{supp} \mu_0 \subset [0,1]$, for each $\varepsilon \in (0,1)$ we can write

$$\int_{(0,\infty)} \min\{1, s\}\, \mu_0(ds) \geq \int_{(\varepsilon/2,1]} s\, \mu_0(ds),$$

thus, to conclude (3.4.39), it is enough to show that

$$\int_{(\varepsilon/2,1]} s\, \mu_0(ds) \gtrsim \log \varepsilon^{-1}. \tag{3.4.40}$$

For the proof, for any $\varepsilon \in (0,1)$ we define the following bounded continuous function

$$f_\varepsilon(s) = \begin{cases} 0 & \text{if } s < \varepsilon/2, \\ 2s - \varepsilon, & \text{if } \varepsilon/2 \leq s < \varepsilon, \\ s & \text{if } \varepsilon \leq s < 1, \\ 1 & \text{if } s \geq 1. \end{cases} \tag{3.4.41}$$

We have, in view of (3.4.38),

$$\int_{(\varepsilon/2,1]} s\, \mu_0(ds) \geq \int_{(0,1]} f_\varepsilon(s)\, \mu_0(ds) = \lim_{n\to\infty} \int_{(0,1]} f_\varepsilon(s)\, \mu_{t_n}(ds) \geq \liminf_{n\to\infty} \int_{(\varepsilon,1]} s\, \mu_{t_n}(ds). \tag{3.4.42}$$

Let us estimate the last integral. We write

$$\int_{(\varepsilon,1]} s\, \mu_t(ds) = t\lambda^{-1} \int_{(\lambda\varepsilon,\lambda]} s\, \nu_1(ds) = \tfrac{1}{2} t\lambda^{-1} \int_{(\lambda\varepsilon,\lambda]} s\, \nu(ds).$$

By the Fubini–Tonelli theorem, we get

$$\int_{[\lambda\varepsilon,\lambda)} s\, \nu(ds) = \int_{\lambda\varepsilon}^{\lambda} u^{-2} \int_{(0,u]} s^2\, \nu(ds)\, du + \lambda K(\lambda) - \lambda\varepsilon K(\lambda\varepsilon).$$

Thus,

$$2\int_{[\varepsilon,1)} s\, \mu_t(ds) = t\lambda^{-1} \int_{\lambda\varepsilon}^{\lambda} K(u)\, du + tK(\lambda) - t\varepsilon K(\lambda\varepsilon). \tag{3.4.43}$$

Setting $z = 1/\lambda$, by (A.1.6) and (3.4.32), we obtain

$$tK(\lambda) \approx t\Phi(z) \approx 1.$$

Moreover, since Φ is a 1-regularly varying function at infinity, we have

$$t\varepsilon K(\lambda\varepsilon) \approx t\varepsilon\Phi(z/\varepsilon) = M\varepsilon \frac{\Phi(z/\varepsilon)}{\Phi(z)} \to M,$$

as z tends to infinity. Therefore, it remains to estimate the integral in (3.4.43). Using (A.1.6) we get

$$\begin{aligned} t\lambda^{-1} \int_{\lambda\varepsilon}^{\lambda} K(u)\, du &\approx \frac{z}{\Phi(z)} \int_{\varepsilon z^{-1}}^{z^{-1}} \Phi(u^{-1})\, du \\ &= \frac{z}{\Phi(z)} \int_{z}^{\varepsilon^{-1} z} u^{-2}\Phi(u)\, du \\ &= \frac{\phi'(z) - \phi'(\varepsilon^{-1} z)}{z\big(-\phi''(z)\big)}. \end{aligned}$$

Since $-\phi''(s) = s^{-1}\ell(s)$ for a certain function ℓ slowly varying at infinity, by [5, Theorem 1.5.6],

$$\frac{\phi'(z) - \phi'(\varepsilon^{-1} z)}{z\big(-\phi''(z)\big)} = \int_1^{\varepsilon^{-1}} \frac{\ell(zt)}{\ell(z)} \frac{dt}{t} \to \log \varepsilon^{-1},$$

as z tends to infinity. Hence,

$$\liminf_{n\to\infty} \int_{(\varepsilon,1]} s\, \mu_{t_n}(ds) \gtrsim \log \varepsilon^{-1},$$

which, by (3.4.42), implies (3.4.40).

Next, let us consider the case (i), that is when $-\phi'' \in \mathrm{WUSC}(\beta - 2, C, x_0)$ with $C \geq 1$ and $\alpha \leq \beta < 1$. We claim that for all $\varepsilon \in (0,1)$,

$$\int_{(0,\varepsilon)} s^2 \, \mu_0(ds) > 0. \tag{3.4.44}$$

To see this, it is enough to show that there is $C > 0$ such that for all $\varepsilon \in (0,1]$ and $t \in (0, 1/\Phi(x_0))$,

$$\int_{(0,\varepsilon)} s^2 \, \mu_t(ds) \geq C\varepsilon^{2-\alpha}. \tag{3.4.45}$$

For the proof, we select a continuous function on $\mathbb{R}$ such that

$$\mathbb{1}_{(-1,1)} \leq \eta \leq \mathbb{1}_{(-2,2)},$$

and for each $\tau > 0$ set

$$\eta_\tau(x) = \eta(\tau^{-1}x).$$

Since for $0 < 2\tau < \varepsilon$,

$$\int_{(0,\infty)} s^2 \big(\eta_\varepsilon(s) - \eta_\tau(s)\big) \, \mu_t(ds) + \int_{(0,2\tau)} s^2 \eta_\tau(s) \, \mu_t(ds) \geq \int_{(0,\varepsilon)} s^2 \, \mu_t(ds),$$

by (3.4.45) and (3.4.38),

$$\int_{(0,\infty)} s^2 \big(\eta_\varepsilon(s) - \eta_\tau(s)\big) \, \mu_0(ds) + \limsup_{n \to \infty} \int_{(0,\infty)} s^2 \eta_\tau(s) \, \mu_{t_n}(ds) \geq C\varepsilon^{2-\alpha}.$$

Since Y_{t_n} and Y_0 are purely non-Gaussian, by [93, Theorem 8.7(2)],

$$\lim_{\tau \to 0^+} \limsup_{n \to \infty} \int_{(-\tau,\tau)} s^2 \, \mu_{t_n}(ds) = 0,$$

thus,

$$\int_{(0,\varepsilon)} s^2 \, \mu_0(ds) \geq C\varepsilon^{2-\alpha},$$

which entails (3.4.44).

We now turn to showing (3.4.45). We have

$$\begin{aligned}
\int_{(0,\varepsilon)} s^2 \, \mu_t(ds) &= t\lambda^{-2} \int_{(0,\lambda\varepsilon)} s^2 \, \nu_1(ds) \\
&= \tfrac{1}{2} t\lambda^{-2} \int_{(0,\lambda\varepsilon)} s^2 \, \nu(ds) \\
&= \tfrac{1}{2} t\varepsilon^2 K(\lambda\varepsilon),
\end{aligned}$$

thus, by (A.1.6) and the weak lower scaling property of Φ,

$$\int_{(0,\varepsilon)} s^2 \, \mu_t(ds) \gtrsim t\varepsilon^2 \Phi(\varepsilon^{-1}\lambda^{-1}) \gtrsim t\varepsilon^{2-\alpha}\Phi(1/\lambda),$$

which, together with the definition of λ, implies (3.4.45).

Since the support of the probability distribution of Y_0 is not the whole real line, by [82, Lemma 2.5], the inequality (3.4.44) implies that

$$\int_{(0,\infty)} \min\{1, s\}\, \mu_0(ds) < \infty$$

and the support of Y_0 equals $[\chi, \infty)$ where

$$\chi = \gamma_0 - \int_{(0,\infty)} sw(s)\, \mu_0(ds) = -\int_{(0,1]} s\, \mu_0(ds).$$

To conclude (3.4.36), it is enough to show that $\chi \leq -\widetilde{\rho}$. Since $\rho(t_n) \leq \frac{1}{2} t_n \lambda_n^{-1} b_{\lambda_n}$, the latter can be deduced from

$$\chi = -\lim_{n\to\infty} \frac{1}{2} t_n \lambda_n^{-1} b_{\lambda_n} = -\lim_{n\to\infty} \int_{(0,1]} s\, \mu_{t_n}(ds) \tag{3.4.46}$$

where the last equality is a consequence of (3.4.35), since

$$\int_{(0,1]} s\, \mu_t(ds) = t\lambda^{-1} \int_{(0,\lambda]} s\, \nu_1(ds) = \tfrac{1}{2} t\lambda^{-1} \int_{(0,\lambda]} s\, \nu(ds). \tag{3.4.47}$$

Therefore, the problem is reduced to showing (3.4.46). By the monotone convergence theorem and (3.4.38) we have

$$\chi = -\lim_{\varepsilon\to 0^+} \int_{(0,1]} f_\varepsilon(s)\, \mu_0(ds) = -\lim_{\varepsilon\to 0^+} \lim_{n\to\infty} \int_{(0,1]} f_\varepsilon(s)\, \mu_{t_n}(ds),$$

and

$$\lim_{\varepsilon\to 0^+} \int_{(0,1]} f_\varepsilon(s)\, \mu_{t_n}(ds) = \int_{(0,1]} s\, \mu_{t_n}(ds), \tag{3.4.48}$$

where f_ε is as in (3.4.41). Hence, we just need to justify the change in the order of limits. In view of the Moore–Osgood theorem [33, Chapter VII], it is enough to show that the limit in (3.4.48) is uniform with respect to $n \in \mathbb{N}$.

We write

$$\begin{aligned}\left| \int_{(0,1]} s\, \mu_t(ds) - \int_{(0,1]} f_\varepsilon(s)\, \mu_t(ds) \right| &\leq \int_{(0,\varepsilon/2]} s\, \mu_t(ds) + \int_{(\varepsilon/2,\varepsilon]} (\varepsilon - s)\, \mu_t(ds) \\ &\leq \int_{(0,\varepsilon]} s\, \mu_t(ds).\end{aligned}$$

By (3.4.47) and the Fubini–Tonelli theorem, we have

$$\begin{aligned}2t^{-1}\lambda \int_{(0,\varepsilon]} s\, \mu_t(ds) = \int_{(0,\lambda\varepsilon]} s\, \nu(ds) &= \int_0^{\lambda\varepsilon} u^{-2} \int_{(0,u]} s^2\, \nu(ds)\, du + \lambda\varepsilon K(\lambda\varepsilon) \\ &\approx \int_0^{\lambda\varepsilon} \Phi(u^{-1})\, du + \lambda\varepsilon\Phi(\lambda^{-1}\varepsilon^{-1}).\end{aligned}$$

By almost monotonicity of Φ,

$$\int_{(0,\varepsilon]} s\, \mu_t(ds) \approx t\lambda^{-1} \int_0^{\lambda\varepsilon} \Phi(u^{-1})\, du + t\varepsilon\Phi(\lambda^{-1}\varepsilon^{-1}) \approx t\lambda^{-1} \int_0^{\lambda\varepsilon} \Phi(u^{-1})\, du.$$

Now, setting $z = \Phi^{-1}(M/t)$, by (3.4.32), we get

$$\begin{aligned} t\lambda^{-1} \int_0^{\lambda\varepsilon} \Phi(u^{-1})\, du &= t\Phi^{-1}(M/t) \int_0^{\varepsilon/\Phi^{-1}(M/t)} \Phi(u^{-1})\, du \\ &\approx \frac{z}{\Phi(z)} \int_0^{\varepsilon z^{-1}} \Phi(u^{-1})\, du \\ &= \frac{z}{\Phi(z)} \int_{\varepsilon^{-1}z}^{\infty} u^{-2}\Phi(u)\, du \\ &= \frac{\phi'(\varepsilon^{-1}z)}{z\big(-\phi''(z)\big)}. \end{aligned}$$

In view of Proposition 3.2.4, by the upper scaling of $-\phi''$, there is $c > 0$ such that for all $z > x_0$,

$$\frac{\phi'(\varepsilon^{-1}z)}{z\big(-\phi''(z)\big)} \le c\varepsilon^{1-\beta}.$$

Hence, the limit in (3.4.48) is uniform with respect to $n \in \mathbb{N}$, which justifies (3.4.46). This completes the proof of (3.4.36) and the lemma follows. □

Now we treat the remaining part the transition density, i.e. the one on the right side of the supremum. In general, based on known estimates concerning various kinds of Lévy process, we expect the decay to be expressed by means of the Lévy density $\nu(x)$. For instance, in the case of unimodal Lévy processes, it is known (see e.g. Bogdan, Grzywny and Ryznar [10, Theorem 21 and Corollary 23]) that

$$p(t,x) \approx p(t,0) \wedge t\nu(x).$$

In particular, for fixed $t > 0$ we have $p(t,x) \approx t\nu(x)$ for x large enough. The following proposition states that the right tail of the transition density enjoys the lower bound of the same form.

Proposition 3.4.9. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ*. Suppose that* $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \ge 0$, *and* $\alpha > 0$, *and assume that one of the following conditions holds true:*

1. $-\phi'' \in \mathrm{WUSC}(\beta-2, C, x_0)$ *for some* $C \ge 1$ *and* $\alpha \le \beta < 1$, *or*
2. $-\phi''$ *is a function regularly varying at infinity with index* -1*. If* $x_0 = 0$, *we also assume that* $-\phi''$ *is regularly varying at zero with index* -1*.*

We also assume that the Lévy measure ν *has an almost decreasing density* $\nu(x)$*. Then there are* $M_0 > 1$, $\rho_0 > 0$ *and* $C > 0$ *such that for all* $t \in (0, 1/\Phi(x_0))$ *satisfying*

$$x \ge 2t\phi'\big(\Phi^{-1}(M_0/t)\big) + \frac{2\rho_0}{\Phi^{-1}(1/t)},$$

we have

$$p(t,x) \ge Ct\nu(x).$$

Proof. Let $\lambda > 0$. We begin by decomposing the Lévy measure $\nu(dx)$. Let $\nu_1(dx) = \nu_1(x)\, dx$ and $\nu_2(dx) = \nu_2(x)\, dx$ where

$$\nu_1(x) = \nu(x) - \nu_2(x) \qquad \text{and} \qquad \nu_2(x) = \tfrac{1}{2}\nu(x)\mathbb{1}_{[\lambda,\infty)}(x).$$

For $u > 0$, we set

$$\phi_1(u) = bu + \int_{(0,\infty)} (1 - e^{-us})\, \nu_1(ds), \qquad \text{and} \qquad \phi_2(u) = \int_{(0,\infty)} (1 - e^{-us})\, \nu_2(ds).$$

Let $\mathbf{T}^{(j)}$ be a Lévy process having the Laplace exponent ϕ_j, for $j \in \{1,2\}$. Since $\frac{1}{2}\nu \leq \nu_1 \leq \nu$, we have

$$\tfrac{1}{2}\phi \leq \phi_1 \leq \phi,$$

and for all $n \in \mathbb{N}$,

$$\tfrac{1}{2}(-1)^{n+1}\phi^{(n)} \leq (-1)^{n+1}\phi_1^{(n)} \leq (-1)^{n+1}\phi^{(n)}. \tag{3.4.49}$$

Thus

$$\tfrac{1}{2}\Phi \leq \Phi_1 \leq \Phi,$$

and so for all $u > 0$,

$$\Phi_1^{-1}(u/2) \leq \Phi^{-1}(u) \leq \Phi_1^{-1}(u). \tag{3.4.50}$$

In particular, $-\phi_1''$ has the weak lower scaling property. Therefore, by Theorem 3.1.1, $T_t^{(1)}$ and T_t are absolutely continuous for all $t > 0$. Let us denote by $p(t,\,\cdot\,)$ and $p^{(1)}(t,\,\cdot\,)$ the densities of T_t and $T_t^{(1)}$, respectively. Observe that $\mathbf{T}^{(2)}$ is a compound Poisson process with the probability distribution denoted by $P_t(dx)$. By [93, Remark 27.3],

$$P_t(dx) \geq t e^{-t\nu_2(\mathbb{R})} \nu_2(x)\, dx. \tag{3.4.51}$$

We apply Theorem 3.4.7 to the process $\mathbf{T}^{(1)}$. For $t > 0$, we set

$$x_t = t\phi_1'\big(\Phi_1^{-1}(M_0/t)\big).$$

Then there are $C > 0$ and $\rho_0 > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \geq 0$ satisfying

$$x_t - \frac{\rho_0}{\Phi_1^{-1}(1/t)} \leq x \leq x_t + \frac{\rho_0}{\Phi_1^{-1}(1/t)},$$

we have

$$p^{(1)}(t,x) \geq C\Phi_1^{-1}(1/t).$$

Therefore, if

$$\lambda = x_t + \frac{\rho_0}{\Phi_1^{-1}(1/t)},$$

then

$$\int_0^\lambda p^{(1)}(t,x)\, dx \gtrsim 1. \tag{3.4.52}$$

Next, if $\lambda \geq \rho_0/\Phi^{-1}(1/t)$ then, by (A.1.6),

$$t\nu_2(\mathbb{R}) = \tfrac{1}{2}t\int_\lambda^\infty \nu(x)\, dx \leq \tfrac{1}{2}th\big(\rho_0/\Phi^{-1}(1/t)\big) \lesssim th\big(1/\Phi^{-1}(1/t)\big) \lesssim 1, \tag{3.4.53}$$

where the penultimate inequality follows either by monotonicity of h or by [43, Lemma 2.1 (4)]. Finally, by (3.4.51) and (3.4.53), for $x \geq 2\lambda$ we can compute

$$\begin{aligned} p(t,x) &= \int_{\mathbb{R}} p^{(1)}(t,x-y) P_t(dy) \\ &\gtrsim t\int_{\mathbb{R}} p^{(1)}(t,x-y)\nu_2(y)\, dy \\ &= \tfrac{1}{2}t\int_\lambda^x p^{(1)}(t,x-y)\nu(y)\, dy. \end{aligned}$$

Hence, by the monotonicity of ν, we get

$$\begin{aligned} p(t,x) &\gtrsim t\nu(x) \int_0^{x-\lambda} p^{(1)}(t,y)\,dy \\ &\geq t\nu(x) \int_0^{\lambda} p^{(1)}(t,y)\,dy \\ &\gtrsim t\nu(x), \end{aligned}$$

where in the last estimate we have used (3.4.52). Using (3.4.49) and (3.4.50), we can easily show that

$$\lambda = x_t + \frac{\rho_0}{\Phi_1^{-1}(1/t)} \leq t\phi'(\Phi^{-1}(M_0/t)) + \frac{\rho_0}{\Phi^{-1}(1/t)},$$

and the proposition follows. □

3.4.3 Sharp two-sided estimates

Having proved both lower and upper estimates, let us now discuss the case when these two coincide. As already hinted in statements of previous results, this will hold under the additional assumption of upper scaling with $\beta < 1$. Note that by virtue of Propositions 3.2.3 and 3.2.2 and Corollary 3.2.8 it is irrelevant whether one imposes scalings on the Laplace exponent ϕ or on its (minus) second derivative $-\phi''$. The formal proofs are rather technical and require some care, but the idea behind is rather simple: under lower and upper scaling it follows from Proposition A.1.5 that both the localisation and the magnitude of regime endpoints in Theorems 3.1.1, 3.4.4 and 3.4.7 and Proposition 3.4.9 are of the same *scale*. Therefore, they may be merged in one consolidated theorem.

First, let us combine Theorems 3.4.3 and 3.4.7 into one statement describing the localisation of the supremum of the density $p(t,x)$.

Theorem 3.4.10. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ. *Suppose that* $\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ *for some* $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$ *and* $0 < \alpha \leq \beta < 1$. *We also assume that* $b = 0$. *Then for all* $0 < \chi_1 < \chi_2$ *there is* $C' \geq 1$ *such that for all* $t \in (0, 1/\Phi(x_0))$ *and* $x > 0$ *satisfying*

$$\chi_1 \leq x\phi^{-1}(1/t) \leq \chi_2,$$

we have

$$C'^{-1}\phi^{-1}(1/t) \leq p(t,x) \leq C'\phi^{-1}(1/t). \tag{3.4.54}$$

Proof. First, let us notice that in view of Corollary 3.2.8 we have that $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0) \cap \mathrm{WUSC}(\beta-2, C, x_0)$. Therefore, the assumptions of Theorems 3.4.3 and 3.4.7 are satisfied.

It is enough to show the first inequality in (3.4.54), since the latter is an easy consequence of (3.4.4) and Proposition 3.4.2. For $t \in (0, 1/\Phi(x_0))$ and $M \geq 1$, we set

$$x_t = t\phi'(\Phi^{-1}(M/t)).$$

By Proposition 3.4.2, there is $C_1 \geq 1$ such that for all $r > \Phi(x_0)$,

$$C_1^{-1}\Phi^{-1}(r) \leq \phi^{-1}(r) \leq C_1\Phi^{-1}(r). \tag{3.4.55}$$

Observe that by Proposition 3.2.4 and (3.4.3), there is $C_2 \geq 1$, such that

$$x_t \leq C_2 M^{1-1/\beta} \frac{1}{\Phi^{-1}(1/t)}. \tag{3.4.56}$$

We select $M \geq 1$ satisfying

$$C_1 C_2 M^{1-1/\beta} < \chi_1.$$

We note that $\beta < 1$ is crucial in this case. Recall that in view of (A.1.4) one always has the lower scaling of Φ^{-1} with the exponent $\frac{1}{2}$, but this property it insufficient here. Let $\rho_1 = \rho_0/2$ where ρ_0 is determined in Theorem 3.4.7. Then, by (3.4.55) and (3.4.56), we have

$$x_t - \frac{\rho_1}{\Phi^{-1}(1/t)} \leq C_1 C_2 M^{1-1/\beta} \frac{1}{\phi^{-1}(1/t)} < \frac{\chi_1}{\phi^{-1}(1/t)}. \tag{3.4.57}$$

Now set $\rho_2 = C_1 \chi_2$. Then, by (3.4.55), we have

$$x_t + \frac{\rho_2}{\Phi^{-1}(1/t)} > \frac{\rho_2}{C_1 \phi^{-1}(1/t)} = \frac{\chi_2}{\phi^{-1}(1/t)}. \tag{3.4.58}$$

Putting (3.4.58) and (3.4.57) together, we conclude that

$$\left[\frac{\chi_1}{\phi^{-1}(1/t)}, \frac{\chi_2}{\phi^{-1}(1/t)}\right] \subseteq \left(x_t - \frac{\rho_1}{\Phi^{-1}(1/t)}, x_t + \frac{\rho_2}{\Phi^{-1}(1/t)}\right).$$

Therefore, by Theorem 3.4.7, for all $t \in (0, 1/\Phi(x_0))$ and $x > 0$ satisfying

$$\chi_1 \leq x\phi^{-1}(1/t) \leq \chi_2,$$

we have

$$p(t,x) \gtrsim \Phi^{-1}(1/t).$$

In view of (3.4.55), this completes the proof of the theorem. □

Next, following [10, Lemma 13], we prove an auxiliary result.

Proposition 3.4.11. *Assume that the Lévy measure ν has an almost decreasing density $\nu(x)$. Suppose that $-\phi'' \in \mathrm{WUSC}(\gamma, C, x_0)$ for some $C \geq 1$, $x_0 \geq 0$ and $\gamma < 0$. Then there is $c \in (0,1]$ such that for all $0 < x < 1/x_0$,*

$$\nu(x) \geq c x^{-3}\big(-\phi''(1/x)\big).$$

Proof. Let $a \in (0,1]$. Recall that by (3.4.17) we have $\nu(s) \leq C_1 s^{-3}\big(-\phi''(1/s)\big)$ for any $s > 0$. Hence, for any $u > 0$,

$$\begin{aligned}
-\phi''(u) &= \int_0^{au^{-1}} s^2 e^{-us} \nu(s)\, ds + \int_{au^{-1}}^{\infty} s^2 e^{-us} \nu(s)\, ds \\
&\leq C_1 \int_0^{au^{-1}} s^{-1} e^{-us} \big(-\phi''(1/s)\big)\, ds + C_2 \nu(au^{-1}) \int_{au^{-1}}^{\infty} s^2 e^{-us}\, ds
\end{aligned} \tag{3.4.59}$$

where C_2 is a constant from the almost monotonicity of ν. If $u > x_0$, then by the scaling property of $-\phi''$ we obtain

$$\begin{aligned}
C_1 \int_0^{au^{-1}} s^{-1} e^{-us} \big(-\phi''(1/s)\big)\, ds &\leq C \int_0^{au^{-1}} s^{-1} e^{-us} (su)^{-\gamma} \big(-\phi''(u)\big)\, ds \\
&\leq C\big(-\phi''(u)\big) \int_0^a s^{-1-\gamma} e^{-s}\, ds.
\end{aligned}$$

By selecting $a \in (0,1]$ such that

$$2C \int_0^a s^{-1-\gamma} e^{-s}\, ds \leq 1,$$

we get

$$\int_0^{au^{-1}} s^{-1} e^{-us} \big(-\phi''(1/s)\big)\, ds \leq \tfrac{1}{2}\big(-\phi''(u)\big).$$

Since

$$\int_{au^{-1}}^{\infty} s^2 e^{-us}\, ds = u^{-3} e^{-a}(a^2+2a+2),$$

by (3.4.59), we obtain

$$\nu(au^{-1}) \geq \frac{e^a}{2(a^2+2a+2)} u^3 \big(-\phi''(u)\big),$$

provided that $u > x_0$. Now, by the monotonicity of $-\phi''$ we get the claim for the case $x_0 = 0$. If $x_0 > 0$, then, by positivity and continuity of ν and $-\phi''$, we may extend the area from $< a/x_0$ to $x < 1/x_0$ at the cost of the constant c. $\square$

In view of Propositions 3.2.3 and 3.2.4, we immediately obtain the following corollary.

Corollary 3.4.12. *Assume that the Lévy measure ν has an almost monotone density $\nu(x)$. Suppose that $b = 0$ and $\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$ and $0 < \alpha \leq \beta < 1$. Then there is $c' \in (0,1]$ such that for all $0 < x < 1/x_0$,*

$$\nu(x) \geq c' x^{-1} \phi(1/x).$$

We are now ready to prove our main result in this subsection. Note that if the scalings are global, then the estimate is global both in space and in time. Such version is displayed in the introduction as Theorem 3.1.2.

Theorem 3.4.13. *Let $\mathbf{T}$ be a subordinator with the Laplace exponent ϕ. Suppose that $\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$, and $0 < \alpha \leq \beta < 1$. We also assume that $b = 0$ and that the Lévy measure ν has an almost decreasing density $\nu(x)$. Then there is $x_1 \in (0, \infty]$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in (0, x_1)$,*

$$p(t,x) \approx \begin{cases} \big(t(-\phi''(w))\big)^{-\frac{1}{2}} \exp\big\{ -t(\phi(w) - w\phi'(w))\big\} & \text{if } 0 < x\phi^{-1}(1/t) \leq 1, \\ t x^{-1} \phi(1/x) & \text{if } 1 < x\phi^{-1}(1/t), \end{cases}$$

where $w = (\phi')^{-1}(x/t)$. If $x_0 = 0$ then $x_1 = \infty$.

Proof. First, let us note that, by Corollary 3.2.8, $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0) \cap \mathrm{WUSC}(\beta-2, C, x_0)$. Therefore, we are in position to apply Proposition 3.4.9. By Corollary 3.3.6, for $\chi_1 = \min\{1, \delta\}$, we have

$$p(t,x) \approx \big(t(-\phi''(w))\big)^{-\frac{1}{2}} \exp\big\{ -t(\phi(w) - w\phi'(w))\big\},$$

whenever $0 < x\phi^{-1}(1/t) \leq \chi_1$. Next, let M_0' be M_0 determined by Proposition 3.4.9. By Proposition 3.2.4, (A.1.6) and monotonicity of Φ^{-1}, for $t \in (0, 1/\Phi(x_0))$, we get

$$t\phi'(\psi^{-1}(1/t)) \lesssim \frac{1}{\psi^{-1}(1/t)} \quad \text{and} \quad t\phi'(\Phi^{-1}(M_0'/t)) \lesssim \frac{1}{\Phi^{-1}(1/t)},$$

thus, by Propositions A.1.5 and 3.4.2, there is $C_1 > 0$ such that

$$2et\phi'(\psi^{-1}(1/t)) \le C_1 \frac{1}{\phi^{-1}(1/t)}$$

and

$$2t\phi'(\Phi^{-1}(M_0'/t)) + \frac{2\rho_0'}{\Phi^{-1}(1/t)} \le C_1 \frac{1}{\phi^{-1}(1/t)},$$

where ρ_0' is the value of ρ_0 determined in Proposition 3.4.9. Let $\chi_2 = \max\{1, C_1, \chi_1\}$. Proposition 3.4.9 and Corollary 3.4.12 yield that if $x\phi^{-1}(1/t) > \chi_2$ and $0 < x < 1/x_0$, then

$$p(t,x) \gtrsim t\nu(x) \gtrsim tx^{-1}\phi(1/x).$$

Furthermore, by (3.4.16), if $x\phi^{-1}(1/t) > \chi_2$, then

$$p(t,x) \lesssim t\eta(x) \lesssim tx^{-1}\phi(1/x),$$

where in the last step we have also used (3.4.2). Lastly, by Theorem 3.4.10, there is $C_2 \ge 1$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x > 0$ satisfying

$$\chi_1 \le x\phi^{-1}(1/t) \le \chi_2,$$

we have

$$C_2^{-1}\phi^{-1}(1/t) \le p(t,x) \le C_2\phi^{-1}(1/t). \tag{3.4.60}$$

We next claim that the following holds true.

Claim 3.4.14. *There exist* $0 < c_1 \le 1 \le c_2$ *such that for all* $t \in (0, c_1/\Phi(x_0))$ *and* $x > 0$ *satisfying*

$$\chi_1 \le x\phi^{-1}(1/t) \le \chi_2,$$

we have

$$t\phi'(\phi^{-1}(c_2/t)) \le x \le t\phi'(\phi^{-1}(c_1/t)). \tag{3.4.61}$$

Indeed, by Proposition 3.4.2, there is $C_3 \ge 1$ such that for $r > \Phi(x_0)$,

$$C_3^{-1}\Phi^{-1}(r) \le \phi^{-1}(r) \le C_3\Phi^{-1}(r).$$

Let $c_2 = (\chi_1 c' C_3^{-2})^{-\beta/(1-\beta)} \in [1, \infty)$, where c' is taken from (3.4.3). Then

$$c_2^{-1}\phi^{-1}(c_2/t) \ge C_3^{-2}c'c_2^{-1+1/\beta}\phi^{-1}(1/t) = \chi_1^{-1}\phi^{-1}(1/t).$$

Consequently, by Proposition 3.2.3,

$$x \ge \frac{\chi_1}{\phi^{-1}(1/t)} \ge t\frac{\phi(\phi^{-1}(c_2/t))}{\phi^{-1}(c_2/t)} \ge t\phi'(\phi^{-1}(c_2/t)). \tag{3.4.62}$$

Moreover, there is $C_4 \ge 1$ such that $C_4 x\phi'(x) \ge \phi(x)$ provided that $x > x_0$. Therefore, if $\chi_2 \le C_4^{-1}$, then

$$\frac{\chi_2}{\phi^{-1}(1/t)} = \chi_2 t\frac{\phi(\phi^{-1}(1/t))}{\phi^{-1}(1/t)} \le t\phi'(\phi^{-1}(1/t)), \tag{3.4.63}$$

which yields (3.4.61) with $c_1 = 1$. If $\chi_2 > C_4^{-1}$, then we set $c_1 = \left(C_4\chi_2 C_3^2 (c')^{-1}\right)^{-\beta/(1-\beta)} \in (0,1]$. Hence, by Proposition 3.4.2, for all $t \in (0, c_1/\Phi(x_0))$,

$$\frac{C_4\chi_2}{c_1}\phi^{-1}(c_1/t) \leq C_4\chi_2 C_3^2 (c')^{-1} c_1^{-1+1/\beta}\phi^{-1}(1/t) = \phi^{-1}(1/t).$$

Therefore,

$$\begin{aligned} x &\leq \frac{\chi_2}{\phi^{-1}(1/t)} \\ &\leq t\frac{\chi_2}{c_1} \cdot \frac{\phi^{-1}(c_1/t)}{\phi^{-1}(1/t)} \cdot \frac{\phi(\phi^{-1}(c_1/t))}{\phi^{-1}(c_1/t)} \\ &\leq t\phi'(\phi^{-1}(c_1/t)), \end{aligned}$$

which, combined with (3.4.62) and (3.4.63), implies (3.4.61).

Now, using Claim 3.4.14 and Propositions A.1.5 and 3.4.2, we deduce that for $t \in (0, c_1/\Phi(x_0))$ and $\chi_1 \leq x\phi^{-1}(1/t) \leq \chi_2$,

$$w \leq \phi^{-1}(c_2/t) \lesssim \phi^{-1}(1/t), \tag{3.4.64}$$

and

$$w \geq \phi^{-1}(c_1/t) \gtrsim \phi^{-1}(1/t). \tag{3.4.65}$$

Hence, $tw\phi'(w) \approx 1$ and

$$\exp\left\{-t(\phi(w) - w\phi'(w))\right\} \approx 1. \tag{3.4.66}$$

Next, by Propositions 3.2.4 and 3.2.1,

$$w^2\left(-\phi''(w)\right) \approx w\phi'(w),$$

thus, by (3.4.64) and (3.4.65), we obtain

$$\frac{1}{\sqrt{t\left(-\phi''(w)\right)}} \approx \frac{w}{\sqrt{tw\phi'(w)}} \approx \phi^{-1}(1/t),$$

which, together with (3.4.66), implies that

$$\left(t(-\phi''(w))\right)^{-\frac{1}{2}} \exp\left\{-t(\phi(w) - w\phi'(w))\right\} \approx \phi^{-1}(1/t),$$

for $t \in (0, c_1/\Phi(x_0))$ and $\chi_1 \leq x\phi^{-1}(1/t) \leq \chi_2$. In view of (3.4.60), the theorem follows in the case $x_0 = 0$. Now, it remains to observe that in the case $x_0 > 0$ we may use positivity and continuity to conclude the claim for all $t \in (0, 1/\Phi(x_0))$. □

We end this section with an example of a direct application of Theorem 3.4.13.

Example 3.4.15. For any $\alpha \in (0,1)$ let us consider the relativistic α-stable subordinator $\mathbf{T}^m$ with mass m. Its Laplace exponent is of the form

$$\phi_m(\lambda) = \left(\lambda + m^{1/\alpha}\right)^\alpha - m, \quad \lambda \geq 0,$$

and its Lévy measure ν_m has a density $\nu_m(x)$ given by the formula

$$\nu_m(x) = \frac{\alpha}{\Gamma(1-\alpha)} e^{-m^{1/\alpha}x} x^{-1-\alpha}, \quad x > 0.$$

Note that if no mass is enforced, then $\mathbf{T}^0$ reduces to the α-stable subordinator.

By elementary calculations one may easily deduce that there is a constant $c_1 > 0$ independent of m such that

$$\phi_m(\lambda) \approx \lambda^\alpha, \quad \lambda \geq c_1 m^{1/\alpha},$$

with the implied constant independent of m. It follows at once that $\phi_m \in \mathrm{WLSC}(\alpha, c, c_1 m^{1/\alpha}) \cap \mathrm{WUSC}(\alpha, C, c_1 m^{1/\alpha})$ for some constants $c, C > 0$ independent of m. Thus, we are in position to apply Theorems 3.1.1 and 3.4.13. Again, straightforward computations yield

$$-\phi_m''(\lambda) = \alpha(1-\alpha)\big(\lambda + m^{1/\alpha}\big)^{\alpha-2}, \quad \lambda > 0,$$

and

$$(\phi_m')^{-1}(y) = \alpha^{1/(1-\alpha)} y^{-1/(1-\alpha)} - m^{1/\alpha}, \quad \lambda > 0.$$

Let us denote the density of T_t^m by $p_m(t, \cdot)$. Then it follows from Theorem 3.1.1 that the asymptotics of $p_m(t,x)$ is of the form

$$E_\alpha^m(x,t) = \frac{1}{\sqrt{2\pi\alpha(1-\alpha)}} \left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(1-\alpha)}} t^{\frac{1}{2(1-\alpha)}} \exp\left\{-(1-\alpha)\left(\frac{x}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}} t^{\frac{1}{1-\alpha}} + mt - xm^{1/\alpha}\right\}.$$

Next, observe that we in fact have $x_1 = 1/x_0$ in Theorem 3.4.13. Another easy calculation show that $\phi_m^{-1}(y) = (y+m)^{1/\alpha} - m^{1/\alpha}$ for $y \geq 0$. With that in mind let us denote

$$D_1 = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \colon 0 < x < c_1^{-1} m^{-1/\alpha} \text{ and } 0 \leq x\left(\left(t^{-1} + m\right)^{1/\alpha} - m^{1/\alpha}\right) \leq 1\right\}$$

and

$$D_2 = \left\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \colon 0 < x < c_1^{-1} m^{-1/\alpha} \text{ and } x\left(\left(t^{-1} + m\right)^{1/\alpha} - m^{1/\alpha}\right) \geq 1\right\}.$$

Then by Theorem 3.4.13,

$$p_m(t,x) \approx \begin{cases} E_\alpha^m(x,t) & \text{on } D_1, \\ tx^{-1}\left(\left(x^{-1} + m^{\frac{1}{\alpha}}\right)^{-\alpha} - m\right) & \text{on } D_2. \end{cases}$$

It goes without saying that for the case $m = 0$ the first condition in the definitions of D_1 and D_2 vanishes and consequently, we obtain global sharp two-sided estimate of the transition density of the α-stable subordinator. Its special case for $t = 1$ is displayed as a preliminary result in the aforementioned article of Hawkes [46], see also Zolatarev [107, Theorem 2.5.2].

3.5 Applications

In the last section of this chapter we present two possible applications of our results. This is just a sample which we present to advocate the relevance of our findings, but we would like to stress here that this short list certainly does not exhaust the topic. One may, for instance, consult Meerschaert and Scheffler [71], Chen [19] or Chen, Kim, Kumagai and Wang [22] to observe that the considered solutions of the generalised fractional-time heat equation are expressed by means of the corresponding (inverse) subordinator and conclude that our results may also be of use there.

Let us start with the concept of subordination in more general setting that $\mathbb{R}^d$.

3.5.1 Subordination

Let $(\mathscr{X},\tau)$ be a locally compact separable metric space with a Radon measure μ having full support on $\mathscr{X}$. Assume that $(X_t\colon t\geq 0)$ is a homogeneous in time Markov process on $\mathscr{X}$ with density $h(t,\,\cdot\,,\,\cdot\,)$, that is

$$\mathbb{P}\Big(X_t\in B\big|X_0=x\Big)=\int_B h(t,x,y)\,\mu(dy),$$

for any Borel set $B\subset\mathscr{X}$, $x\in\mathscr{X}$ and $t>0$. Assume that for all $t>0$ and $x,y\in\mathscr{X}$,

$$t^{-\frac{n}{\gamma}}\Psi_1\Big(\tau(x,y)t^{-\frac{1}{\gamma}}\Big)\leq h(t,x,y)\leq t^{-\frac{n}{\gamma}}\Psi_2\Big(\tau(x,y)t^{-\frac{1}{\gamma}}\Big) \tag{3.5.1}$$

where n and γ are some positive constants, Ψ_1 and Ψ_2 are non-negative, non-increasing functions on $[0,\infty)$ such that $\Psi_1(1)>0$ and

$$\sup_{s\geq 0}\Psi_2(s)(1+s)^{n+\gamma}<\infty. \tag{3.5.2}$$

By $H(t,x,y)$ we denote the heat kernel for the subordinate process $\big(X_{T_t}\colon t\geq 0\big)$, that is

$$H(t,x,y)=\int_0^\infty h(s,x,y)\,G(t,ds),$$

where

$$G(t,s)=\mathbb{P}(T_t\geq s).$$

Suppose that $\phi\in\mathrm{WLSC}(\alpha,c,x_0)\cap\mathrm{WUSC}(\beta,C,x_0)$ for some $c\in(0,1]$, $C\geq 1$, $x_0>0$ and $0<\alpha\leq\beta<1$. We also assume that

$$\lim_{x\to\infty}\phi'(x)=b=0,$$

and that the Lévy measure ν has an almost monotone density $\nu(x)$.

Claim 3.5.1. *For all $x,y\in\mathscr{X}$ satisfying $\tau(x,y)^{-\gamma}>x_0$, and any $t\in(0,1/\Phi(x_0))$,*

$$H(t,x,y)\approx\begin{cases} t\phi(\tau(x,y)^{-\gamma})\tau(x,y)^{-n} & \text{if } 0<t\phi(\tau(x,y)^{-\gamma})\leq 1,\\ (\phi^{-1}(1/t))^{\frac{n}{\gamma}} & \text{if } 1\leq t\phi(\tau(x,y)^{-\gamma}).\end{cases}$$

Indeed, by Proposition 3.2.3, $\phi'\in\mathrm{WLSC}(\alpha-1,c,x_0)\cap\mathrm{WUSC}(\beta-1,C,x_0)$. Let $0<r<\phi'(x_0^+)$. If $0<\lambda\leq C$, then, by setting

$$D=C^{\frac{1}{1-\beta}}\lambda^{-\frac{1}{1-\beta}},$$

we see that the weak upper scaling property of ϕ' implies that

$$\lambda r=\lambda\phi'\big((\phi')^{-1}(r)\big)\geq\phi'\big(D(\phi')^{-1}(r)\big).$$

Therefore,

$$(\phi')^{-1}(\lambda r)\leq C^{\frac{1}{1-\beta}}\lambda^{-\frac{1}{1-\beta}}(\phi')^{-1}(r). \tag{3.5.3}$$

Analogously, we can prove the lower estimate: if $0<\lambda\leq c$, then, by setting

$$D=c^{\frac{1}{1-\alpha}}\lambda^{-\frac{1}{1-\alpha}},$$

we obtain

$$\lambda r = \lambda \phi'\big((\phi')^{-1}(r)\big) \leq \phi'\big(D(\phi')^{-1}(r)\big),$$

and consequently,

$$(\phi')^{-1}(\lambda r) \geq c^{\frac{1}{1-\alpha}} \lambda^{-\frac{1}{1-\alpha}} (\phi')^{-1}(r). \tag{3.5.4}$$

Since $(\phi')^{-1}$ is non-increasing, the last inequality is valid for all $0 < \lambda \leq 1$. Let

$$\begin{aligned} H(t,x,y) &= \left(\int_0^{\frac{1}{\phi^{-1}(1/t)}} + \int_{\frac{1}{\phi^{-1}(1/t)}}^{\infty} \right) h(s,x,y)\, G(t,ds) \\ &= I_1(t,x,y) + I_2(t,x,y). \end{aligned}$$

By Theorem 3.4.13,

$$I_1 \approx \frac{1}{\phi^{-1}(1/t)} \int_0^1 h\Big(\frac{u}{\phi^{-1}(1/t)}, x, y\Big) \frac{1}{\sqrt{t(-\phi''(w))}} \exp\Big(-t\big(\phi(w) - w\phi'(w)\big)\Big)\, du \tag{3.5.5}$$

where

$$w = (\phi')^{-1}\Big(\frac{u}{t\phi^{-1}(1/t)}\Big).$$

Recall that, by Proposition 3.2.3, for all $r > x_0$, we have

$$r\phi'(r) \leq \phi(r) \leq C_1 r \phi'(r). \tag{3.5.6}$$

We can assume that

$$t\phi\Big(2(CC_1)^{\frac{1}{1-\beta}} x_0\Big) < 1.$$

By (3.5.6) and the weak upper scaling of ϕ', we get

$$\phi'\Big(\phi^{-1}(1/t)\Big) \leq \frac{1}{t\phi^{-1}(1/t)} \leq C_1 \phi'(\phi^{-1}(1/t)) \leq \phi'\Big((CC_1)^{-\frac{1}{1-\beta}} \phi^{-1}(1/t)\Big),$$

thus,

$$(\phi')^{-1}\Big(\frac{1}{t\phi^{-1}(1/t)}\Big) \approx \phi^{-1}(1/t).$$

Hence, by (3.5.3) and (3.5.4), we obtain

$$u^{-\frac{1}{1-\alpha}} \phi^{-1}(1/t) \lesssim w \lesssim u^{-\frac{1}{1-\beta}} \phi^{-1}(1/t), \quad u \in (0,1]. \tag{3.5.7}$$

Moreover, since $w > x_0$, by (3.5.6) and Proposition 3.4.2,

$$\begin{aligned} w\phi'(w) \gtrsim \phi(w) - w\phi'(w) &= \int_0^w \Phi(u) \frac{du}{u} \\ &\geq \int_{x_0}^w \Phi(u) \frac{du}{u} \\ &\gtrsim w\phi'(w). \end{aligned}$$

Thus, (3.5.7) entails that

$$u^{-\frac{\alpha}{1-\alpha}} \lesssim t\big(\phi(w) - w\phi'(w)\big) \lesssim u^{-\frac{\beta}{1-\beta}}, \quad u \in (0,1]. \tag{3.5.8}$$

Next, by Proposition 3.4.2 and (3.5.6), we get

$$\frac{1}{\sqrt{t(-\phi''(w))}} \approx \frac{w}{\sqrt{t\phi(w)}} \approx \sqrt{u^{-1}\phi^{-1}(1/t)w}.$$

Therefore, by (3.5.7),

$$u^{-\frac{2-\alpha}{2(1-\alpha)}}\phi^{-1}(1/t) \lesssim \frac{1}{\sqrt{t(-\phi''(w))}} \lesssim u^{-\frac{2-\beta}{2(1-\beta)}}\phi^{-1}(1/t), \quad u \in (0,1]. \tag{3.5.9}$$

Now, by (3.5.5) and (3.5.1) together with (3.5.8) and (3.5.9), we can estimate

$$I_1 \lesssim (\phi^{-1}(1/t))^{\frac{n}{\gamma}} \int_0^1 \Psi_2\Big(u^{-\frac{1}{\gamma}}A^{\frac{1}{\gamma}}\Big)u^{-\frac{n}{\gamma}-\frac{2-\beta}{2(1-\beta)}} \exp\Big(-C''u^{-\frac{\alpha}{1-\alpha}}\Big)\,du \tag{3.5.10}$$

and

$$I_1 \gtrsim (\phi^{-1}(1/t))^{\frac{n}{\gamma}} \int_0^1 \Psi_1\Big(u^{-\frac{1}{\gamma}}A^{\frac{1}{\gamma}}\Big)u^{-\frac{n}{\gamma}-\frac{2-\alpha}{2(1-\alpha)}} \exp\Big(-C'u^{-\frac{\beta}{1-\beta}}\Big)\,du \tag{3.5.11}$$

where we have set

$$A = \tau(x,y)^{\gamma}\phi^{-1}(1/t).$$

Suppose that $A \le 1$. Since Ψ_1 and Ψ_2 are non-increasing, by (3.5.10) and (3.5.11), we easily see that

$$I_1 \approx (\phi^{-1}(1/t))^{\frac{n}{\gamma}}.$$

We also have

$$I_2 \lesssim \int_{\frac{1}{\phi^{-1}(1/t)}}^{\infty} s^{-\frac{n}{\gamma}}p(t,s)\,ds \lesssim (\phi^{-1}(1/t))^{\frac{n}{\gamma}}.$$

Therefore,

$$H(t,x,y) \approx (\phi^{-1}(1/t))^{\frac{n}{\gamma}}.$$

We now turn to the case $A > 1$. By (3.5.2) and (3.5.10),

$$I_1 \lesssim (\phi^{-1}(1/t))^{\frac{n}{\gamma}}A^{-\frac{n}{\gamma}-1}\int_0^1 u^{-\frac{\beta}{2(1-\beta)}} \exp\Big(-C''u^{-\frac{\alpha}{1-\alpha}}\Big)\,du \lesssim A^{-1}\tau(x,y)^{-n}. \tag{3.5.12}$$

Moreover, by (3.2.3) we have

$$t\phi(\tau(x,y)^{-\gamma}) = t\phi(A^{-1}\phi^{-1}(1/t)) \ge A^{-1},$$

hence, by (3.5.12),

$$I_1 \lesssim t\phi(\tau(x,y)^{-\gamma})\tau(x,y)^{-n}.$$

It remains to estimate I_2. Let us observe that for all $r > x_0$, if $u \ge 1$, then, by the weak upper scaling of ϕ, we have

$$\phi(r) \le \phi(ru) \le Cu^{\beta}\phi(r).$$

On the other hand, if $0 < u \le 1$, then, by (3.2.3) and the monotonicity of ϕ, we get

$$u\phi(r) \le \phi(ru) \le \phi(r).$$

Therefore, for all $u > 0$ and $r > x_0$,

$$\min\{1, u\}\phi(r) \le \phi(ru) \le C \max\{1, u^\beta\}\phi(r). \tag{3.5.13}$$

Since $\tau(x, y)^{-\gamma} > x_0$, by Theorem 3.4.13, (3.5.1) and estimates (3.5.13), we get

$$I_2 \lesssim t\phi(\tau(x, y)^{-\gamma})\tau(x, y)^{-n} \int_{1/A}^{\infty} \Psi_2\Big(u^{-\frac{1}{\gamma}}\Big) u^{-\frac{n}{\gamma}-1} \max\Big\{1, u^{-\beta}\Big\}\, du$$

and

$$I_2 \gtrsim t\phi(\tau(x, y)^{-\gamma})\tau(x, y)^{-n} \int_{1/A}^{\infty} \Psi_1\Big(u^{-\frac{1}{\gamma}}\Big) u^{-\frac{n}{\gamma}-1} \min\{1, u\}\, du.$$

Finally, by (3.5.2), we have

$$\int_0^1 \Psi_2\Big(u^{-\frac{1}{\gamma}}\Big) u^{-\frac{n}{\gamma}-\beta-1}\, du \lesssim \int_0^1 u^{-\beta}\, du < \infty,$$

thus,

$$I_2 \approx t\phi(\tau(x, y)^{-\gamma})\tau(x, y)^{-n},$$

proving the claim.

Example 3.5.2. Let $(\mathscr{X}, \tau)$ be a nested fractal with the geodesic metric on $\mathscr{X}$. Let d_w and d_f be the walk dimension and the Hausdorff dimension of $\mathscr{X}$, respectively. Let $(X_t : t \ge 0)$ be the diffusion on $\mathscr{X}$ constructed in [2, Section 7]. By [2, Theorem 8.18], the corresponding heat kernel satisfies (3.5.1) with $n = d_f$, $\gamma = d_w$, and

$$\Psi_1(s) = \Psi_2(s) = \exp\Big(-s^{\frac{\gamma}{\gamma-1}}\Big).$$

Let $\mathbf{T}$ be a subordinator with the Laplace exponent

$$\phi(s) = s^\alpha \log^\sigma(2 + s),$$

where $\alpha \in (0, 1)$ and $\sigma \in \mathbb{R}$. Then, by Claim 3.5.1, the process $(X_{T_t} : t \ge 0)$ has density $H(t, x, y)$ such that for all $x, y \in \mathscr{X}$ and $t > 0$,

- if $t > \tau(x, y)^{\alpha\gamma} \log^{-\sigma}\big(2 + \tau(x, y)^{-\gamma}\big)$, then
$$H(t, x, y) \approx t^{-\frac{n}{\alpha\gamma}} \log^{-\frac{\sigma n}{\alpha\gamma}}\big(2 + t^{-1}\big),$$
- if $t < \tau(x, y)^{\alpha\gamma} \log^{-\sigma}\big(2 + \tau(x, y)^{-\gamma}\big)$, then
$$H(t, x, y) \approx t\tau(x, y)^{-\alpha\gamma-n} \log^{\sigma}\big(2 + \tau(x, y)^{-\gamma}\big).$$

Example 3.5.3. Let $(\mathscr{X}, \tau)$ be a complete manifold without boundary, having non-negative Ricci curvature. Then, by [70], the heat kernel corresponding to the Laplace–Beltrami operator on $\mathscr{X}$ satisfies estimates (3.5.1) with

$$\Psi_1(s) = e^{-C_1 s^2}, \qquad \Psi_2(s) = e^{-C_2 s^2}.$$

Now, one can take $\mathbf{T}$ with a Lévy-Khintchine exponent regularly varying at infinity with index $\alpha \in (0, 1)$ and apply Claim 3.5.1 to obtain the asymptotic behaviour of subordinate process.

3.5.2 Green function estimates

Let $\mathbf{T} = (T_t : t \geq 0)$ be a subordinator with the Laplace exponent ϕ. If $-\phi''$ has the weak lower scaling property of index $\alpha - 2$ for some $\alpha > 0$, then, by virtue of Proposition A.1.8 and the Hartman-Wintner condition (HW), the probability distribution of T_t has a density $p(t, \cdot)$. Our goal is to derive sharp estimates of the Green function U based on results from Sections 3.3 and 3.4. Let us recall that the Green function is defined by

$$U(x) = \int_0^\infty p(t,x)\,dt, \qquad x > 0.$$

We define an auxiliary function

$$f(x) = \frac{\Phi(x)}{\phi'(x)}, \qquad x > 0.$$

Notice that, by (3.2.2) and Proposition 3.2.3, for all $x > x_0$,

$$f^*(x) \lesssim x. \tag{3.5.14}$$

In view of (A.1.6) and Proposition A.1.5, the function Φ is almost increasing, thus by monotonicity of ϕ', f is almost increasing as well. Therefore, there is $c_0 \in (0, 1]$ such that for all $x > x_0$,

$$c_0 f^*(x) \leq f(x) \leq f^*(x). \tag{3.5.15}$$

Moreover, f has the doubling property on (x_0, ∞). Since Φ belongs to $\mathrm{WLSC}(\alpha, c, x_0)$, by monotonicity of ϕ' we conclude that f belongs to $\mathrm{WLSC}(\alpha, c, x_0)$ and consequently, $f^* \in \mathrm{WLSC}(\alpha, c, x_0)$. We also have that $f^{-1} \in \mathrm{WUSC}(1/\alpha, C, f^*(x_0))$ for some $C \geq 1$. The proof of this claim is not difficult, especially at this stage, but we provide it nonetheless for the convenience of the reader. For any $x > 0$, let u be such that $f^{-1}(x) = u$ and set $w = (\lambda/c)^{1/\alpha} \geq 1$. Now, by scaling property of f^* and the same reasoning as in Remark A.1.7,

$$f^{-1}(\lambda x) = f^{-1}(\lambda f^*(u)) \leq f^{-1}(f^*(wu)) \lesssim wu \lesssim \lambda^{1/\alpha} f^{-1}(x),$$

if only $x > f^*(x_0)$ and the claim follows. Moreover, since f^{-1} is increasing, we infer that f^{-1} also has a doubling property on $(f^*(x_0), \infty)$.

Proposition 3.5.4. *Suppose that $b = 0$ and $-\phi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\alpha > 0$. Then for each $A > 0$ and $M > 0$ there is $C \geq 1$ so that for all $x < A/x_0$,*

$$C^{-1} \frac{1}{x\phi(1/x)} \leq \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^\infty p(t,x)\,dt \leq C \frac{1}{x\phi(1/x)}.$$

In particular, for each $A > 0$ there is $C > 0$ such that for all $x < A/x_0$,

$$U(x) \geq C \frac{1}{x\phi(1/x)}.$$

Proof. For $M > 0$ and $x > 0$ we set

$$I_M(x) = \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^\infty p(t,x)\,dt.$$

Let us first show that for each $M > 0$ there are $A_M > 0$ and $C \geq 1$ such that for all $x < A_M/x_0$,

$$C^{-1} \frac{1}{x\phi(1/x)} \leq \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\infty} p(t,x)\, dt \leq C \frac{1}{x\phi(1/x)}. \tag{3.5.16}$$

Let

$$A_M = \min\left\{M, c_0^{-1} M_0\right\} \cdot \min\left\{1, \frac{x_0}{f^*(x_0)}\right\}$$

where M_0 is determined in Corollary 3.3.4 and c_0 is taken from (3.5.15). We claim that the following holds true.

Claim 3.5.5. *For each $M > 0$ there is $C \geq 1$ so that for all $x < A_M/x_0$,*

$$C^{-1} \frac{1}{\phi'(f^{-1}(1/x))} \leq I_M(x) \leq C \frac{1}{\phi'(f^{-1}(1/x))}. \tag{3.5.17}$$

Indeed, suppose that

$$t > \frac{x}{\phi'(f^{-1}(M_1/x))} \tag{3.5.18}$$

with $M_1 = c_0^{-1} M_0$. Notice that for $x < A_M/x_0$, we have $x < M_1/f^*(x_0)$. Hence, $x_0 \leq f^{-1}(M_1/x)$ and by monotonicity of ϕ', we obtain

$$\frac{x}{t} \leq \phi'(f^{-1}(M_1/x)) \leq \phi'(x_0). \tag{3.5.19}$$

Moreover, for $w = (\phi')^{-1}(x/t)$, the condition (3.5.18) implies that

$$f^*(w) \geq M_1/x,$$

which together with (3.5.15) give

$$t\Phi(w) = xf(w) \geq c_0 x f^*(w) \geq M_0. \tag{3.5.20}$$

Now, to justify the claim, let us first consider $M \geq M_1$. In view of (3.5.19) and (3.5.20) we can apply Corollary 3.3.4 to get

$$I_M(x) \approx \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\infty} \frac{1}{\sqrt{t(-\phi''(w))}} \exp\Big\{ - t(\phi(w) - w\phi'(w))\Big\}\, dt.$$

Since, by Proposition A.1.3 and Corollary A.1.2, for all $w > x_0$,

$$\phi(w) - w\phi'(w) \approx h(1/w) \approx K(1/w) \approx w^2\big(- \phi''(w)\big),$$

after the change of variables $t = x/\phi'(s)$ we can find $C_2 \geq 1$ such that for all $x < A_M/x_0$,

$$\begin{aligned} \int_{f^{-1}(M/x)}^{\infty} \exp\{-C_2 x f(s)\} \sqrt{xf(s)} \frac{ds}{s\phi'(s)} &\lesssim I_M(x) \\ &\lesssim \int_{f^{-1}(M/x)}^{\infty} \exp\{-C_2^{-1} x f(s)\} \sqrt{xf(s)} \frac{ds}{s\phi'(s)}. \end{aligned} \tag{3.5.21}$$

Recall that f^{-1} has the doubling property on $(f^*(x_0), \infty)$. Using now Proposition 3.2.3 and (3.5.15), we get

$$\begin{aligned} I_M(x) &\gtrsim \int_{f^{-1}(M/x)}^{2f^{-1}(M/x)} \exp\{-C_2 x f^*(s)\} \sqrt{x f^*(s)} \frac{ds}{\phi(s)} \\ &\gtrsim \frac{1}{\phi(f^{-1}(M/x))} f^{-1}(M/x) \\ &\gtrsim \frac{1}{\phi'(f^{-1}(M/x))} \end{aligned} \tag{3.5.22}$$

where the implicit constants may depend on M. Therefore, by monotonicity of f^{-1} and ϕ', the estimate (3.5.22) gives

$$I_M(x) \gtrsim \frac{1}{\phi'(f^{-1}(1/x))}. \tag{3.5.23}$$

This proves the first inequality in (3.5.17).

We next observe that (3.5.14) entails that $f^{-1}(s) \gtrsim s$ for $s > f^*(x_0)$, thus, by (3.5.23),

$$U(x) \geq I_{M_1}(x) \gtrsim \frac{1}{\phi'(1/x)} \gtrsim \frac{1}{x\phi(1/x)} \tag{3.5.24}$$

where the last estimate follows by Proposition 3.2.3.

We next show the second inequality in (3.5.17). By (3.5.21), Proposition 3.2.3 and monotonicity of ϕ,

$$\begin{aligned} I_M(x) &\lesssim \int_{f^{-1}(M/x)}^{\infty} \exp\big\{-C_2^{-1} x f(s)\big\} \sqrt{x f(s)} \frac{ds}{\phi(s)} \\ &\leq \frac{1}{\phi(f^{-1}(M/x))} \int_{f^{-1}(M/x)}^{\infty} \exp\big\{-C_2^{-1} x f(s)\big\} \sqrt{x f(s)}\, ds \\ &\leq \frac{1}{\phi(f^{-1}(M/x))} \int_{f^{-1}(M/x)}^{\infty} \exp\Big\{-\tfrac{1}{2C_2} x f(s)\Big\}\, ds \end{aligned}$$

where in the last inequality we have used

$$\exp\big\{-C_2^{-1} x f(s)\big\} \sqrt{x f(s)} \leq \exp\Big\{-\tfrac{1}{2C_2} x f(s)\Big\}.$$

Since $\Phi \in \mathrm{WLSC}(\alpha, c, x_0)$, by [10, Lemma 16],

$$\int_{\mathbb{R}} \exp\big\{-C_2^{-1} x f(|s|)\big\}\, ds \lesssim f^{-1}(M_1/x).$$

Finally, the doubling property of f^{-1}, monotonicity of ϕ, and Proposition 3.2.3 give

$$I_M(x) \lesssim \frac{1}{\phi(f^{-1}(M/x))} f^{-1}(M_1/x) \lesssim \frac{1}{\phi'(f^{-1}(1/x))}$$

where the implied constant may depend on M. This finishes the proof of (3.5.17) for $M \geq M_1$.

We next consider $0 < M < M_1$. By monotonicity, the lower estimate remains valid for all $M > 0$. Therefore, it is enough to show that for each $0 < M < M_1$, there is $C \geq 1$ such that for all $x < A_M/x_0$,

$$\int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\frac{x}{\phi'(f^{-1}(M_1/x))}} p(t,x)\, dt \leq C \frac{1}{\phi'(f^{-1}(1/x))}.$$

By [43, Theorem 3.1], there is $t_0 > 0$ such that for all $0 < t < t_0$,

$$p(t,x) \lesssim \Phi^{-1}(1/t).$$

If $x_0 = 0$, then $t_0 = \infty$. Since Φ is almost increasing, we have

$$\frac{x}{\phi'(f^{-1}(M_1/x))} \leq \frac{M_1}{\Phi(f^{-1}(M_1/x))} \lesssim \frac{M_1}{\Phi(f^{-1}(M_1 x_0/A))}.$$

Hence, by continuity and positivity of $p(t,x)$ and $\Phi^{-1}(1/t)$, we can take

$$t_0 \geq \frac{x}{\phi'(f^{-1}(M_1/x))}.$$

Therefore, by the change of variables $t = x/\phi'(s)$, we get

$$\begin{aligned}\int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\frac{x}{\phi'(f^{-1}(M_1/x))}} p(t,x)\,dt &\lesssim \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\frac{x}{\phi'(f^{-1}(M_1/x))}} \Phi^{-1}(1/t)\,dt \\ &= x\int_{f^{-1}(M/x)}^{f^{-1}(M_1/x)} \Phi^{-1}\Big(\frac{\phi'(s)}{x}\Big) f(s)\frac{ds}{s^2\phi'(s)}.\end{aligned}$$

Next, by monotonicity and the doubling property of f^{-1} and ϕ', we obtain

$$\begin{aligned}\int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\frac{x}{\phi'(f^{-1}(M_1/x))}} p(t,x)\,dt &\lesssim \frac{1}{\big(f^{-1}(M/x)\big)^2}\cdot\frac{1}{\phi'(f^{-1}(M_1/x))}\int_{f^{-1}(M/x)}^{f^{-1}(M_1/x)} \Phi^{-1}\Big(\frac{\phi'(s)}{x}\Big)\,ds \\ &\lesssim \frac{1}{\big(f^{-1}(1/x)\big)^2}\cdot\frac{1}{\phi'(f^{-1}(1/x))}\int_{f^{-1}(M/x)}^{f^{-1}(M_1/x)} \Phi^{-1}\Big(\frac{\phi'(s)}{x}\Big)\,ds.\end{aligned} \tag{3.5.25}$$

Since, by (3.5.15), for $s \geq f^{-1}(M/x)$, we have

$$\frac{\phi'(s)}{x} = \frac{\Phi(s)}{xf(s)} \lesssim \Phi^*(s),$$

by monotonicity of Φ^{-1}, Proposition A.1.5, Remark A.1.7, and the doubling property of f^{-1} and Φ^{-1}, we get

$$\int_{f^{-1}(M/x)}^{f^{-1}(M_1/x)} \Phi^{-1}\Big(\frac{\phi'(s)}{x}\Big)\,ds \lesssim \big(f^{-1}(1/x)\big)^2,$$

which together with (3.5.25) give (3.5.17) for $0 < M < M_1$. This completes the proof of Claim 3.5.5.

Our next task is to deduce (3.5.16) from Claim 3.5.5. By Lemma 3.2.9 and Proposition 3.2.3, there is a complete Bernstein function $\widetilde{\phi}$ such that $\widetilde{\phi} \approx \phi$ and

$$f(x) \approx \widetilde{f}(x) = \frac{x^2\big(-\widetilde{\phi}''(x)\big)}{\widetilde{\phi}'(x)},$$

for all $x > x_0$. Let $\widetilde{\mathbf{T}}$ be a subordinator with the Laplace exponent $\widetilde{\phi}$. By $\widetilde{p}(t,\,\cdot\,)$ we denote the density of the probability distribution of $\widetilde{T}_t$. We set

$$\widetilde{U}(x) = \int_0^\infty \widetilde{p}(t,x)\,dt$$

and

$$\widetilde{I}_M(x) = \int_{\frac{x}{\widetilde{\phi}'(\widetilde{f}^{-1}(M/x))}}^{\infty} \widetilde{p}(t,x)\,dt.$$

Fix $M > 0$. By Claim 3.5.5, there is $A_M > 0$ such that for all $x < A_M/x_0$,

$$\widetilde{I}_M(x) \approx \frac{1}{\widetilde{\phi}'(\widetilde{f}^{-1}(1/x))} \approx \frac{1}{\phi'(f^{-1}(1/x))}. \tag{3.5.26}$$

On the other hand, since $\widetilde{\phi}$ is the complete Bernstein function, by (3.5.24) and [61, Corollary 2.6], there is $C_3 \geq 1$ such that

$$C_3^{-1}\frac{1}{x\widetilde{\phi}(1/x)} \leq \widetilde{I}_M(x) \leq \widetilde{U}(x) \leq C_3\frac{1}{x\widetilde{\phi}(1/x)}.$$

Therefore, by (3.5.26), for $x < A_M/x_0$,

$$\frac{1}{\phi'(f^{-1}(1/x))} \approx \widetilde{I}_M(x) \approx \frac{1}{x\widetilde{\phi}(1/x)} \approx \frac{1}{x\phi(1/x)}, \tag{3.5.27}$$

and (3.5.16) follows for all $A \leq A_M$. Let us now consider $A > A_M$. Observe that the functions

$$\left[\frac{A_M}{x_0}, \frac{A}{x_0}\right] \ni x \mapsto \frac{1}{x\phi(1/x)}$$

and

$$\left[\frac{A_M}{x_0}, \frac{A}{x_0}\right] \ni x \mapsto \int_{\frac{x}{\phi'(f^{-1}(M/x))}}^{\infty} p(t,x)\,dt$$

are both positive and continuous, thus they are bounded for each A. Therefore, at the possible expense of the constant, we can conclude the proof of the proposition. ☐

Proposition 3.5.6. *Suppose that $b = 0$, $-\phi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$, and that the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure with an almost decreasing density $\nu(x)$. Then there is $\varepsilon \in (0,1)$ such that for each $A > 0$, there is $C \geq 1$ such that for all $x < A/x_0$,*

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \leq C\frac{1}{x\phi(1/x)}.$$

Proof. In view of (3.5.27), it is enough to show that for some $\varepsilon \in (0,1)$ and all $A > 0$ there is $C \geq 1$, such that for all $x < A/x_0$,

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \leq C\frac{1}{\phi'(f^{-1}(1/x))}. \tag{3.5.28}$$

Let $\varepsilon \in (0,1)$ and

$$A = \min\left\{1, \frac{x_0}{f^*(x_0)}\right\}.$$

Suppose that

$$t \leq \frac{x}{\phi'(f^{-1}(1/x))}\varepsilon,$$

that is

$$t \le \frac{1}{\Phi^*(f^{-1}(1/x))}\varepsilon.$$

Hence, by monotonicity of Φ^{-1} and ϕ',

$$x \ge \frac{1}{f^*(\Phi^{-1}(\varepsilon/t))} = \frac{t}{\varepsilon}\phi'(\Phi^{-1}(\varepsilon/t)) \ge \frac{t}{\varepsilon}\phi'(\Phi^{-1}(1/t)).$$

By Proposition A.1.5 and the scaling property of ϕ', there are $c \in (0,1]$ and $C \ge 1$ such that

$$x \ge \frac{t}{\varepsilon}\phi'(C\psi^{-1}(1/t)) \ge \frac{t}{\varepsilon}cC^{\alpha-1}\phi'(\psi^{-1}(1/t)).$$

Therefore, by taking $\varepsilon = (2e)^{-1}cC^{\alpha-1}$, we get

$$x \ge 2et\phi'(\psi^{-1}(1/t)).$$

Since $\nu(x)$ is a almost decreasing density of $\nu(dx)$, by Theorem 3.4.4, we get

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \lesssim \frac{\Phi(1/x)}{x}\left(\frac{x}{\phi'(f^{-1}(1/x))}\right)^2.$$

By (3.5.14), $f^{-1}(s) \gtrsim s$ for $s > f^*(x_0)$, thus using (A.1.2),

$$\frac{x\Phi(1/x)}{\phi'(f^{-1}(1/x))} \le \frac{\Phi(1/x)}{\Phi(f^{-1}(1/x))} \lesssim 1,$$

which entails (3.5.28). The extension to arbitrary A follows by continuity and positivity argument as in the proof of Proposition 3.5.4. □

It is possible to get the same conclusion as in Proposition 3.5.6 without imposing the existence of the almost decreasing density of $\nu(dx)$; however, instead we need to assume the weak upper scaling of $-\phi''$.

Proposition 3.5.7. *Suppose that $b = 0$ and $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0) \cap \mathrm{WUSC}(\beta-2, C, x_0)$ for some $c \in (0,1]$, $C \ge 1$, $x_0 \ge 0$ and $\frac{1}{2} < \alpha \le \beta < 1$. Then there is $\varepsilon \in (0,1)$ such that for each $A > 0$, there is $C_1 \ge 1$, so that for all $x < A/x_0$,*

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \le C_1\frac{1}{x\phi(1/x)}. \tag{3.5.29}$$

Proof. Let

$$A = \min\left\{1, \frac{x_0}{f^*(x_0)}\right\}.$$

By repeating the same reasoning as in the proof of Proposition 3.5.6, we can see that the condition

$$t \le \frac{x}{\phi'(f^{-1}(1/x))}\varepsilon$$

implies

$$x \ge 2et\phi'(\psi^{-1}(1/t))$$

for $\varepsilon = (2e)^{-1}cC^{\alpha-1}$. Therefore, we can apply Theorem 3.4.3 to get

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \lesssim \Phi(1/x) \int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} t\Phi^{-1}(1/t)\,dt \tag{3.5.30}$$

where the implied constant may depend on ε. Since $\alpha > \frac{1}{2}$, by Proposition A.1.5, [1, Theorem 3], and the doubling property of Φ^{-1}, we obtain

$$\begin{aligned} \int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} t\Phi^{-1}(1/t)\,dt &\lesssim \left(\frac{x}{\phi'(f^{-1}(1/x))}\right)^2 \Phi^{-1}\left(\frac{\phi'(f^{-1}(1/x))}{\varepsilon x}\right) \\ &\lesssim \left(\frac{x}{\phi'(f^{-1}(1/x))}\right)^2 \Phi^{-1}\left(\frac{\phi'(f^{-1}(1/x))}{x}\right). \end{aligned} \tag{3.5.31}$$

In view of (3.5.15), we have

$$\frac{\phi'(f^{-1}(1/x))}{x} = \frac{\Phi^*(f^{-1}(1/x))}{xf^*(f^{-1}(1/x))} \lesssim \Phi^*(f^{-1}(1/x)),$$

thus, by Proposition A.1.5 and Remark A.1.7,

$$\frac{\Phi(1/x)x^2}{\phi'(f^{-1}(1/x))}\Phi^{-1}\left(\frac{\phi'(f^{-1}(1/x))}{x}\right) \lesssim \frac{\Phi^*(1/x)x^2}{\phi'(f^{-1}(1/x))} f^{-1}(1/x) = xf^{-1}(1/x)\frac{\Phi^*(1/x)}{\Phi^*(f^{-1}(1/x))}.$$

In view of Propositions 3.2.3 and 3.4.2, we have $f(s) \approx s$ for $s > x_0$, thus, $f^{-1}(s) \approx s$ for $s > f^*(x_0)$. Hence,

$$\frac{\Phi(1/x)x^2}{\phi'(f^{-1}(1/x))}\Phi^{-1}\left(\frac{\phi'(f^{-1}(1/x))}{x}\right) \lesssim 1.$$

Therefore, by (3.5.30) and (3.5.31), we conclude that

$$\int_0^{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon} p(t,x)\,dt \lesssim \frac{1}{\phi'(f^{-1}(1/x))},$$

which, by Proposition 3.5.4 and (3.5.27), entails (3.5.29). The extension to arbitrary A follows by positivity and continuity argument. ☐

Combining Propositions 3.5.4 – 3.5.7 we obtain a final result.

Theorem 3.5.8. *Let* $\mathbf{T}$ *be a subordinator with the Laplace exponent* ϕ. *Suppose that*

$$\phi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$$

for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$, *and* $0 < \alpha \leq \beta < 1$. *We assume that one of the following conditions holds:*

1. *The Lévy measure* ν *is absolutely continuous with respect to the Lebesgue measure with almost decreasing density* $\nu(x)$, *or*

2. $\alpha > \frac{1}{2}$.

Then for each $A > 0$ *there is* $C_1 \geq 1$ *such that for all* $x < A/x_0$,

$$C_1^{-1}\frac{1}{x\phi(1/x)} \leq U(x) \leq C_1\frac{1}{x\phi(1/x)}.$$

Proof. By Corollary 3.2.8, $-\phi'' \in \mathrm{WLSC}(\alpha-2, c, x_0) \cap \mathrm{WUSC}(\beta-2, C, x_0)$. Let $p(t, \cdot)$ be the transition density of T_t. In view of Propositions 3.5.4, 3.5.6 and 3.5.7, and (3.5.27), it is enough to show that for each $A > 0$ and $\varepsilon \in (0,1)$ there is $C_1 > 0$ such that for all $x < A/x_0$,

$$\int_{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon}^{\frac{x}{\phi'(f^{-1}(1/x))}} p(t,x)\,dt \le C_1 \frac{1}{\phi'(f^{-1}(1/x))}. \tag{3.5.32}$$

By [43, Theorem 3.1], there is $t_0 > 0$ such that for all $t \in (0, t_0)$,

$$p(t,x) \lesssim \Phi^{-1}(1/t).$$

If $x_0 = 0$, then $t_0 = \infty$. We can take

$$t_0 \ge \frac{x}{\phi'(f^{-1}(1/x))}.$$

Therefore, by monotonicity of Φ^{-1}, we get

$$\begin{aligned}\int_{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon}^{\frac{x}{\phi'(f^{-1}(1/x))}} p(t,x)\,dt &\lesssim \int_{\frac{x}{\phi'(f^{-1}(1/x))}\varepsilon}^{\frac{x}{\phi'(f^{-1}(1/x))}} \Phi^{-1}(1/t)\,dt \\ &\le \frac{x}{\phi'(f^{-1}(1/x))}\Phi^{-1}\bigg(\frac{\phi'(f^{-1}(1/x))}{\varepsilon x}\bigg).\end{aligned}$$

By the doubling property of Φ^{-1}, definition of f and Remark A.1.7,

$$\begin{aligned}\Phi^{-1}\bigg(\frac{\phi'(f^{-1}(1/x))}{\varepsilon x}\bigg) &\lesssim \Phi^{-1}\bigg(\frac{\phi'(f^{-1}(1/x))}{x}\bigg) \\ &\lesssim \Phi^{-1}\bigg(\frac{\Phi^*(f^{-1}(1/x))}{x f^*(f^{-1}(1/x))}\bigg) \\ &\le f^{-1}(1/x) \\ &\lesssim \frac{1}{x},\end{aligned}$$

since by the weak upper scaling property of $-\phi''$, $f(s) \approx s$ for all $s > f^*(x_0)$. Consequently, we obtain (3.5.32) and the theorem follows. □

Chapter 4

Transition densities of spectrally positive Lévy processes

4.1 Introduction

In this chapter we continue the approach proposed in Chapter 3 and adapt it to spectrally positive Lévy processes. The content of this part is taken from the author's article [69].

To be precise, our aim here is to discuss the behaviour of the transition densities of spectrally one-sided Lévy processes of unbounded variation. Recall that by spectrally one-sided we mean that the process **X** jumps only in one direction. In other words, the Lévy measure of **X** is supported either on $(-\infty, 0)$ or on $(0, \infty)$. Note that by [93, Theorems 21.9 and 24.10], **X** is of unbounded variation either when $\sigma > 0$ or the following condition is satisfied:

$$\int_{(0,\infty)} (1 \wedge x)\, \nu(dx) = \infty. \tag{4.1.1}$$

Therefore, in this chapter we exclude the case when **X** is a subordinator with drift. The independent sum of Brownian motion and a subordinator, however, is included. Spectrally one-sided Lévy processes, due to their specific structure, find natural applications in financial models, in particular insurance risk modelling and queue theory and therefore, they have been intensively analysed from that point of view. The prominent example here and, at the same time, one of the first questions one would like to ask in the financial setting, is the so-called exit problem — the identification of the distribution of the pair (τ_I, X_{τ_I}) where I is an open interval — which has been intensively discussed over last decades. One should list here prominent works of Zolotarev [107], Takács [101], Emery [28] and Rogers [90]. Vast majority of results is expressed by means of the so-called scale functions which have been of independent interest later on, see e.g. Hubalek and Kyprianou [48] or Kuznetsov, Kyprianou and Rivero [66]. Also, specific structure of spectrally one-sided processes considerably simplifies the fluctuation theory for these processes, which in the general case is rather implicit. For details we refer to books of Bertoin [4, Sections VI and VII], Kyprianou [68] or Sato [93]. This short list is far from being complete and for further discussion we refer to the works above and the references therein.

The abundant number of articles related to the financial applications stays in stark contrast with the fact that surprisingly little is known about transition densities of general spectrally one-sided Lévy processes, although it seems that such knowledge could potentially be an important and useful tool. One may find e.g. asymptotic series expansion for the special case of stable processes in the book of Zolotarev [108, Theorem 2.5.2] or the asymptotics with a fixed time variable under rather implicit assumptions in a recent article by Patie and Vaidyanathan [79], but, to the author's best knowledge, general results have not been obtained. In this chapter we aim at filling this gap in the theory and analysing the behaviour of the transition densities of spectrally one-sided Lévy processes in a feasibly wide generality.

Since we follow the approach from the previous chapter, the generality level also will be expressed by means of scaling properties. From theoretical point of view, the absence of negative (positive) jumps allows us to exploit techniques involving the Laplace transform, which can be easily proved to exist (see e.g. the book of Bertoin [4, Section VII]). We exploit that property in the first part, where we concentrate on derivation of the asymptotic behaviour of the transition density, which is covered by the following result (cf. Theorem 3.1.1) under the assumption of weak lower scaling property of φ'' with the scaling index $\alpha-2$ where $\alpha > 0$. Recall that in view of Proposition A.1.8 we see that the Hartman-Wintner condition is satisfied and consequently, for any $t > 0$ the distribution of X_t is absolutely continuous with respect to the Lebesgue measure with a density $p(t, \cdot)$.

Theorem 4.1.1. *Let* $\mathbf{X}$ *be a spectrally positive Lévy process of unbounded variation with the Laplace exponent* φ*. Suppose that* $\varphi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ *for some* $c \in (0, 1]$*,* $x_0 \geq 0$ *and* $\alpha > 0$*. Then for each* $\varepsilon > 0$ *there is* $M_0 > 0$ *such that*

$$\left| p(t, -t\varphi'(w)) \sqrt{2\pi t\varphi''(w)} \exp\left\{ t(w\varphi'(w) - \varphi(w)) \right\} - 1 \right| \leq \varepsilon,$$

provided that $w > x_0$ *and* $tw^2\varphi''(w) > M_0$*.*

Note that the signs in the exponent, in the radicand and in the argument of the heat kernel are changed compared to Theorem 3.1.1. This phenomenon is partially due to the fact that the integral representation of φ is slightly different than the one of a Bernstein function ϕ. In particular, we have that φ is convex while, ϕ is concave, see (2.2.11) and (3.2.1). An immediate consequence of the results above is the following approximation which is valid in some region of time and space:

$$p(t, x) \approx \frac{1}{\sqrt{t\varphi''(w)}} \exp\left\{ -t(w\varphi'(w) - \varphi(w)) \right\}$$

where $w = (\varphi')^{-1}(-x/t)$. We repeat the remark that the result is very general, as the only assumption is the lower scaling property with the index $\alpha > 0$. In particular, we assume neither upper scaling property nor absolute continuity of the Lévy measure ν. Observe that the independent sum of Brownian motion and a subordinator is admissible. The case $\alpha = 2$ is also included. Recall that without Gaussian component, the condition of having unbounded variation is tantamount to satisfying the integral condition (4.1.1). In fact, if this is the case, the assumption $\alpha > 0$ may seem superfluous at first sight, as the integrability condition (4.1.1), roughly speaking, requires enough singularity of order at least 1. One may, however, construct a bit pathological example of a Lévy process of unbounded variation but with lower scaling index strictly smaller than 1. The reader is referred for details to Remark 4.3.3 and Example 4.5.7, but we highlight here that such processes are also included. Let us also repeat after Chapter 3 that in some cases it is convenient to impose scaling condition on the real part of the characteristic exponent or on the tail of the Lévy measure instead of on the second derivative of the Laplace exponent. These two in fact imply the scaling of φ'' and we state that result in Propositions A.1.8 and A.1.9. Moreover, if the scaling of $\operatorname{Re}\psi$ holds true, then by (A.1.6) and (A.1.7), we have, for some $x_0 \geq 0$,

$$x^2\varphi''(x) \approx \operatorname{Re}\psi(x), \quad x \geq x_0.$$

Admittedly, one of the strengths of Theorem 4.1.1 lies in the fact that the expression in the exponent is given explicitly and there is no hidden, unknown constant. Nonetheless, in view of the equation above, one can, if necessary, substitute the Laplace exponent with the characteristic exponent at the cost of losing the exact formula and an implicit constant which will appear instead.

Next, we restrict ourselves to processes of unbounded variation without Gaussian component and focus on upper and lower estimates of the transition density. While the former, covered by Theorem 4.4.2, does not require additional assumptions and is independent of previous results, the latter, consisting of Lemma 4.4.4 and Lemma 4.4.5, requires apparently stronger conditions, i.e. $\alpha \geq 1$ and $\alpha > 1$, respectively. They provide local and right tail lower estimates of the transition density and the proof of the latter relies strongly on Theorem 4.1.1. As above, we point out that the condition $\alpha \geq 1$ is not very restrictive in the class of processes with unbounded variation. Finally, we merge all the results mentioned above in order to obtain sharp two-sided estimates. This result is displayed in Theorem 4.5.3 and below we present its version when the obtained estimates are global both in space and in time. Let

$$\theta_1 = \inf\{s > 0\colon \varphi'(s) > 0\}.$$

Theorem 4.1.2. *Let* $\mathbf{X}$ *be a spectrally positive Lévy process of infinite variation with the Laplace exponent* φ *such that* $\theta_1 = 0$ *and* $\varphi'(0) = 0$. *Suppose that* $\sigma = 0$ *and* $\varphi \in \mathrm{WLSC}(\alpha, c) \cap \mathrm{WUSC}(\beta, C)$ *for some* $c \in (0,1]$, $C \geq 1$ *and* $1 < \alpha \leq \beta < 2$. *We also assume that the Lévy measure* ν *has an almost decreasing density* $\nu(x)$. *Then for all* $t \in (0,\infty)$ *and* $x \in \mathbb{R}$,

$$p(t,x) \approx \begin{cases} \big(t\varphi''(w)\big)^{-\frac{1}{2}} \exp\big\{ -t\big(w\varphi'(w) - \varphi(w)\big)\big\}, & \text{if} \quad x\varphi^{-1}(1/t) \leq -1, \\ \varphi^{-1}(1/t), & \text{if} \quad -1 < x\varphi^{-1}(1/t) \leq 1, \\ tx^{-1}\varphi(1/x), & \text{if} \quad x\varphi^{-1}(1/t) > 1 \end{cases}$$

where $w = (\varphi')^{-1}(-x/t)$.

Again, the reader may compare the result above with Theorem 3.1.2 and easily spot major similarities and differences. Note that here we require both lower and upper scaling condition with indices strictly separated from 1 and 2, i.e. $1 < \alpha \leq \beta < 2$. By inspecting the proofs of Lemmas 4.4.4 and 4.4.5, Theorem 4.5.1, and Proposition 4.5.2, we see that covering the limit cases using our methods is not possible, and it is not very surprising, as they usually require more sophisticated methods or sometimes even a completely different approach. Nonetheless, the asymptotic behaviour displayed by Theorem 4.1.1 covers both $\alpha = 1$ and $\alpha = 2$.

Finally, we should warn the reader that the content of this chapter displays significant similarities to the one presented in Chapter 3. For instance, it may be easily observed that the proof of Theorem 4.1.1 follows exactly the same pattern as the one of Theorem 3.1.1 with some mild changes due to convexity of the Laplace exponent φ, see also Remark 3.3.3. The same rule applies to other, both preliminary and main, results of this chapter. With the intention to make this part self-contained, we provide all proofs and argumentations in detail, but one will easily spot multiple similarities to the results of Chapter 3. For the convenience of the reader, we keep track of these connections and provide references to appropriate analogous results as well as point out significant differences throughout the chapter.

4.2 Properties of the Laplace exponent φ

In this section we concentrate on the Laplace exponent φ and its properties. Recall that

$$\varphi(\lambda) = \sigma^2\lambda^2 - \gamma\lambda + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{x<1})\, \nu(dx), \quad \lambda \geq 0.$$

Note that here, contrary to the previous chapter, we allow γ to be negative. Just for the record we note that $\varphi(\lambda) \to \infty$ as $\lambda \to \infty$. Furthermore, by differentiating the identity above, we conclude that also $\varphi'(\lambda) \to \infty$ as $\lambda \to \infty$. The content of this part mirrors Section 3.2 devoted to Bernstein functions in Chapter 3, but there are slight differences which will play some role later on. In general, the proofs we provide are similar, but since φ is convex and not necessarily positive in the neighbourhood of the origin, we need to keep track on the sign of φ and because of that some minor technical difficulties appear. For this reason, we let θ_0 be the largest root of φ. Note that there is always a root of φ at $\lambda = 0$ and, due to convexity of φ, at most one root for $\lambda > 0$, precisely in the case $\theta_1 > 0$. We then have $\theta_1 \leq \theta_0$ and the equality may occur only for the case $\theta_0 = \theta_1 = 0$. To summarise, the following identities hold:

1. $\theta_0 > 0 \iff \theta_1 > 0 \iff \theta_1 < \theta_0 \iff \varphi'(0) < 0 \implies \varphi'(\theta_1) = 0,$
2. $\theta_0 = 0 \iff \theta_1 = 0 \iff \theta_1 = \theta_0 \iff \varphi'(0) \geq 0.$

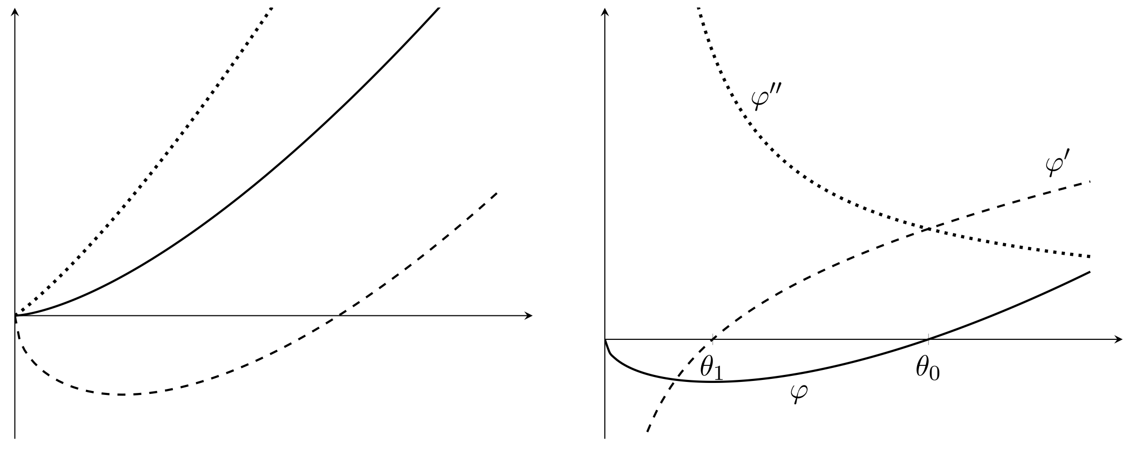
See Figure 4.2.1 for some glimpse on these relations. By the Wiener-Hopf factorization we get that φ is necessarily of the form

$$\varphi(\lambda) = (\lambda - \theta_0)\phi(\lambda), \tag{4.2.1}$$

where ϕ is the Laplace exponent of a (possibly killed) subordinator, known as an ascending ladder height process, with the Lévy measure ω given by

$$\omega((x,\infty)) = e^{\theta_0 x} \int_x^\infty e^{-\theta_0 u} \nu((u,\infty))\, du.$$

See Hubalek and Kyprianou [48, Section 4] and the references therein. Note here that ϕ is a Bernstein function.



(a) Three different possible versions of φ depending on the value of $\varphi'(0^+)$

(b) Visualisation of behaviour of φ, φ' and φ'' in the case $\theta_0 > 0$ and $\varphi'(0^+) = -\infty$

Figure 4.2.1: The visualisation of possible behaviour of the Laplace exponent and its derivatives. The dashed line in Figure 4.2.1a corresponds to the first identity above while the solid and dotted lines — to the second. Figure 4.2.1b is the extension of the case depicted by dashed line in Figure 4.2.1a.

Remark 4.2.1. Observe that we may interpret (4.2.1) as follows: φ is, roughly speaking, *one order higher than* ϕ in the sense of power of λ. Therefore, one may expect various phenomena which occurred for Bernstein functions to appear for the first derivative of φ. This rule will be visible in the forthcoming properties.

First, let us observe that by differentiating (2.2.11) and using the fact that for all $x > 0$,

$$xe^{-x} \le 1 - e^{-x},$$

we get that for all $\lambda \ge 0$,

$$\varphi(\lambda) \le \lambda\varphi'(\lambda). \tag{4.2.2}$$

We observe that the inequality in (3.2.2) is reversed. Furthermore, if $\theta_0 > 0$, then $-\varphi$ is positive and concave on $(0, \theta_1)$. Thus, for all $x \le \theta_1$ and $\lambda \le 1$,

$$-\varphi(x) - \big(-\varphi(\lambda x)\big) \le (1-\lambda)x\big(-\varphi'(\lambda x)\big).$$

Thus, by (4.2.2), for all $x \le \theta_1$ and $\lambda \le 1$,

$$\lambda\big(-\varphi(x)\big) \le -\varphi(\lambda x). \tag{4.2.3}$$

The following proposition is the analogue of (3.2.3).

Proposition 4.2.2. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation. There are $C_1, C_2 \ge 1$ such that $\varphi' \in \mathrm{WUSC}(1, C_1, 2\theta_1 \wedge \theta_0)$ and $\varphi \in \mathrm{WUSC}(2, C_2, 2\theta_0)$. Furthermore, if $\theta_0 = 0$ and $\varphi'(0) = 0$, then $C_1 = C_2 = 1$, i.e. for all $x > 0$ and $\lambda \ge 1$,*

$$\varphi'(\lambda x) \le \lambda\varphi'(x) \qquad \text{and} \qquad \varphi(\lambda x) \le \lambda^2\varphi(x).$$

Proof. Let $\lambda \ge 1$. First, observe that by monotonicity of φ'', for all $x > \theta_1$,

$$\varphi'(\lambda x) - \varphi'(\lambda\theta_1) = \int_{\lambda\theta_1}^{\lambda x} \varphi''(s)\,ds = \lambda\int_{\theta_1}^{x} \varphi''(\lambda s)\,ds \le \lambda\int_{\theta_1}^{x} \varphi''(s)\,ds = \lambda\big(\varphi'(x) - \varphi'(\theta_1)\big).$$

Thus, we get the claim for φ' in the case $\theta_1 = 0$ and $\varphi'(0) = 0$. If $\theta_1 = 0$ but $\varphi'(0) > 0$, then we clearly have $\varphi'(0) \le \lambda\varphi'(x)$ for all $x > 0$ and $\lambda \ge 1$, and we get the claim with $C_1 = 2$. Finally, if $\theta_1 > 0$, then it remains to prove that there is $c > 0$ such that for all $x > 2\theta_1 \wedge \theta_0$ and $\lambda \ge 1$,

$$\varphi'(\lambda\theta_1) \le c\lambda\varphi'(x). \tag{4.2.4}$$

Since $\varphi'(\theta_1) = 0$, by monotonicity of φ'' and φ', we obtain

$$\varphi'(\lambda\theta_1) = \int_{\theta_1}^{\lambda\theta_1} \varphi''(s)\,ds \le (\lambda-1)\theta_1\varphi''(\theta_1) \le \lambda\theta_1\varphi''(\theta_1) \le c\lambda\varphi'(x)$$

where $c = (\theta_1\varphi''(\theta_1))/\varphi'(2\theta_1 \wedge \theta_0)$, and (4.2.4) follows. Now, with the first part proved, a similar argument applies to the second and therefore it is omitted. □

The following property is remarkable in a sense that it holds for every Laplace exponent φ. This is not the case for Bernstein functions, cf. Proposition 3.2.3, but, in view of Remark 4.2.1, this comparison is not exactly adequate. In this spirit we should rather say that the second part of the following Proposition corresponds to (3.2.2) with $n = 1$.

Proposition 4.2.3. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation. There is $C = C(\varphi) \geq 1$ such that for all $x > 2\theta_0$ we have*

$$\varphi(x) \leq x\varphi'(x) \leq C\varphi(x). \tag{4.2.5}$$

Furthermore, for all $x > 2\theta_1$,

$$2\varphi'(x) \geq x\varphi''(x).$$

Proof. We first observe that the first inequality of (4.2.5) follows from (4.2.2). Let $x > 2\theta_0$ and $1 \leq b < a$. By monotonicity of φ',

$$\varphi(ax) - \varphi(bx) \geq x(a-b)\varphi'(bx).$$

Put $b = 1$ and $a = 2$. By Proposition 4.2.2,

$$\frac{x\varphi'(x)}{\varphi(x)} \leq \frac{\varphi(2x)}{\varphi(x)} - 1 \leq 4\tilde{C} - 1$$

where $\tilde{C}$ is taken from Proposition 4.2.2, and the first part follows. For the proof of the second part, it remains to observe that by monotonicity of φ'', for $x > 2\theta_1$,

$$\varphi'(x) \geq \varphi'(x) - \varphi'(\theta_1) = \int_{\theta_1}^{x} \varphi''(s)\, ds \geq (x - \theta_1)\varphi''(x).$$

Thus, for $x > 2\theta_1$, we get the claim. □

Corollary 4.2.4. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation. There is $c = c(\varphi) > 0$ such that for all $x \in (0, \theta_0/2) \cup (2\theta_0, \infty)$,*

$$|\varphi(x)| \geq cx^2\varphi''(x).$$

The implied constant c depends only on θ_0.

Proof. In view of Proposition 4.2.3, it remains to prove that if $\theta_0 > 0$, then there is $c > 0$ such that for all $x < \theta_0/2$,

$$-\varphi(x) \geq cx^2\varphi''(x). \tag{4.2.6}$$

From (4.2.1), we have

$$\varphi''(x) = 2\phi'(x) + (\theta_0 - x)\big(-\phi''(x)\big).$$

Hence, by (3.2.2), we have, for $x < \theta_0/2$,

$$-\varphi(x) = (\theta_0 - x)\phi(x) \gtrsim \phi(x) \geq x\phi'(x) \gtrsim x^2\phi'(x). \tag{4.2.7}$$

Moreover,

$$(\theta_0 - x)\phi(x) \gtrsim (\theta_0 - x)x^2\big(-\phi''(x)\big),$$

which together with (4.2.7) imply (4.2.6), and the claim follows. □

Now, we deduce some properties of φ and its derivatives which follow from scaling properties. The first one is the analogue of Remark 3.2.5. Recall that the condition $\varphi'(\theta_1) = 0$ corresponds to the case $\mathbb{E}X_1 \geq 0$, cf. (2.2.12). We note that this assumption is relevant, since if $\varphi'(\theta_1) > 0$, then we necessarily have $\theta_1 = 0$ and the extension argument of the scaling property of φ' at the end of the proof may fail if $x_0 = 0$.

Proposition 4.2.5. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation such that $\varphi'(\theta_1) = 0$. Suppose $\varphi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\alpha > 0$. Then $\varphi' \in \mathrm{WLSC}(\alpha - 1, c, x_0 \vee \theta_1)$ and $\varphi \in \mathrm{WLSC}(\alpha, c, x_0 \vee \theta_0)$.*

Proof. We proceed as in the proof of Proposition 4.2.2. Let $\lambda \geq 1$ and $x > \theta_1 \vee x_0$. By the weak scaling property of φ'',

$$\varphi'(\lambda x) \geq \varphi'(\lambda x) - \varphi'(\lambda(\theta_1 \vee x_0)) = \int_{\lambda(\theta_1 \vee x_0)}^{\lambda x} \varphi''(s)\, ds = \lambda \int_{\theta_1 \vee x_0}^{x} \varphi''(\lambda s)\, ds$$
$$\geq c\lambda^{\alpha-1} \int_{\theta_1 \vee x_0}^{x} \varphi''(s)\, ds = c\lambda^{\alpha-1}(\varphi'(x) - \varphi'(\theta_1 \vee x_0)),$$

and the claim follows for the case $\theta_1 \geq x_0$. For the proof of the remaining case, we note that φ' is positive (and obviously increasing) on $[x_0, \infty)$. Since $\varphi'(\lambda) \to \infty$ as $\lambda \to \infty$, there is $x_1 > x_0$ such that $\varphi'(x) \geq 2\varphi'(x_0)$ for all $x > x_1$, and consequently, $\varphi' \in \mathrm{WLSC}(\alpha - 1, \tilde{c}, x_1)$ for some $\tilde{c} \in (0, 1]$. Finally, using continuity and positivity of φ', we can extend the scaling area to (x_0, ∞) at the expense of the constant. The proof of the weak scaling property of φ follows by an analogous argument. □

The next proposition to some extent corresponds to Propositions 3.2.2 and 3.2.3. Note that the additional condition on the scaling exponent τ appears here. It may seem innocent and harmless at first glance, but it will have some impact on main results of Section 4.4.

Proposition 4.2.6. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation such that $\varphi'(\theta_1) = 0$. If $\varphi'' \in \mathrm{WLSC}(\tau - 1, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\tau > 0$, then $\varphi' \in \mathrm{WLSC}(\tau, c', x_0 \vee \theta_1)$ for some $c' \in (0, 1]$, $x_0 \geq 0$ and $\tau > 0$. Conversely, if $\varphi' \in \mathrm{WLSC}(\tau, c', x_1)$ for some $c' \in (0, 1]$, $x_1 \geq \theta_1$ and $\tau > 0$, then also $\varphi'' \in \mathrm{WLSC}(\tau - 1, c, x_1)$ for some $c \in (0, 1]$. Furthermore, if $\varphi'' \in \mathrm{WLSC}(\tau - 1, c, x_0)$, then there is $C \geq 1$ such that for all $x > x_0 \vee 2\theta_1$,*

$$C^{-1}\varphi'(x) \leq x\varphi''(x) \leq C\varphi'(x). \tag{4.2.8}$$

Proof. First, observe that, in view of Proposition 4.2.5, it is enough to prove (4.2.8) and deduce scaling property of φ'' from the scaling property of φ'.

Therefore, assume that $\varphi' \in \mathrm{WLSC}(\tau, c, x_1)$. By monotonicity of φ'', for $0 < b < a$,

$$\frac{\varphi'(ax) - \varphi'(bx)}{\varphi'(x)} \leq \frac{x(a-b)\varphi''(bx)}{\varphi'(x)}, \quad x > x_1.$$

Put $b = 1$. Then by the scaling property of φ',

$$\frac{x(a-1)\varphi''(x)}{\varphi'(x)} \geq \frac{\varphi'(ax)}{\varphi'(x)} - 1 \geq ca^{\tau} - 1$$

for all $x > x_1$. Thus, for $a = 2^{1/\tau}c^{-1/\tau}$, we obtain that $\varphi'(x) \lesssim x\varphi''(x)$ for all $x > x_1$, which, combined with Proposition 4.2.3, yields (4.2.8) for $x > x_1 \vee 2\theta_1$, and the scaling property of φ'' follows for $x > x_1 \vee 2\theta_1$. Furthermore, observe that if $x_1 = 0$, then we necessarily have $\theta_1 = 0$. If this is not the case, then observe that by the positivity and continuity of φ'', we may extend the scaling property on the set (x_1, ∞) as claimed. Moreover, another application of Proposition 4.2.3 yields that if $\varphi'' \in \mathrm{WLSC}(\tau - 1, c, x_0)$, then we in fact have $x_1 = x_0 \vee \theta_1$. This completes the proof. □

Combining Propositions 4.2.3 and 4.2.6, we immediately obtain the following corollary.

Corollary 4.2.7. *Let φ be the Laplace exponent of a spectrally positive Lévy process of infinite variation such that $\varphi'(\theta_1) = 0$. If $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 1$, then $\varphi \in \mathrm{WLSC}(\alpha, c', x_0 \vee \theta_0)$ for some $c' \in (0,1]$. Conversely, if $\varphi \in \mathrm{WLSC}(\alpha, c', x_1)$ for some $c' \in (0,1]$, $x_1 \geq \theta_0$ and $\alpha > 1$, then $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_1)$ for some $c \in (0,1]$. Furthermore, if $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$, then there is $C \geq 1$ such that for all $x > x_0 \vee 2\theta_0$,*

$$C'^{-1}\varphi(x) \leq x^2\varphi''(x) \leq C'\varphi(x).$$

At the end of this section let us stress one vital observation. Note that in the preceding preliminary results we do **not** impose the upper scaling property and the only restriction appearing e.g. in Corollary 4.2.7 is the lower scaling condition with $\alpha > 1$. As we have seen in Chapter 3, at some point it was important to assume the upper scaling condition with $\beta < 1$ or, informally speaking, to be *separated from* or even to be *below* the limit case $\beta = 1$. The content of this section suggests that in case of spectrally positive Lévy processes of unbounded variation, this *separation* will also play a role but with the difference that here we will assume the lower scaling property with $\alpha > 1$, which may be understood as *separation from* or *being over* the limit case $\alpha = 1$. These intuitions will come into play e.g. in the proof of lower estimate of the transition density in Section 4.4, see the comment before Lemma 4.4.5.

4.3 Asymptotic behaviour of density

This section is devoted to the proof of Theorem 4.1.1 and its consequences. We follow almost directly the reasoning presented in Chapter 3 with some mild modification resulting from the convexity of φ. There is, however, one substantial difference. Namely, if $\mathbf{T} = (T_t\colon t \geq 0)$ is a subordinator, then for every $t \geq 0$ its distribution is supported on the positive half-line. This is not the case for spectrally positive Lévy processes of unbounded variation, where, for every $t > 0$, the support of the distribution of X_t is the whole real line. This contrast is reflected in regions of asymptotics in Theorems 3.1.1 and 4.1.1, but perhaps the best insight one may achieve through comparison of Corollaries 3.3.6 and 4.3.5. We shall comment on it later on.

Before embarking on our main results, let us prove one key lemma which provides control on the real part of the holomorphic extension of the Laplace exponent. Its proof follows the proof of Lemma 3.3.1.

Lemma 4.3.1. *Suppose that $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. Then there exists $C > 0$ such that for all $w > x_0$ and $\lambda \in \mathbb{R}$,*

$$\operatorname{Re}\left(\varphi(w) - \varphi(w+i\lambda)\right) \geq C\lambda^2\left(\varphi''(|\lambda| \vee w)\right).$$

Proof. By the integral representation (2.2.11), for $\lambda \in \mathbb{R}$, we have

$$\operatorname{Re}\left(\varphi(w) - \varphi(w+i\lambda)\right) = \sigma^2\lambda^2 + \int_{(0,\infty)} (1-\cos\lambda s)e^{-ws}\,\nu(ds).$$

In particular, we see that the expression above is symmetric in λ. Thus, it is sufficient to consider $\lambda > 0$. Moreover, we infer that

$$\operatorname{Re}\left(\varphi(w) - \varphi(w+i\lambda)\right) \gtrsim \lambda^2\bigg(\sigma^2 + \int_{(0,1/\lambda)} s^2 e^{-ws}\,\nu(ds)\bigg). \tag{4.3.1}$$

Due to Lemma A.1.1, we obtain, for $\lambda \geq w$,

$$\operatorname{Re}\left(\varphi(w) - \varphi(w+i\lambda)\right) \gtrsim \lambda^2\bigg(\sigma^2 + \int_{(0,1/\lambda)} s^2\,\nu(ds)\bigg) \gtrsim \lambda^2\varphi''(\lambda).$$

If $w > \lambda > 0$, then by (4.3.1),

$$\operatorname{Re}\big(\varphi(w) - \varphi(w + i\lambda)\big) \gtrsim \lambda^2\Big(\sigma^2 + \int_{(0,1/w)} s^2 e^{-ws}\,\nu(ds)\Big) \geq e^{-1}\lambda^2\Big(\sigma^2 + \int_{(0,1/w)} s^2\,\nu(ds)\Big),$$

which, together with Lemma A.1.1 ends the proof. □

Proof of Theorem 4.1.1. Let $x = -t\varphi'(w)$ and $M > 0$. We first show that

$$p(t,x) = \frac{1}{2\pi} \cdot \frac{e^{-\Theta(x/t,0)}}{\sqrt{t\varphi''(w)}} \int_{\mathbb{R}} \exp\bigg\{ - t\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\Big) - \Theta\Big(\frac{x}{t}, 0\Big)\bigg)\bigg\}\,du, \tag{4.3.2}$$

provided that $w > x_0$ and $tw^2\varphi''(w) > M$, where for $\lambda > 0$ we have set

$$\Theta(x/t, \lambda) = -\big(\varphi(w + i\lambda) + \frac{x}{t}(w + i\lambda)\big). \tag{4.3.3}$$

To this end let us recall that, by the Mellin's inversion formula, if the limit

$$\lim_{L\to\infty} \frac{1}{2\pi i} \int_{w-iL}^{w+iL} e^{t\varphi(\lambda)+\lambda x}\,d\lambda \quad \text{exists}, \tag{4.3.4}$$

then the probability density $p(t,\,\cdot\,)$ satisfies

$$p(t,x) = \lim_{L\to\infty} \frac{1}{2\pi i} \int_{w-iL}^{w+iL} e^{t\varphi(\lambda)+\lambda x}\,d\lambda.$$

Using change of variables twice, we obtain

$$\begin{aligned}
\frac{1}{2\pi i} \int_{w-iL}^{w+iL} e^{t\varphi(\lambda)+\lambda x}\,d\lambda &= \frac{1}{2\pi}\int_{-L}^{L} e^{-t\Theta(x/t,\lambda)}\,d\lambda \\
&= \frac{e^{-t\Theta(x/t,0)}}{2\pi} \int_{-L}^{L} \exp\Big\{ - t\big(\Theta(x/t,\lambda) - \Theta(x/t,0)\big)\Big\}\,d\lambda \\
&= \frac{e^{-t\Theta(x/t,0)}}{2\pi\sqrt{t\varphi''(w)}} \int_{-L\sqrt{t\varphi''(w)}}^{L\sqrt{t\varphi''(w)}} \exp\bigg\{ - t\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\Big) - \Theta\Big(\frac{x}{t},0\Big)\bigg)\bigg\}\,du.
\end{aligned}$$

Next, we observe that there is $C > 0$, not depending on M, such that for all $u \in \mathbb{R}$,

$$t\operatorname{Re}\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\Big) - \Theta\Big(\frac{x}{t},0\Big)\bigg) \geq C\Big(u^2 \wedge \big(|u|^\alpha M^{1-\alpha/2}\big)\Big), \tag{4.3.5}$$

provided that $w > x_0$ and $tw^2\varphi''(w) > M$. Indeed, by (4.3.3) and Lemma 4.3.1, for $w > x_0$, we get

$$t\operatorname{Re}\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\Big) - \Theta\Big(\frac{x}{t},0\Big)\bigg) \gtrsim \frac{|u|^2}{\varphi''(w)}\varphi''\Big(\frac{|u|}{\sqrt{t\varphi''(w)}} \vee w\Big),$$

and (4.3.5) follows by the scaling property of φ''. Hence, (4.3.4) follows from the dominated convergence theorem. Consequently, Mellin's inversion formula yields (4.3.2).

Next, we prove that for each $\varepsilon > 0$ there is $M_0 > 0$ such that

$$\left| \int_{\mathbb{R}} \exp\bigg\{ - t\bigg(\Theta\Big(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\Big) - \Theta\Big(\frac{x}{t},0\Big)\bigg)\bigg\}\,du - \int_{\mathbb{R}} e^{-\frac{1}{2}u^2}\,du \right| \leq \varepsilon, \tag{4.3.6}$$

provided that $w > x_0$ and $tw^2\varphi''(w) > M_0$. In view of (4.3.5), by taking $M_0 > 1$ sufficiently large, we get

$$\left| \int_{|u| \geq M_0^{1/4}} \exp\left\{ -t\left(\Theta\left(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\right) - \Theta\left(\frac{x}{t}, 0\right)\right)\right\} du \right| \leq \int_{|u| \geq M_0^{1/4}} e^{-C|u|^\alpha} \, du \leq \varepsilon, \quad (4.3.7)$$

and

$$\int_{|u| \geq M_0^{1/4}} e^{-\frac{1}{2}u^2} \, du \leq \varepsilon. \quad (4.3.8)$$

Next, let us observe that there is $C > 0$ such that

$$\left| t\left(\Theta\left(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\right) - \Theta\left(\frac{x}{t}, 0\right)\right) - \frac{1}{2}|u|^2 \right| \leq C|u|^3 M_0^{-\frac{1}{2}}. \quad (4.3.9)$$

Indeed, since $\partial_\lambda \Theta(x/t, 0) = 0$, by Taylor's formula we get

$$\begin{aligned} \left| t\left(\Theta\left(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\right) - \Theta\left(\frac{x}{t}, 0\right)\right) - \frac{1}{2}|u|^2 \right| &= \left| \frac{1}{2}\partial_\lambda^2 \Theta\left(\frac{x}{t}, \xi\right) \frac{|u|^2}{\varphi''(w)} - \frac{1}{2}|u|^2 \right| \\ &= \frac{|u|^2}{2\varphi''(w)} |\varphi''(w) - \varphi''(w + i\xi)|, \end{aligned} \quad (4.3.10)$$

where ξ is some number satisfying $|\xi| \leq \frac{|u|}{\sqrt{t\varphi''(w)}}$. We also have

$$\begin{aligned} |\varphi''(w) - \varphi''(w + i\xi)| &\leq \int_{(0,\infty)} s^2 e^{-ws} |e^{-i\xi s} - 1| \, \nu(ds) \\ &\leq 2|\xi| \int_{(0,\infty)} s^3 e^{-ws} \, \nu(ds) \\ &= 2|\xi|(-\varphi'''(w)). \end{aligned}$$

Since φ'' is completely monotone and has a doubling property, by Proposition 3.2.1, we have, for $w > x_0$,

$$\varphi''(w) \gtrsim w(-\varphi'''(w)),$$

which together with the estimate of $|\xi|$ yield

$$|\varphi''(w) - \varphi''(w + i\xi)| \leq C\frac{|u|}{\sqrt{t\varphi''(w)}} \cdot \frac{\varphi''(w)}{w} \leq CM_0^{-\frac{1}{2}}|u|\varphi''(w),$$

if only $tw^2\varphi''(w) > M_0$, proving (4.3.9) through (4.3.10). Finally, since for any $z \in \mathbb{C}$,

$$|e^z - 1| \leq |z| e^{|z|},$$

(4.3.9) implies

$$\begin{aligned} &\left| \int_{|u| < M_0^{1/4}} \exp\left\{ -t\left(\Theta\left(\frac{x}{t}, \frac{u}{\sqrt{t\varphi''(w)}}\right) - \Theta\left(\frac{x}{t}, 0\right)\right)\right\} du - \int_{|u| < M_0^{1/4}} e^{-\frac{1}{2}u^2} \, du \right| \\ &\qquad \leq CM_0^{-\frac{1}{2}} \int_{|u| < M_0^{1/4}} \exp\left\{ -\frac{1}{2}|u|^2 + CM_0^{-\frac{1}{2}}|u|^3 \right\} |u|^3 \, du \leq \varepsilon, \end{aligned}$$

provided that M_0 is sufficiently large, which together with (4.3.7) and (4.3.8) complete the proof of (4.3.6) and the theorem follows. □

Remark 4.3.2. Let us repeat after Remark 3.3.2 that if $x_0 = 0$, then the constant M_0 in Theorem 4.1.1 depends only on α and c. If $x_0 > 0$, then it also depends on

$$\sup_{x\in[x_0,2x_0]} \frac{x(-\varphi'''(x))}{\varphi''(x)}.$$

Remark 4.3.3. Suppose $\sigma = 0$. It is known (see e.g. [43, Lemma 2.9]) that the scaling property with the scaling index $\alpha \geq 1$ implies unbounded variation. Theorem 4.1.1, however, holds in greater generality. Namely, there are processes of unbounded variation which satisfy scaling condition with α strictly smaller than 1 and one may construct a Lévy measure which satisfies (4.1.1) and whose corresponding second derivative of the Laplace exponent has lower and upper Matuszewska indices of order $-\frac{3}{2}$ and $-\frac{1}{2}$, respectively. Thus, in this example the lower scaling condition for φ'' holds only for $\alpha < 1$. An example of such process is constructed in Example 4.5.7. We also note that the Gaussian component is not excluded.

The following corollary is an immediate consequence of Theorem 4.1.1. The reader may compare it with Corollary 3.3.4 and conclude that in this case there is no lower bound on x. From the analytical point of view one may notice that, contrary to the subordinators setting, φ' is strictly increasing and $\lim_{\lambda\to\infty}\varphi'(\lambda) = \infty$; hence, there is no additional restriction on x. This coincides with the fact the support of the distribution of X_t is the whole real line and consequently, no additional lower bound on x is required.

Corollary 4.3.4. *Suppose that $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. Then there is $M_0 > 0$ such that*

$$p(t,x) \approx \frac{1}{\sqrt{t\varphi''(w)}} \exp\Big\{-t\big(w\varphi'(w) - \varphi(w)\big)\Big\},$$

uniformly on the set

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R} \colon x < -t\varphi'(x_0) \text{ and } tw^2\varphi''(w) > M_0\}$$

where $w = (\varphi')^{-1}(-x/t)$.

As aforementioned at the beginning of this section, we encourage the reader to compare the next result with Corollary 3.3.6. It is clear that both of them describe some region of time and space, and this area may be informally described as *small times* (unless we have global scalings and $\theta_0 = 0$ in case of spectrally positive Lévy process of unbounded variation) and *small x with respect to time.* Now, in case of subordinators, the concept of *smallness* of x is understood as closeness to the origin. Here, the situation is somewhat different, as by *small* we mean simply *smaller than* without any other restrictions. That is to say, for a fixed t the set described by Corollary 4.3.4 is a half-line. This contrast is in line with our expectations as in both cases we *cover* the left endpoint of the support of the probability distribution. We note in passing that in this way the reader may also perceive the idea of *separation* from the limit case $\alpha = 1$ ($\beta = 1$) in yet another setting.

Corollary 4.3.5. *Suppose that $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 1$. Assume also that $\varphi'(\theta_1) = 0$. Then there is $M > 0$ such that*

$$p(t,x) \approx \frac{1}{\sqrt{t\varphi''(w)}} \exp\Big\{-t\big(w\varphi'(w) - \varphi(w)\big)\Big\},$$

uniformly on the set

$$\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}\colon\ -x\varphi^{-1}(1/t) > M \text{ and } 0 \leq t\varphi(x_0 \vee 2\theta_0) \leq 1\}, \tag{4.3.11}$$

where $w = (\varphi')^{-1}(-x/t)$.

Remark 4.3.6. As already stated, the condition $\varphi'(\theta_1) = 0$ covers the case $\mathbb{E}X_1 \in [0,\infty]$. For the case $\mathbb{E}X_1 > 0$ it is not, however, optimal, because we do not treat positive x which may be in the area from Corollary 4.3.4. We also note that $\alpha = 2$ is included.

Proof. We verify that for (t,x) belonging to the set (4.3.11), we have $w > x_0$ and $tw^2\varphi''(w) > M_0$ where $w = (\varphi')^{-1}(-x/t)$. Let $M \geq C_1C_2$ where C_1, C_2 are taken from Propositions 4.2.3 and 4.2.2, respectively. By Propositions 4.2.3 and 4.2.2,

$$\frac{-x}{t} > M\frac{1}{t\varphi^{-1}(1/t)} = M\frac{\varphi(\varphi^{-1}(1/t))}{\varphi^{-1}(1/t)} \geq C_1(C_1^{-1}C_2^{-1}M)\varphi'(\varphi^{-1}(1/t)) \geq \varphi'(C_1^{-1}C_2^{-1}M\varphi^{-1}(1/t)).$$

Thus,

$$w = (\varphi')^{-1}(-x/t) > C_1^{-1}C_2^{-1}M\varphi^{-1}(1/t). \tag{4.3.12}$$

In particular, $w > x_0 \vee 2\theta_0$. Next, by Corollary 4.2.7, there is $c_1 \in (0,1]$ such that $tw^2\varphi''(w) \geq c_1t\varphi(w)$. Since, by Proposition 4.2.5, there is $c_2 \in (0,1]$ such that $\varphi \in \mathrm{WLSC}(\alpha, c_2, x_0 \vee \theta_0)$, we obtain

$$t\varphi(w) \geq c_2\left(\frac{w}{\varphi^{-1}(1/t)}\right)^{\alpha}.$$

Therefore, in view of (4.3.12), we get that

$$tw^2\varphi''(w) > c_1c_2(C_1^{-1}C_2^{-1}M)^{\alpha} > M_0,$$

provided that M is sufficiently large. Applying Theorem 4.1.1 yields the desired result. □

4.4 Upper and lower estimates of the density

Now, let us turn our attention to upper an lower estimates of the density of $\mathbf{X}$. As already stated, we are not aware of works that would treat transition densities of spectrally positive Lévy processes of unbounded variation specifically. The overview of general research in this area was already presented in Chapter 3, so we will excuse ourselves and only refer to the beginning of Section 3.4 therein. For the sake of clarity, we note that we always assume $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. Then, by Proposition A.1.8 and the Hartman-Wintner condition (HW), the probability distribution of X_t has a density $p(t,\cdot)$. We use the notation from Chapter 3 and set

$$\Phi(x) = x^2\varphi''(x), \quad x > 0.$$

Clearly, $\Phi \in \mathrm{WLSC}(\alpha, c, x_0)$. By Φ^{-1} we denote its right-sided inverse, i.e.

$$\Phi^{-1}(s) = \sup\{r > 0\colon \Phi^*(r) = s\},$$

where

$$\Phi^*(r) = \sup_{0<s\leq r} \Phi(s).$$

Again, various consequences of the lower scaling property of φ'' are derived in Appendix A in order to avoid unnecessary duplication. In the beginning we establish a version of Proposition 3.4.2. Note that the analogous separation from the limit case $\alpha = 1$ also plays a role here.

Proposition 4.4.1. *Suppose that* $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$ *and* $\alpha > 1$. *Assume also that* $\varphi'(\theta_1) = 0$. *Then for all* $x > x_0 \vee 2\theta_0$,

$$\Phi^*(x) \approx \varphi(x),$$

and for all $r > \Phi(x_0 \vee 2\theta_0)$,

$$\Phi^{-1}(r) \approx \varphi^{-1}(r).$$

Proof. Corollary 4.2.7 and Remark A.1.6 yield the first part. The proof of the second is omitted due to similarity to the proof of Proposition A.1.5. □

4.4.1 Upper estimates

From this moment on, we additionally assume that $\sigma = 0$. As explained in the introduction, that is equivalent to saying that **X** satisfies the integral condition (4.1.1). Recall that, since φ'' is positive and continuous on $(0, \infty)$, if $x_0 > 0$, then, at the cost of the constant c, we can extend area of comparability to any $x_1 \in (0, x_0)$ so that $\varphi'' \in \mathrm{WLSC}(\alpha-2, c', x_1)$, where c' depends on x_1. Thus, if $\theta_1 > 0$ and $x_0 > 0$, then we may and do assume that x_0 is shifted so that $x_0 \leq \theta_1$. With this in mind, let us define $\eta\colon [0,\infty) \mapsto [0,\infty]$,

$$\eta(s) = \begin{cases} \infty & \text{if } s = 0, \\ s^{-1}\Phi^*(1/s) & \text{if } 0 < s < x_0^{-1}, \\ As^{-1}|\varphi(1/s)| & \text{if } x_0^{-1} \leq s, \end{cases}$$

where $A = \Phi^*(x_0)/|\varphi(x_0)|$.

Let us comment on the function η. As we will see in Theorem 4.4.2, it will play a role of majorant on the transition density. In such setting it is clear that η must be non-negative. Moreover, in the proof we will need it to be monotone. Now, if $\theta_1 > 0$, then we know that φ is indeed monotone for $x \in (0, \theta_1)$, but it is also negative. Thus, a change of sign in the definition of η is required. On the other hand, if $\theta_1 = 0$, then $\varphi \geq 0$ and there is no need for absolute value and shifting of x_0. In general, however, φ may be negative in a neighbourhood of the origin and may change sign at θ_0, so one has to be careful in expanding scaling area to the proper place. Note that, by Corollary 4.2.4 and Remark A.1.6, we have $A \leq c'$ where c' depends only on θ_1.

Recall that

$$b_r = \gamma + \int_{(0,\infty)} s(\mathbb{1}_{s<r} - \mathbb{1}_{s<1})\, \nu(ds).$$

The next theorem provides the upper bound on the transition density. The reader may easily verify that it is a counterpart of Theorem 3.4.4.

Theorem 4.4.2. *Let* **X** *be a spectrally positive Lévy process of infinite variation with Lévy-Khintchine exponent* ψ *and Laplace exponent* φ. *Suppose that* $\sigma = 0$ *and* $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$ *and* $\alpha > 0$. *We also assume that the Lévy measure* ν *has an almost decreasing density* $\nu(x)$. *Then there is* $C > 0$ *such that for all* $t \in (0, 1/\Phi(x_0))$ *and* $x \in \mathbb{R}$,

$$p\big(t, x + tb_{1/\psi^{-1}(1/t)}\big) \leq C \min\big\{\Phi^{-1}(1/t), t\eta(|x|)\big\}.$$

Proof. In the first step we verify the assumptions of [42, Theorem 5.2]. First, observe that for any $\lambda > 0$,

$$\varphi''(\lambda) \geq \int_0^{1/\lambda} s^2 e^{-\lambda s} \nu(s)\, ds \gtrsim \nu(1/\lambda)\lambda^{-3}, \tag{4.4.1}$$

thus, by Corollary 4.2.4, $\nu(x) \lesssim \eta(x)$ for all $x > 0$. Since η is non-increasing, we conclude that the first assumption is satisfied. Next, we claim that η has a doubling property on $(0, \infty)$. Indeed, since φ'' is non-increasing, by Remark A.1.6 we have for $0 < s < x_0^{-1}$ that,

$$\eta(\tfrac{1}{2}s) \approx s^{-3}\varphi''(2/s) \lesssim s^{-3}\varphi''(1/s) \approx \eta(s).$$

This completes the argument for the case $x_0 = 0$. If $x_0 > 0$, then Proposition 4.2.2 (or (4.2.3) if $\theta_0 > 0$) yields the claim for $s > 2x_0^{-1}$. Lastly, the function

$$[\tfrac{1}{2}x_0, x_0] \ni x \mapsto \frac{\Phi^*(2x)}{|\varphi(x)|}$$

is continuous, hence bounded.

Therefore, since $s \wedge |x| - \frac{1}{2}|x| \geq \frac{1}{2}s$ for $s > 0$ and $x \in \mathbb{R}$, the doubling property of η and (A.1.6) imply the second assumption. Finally, since ψ^* has the weak lower scaling property and satisfies (A.1.7), by [43, Theorem 3.1] and Proposition A.1.4, there are $C > 0$ and $t_1 \in (0, \infty]$ such that for all $t \in (0, t_1)$,

$$\int_{\mathbb{R}} e^{-t \operatorname{Re} \psi(\xi)} \, d\xi \leq C\psi^{-1}(1/t),$$

with $t_1 = \infty$, whenever $x_0 = 0$. Note that if $t_1 < 48/\Phi(x_0)$, then using positivity and monotonicity we can expand the estimate for $t_1 \leq t < 48/\Phi(x_0)$, and the first step is finished.

Therefore, by [42, Theorem 5.2], there is $C > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in \mathbb{R}$,

$$p\big(t, x + tb_{1/\psi^{-1}(1/t)}\big) \leq C\psi^{-1}(1/t) \cdot \min\Big\{1, t\big(\psi^{-1}(1/t)\big)^{-1}\eta(|x|) + \big(1 + |x|\psi^{-1}(1/t)\big)^{-3}\Big\}.$$

Now, it suffices to prove that

$$\frac{\psi^{-1}(1/t)}{\big(1 + |x|\psi^{-1}(1/t)\big)^3} \lesssim t\eta(|x|), \tag{4.4.2}$$

whenever $t\eta(|x|) \leq \frac{A}{c'}\Phi^{-1}(1/t)$.

First, let us observe that for any $\varepsilon \in (0, 1]$, the condition $t\eta(|x|) \leq \frac{A\varepsilon}{c'}\Phi^{-1}(1/t)$ implies

$$t\Phi^*(1/|x|) \leq \varepsilon|x|\Phi^{-1}(1/t). \tag{4.4.3}$$

Indeed, by Corollary 4.2.4 and Remark A.1.6, we have $|x|\eta(|x|) \geq \frac{A}{c'}\Phi^*(1/|x|)$, which entails (4.4.3). Furthermore, we have $\varepsilon^{1/3}|x|\Phi^{-1}(1/t) \geq 1$, since otherwise by (A.1.3) we would have

$$1 < t\Phi^*\bigg(\frac{1}{\varepsilon^{1/3}|x|}\bigg) \leq \frac{1}{\varepsilon^{2/3}} t\Phi^*(1/|x|),$$

which in turn would yield

$$\varepsilon^{1/3}|x|\Phi^{-1}(1/t) < \varepsilon^{-2/3} t\Phi^*(1/|x|),$$

contrary to (4.4.3).

Now, we suppose $t\eta(|x|) \leq \frac{A}{c'}\Phi^{-1}(1/t)$. Since $|x|\Phi^{-1}(1/t) \geq 1$, by (A.1.3) we infer that

$$t\Phi^*(1/|x|) = \frac{\Phi^*(1/|x|)}{\Phi^*\big(|x|\Phi^{-1}(1/t) \cdot 1/|x|\big)} \geq \frac{1}{\big(|x|\Phi^{-1}(1/t)\big)^2} \geq \frac{|x|\Phi^{-1}(1/t)}{\big(1 + |x|\Phi^{-1}(1/t)\big)^3},$$

It remains to notice that Proposition A.1.5 entails (4.4.2), and the proof is completed. □

Remark 4.4.3. In the statement of Theorem 4.4.2 we may replace $b_{1/\psi^{-1}(1/t)}$ with $b_{1/\Phi^{-1}(1/t)}$. Indeed, if $0 < r_1 \leq r_2 < 1/\Phi(x_0)$, then by (A.1.6) and Proposition A.1.5,

$$|b_{r_1} - b_{r_2}| \leq \int_{(r_1,r_2]} s\,\nu(ds) \leq r_1^{-1} r_2^2 h(r_2) \lesssim r_1^{-1} r_2^2 \psi^*(r_2^{-1}) \lesssim r_1^{-1} r_2^2 \Phi^*(r_2^{-1}).$$

Thus, again by Proposition A.1.5, there is $C \geq 1$ such that for all $t \in (0, 1/\Phi(x_0))$,

$$\left| b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)} \right| \leq \frac{C}{t\Phi^{-1}(1/t)}.$$

Now, recall that if $t\eta(|x|) \leq \frac{A\varepsilon}{c'}\Phi^{-1}(1/t)$, then, by the proof of Theorem 4.4.2, we have $|x|\Phi^{-1}(1/t) \geq \varepsilon^{-1/3}$. Therefore, by taking $\varepsilon = (2C)^{-3}$, we obtain $|x| \geq \frac{2C}{\Phi^{-1}(1/t)}$, and, by monotonicity and doubling property of η, we conclude that

$$\eta\Big(\Big|x + t\Big(b_{1/\psi^{-1}(1/t)} - b_{1/\Phi^{-1}(1/t)}\Big)\Big|\Big) \lesssim \eta(|x|).$$

4.4.2 Lower estimates

We begin with an estimate which, together with (A.1.6), (A.1.7), [43, Theorem 3.1], and Propositions A.1.4 and A.1.5 (or Theorem 4.4.2 if the Lévy measure ν has an almost decreasing density), will allow us to localize the supremum of $p(t, \cdot)$. It is the analogue of Theorem 3.4.7 but here the main work is already done. Recall that the key point in the proof of Theorem 3.4.7 was to prove that the support of the probability distribution of a limit random variable contains the appropriate half-line. In our setting this problem is resolved by Grzywny and Szczypkowski [43] in Theorem 5.4. We encourage the reader to consult its proof for a conclusion that the analogous obstacle in the form of the support of the limit random variable was resolved precisely under the assumption $\alpha \geq 1$. In the following lemma we apply this result.

Lemma 4.4.4. *Let $\mathbf{X}$ be a spectrally positive Lévy process of infinite variation with Laplace exponent φ. Suppose that $\sigma = 0$ and $\varphi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\alpha \geq 1$. Then there is $M_0 > 1$ such that for each $M \geq M_0$ and $\rho_1, \rho_2 > 0$ there exists $C > 0$, so that for all $t \in (0, 1/\Phi(x_0))$ and any $x \in \mathbb{R}$ satisfying*

$$-\frac{\rho_1}{\Phi^{-1}(1/t)} \leq x + t\varphi'(\Phi^{-1}(M/t)) \leq \frac{\rho_2}{\Phi^{-1}(1/t)},$$

we have

$$p(t, x) \geq C\Phi^{-1}(1/t).$$

Proof. Without loss of generality we may assume that $b = 0$. By [43, Theorem 5.4], for any $\theta > 0$ there is $c > 0$ such that for all $t \in (0, 1/\Phi(x_0))$ and $|x| \leq \theta h^{-1}(1/t)$,

$$p\big(t, x + tb_{h^{-1}(1/t)}\big) \geq c\big(h^{-1}(1/t)\big)^{-1}.$$

Since, by Propositions A.1.4 and A.1.5, we have $h^{-1}(1/t) \approx 1/\Phi^{-1}(1/t)$ for all $t \in (0, 1/\Phi(x_0))$, it suffices to prove that

$$\left| t\varphi'(\Phi^{-1}(M/t)) + tb_{h^{-1}(1/t)} \right| \leq \frac{c_1}{\Phi^{-1}(1/t)}$$

for some $c_1 > 0$. To this end, observe that for $\lambda_1, \lambda_2 > 0$, we have

$$|\varphi'(\lambda_1) + b_{\lambda_2}| = \left| \int_0^\infty s(\mathbb{1}_{s<\lambda_2} - e^{-\lambda_1 s})\, \nu(\mathrm{d}s) \right| \lesssim \lambda_1 \int_0^{\lambda_2} s^2\, \nu(\mathrm{d}s) + \int_{\lambda_2}^\infty s e^{-\lambda_1 s}\, \nu(\mathrm{d}s)$$
$$\lesssim \lambda_1 \lambda_2^2 K(\lambda_2) + \lambda_1^{-1} h(\lambda_2).$$

Now, put $\lambda_1 = \Phi^{-1}(M/t)$ and $\lambda_2 = h^{-1}(1/t)$. Then, using again Propositions A.1.4 and A.1.5, we infer that

$$|\varphi'(\lambda_1) + b_{\lambda_2}| \lesssim \frac{1}{t\Phi^{-1}(1/t)},$$

and the proof is completed. □

In order to complete the picture, it remains to investigate the transition density on the right side of its supremum. After reading Chapter 3, it is not surprising that the expected expression by means of the Lévy density $\nu(x)$ appears also in case of spectrally one-sided Lévy processes as the following lemma states. Note, however, that here the assumption on the scaling exponent α is even more restrictive than in Lemma 4.4.4. It stays in contrast to the situation in previous chapter, where the assumption on α in corresponding Theorem 3.4.7 and Proposition 3.4.9 are the same, but for the sake of clarity we should note that in both of these results additional assumptions on $-\phi''$ appear.

Lemma 4.4.5. *Let* $\mathbf{X}$ *be a spectrally positive Lévy process of infinite variation with the Laplace exponent* φ*. Suppose that* $\sigma = 0$ *and* $\varphi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ *for some* $x_0 \geq 0$, $c \in (0, 1]$ *and* $\alpha > 1$*. We also assume that the Lévy measure* ν *has an almost decreasing density* $\nu(x)$*. Then there are* $M_0 > 1$, $\rho_0 > 0$ *and* $C > 0$ *such that for all* $t \in (0, 1/\Phi(x_0 \vee 2\theta_1))$ *and*

$$x \geq \frac{\rho_0}{\Phi^{-1}(1/t)},$$

we have

$$p(t, x) \geq Ct\nu(x).$$

Proof. Without loss of generality we may and do assume that $b = 0$. Let $\lambda > 0$. We decompose the Lévy measure $\nu(dx)$ as follows: let $\nu_1(dx) = \nu_1(x)\, dx$ and $\nu_2(dx) = \nu_2(x)\, dx$, where

$$\nu_1(x) = \frac{1}{2}\nu(x)\mathbb{1}_{[\lambda,\infty)}(x) \qquad \text{and} \qquad \nu_2(x) = \nu(x) - \nu_1(x).$$

For $u > 0$ we set

$$\begin{aligned} \varphi_1(u) &= \int_0^\infty (e^{-us} - 1)\nu_1(s)\, ds, \\ \varphi_2(u) &= \int_0^\infty (e^{-us} - 1 + us\mathbb{1}_{s<1})\nu_2(s)\, ds + u\int_0^1 s\nu_1(s)\, ds = \varphi(u) - \varphi_1(u). \end{aligned} \tag{4.4.4}$$

Let $\mathbf{X}^{(j)}$ be spectrally positive Lévy processes having the Laplace exponent φ_j, $j \in \{1, 2\}$. First, we observe that $\frac{1}{2}\nu \leq \nu_2 \leq \nu$, thus, for every $u > 0$,

$$\frac{1}{2}\varphi''(u) \leq \varphi_2''(u) \leq \varphi''(u),$$

and consequently,

$$\frac{1}{2}\Phi(u) \leq \Phi_2(u) \leq \Phi(u). \tag{4.4.5}$$

In particular, since $\varphi'' \in \mathrm{WLSC}(\alpha-2, c, x_0)$, by Proposition A.1.8 and the Hartman-Wintner condition (HW), random variables X_t and $X_t^{(2)}$ are absolutely continuous for all $t > 0$. By $p(t,\cdot)$ and $p^{(2)}(t,\cdot)$ we denote its densities. Observe that $\mathbf{X}^{(1)}$ is in fact a compound Poisson process. If we denote its probability distribution by $P_t^{(1)}(dx)$, then, by [93, Remark 27.3],

$$P_t^{(1)}(dx) \geq te^{-\nu_1(\mathbb{R})}\nu_1(x)\,dx. \tag{4.4.6}$$

Note that if $\lambda \geq c_1/\Phi^{-1}(1/t)$ for some $c_1 > 0$, then by (A.1.6),

$$t\nu_1(\mathbb{R}) = \frac{1}{2}t\int_\lambda^\infty \nu(x)\,dx \leq \frac{1}{2}th(c_1/\Phi^{-1}(1/t)) \lesssim th\big(1/\Phi^{-1}(1/t)\big) \lesssim 1 \tag{4.4.7}$$

where in the third line we use [43, Lemma 2.1].

Now, denote

$$x_t = -t\varphi_2'(\Phi_2^{-1}(M_0/t))$$

where M_0 is taken from Lemma 4.4.4. We claim that there is $\rho_0 > 0$ such that for all $t \in (0, 1/\Phi_2(x_0 \vee 2\theta_1))$,

$$\frac{\rho_0}{\Phi^{-1}(1/t)} \geq -x_t. \tag{4.4.8}$$

Indeed, observe that, by (4.4.5), for any $s > 0$,

$$\Phi_2^{-1}(s) \geq \Phi^{-1}(s). \tag{4.4.9}$$

Thus, using Proposition 4.2.6 and monotonicity of Φ^{-1}, we conclude that there is $c_2 > 0$ such that

$$t\varphi_2'(\Phi_2^{-1}(M_0/t)) \leq c_2 t\frac{M_0}{t}\frac{1}{\Phi_2^{-1}(M_0/t)} \leq \frac{c_2M_0}{\Phi^{-1}(M_0/t)} \leq \frac{c_2M_0}{\Phi^{-1}(1/t)},$$

and (4.4.8) follows with $\rho_0 = c_2M_0$.

Now, we apply Lemma 4.4.4 for $\mathbf{X}^{(2)}$. For all $\rho > 0$ there is $C > 0$ such that for all $t \in (0, 1/\Phi_2(x_0 \vee 2\theta_1))$ and $x \in \mathbb{R}$ satisfying

$$x_t - \frac{\rho}{\Phi_2^{-1}(1/t)} \leq x \leq x_t + \frac{\rho}{\Phi_2^{-1}(1/t)},$$

we have

$$p^{(2)}(t,x) \geq C\Phi_2^{-1}(1/t). \tag{4.4.10}$$

Note that, since $\varphi_2' \geq \varphi'$, we have that θ_1 for φ_2 cannot be bigger that θ_1 for φ. Thus, if $x_0 \vee 2\theta_1 > 0$ in (4.4.10), then we may easily extend the above for $t \in (0, 1/\Phi(x_0 \vee 2\theta_1))$. Let ρ_0 be taken from (4.4.8) and set $\lambda = x_t + \frac{\rho}{\Phi_2^{-1}(1/t)}$ where $\rho = \frac{3}{2}\rho_0$. Then it follows that $\lambda \geq \frac{1}{2}\frac{\rho_0}{\Phi_2^{-1}(1/t)}$, and consequently,

$$\int_0^\lambda p^{(2)}(t,x)\,dx \gtrsim 1. \tag{4.4.11}$$

Thus, using (4.4.6) and (4.4.7), for $x \geq 2\lambda$, we have

$$p(t,x) = \int_{\mathbb{R}} p^{(2)}(t,x-y)P_t^{(1)}(dy) \gtrsim t\int_{\mathbb{R}} p^{(2)}(t,x-y)\nu_1(y)\,dy \geq \frac{1}{2}t\int_\lambda^x p^{(2)}(t,x-y)\nu(y)\,dy.$$

Finally, using almost monotonicity of ν and (4.4.11), we get

$$p(t,x) \gtrsim t\nu(x)\int_0^\lambda p^{(2)}(t,y)\,dy \gtrsim t\nu(x).$$

Now, it remains to observe that by (4.4.4), for any $u > 0$,

$$\varphi_2'(u) \geq \varphi'(u).$$

Thus, by (4.4.9),

$$\lambda = -t\varphi_2'(\Phi_2^{-1}(M_0/t)) + \frac{\rho}{\Phi^{-1}(1/t)} \leq -t\varphi'(\Phi^{-1}(M_0/t)) + \frac{\rho}{\Phi^{-1}(1/t)}.$$

The proof is completed. □

4.5 Sharp two-sided estimates

This section is devoted to derivation of sharp two-sided estimates. As mentioned in the introduction, we will require here the upper scaling condition as well in order to express the Lévy density by means of Laplace exponent φ. First, however, thanks to strict separation from the limit case $\alpha = 1$, we are able to provide simpler expression for the localization of $\sup_{x\in\mathbb{R}} p(t,x)$. The following result is the counterpart of Theorem 3.4.10.

Theorem 4.5.1. *Let* $\mathbf{X}$ *be a spectrally positive Lévy process of infinite variation with the Laplace exponent* φ. *Suppose that* $\sigma = 0$ *and* $\varphi \in \mathrm{WLSC}(\alpha, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$ *and* $\alpha > 1$. *We assume also that* $\varphi'(\theta_1) = 0$. *Then for all* $-\infty < \chi_1 < \chi_2 < \infty$ *there is* $C > 0$ *such that for all* $t \in (0, 1/\Phi(x_0 \vee 2\theta_0))$ *and* $x \in \mathbb{R}$ *satisfying*

$$\chi_1 < x\varphi^{-1}(1/t) < \chi_2,$$

we have

$$C^{-1}\varphi^{-1}(1/t) \leq p(t,x) \leq C\varphi^{-1}(1/t). \tag{4.5.1}$$

Proof. First, let us note that, by Proposition 4.4.1, there is $C' \geq 1$ such that for all $r \in (0, 1/\Phi(x_0 \vee 2\theta_0))$,

$$C'^{-1}\Phi^{-1}(r) \leq \varphi^{-1}(r) \leq C'\Phi^{-1}(r). \tag{4.5.2}$$

Thus, in view of (A.1.6), (A.1.7), [43, Theorem 3.1], and Propositions A.1.4 and A.1.5, it is enough to prove the first inequality in (4.5.1). Next, we observe that assumptions of Lemma 4.4.4 are satisfied. Let M_0 be taken from Lemma 4.4.4; for fixed $M > M_0$ and $t \in (0, 1/\Phi(x_0 \vee 2\theta_0))$ we set

$$x_t = -t\varphi'(\Phi^{-1}(M/t)).$$

By Propositions 4.2.3 and A.1.5, and (4.5.2), there is $c_1 \in (0,1]$ such that

$$t\varphi'(\Phi^{-1}(M/t)) \geq \frac{c_1}{\varphi^{-1}(1/t)}.$$

Furthermore, by Proposition 4.2.6, (A.1.4) and (4.5.2), there is $C_1 \geq 1$ such that

$$t\varphi'(\Phi^{-1}(M/t)) \leq \frac{C_1}{\varphi^{-1}(1/t)}.$$

Now, we apply Lemma 4.4.4. Pick ρ_1 and ρ_2 so that

$$-c_1 - \frac{\rho_1}{C'} \leq \chi_1 \qquad \text{and} \qquad -C_1 + \frac{\rho_2}{C'} \geq \chi_2.$$

Then is is clear that

$$\left[\frac{\chi_1}{\varphi^{-1}(1/t)}, \frac{\chi_2}{\varphi^{-1}(1/t)}\right] \subset \left(x_t - \frac{\rho_1}{\Phi^{-1}(1/t)}, x_t + \frac{\rho_2}{\Phi^{-1}(1/t)}\right).$$

Hence, by Lemma 4.4.4 and (4.5.2), for all $t \in (0, 1/\Phi(x_0 \vee 2\theta_0))$ and $x \in \mathbb{R}$ satisfying

$$\chi_1 \leq x\varphi^{-1}(1/t) \leq \chi_2,$$

we have

$$p(t,x) \gtrsim \varphi^{-1}(1/t),$$

and the theorem follows. □

Proceeding exactly as in the proof of Proposition 3.4.11 and applying Corollary 4.2.7 yields the following.

Proposition 4.5.2. *Assume that the Lévy measure ν has an almost decreasing density $\nu(x)$. Suppose that $\varphi'(\theta_1) = 0$ and $\varphi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq \theta_0$ and $1 < \alpha \leq \beta < 2$. Then there is $c' \in (0,1]$ such that for all $0 < x < x_0^{-1} \wedge (2\theta_0)^{-1}$,*

$$\nu(x) \geq c' x^{-1}\varphi(1/x).$$

Now we are ready to prove our main result of this section. Recall that its special version when $x_0 = 0$ is depicted by Theorem 4.1.2 in the introduction of this chapter. In such case the estimates are in fact global both in space and in time.

Theorem 4.5.3. *Let $\mathbf{X}$ be a spectrally positive Lévy process of infinite variation with the Laplace exponent φ such that $\theta_1 = 0$ and $\varphi'(0) = 0$. Suppose that $\sigma = 0$ and $\varphi \in \mathrm{WLSC}(\alpha, c, x_0) \cap \mathrm{WUSC}(\beta, C, x_0)$ for some $c \in (0,1]$, $C \geq 1$, $x_0 \geq 0$, and $1 < \alpha \leq \beta < 2$. We also assume that the Lévy measure ν has an almost decreasing density $\nu(x)$. Then there is $x_1 \in (0, \infty]$ such that for all $t \in (0, 1/\Phi(x_0))$ and $x \in (-\infty, x_1)$,*

$$p(t,x) \approx \begin{cases} \big(t\varphi''(w)\big)^{-\frac{1}{2}} \exp\big\{-t\big(w\varphi'(w) - \varphi(w)\big)\big\}, & \text{if} \quad x\varphi^{-1}(1/t) \leq -1, \\ \varphi^{-1}(1/t), & \text{if} \quad -1 < x\varphi^{-1}(1/t) \leq 1, \\ tx^{-1}\varphi(1/x), & \text{if} \quad x\varphi^{-1}(1/t) > 1. \end{cases}$$

where $w = (\varphi')^{-1}(-x/t)$. If $x_0 = 0$, then $x_1 = \infty$.

Proof. Set $x_1 = x_0^{-1}$. First, we note that in view of Propositions 4.2.3 and 4.2.6, $\varphi'' \in \mathrm{WLSC}(\alpha - 2, c, x_0)$. Hence, by Corollary 4.3.5, for $\chi_1 = -M \wedge -1$,

$$p(t,x) \approx \big(t\varphi''(w)\big)^{\frac{1}{2}} \exp\big\{-t\big(w\varphi'(w) - \varphi(w)\big)\big\} \tag{4.5.3}$$

if only $x\varphi^{-1}(1/t) < \chi_1$. In fact, if $\chi_1 < -1$, then we also have

$$\big(t\varphi''(w)\big)^{\frac{1}{2}} \exp\big\{-t\big(w\varphi'(w) - \varphi(w)\big)\big\} \approx \varphi^{-1}(1/t) \tag{4.5.4}$$

for $\chi_1 \leq x\varphi^{-1}(1/t) \leq -1$. To show this, we first prove the following.

Claim 4.5.4. *There exist $0 < c_1 \leq 1 \leq c_2$ such that for all $t \in (0, c_1/\Phi(x_0))$ and $x \in (-\infty, x_1)$ satisfying*

$$\chi_1 \leq x\varphi^{-1}(1/t) \leq -1,$$

we have

$$-t\varphi'(\varphi^{-1}(c_2/t)) \leq x \leq -t\varphi'(\varphi^{-1}(c_1/t)).$$

Indeed, by Proposition 4.4.1, there is $C_1 \geq 1$ such that for all $r > \Phi(x_0)$,

$$C_1^{-1}\Phi^{-1}(r) \leq \varphi^{-1}(r) \leq C_1\Phi^{-1}(r).$$

Let $c_2 = (-\chi_1 C' C_1^2)^{\alpha/(\alpha-1)} \in [1, \infty)$ where C' is taken from (A.1.13). Then it follows that

$$c_2^{-1}\varphi^{-1}(c_2/t) \leq c_2^{(1-\alpha)/\alpha} C_1^2 C' \varphi^{-1}(1/t) = (-\chi_1)^{-1}\varphi^{-1}(1/t).$$

Thus, by Proposition 4.2.3,

$$x \geq -\frac{-\chi_1}{\varphi^{-1}(1/t)} \geq -t\frac{\varphi(\varphi^{-1}(c_2/t))}{\varphi^{-1}(c_2/t)} \geq -t\varphi'(\varphi^{-1}(c_2/t)).$$

Moreover, also by Proposition 4.2.3, with $c_1 = C^{-\alpha/(\alpha-1)}$ we have, for $t \in (0, c_1/\varphi(x_0))$,

$$t\varphi'(\varphi^{-1}(c_1/t)) \leq \frac{Cc_1}{\varphi^{-1}(1/t)} \cdot \frac{\varphi^{-1}(1/t)}{\varphi^{-1}(c_1/t)} \leq \frac{Cc_1^{(\alpha-1)/\alpha}}{\varphi^{-1}(1/t)},$$

thus,

$$x \leq -\frac{1}{\varphi^{-1}(1/t)} \leq -t\varphi'(\varphi^{-1}(c_1/t)),$$

and the claim follows.

Now, using Claim 4.5.4, Proposition A.1.5, (A.1.4), and Proposition 4.4.1, we get that for all $t \in (0, c_1/\Phi(x_0))$,

$$w \approx \varphi^{-1}(1/t). \tag{4.5.5}$$

Hence, in view of Proposition 4.2.3, $tw\varphi'(w) \approx 1$ and consequently,

$$\exp\big\{-t(w\varphi'(w) - \varphi(w))\big\} \approx 1.$$

Furthermore, by Proposition 4.2.6,

$$\varphi''(w) \approx w\varphi'(w),$$

which, combined with (4.5.5), yields

$$\frac{1}{\sqrt{t\varphi''(w)}} \approx \frac{w}{\sqrt{w\varphi'(w)}} \approx \varphi^{-1}(1/t),$$

and (4.5.4) follows for $t \in (0, c_1/\Phi(x_0))$.

Next, recall that $\theta_1 = 0$ and $\varphi'(0) = 0$. Therefore, in view of (2.2.12) and (2.2.5), we in fact have

$$b_r = -\int_r^\infty s\nu(s)\,ds.$$

Now, let $x > 1/\varphi^{-1}(1/t)$. By Theorem 4.4.2, Remark 4.4.3 and monotonicity of η,

$$p(t,x) \lesssim t\eta(x - tb_{1/\Phi^{-1}(1/t)}) \leq t\eta(x).$$

Thus, by Corollary 4.2.7, for all $t \in (0, 1/\Phi(x_0))$ and $x \in (0, x_1)$ such that $x\varphi^{-1}(1/t) > 1$,

$$p(t,x) \lesssim tx^{-1}\varphi(1/x). \tag{4.5.6}$$

Next, by Lemma 4.4.5 and Proposition 4.5.2, there are $M_0 > 0$, $\rho_0 > 0$ and $c > 0$ such that for any $t \in (0, 1/\Phi(x_0))$ and $x \in (0, x_1)$ satisfying $x\varphi^{-1}(1/t) \geq \rho_0$, we have

$$p(t, x) \geq ctx^{-1}\varphi(1/x). \tag{4.5.7}$$

Thus, if we set $\chi_2 = 1 \vee \rho_0$, then by (4.5.6) and (4.5.7), for all $t \in (0, 1/\Phi(x_0))$ and $x \in (0, x_1)$ such that $x\varphi^{-1}(1/t) \geq \chi_2$,

$$p(t, x) \approx tx^{-1}\varphi(1/x). \tag{4.5.8}$$

Finally, by Theorem 4.5.1, for all $t \in (0, 1/\Phi(x_0))$ and $x \in (-\infty, x_1)$ satisfying $\chi_1 < x\varphi^{-1}(1/t) < \chi_2$, we have

$$p(t, x) \approx \varphi^{-1}(1/t).$$

It remains to notice that if $\chi_2 > 1$, then, by scaling properties of φ, for all $t \in (0, 1/\Phi(x_0))$ and $x \in (0, x_1)$ satisfying $1 \leq x\varphi^{-1}(1/t) \leq \chi_2$, we have

$$tx^{-1}\varphi(1/x) \approx \varphi^{-1}(1/t),$$

which, combined with (4.5.3), (4.5.4) and (4.5.8), finishes the proof for the case $x_0 = 0$. If $x_0 > 0$, then we can use positivity and continuity to extend the time range from $c_1/\Phi(x_0)$ to $1/\Phi(x_0)$. □

Remark 4.5.5. Taking into account (4.4.1), Corollary 4.2.7 and Proposition 4.5.2, we may observe that, in fact,

$$\nu(x) \approx x^{-1}\varphi(1/x)$$

for all $0 < x < x_0^{-1} \wedge (2\theta_0)^{-1}$. Therefore, by inspecting the proof of Theorem 4.5.3, one can show that the term $tx^{-1}\varphi(1/x)$ in the thesis may be replaced by $t\nu(x)$.

Example 4.5.6. Let $\mathbf{X}$ be a spectrally positive α-stable process with the Laplace exponent $\varphi(\lambda) = \lambda^\alpha$ where $\alpha > 1$. Then it is clear that we have $\varphi''(\lambda) = \alpha(\alpha-1)\lambda^{\alpha-2}$ and $(\varphi')^{-1}(y) = (y/\alpha)^{1/(\alpha-1)}$. Consequently, from Theorem 4.1.1 we get that the asymptotics of $p(t, x)$ is of the form

$$E_\alpha(x,t) = \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}}\left(\frac{-x}{\alpha}\right)^{(2-\alpha)/2(\alpha-1)} t^{-1/2(\alpha-1)} \exp\left\{-(\alpha-1)t^{-1/(\alpha-1)}\left(\frac{-x}{\alpha}\right)^{\alpha/(\alpha-1)}\right\},$$

which after setting $t = 1$ coincides with Zolotarev [107, Theorem 2.5.2]. Moreover, by Theorem 4.5.3,

$$p(t,x) \approx \begin{cases} E_\alpha(x,t) & \text{if} \quad xt^{-1/\alpha} \leq -1, \\ t^{-1/\alpha}, & \text{if} \quad -1 < xt^{-1/\alpha} \leq 1, \\ tx^{-1-\alpha}, & \text{if} \quad xt^{-1/\alpha} > 1. \end{cases}$$

For $\alpha = 1$, in view of [92, Proposition 1.2.12], we have $\varphi(\lambda) = \lambda \ln \lambda$. Therefore, $\varphi''(\lambda) = \lambda^{-1}$ and $(\varphi')^{-1}(y) = e^{y-1}$. By Theorem 4.1.1, we get the following form of the asymptotics:

$$\frac{1}{\sqrt{2\pi t}} \exp\left\{\frac{-x/t - 1}{2} - te^{-x/t-1}\right\},$$

which, again, after substituting $t = 1$ coincides with Zolotarev [107, Theorem 2.5.2]. Unfortunately, Theorem 4.5.3 cannot be applied due to scaling condition with $\alpha = 1$ only. Let us also note that, for the case of Brownian motion, using Theorem 4.1.1 it is straightforward to obtain the Gaussian density in the asymptotics.

Lastly, let us justify Remark 4.3.3 by constructing an example of a spectrally positive Lévy process of unbounded variation for which the lower scaling property holds only with $\alpha < 1$.

Example 4.5.7. Let us consider a measure $\nu(dx)$ having density defined as follows: for $x \in (0, 1/2]$ set

$$\nu(x) = \begin{cases} c_k x^{-5/2}, & x \in \left[\big((2k+1)!\big)^{-1}, \big((2k)!\big)^{-1}\right], \\ c_k \sqrt{(2k+1)!}\, x^{-3/2}, & x \in \left[\big((2k+2)!\big)^{-1}, \big((2k+1)!\big)^{-1}\right], \end{cases} \tag{4.5.9}$$

where $c_k = \big((2k)!!\big)^{-1/2}$. For $x > 1/2$ we put $\nu(x) = 0$. By arguing as in the proofs of Propositions A.1.9 and 3.4.11, we conclude that

$$\varphi''(x) \approx x^{-3}\nu(1/x) \approx \begin{cases} c_k x^{-1/2}, & x \in \big[(2k)!, (2k+1)!\big], \\ c_k \sqrt{(2k+1)!} x^{-3/2}, & x \in \big[(2k+1)!, (2k+2)!\big]. \end{cases}$$

By construction, φ'' has lower and upper Matuszewska indices equal to $-3/2$ and $-1/2$, respectively, and consequently, the lower scaling property holds only with $\alpha < 1$.

Now let us define a Lévy process $\mathbf{X}$ by setting $\sigma = 0$, $b = 0$ and ν as in (4.5.9). Then we have

$$\begin{aligned} \int_0^1 x\nu(x)\,dx &\geq \sum_{k=1}^{\infty} c_k \int_{1/(2k+1)!}^{1/(2k)!} x^{-3/2}\,dx = 2\sum_{k=1}^{\infty} c_k \Big(\sqrt{(2k+1)!} - \sqrt{(2k)!}\Big) \\ &\geq \frac{1}{3}\sum_{k=1}^{\infty} c_k \sqrt{(2k+1)!} = \infty. \end{aligned}$$

Therefore, $\mathbf{X}$ is of unbounded variation.

Chapter 5

Hitting probabilities for Lévy processes on the real line

5.1 Introduction

The aim of this chapter is to discuss the distribution of the first hitting time of a point or a bounded interval for non-symmetric Lévy processes which satisfy the following integral condition

$$\int_0^\infty \frac{d\xi}{1 + \operatorname{Re}\psi(\xi)} < \infty, \tag{5.1.1}$$

and to derive sharp two-sided estimates of the tail probability of the first hitting time of a point or a bounded interval as well as its asymptotic behaviour. Its content is taken from the article of Grzywny, the author and Miśta [37].

Let us first provide a brief historical survey. The first studies on the first hitting time of a point or a compact set concerned α-stable processes. The asymptotic behaviour in the case of recurrent α-stable process, i.e. $1 < \alpha \leq 2$, for arbitrary compact sets was derived by Port [85]. Next, in Yano, Yano and Yor [106], the authors discuss the law of the first hitting time of a point for the symmetric α-stable processes with $1 < \alpha \leq 2$. A series representation of the density of the first hitting time of a point in the case of spectrally positive α-stable Lévy processes, $1 < \alpha < 2$, was obtained by Peskir [80] and Simon [96]. This result was extended to general α-stable processes with $1 < \alpha < 2$ in the work of Kuznetsov, Kyprianou, Pardo and Watson [65]. We note in passing that, in case of spectrally negative Lévy processes starting from the left side of the interval, the first hitting time is equal to the first passage time through the left end, and, in consequence, one may apply tools from the fluctuation theory to handle the problem.

The general symmetric case is much harder to handle and, in principle, requires some regularity assumptions on the characteristic exponent of the process. In [67] Kwaśnicki derived under mild assumptions an integral representation of the distribution function of the first hitting time of a point by means of eigenfunctions of the semigroup of the process killed upon hitting the origin. This idea was later adopted in the article by Juszczyszyn and Kwaśnicki [55] to obtain the asymptotic expansion of the distribution function (and its derivatives) of the first hitting time of a point for symmetric Lévy processes with completely monotone jumps. Recently, in Mucha [76], the ideas from [67] were extended to non-symmetric Lévy processes. A different strategy was proposed by Grzywny and Ryznar [40], where the authors prove and apply the global Harnack inequality in order to obtain sharp estimates of the tail probability of the first hitting time of points and bounded intervals for symmetric processes under global lower scaling condition imposed on the characteristic exponent. The approach applied in this chapter follows [40] as we generalize the ideas developed therein to non-symmetric Lévy processes.

Let us now describe our setting and indicate main difficulties. As one may expect, the non-symmetry induces some technical problems. In the case of recurrent symmetric processes presented in [40], the tail probability of the first hitting time may be described by the compensated potential kernel which is given by

$$S(x) = \int_0^\infty (p(s,0) - p(s,x))\, ds.$$

Here, $p(s, \cdot\,)$ is the heat kernel of the process which exists due to the integral condition. For the short proof observe that, since $e^{-x} \le (1+x)^{-1}$ for $x \ge 0$, the assumption (5.1.1) implies that $\left|e^{-t\psi(\,\cdot\,)}\right| = e^{-t\,\mathrm{Re}\,\psi(\,\cdot\,)}$ is integrable. Thus, by the Fourier inversion formula, the transition density $p(t, \cdot\,)$ of X_t exists for all $t > 0$ and is given by

$$p(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{Re}\left[e^{-t\psi(\xi) - i\xi x}\right] d\xi, \quad x \in \mathbb{R}.$$

The first natural guess would be to follow the approach from [40] and define initially the *compensated λ-potential kernel* by setting for $\lambda > 0$

$$S^\lambda(x) = U^\lambda(0) - U^\lambda(x) = \int_0^\infty e^{-\lambda t}(p(s,0) - p(s,x))\, ds, \quad x \in \mathbb{R}. \tag{5.1.2}$$

Here, we should warn the reader that in general non-symmetric case it is not a priori obvious that S^λ is positive. However, by [4, Corollary II.18 and Theorem II.19], one may conclude that the function $x \mapsto \mathbb{E}_x e^{-\lambda T_0}$ is continuous with respect to x, and for any $x \in \mathbb{R}$, we have

$$U^\lambda(x) = U^\lambda(0)\mathbb{E}_{-x} e^{-\lambda T_0}.$$

Therefore, we obtain that $S^\lambda \ge 0$ for all $\lambda \ge 0$. The next natural move would be to pass with λ to 0 and define *the compensated potential kernel* by

$$S(x) = \lim_{\lambda \to 0^+} S^\lambda(x), \quad x \in \mathbb{R}.$$

However, in our setting, one of substantial difficulties one has to overcome is the fact that we do not a priori know if S exists, and even if it does exist, it may vanish on the whole half-line, an example being a one-dimensional, completely asymmetric point recurrent stable process, i.e. with the stability index $\alpha \in (1,2)$ and skewness parameter $\beta = \pm 1$ (see the formula for the compensated potential kernel in Port [85, page 372]). For symmetric Lévy processes its existence is an easy consequence of the monotone convergence theorem, but in our case we are, in general, forced to adopt a different method. Instead, we propose an approach based on the symmetrized compensated potential kernel which we introduce right now. To this end, let us first define the *symmetrized compensated λ-potential kernel*

$$H^\lambda(x) = S^\lambda(x) + S^\lambda(-x).$$

We note that, by [4, Theorem II.19], we have

$$H^\lambda(x) = \frac{1}{\pi}\int_{\mathbb{R}} (1 - \cos xs)\, \mathrm{Re}\left[\frac{1}{\lambda + \psi(s)}\right] ds.$$

Now, recall that $\operatorname{Re}\psi(\xi) \gtrsim \xi^2$ for $|\xi| \leq 1$ (see e.g. [93, proof of Theorem 37.8]), which together with (5.1.1) justify the application of the dominated convergence theorem. Thus, the *symmetrized compensated potential kernel*

$$H(x) = \lim_{\lambda \to 0^+} H^\lambda(x), \quad x \in \mathbb{R},$$

is well defined and

$$H(x) = \frac{1}{\pi} \int_0^\infty (1 - \cos xs) \operatorname{Re}\left[\frac{1}{\psi(s)}\right] ds,$$

and it turns out to be the proper object for description of the behaviour of the first hitting time. Its huge advantage is the fact that the integrability condition we assume in the whole chapter ensures that H is well defined, and therefore, it can serve our purpose.

Let us briefly describe main results of this chapter. Our first goal is the asymptotic behaviour of the first hitting time of arbitrary compact sets which contain the origin, and this task is achieved by Theorem 5.3.7 and Corollary 5.3.8. Note that the asymptotic behaviour is described by means of compensated potential kernel S, so, in particular, one must ensure that S is well defined. The obtained asymptotics hold true if $\operatorname{Re}\psi$ varies regularly with parameter $\alpha \in (1, 2]$ and $\operatorname{Im}\psi$ displays a similar behaviour. The question about situations in which such condition holds true is non-trivial itself, and we devote Section 5.2 to provide the answer. By virtue of Theorem 5.2.4, this turns out to be true if the Lévy measure is of the special form

$$\nu(dx) = C_d \mathbb{1}_{x<0}\nu_0(dx) + C_u \mathbb{1}_{x>0}\nu_0(dx),$$

where $\nu_0(dx)$ is a symmetric Lévy measure. An important class of processes which clearly exhibit such behaviour are spectrally one-sided Lévy processes discussed exclusively in the previous chapter; in such case, we simply have either $C_d = 0$ (spectrally positive case) or $C_u = 0$ (spectrally negative case). For the sake of completeness, we also note that $C_u = C_d$ gives rise to a symmetric process. Another urgent question about well-definiteness of S is answered in the first half of Section 5.3, see in particular Remark 5.3.4 and Corollary 5.3.5.

Next, we turn our attention to derivation of sharp two-sided estimates of the first hitting time. To this end, we first prove the global scale invariant Harnack inequality under weak lower scaling condition on the real part of the characteristic exponent with $\alpha \in (1, 2]$. We then apply those results to obtain estimates of the tail of the first hitting time of points and intervals. They are derived under assumptions of global lower scaling property, zero mean, and a rather vague postulate about control of S^λ from below by H for small λ (see (5.1.2) for definition). The estimates are expressed by means of symmetrized compensated potential kernel H and renewal function for the dual process $\widehat{V}$, and therefore, in particular, do not require the existence of the compensated potential kernel S. We remark here that if the process is symmetric, then our assumptions reduce to those obtained in [40]. Since the third assumption is not a priori obvious for general non-symmetric processes, in Subsection 5.5.1 we present an example of a wide class of processes for which such property holds true. Furthermore, if we restrict ourselves to the special case of spectrally negative Lévy processes, then, due to their specific structure, we are able to prove sharp two-sided estimates on both sides of the interval and for any $t > 0$ (see Corollary 5.5.14).

At the end of this section let us stress the role of compensated potential kernels H and S. Under assumptions of the global scaling property of the real part of the characteristic exponent (which will be frequently imposed in this chapter), its control over the imaginary part (see Grzywny [35, Lemma 12]) and vanishing of the first moment, i.e. $\mathbb{E}X_1 = 0$, one can show that

$\int_0^1 \mathrm{Re}(1/\psi(\xi))d\xi = \infty$ and, in view of [4, Theorem I.17], conclude that $\mathbf{X}$ is recurrent, and consequently, $U(x) = \infty$ for all $x \in \mathbb{R}$. As the processes we study in this paper often satisfy such assumptions, we are in dire need of some object alternative to the (infinite) potential kernel. The symmetrized compensated potential kernel H can be of usage here (especially in Section 5.5), but it appears that the (ordinary) compensated kernel S is more appropriate for description of asymptotic behaviour of the hitting probability. The only problem is that, in general, we are not able to determine whether it exists. We devote Sections 5.3 and 5.5 to the detailed discussion on the subject. Let us also repeat the remark stated already in previous chapters that, although our main object to operate with is the real part of the characteristic exponent, one can work with the tail of the Lévy measure instead, since by the proof of Proposition A.1.9 and [43, Lemma 2.3, Theorem 3.1 and Remark 3.2], scaling property of the latter implies scaling of the former.

5.1.1 Preliminary results

We devote this subsection to the proof of some preliminary results, in particular, certain basic properties of compensated potential kernels defined above. As already stated, it is not easy to show even the existence of S, let alone its further properties. However, if it does exist, then one can express the probability of not hitting the origin by means of S and the quantity κ defined as

$$\kappa = \lim_{\lambda \to 0^+} \frac{1}{U^\lambda(0)} = \lim_{\lambda \to 0^+} \frac{1}{2\pi} \left(\int_{\mathbb{R}} \mathrm{Re}\left[\frac{1}{\lambda + \psi(\xi)} \right] d\xi \right)^{-1}. \tag{5.1.3}$$

Clearly, $\kappa \in [0, \infty)$. Moreover, from [4, Theorem I.17] it follows that the process is transient if $\kappa > 0$. We then have the following simple result.

Proposition 5.1.1. *Suppose that S exists. Then*

$$\mathbb{P}_x(T_0 = \infty) = \kappa S(-x).$$

If $\kappa = 0$, then $\mathbb{P}_x(T_0 < \infty) = 1$ for all $x \in \mathbb{R}$.

Note that it follows that if S exists and $\mathbb{P}_x(T_0 < \infty) = 1$, then $\mathbf{X}$ is recurrent.

Proof. Observe that

$$\lim_{\lambda \to 0^+} \mathbb{E}_x e^{-\lambda T_0} = \mathbb{P}_x(T_0 < \infty).$$

On the other hand,

$$\mathbb{E}_x e^{-\lambda T_0} = \frac{U^\lambda(-x)}{U^\lambda(0)} = 1 - \frac{U^\lambda(0) - U^\lambda(-x)}{U^\lambda(0)} \to 1 - \kappa S(-x)$$

as $\lambda \to 0^+$. □

Next, let us turn to the properties of the symmetrized compensated potential kernel H. First, proceeding as in the proof of [40, Proposition 2.2], one may prove that H is subadditive on $\mathbb{R}$. See also Pantí [78, Proposition 3.7] for a similar result, although derived under different assumptions.

In what follows, we will usually assume global weak lower scaling condition with $\alpha > 1$ imposed on the real part of the characteristic exponent. Clearly, such condition implies (5.1.1). The next proposition is crucial for our development.

Proposition 5.1.2. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *There is* $c \geq 1$ *such that for all* $r > 0$,

$$c^{-1}\frac{1}{rh(r)} \leq H(r) \leq c\frac{1}{rh(r)}.$$

In particular, $H \in \mathrm{WLSC}(\alpha-1,\tilde{\chi})$ *for some* $\tilde{\chi} \in (0,1]$.

Proof. Observe that by [35, Lemma 12], for any $r > 0$,

$$H(r) \approx \int_0^\infty (1-\cos rs)\frac{1}{\operatorname{Re}\psi(s)}\,ds.$$

Now, the claim follows by [35, Lemma 13]. □

Next, we show that the symmetrized compensated potential kernel H provides the upper bound on the Green function of the singleton $\{0\}$ and on the expected first exit time from the interval $(-R,R)$. These results will be useful in the proof of Harnack inequality in Section 5.4.

Proposition 5.1.3. *For any* $x, y \in \mathbb{R}$ *we have*

$$G_{\{0\}}(x,y) \leq H(x) \wedge H(y).$$

Proof. By (2.2.13), for any $\lambda > 0$ we have

$$\begin{aligned} G^\lambda_{\{0\}}(x,y) &= U^\lambda(y-x) - \mathbb{E}_x e^{-\lambda T_0} U^\lambda(y-T_0) \\ &= U^\lambda(y-x) - U^\lambda(y)\frac{U^\lambda(-x)}{U^\lambda(0)} \\ &= -S^\lambda(y-x) + S^\lambda(-x) + S^\lambda(y) - \frac{S^\lambda(-x)S^\lambda(y)}{U^\lambda(0)}. \end{aligned}$$

Recall that $S^\lambda \geq 0$. Proceeding as in the proof of [40, Proposition 2.2], we get that S^λ is subadditive on $\mathbb{R}$. Thus, $S^\lambda(y) \leq S^\lambda(x) + S^\lambda(y-x)$ and consequently,

$$G^\lambda_{\{0\}}(x,y) \leq -S^\lambda(y-x) + S^\lambda(-x) + S^\lambda(y) \leq S^\lambda(x) + S^\lambda(-x).$$

Similarly, we have $S^\lambda(-x) \leq S^\lambda(y-x) + S^\lambda(-y)$, and the claim follows. □

Proposition 5.1.4. *For any* $|x| \in (0,R)$ *we have*

$$\mathbb{E}_x\left[\tau_{(-R,R)} \wedge T_0\right] \leq 2RH(x).$$

Proof. By Proposition 5.1.3,

$$\mathbb{E}_x\left[\tau_{(-R,R)} \wedge T_0\right] = \int_{-R}^R G_{(-R,0)\cup(0,R)}(x,y)\,dy \leq \int_{-R}^R G_{\{0\}}(x,y)\,dy \leq 2RH(x).$$

□

We end this section with a sharp estimate of the probability that the process, when exiting the interval $(0,R)$, chooses the right end. Such result seems to be interesting in and of itself, as we provide sharp two-sided bound which is an analogue of the estimate for the symmetric case (see e.g. Grzywny and Ryznar [39, Proposition 3.7]). Regardless of its cognitive value, such estimate will prove to be useful in Section 5.4. Let us note in passing that a number of helpful results concerning $\widehat{V}$, which is of significant importance and usage in the case of non-symmetric processes, is derived by Grzywny [35].

Proposition 5.1.5. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha, \chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0, 1]$. *Then there is* $c \in (0, 1]$ *such that for any* $R > 0$ *and* $0 < x < R$,

$$c\frac{\widehat{V}(x)}{\widehat{V}(R)} \leq \mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big) \leq \frac{\widehat{V}(x)}{\widehat{V}(R)}.$$

Proof. Fix $R > 0$ and let $x \in (0, R)$. First, from the proof of [35, Theorem 9] we get the following estimate:

$$\mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big) \leq \frac{\widehat{V}(x)}{\widehat{V}(R)}.$$

Next, we claim that there is c_1 such that

$$c_1\frac{\widehat{V}(x)}{\widehat{V}(h^{-1}(1/t))} - \frac{\widehat{V}(x)V(R)}{t} \leq \mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big) \tag{5.1.4}$$

if only $t \geq 1/h(R)$. Indeed, observe that by Markov inequality and [35, Proposition 4],

$$\mathbb{P}_x\big(\tau_{(0,\infty)} > t\big) \leq \mathbb{P}_x\big(\tau_{(0,R)} > t\big) + \mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big) \leq \frac{\widehat{V}(x)V(R)}{t} + \mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big).$$

Thus, using [35, Theorem 6] we get (5.1.4) for $t \geq 1/h(R)$ as claimed.

Now we specify $t > 0$. Set $t = a/h(R)$ where $a \geq 1$. By [43, Lemma 2.3], there is $c_2 \geq 1$ such that

$$h^{-1}(1/t) = h^{-1}\big(h(R)/a\big) \leq c_2 a^{1/\alpha} R.$$

Taking into account subadditivity and monotonicity of $\widehat{V}$, we infer that

$$\widehat{V}\big(h^{-1}(1/t)\big) \leq \widehat{V}\big(c_2 a^{1/\alpha} R\big) \leq 2c_2 a^{1/\alpha}\widehat{V}(R)$$

if only $a \geq c_2^{-\alpha}$. Furthermore, by [35, Corollary 5], there is $c_3 \geq 1$ such that

$$h(R) \leq \frac{c_3}{V(R)\widehat{V}(R)}.$$

It follows that

$$\begin{aligned}
\mathbb{P}_x\big(\tau_{(0,R)} < \tau_{(0,\infty)}\big) &\geq 2^{-1}c_1 c_2^{-1} a^{-1/\alpha}\frac{\widehat{V}(x)}{\widehat{V}(R)} - \frac{c_3}{a}\frac{\widehat{V}(x)}{\widehat{V}(R)} \\
&= \frac{\widehat{V}(x)}{\widehat{V}(R)}\frac{1}{a}\Big(2^{-1}c_1 c_2^{-1} a^{(\alpha-1)/\alpha} - c_3\Big) \\
&\geq \frac{1}{a}\frac{\widehat{V}(x)}{\widehat{V}(R)}
\end{aligned}$$

if a is big enough, and the claim follows. ☐

5.2 Regular variation

In this section, we aim to prove that the regular variation of the real part of the characteristic exponent implies regular variation of its imaginary part if we impose some condition on the structure of the Lévy measure. The main result here is Theorem 5.2.4 which provides an important example for the asymptotic relation in Theorem 5.3.7. As already stated, this class is described by the condition (5.2.2) which imposes some control on the non-symmetry of the Lévy measure ν. Meanwhile, in Proposition 5.2.1 and Lemma 5.2.3 we prove regular variation of some transformations which seem to be of some value in and of itself. For instance, the former immediately implies that regular variation of the Lévy measure implies regular variation of the real part of the characteristic exponent. We note in passing that these preliminary results may be perceived as some versions of Tauberian and Abelian theorems presented in Chapter 4 of the book by Bingham, Goldie and Teugels [5].

Let us first note same basic observations. It is clear from the Lévy-Khintchine representation (2.2.1) that

$$\operatorname{Re}\psi(\xi) = \sigma^2\xi^2 + \int_{\mathbb{R}} (1 - \cos\xi z)\,\nu(dz), \quad \xi \in \mathbb{R},$$

and

$$\operatorname{Im}\psi(\xi) = -\gamma\xi + \int_{\mathbb{R}} (\xi z \mathbb{1}_{|z|<1} - \sin\xi z)\,\nu(dz), \quad \xi \in \mathbb{R}.$$

Observe that $\operatorname{Re}\psi$ is symmetric even if $\mathbf{X}$ is not symmetric. If we assume that $\int_{(-1,1)^c} |z|\,\nu(dz) < \infty$, then we can also write

$$\operatorname{Im}\psi(\xi) = -\gamma_1\xi + \int_{\mathbb{R}} (\xi z - \sin\xi z)\,\nu(dz), \tag{5.2.1}$$

where

$$\gamma_1 = \gamma + \int_{(-1,1)^c} z\,\nu(dz).$$

In particular, if $\mathbb{E}X_1 = 0$, then $\gamma_1 = 0$.

Next, following [5, Chapter 4], we introduce notions of Mellin transform and Mellin convolution. Let $f\colon (0,\infty) \mapsto \mathbb{R}$. The *Mellin transform* $\mathcal{M}$ of the function f is defined by

$$\mathcal{M}f(z) = \int_0^\infty t^{-z}k(t)\,\frac{dt}{t},$$

for $z \in \mathbb{C}$ such that the integral converges. For $f, g\colon (0,\infty) \mapsto \mathbb{R}$ we define its *Mellin convolution* by

$$g \overset{m}{*} f(x) = \int_0^\infty g(x/t)f(t)\,\frac{dt}{t}, \quad x > 0.$$

Finally, by

$$f \overset{s}{*} g(x) = \int_0^\infty f(x/t)\,dg(t), \quad x > 0,$$

we denote the *Mellin-Stieltjes convolution* for functions f and g such that the Stieltjes integral is well defined.

Let us start with the observation on the equivalence between tail measure and real part of the characteristic exponent behaviour.

Proposition 5.2.1. *Let $\alpha \in (0,2)$. We have $\operatorname{Re}\psi \in \mathcal{R}_\alpha^\infty$ ($\operatorname{Re}\psi \in \mathcal{R}_\alpha^0$) if and only if $t \mapsto \nu(\{s\colon |s| \geq t\})$ is regularly varying at 0 (at infinity) with the exponent $-\alpha$. Moreover,*

$$\nu(\{s\colon |s| \geq t\}) \sim \frac{\Gamma(1+\alpha)}{\mathrm{B}\Big(1-\frac{\alpha}{2}, 1+\frac{\alpha}{2}\Big)}\operatorname{Re}\psi(1/t), \quad t \to 0^+ \quad (t \to \infty),$$

where B *is the Euler beta function.*

Proof. We compute the Laplace transform of the function $\operatorname{Re}\psi$. By Fubini's theorem,

$$\begin{aligned}\mathcal{L}(\operatorname{Re}\psi)(\lambda) &= \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}}(1-\cos tx)\,\nu(dx)\,dt = \int_{\mathbb{R}}\int_0^\infty e^{-\lambda t}(1-\cos tx)\,dt\,\nu(dx)\\ &=\frac{1}{\lambda}\int_{\mathbb{R}}\frac{x^2}{\lambda^2+x^2}\,\nu(dx).\end{aligned}$$

Next,

$$\begin{aligned}\frac{\lambda}{2}\mathcal{L}(\operatorname{Re}\psi)(\lambda) &=\frac{1}{2}\Big(\int_{-\infty}^0\frac{x^2}{\lambda^2+x^2}\,\nu(dx)+\int_0^\infty\frac{x^2}{\lambda^2+x^2}\,\nu(dx)\Big)\\ &=-\int_{-\infty}^0\int_x^0\frac{\lambda^2 t}{(\lambda^2+t^2)^2}\,dt\,\nu(dx)+\int_0^\infty\int_0^x\frac{\lambda^2 t}{(\lambda^2+t^2)^2}\,dt\,\nu(dx)\\ &=-\int_{-\infty}^0\frac{(\lambda t)^2}{(\lambda^2+t^2)^2}\nu((-\infty,t])\frac{dt}{t}+\int_0^\infty\frac{(\lambda t)^2}{(\lambda^2+t^2)^2}\nu([t,\infty))\frac{dt}{t}\\ &=\int_0^\infty\frac{(\frac{\lambda}{t})^2}{\left(1+(\frac{\lambda}{t})^2\right)^2}\big(\nu((-\infty,-t])+\nu([t,\infty))\big)\frac{dt}{t}.\end{aligned}$$

Assume first that $\operatorname{Re}\psi\in\mathcal{R}_\alpha^\infty$ with $\alpha\in(0,2)$. Let $\nu_1=\nu\mathbb{1}_{[-1,1]}$ and ψ_1 be the characteristic exponent corresponding to the triplet $(0,0,\nu_1)$. We claim that $\operatorname{Re}\psi\sim\operatorname{Re}\psi_1$ at infinity. Indeed, since

$$0\le\int_{|x|>1}(1-\cos(xz))\,\nu(dx)\le 2\nu(\{x\colon |x|>1\}),\quad z\in\mathbb{R},$$

and $\lim_{z\to\infty}\operatorname{Re}\psi(z)=\infty$, we get that

$$\lim_{z\to\infty}\frac{\operatorname{Re}\psi(z)-\operatorname{Re}\psi_1(z)}{\operatorname{Re}\psi(z)}=0,$$

and the claim follows. Next, using the Abel theorem ([5, Theorem 1.7.1]), one can observe that

$$\lambda^{-1}\mathcal{L}(\operatorname{Re}\psi_1)(1/\lambda)\sim\operatorname{Re}\psi(\lambda)\Gamma(1+\alpha),\quad \lambda\to\infty.$$

Now, for $t>0$, let us define $g(t)=\nu_1(\{s\colon |s|\ge 1/t\})$ and $k(t)=\frac{t^2}{(1+t^2)^2}$. Observe that the Laplace transform of $\operatorname{Re}\psi_1$ may be expressed by means of Mellin convolution of k and g:

$$\frac{1}{2\lambda}\mathcal{L}(\operatorname{Re}\psi_1)(1/\lambda)=\int_0^\infty k(1/(t\lambda))g(1/t)\,\frac{dt}{t}=\int_0^\infty k(t/\lambda)g(t)\,\frac{dt}{t}=k\overset{m}{*}g(\lambda),$$

where in the last equality we used the fact $k(t)=k(1/t)$ for $t>0$. In order to prove that $g(t)$ is regularly varying function, we will use [5, Theorem 4.9.1] for the function $g_1(t)=\int_0^t g(s)\,\frac{ds}{s}$ and convolution $k\overset{s}{*}g_1(\lambda)=k\overset{m}{*}g(\lambda)$. Set σ such that $-2<\sigma<-\alpha$ and $\tau=0$. Observe that

$$\begin{aligned}\|k\|_{\sigma,\tau} &= \sum_{-\infty<n<\infty}\max(e^{-\sigma n},e^{-\tau n})\sup_{e^n\le x\le e^{n+1}}|k(x)|\\ &\le\sum_{n\le -1}e^{2n}+\sum_{n\ge 0}\frac{e^{-\sigma n}}{e^{2n}}<\infty.\end{aligned}$$

See [5, p. 210, eq. (4.4.3)] for the first appearance and introduction of the $\|k\|_{\sigma,\tau}$. Moreover, by [77, Table 1.2 (2.19)],

$$\mathcal{M}k(z)=\int_0^\infty\frac{t^2}{(1+t^2)^2}t^{z-1}\,dt=\frac{1}{2}\int_0^\infty\frac{1}{(1+s)^2}s^{(z/2+1)-1}\,ds=\frac{\Gamma(1-\frac{z}{2})\Gamma(1+\frac{z}{2})}{2},$$

if only $\operatorname{Re} z \in (-2, 2)$. Since $(\sigma, \tau) \subset (-2, 2)$ and neither $\Gamma(1 + \frac{z}{2})$ nor $\Gamma(1 - \frac{z}{2})$ have any roots for $\operatorname{Re} z \in (\sigma, \tau)$, the Wiener condition $\mathcal{M}k(z) \neq 0$ is satisfied. Notice that g_1 is non-decreasing on $(0, \infty)$ and vanishes on $(0, 1)$. Hence, $g_1(t) = \mathcal{O}(t^\sigma)$ at 0^+. It follows that the kernel k and the function g_1 satisfy assumptions of [5, Theorem 4.9.1], hence,

$$g_1(t) \sim \frac{\Gamma(1+\alpha)}{\mathrm{B}\Big(1 - \frac{\alpha}{2}, 1 + \frac{\alpha}{2}\Big)} \frac{\operatorname{Re} \psi(t)}{\alpha}, \quad t \to \infty.$$

By the monotone density theorem [5, Theorem 1.7.2] and the fact that $g(1/t) \sim \nu(\{s \colon |s| \geq t\})$ as t goes to 0^+, we obtain that

$$\nu(\{s \colon |s| \geq t\}) \sim \frac{\Gamma(1+\alpha)}{\mathrm{B}\Big(1 - \frac{\alpha}{2}, 1 + \frac{\alpha}{2}\Big)} \operatorname{Re} \psi(1/t), \quad t \to 0^+.$$

In particular, $t \mapsto \nu(\{s \colon |s| \geq t\})$ is regularly varying function at 0 with index $-\alpha$.

Now assume that $t \mapsto \nu(\{s \colon |s| \geq t\})$ is regularly varying function at 0 with index $-\alpha$. Again, instead of ψ, one can consider ψ_1. Since

$$\operatorname{Re} \psi_1(z) = z \int_0^1 \sin(xz) \nu(\{s \colon |s| \geq x\}) \, dx = \int_0^z \sin(x) \nu(\{s \colon |s| \geq x/z\}) \, dx,$$

one can use the Potter bounds to justify that

$$\lim_{z \to \infty} \frac{\operatorname{Re} \psi_1(z)}{\nu(\{s \colon |s| \geq 1/z\})} = \int_0^\infty \frac{\sin x}{x^\alpha} \, dx,$$

which finishes the proof in this case.

If $\operatorname{Re} \psi \in \mathcal{R}^0_\alpha$, then one can modify the above prove to obtain the behaviour of the tail of ν at infinity. □

We remark that, in fact, equivalence of regular variation of $\operatorname{Re} \psi$ at the origin and regular variation of the tail of ν at infinity can be easily obtained from Pitman [83]. Our proof, however, works in both cases.

We note one important observation: if the Lévy measure is of the special form

$$\nu(dx) = C_d \mathbb{1}_{x<0} \nu_0(dx) + C_u \mathbb{1}_{x>0} \nu_0(dx), \tag{5.2.2}$$

where $\nu_0(dx)$ is a symmetric Lévy measure, the theorem above provides the behaviour of the one-sided tail of ν as well. For instance, if $\operatorname{Re} \psi \in \mathcal{R}^0_\alpha$ with the exponent $\alpha \in (0, 2)$, then

$$\nu_0([t, \infty)) \sim \frac{1}{C_u + C_d} \frac{\Gamma(1+\alpha)}{\mathrm{B}\Big(1 - \frac{\alpha}{2}, 1 + \frac{\alpha}{2}\Big)} \operatorname{Re} \psi(1/t), \quad t \to \infty.$$

This is the case for stable processes where even the equality holds true.

The next proposition states that regular variation of $\operatorname{Re} \psi$ at the origin with $\alpha > 1$ implies the existence of the first moment.

Proposition 5.2.2. *If* $\operatorname{Re} \psi \in \mathcal{R}^0_\alpha$ *with* $\alpha > 1$, *then*

$$\int_{(-1,1)^c} |x| \, \nu(dx) < \infty.$$

Proof. If $\operatorname{Re}\psi$ varies regularly at the origin with a positive exponent, then $\operatorname{Re}\psi \sim \psi^*$ at the origin. Due to (2.2.6) we have then that $h(r) \approx \operatorname{Re}\psi(1/r)$ for large $r > R_0$. By the Potter bounds we get

$$\begin{aligned}\int_{|x|>1} |x|\,\nu(dx) &= \int_0^\infty \nu(\{s\colon |s| > 1 \vee u\})\,du \\ &\leq \int_0^\infty h(1\vee u)\,du \\ &\leq h(1)R_0 + c\int_{R_0}^\infty \operatorname{Re}\psi(1/u)\,du \\ &< \infty.\end{aligned}$$

The claim follows immediately. □

Lemma 5.2.3. *Assume that a function $f\colon [0,\infty) \mapsto [0,\infty)$ is regularly varying at infinity with parameter $-\alpha$, where $\alpha \in (1,2)$, and satisfies the following integrability condition*

$$\int_0^\infty (1 \wedge s^2) f(s)\,ds < \infty.$$

Then the transformation

$$x \mapsto \int_0^\infty (1-\cos xs) f(s)\,ds, \quad x \geq 0,$$

is regularly varying at the origin with the parameter $\alpha - 1$ and satisfies

$$\int_0^\infty (1-\cos xs) f(s)\,ds \sim -\frac{f(1/x)}{x}\frac{\pi}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}, \quad x \to 0^+.$$

Proof. Let $x > 0$. For any $s_0 > 0$ we have

$$\begin{aligned}\frac{x}{f(1/x)}\int_0^\infty (1-\cos xs) f(s)\,ds &= \frac{x}{f(1/x)}\left(\int_0^{s_0} (1-\cos xs) f(s)\,ds + \int_{s_0}^\infty (1-\cos xs) f(s)\,ds\right) \\ &= \frac{x}{f(1/x)}\int_0^{s_0} (1-\cos xs) f(s)\,ds + \int_{s_0 x}^\infty (1-\cos s)\frac{f(s/x)}{f(1/x)}\,ds.\end{aligned}$$

Now we specify s_0. Fix $\delta > 0$ such that $\alpha - \delta > 1$ and $M \geq 1$. By the Potter bounds, there is s_0 such that for $s > s_0 x$ we have $\frac{f(s/x)}{f(1/x)} < Ms^{-\alpha+\delta}$ if only $0 < x < s_0^{-1}$. Moreover, also by the Potter bounds, for some fixed $C > 1$ and $\alpha < \rho < 2$ there exists $\varepsilon > 0$ such that $Cx^\rho \leq f(1/x)$, whenever $0 < x < \varepsilon$. It follows that

$$\frac{x}{f(1/x)}\int_0^A (1-\cos xs) f(s)\,ds \leq C\frac{x^3}{x^\rho}\int_0^A s^2 f(s)\,ds \to 0$$

as $x \to 0^+$. Therefore, by the dominated convergence theorem,

$$\begin{aligned}\lim_{x\to\infty} \frac{x}{f(1/x)}\int_0^\infty (1-\cos xs) f(s)\,ds &= \int_0^\infty (1-\cos s)s^{-\alpha}\,ds \\ &= \frac{\pi}{2\Gamma(\alpha)\sin\left(\frac{\pi(\alpha-1)}{2}\right)} \\ &= -\frac{\pi}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}\end{aligned}$$

where the second equality follows from [93, Theorem 14.15]. □

We are now ready to prove the main result of this section. It shows that, under the assumption (5.2.2), regular variation of the real part of the Lévy exponent implies regular variation of the imaginary part. This is the case for instance for spectrally one-sided Lévy processes.

Theorem 5.2.4. *Assume that the Lévy measure $\nu(dx)$ satisfies (5.2.2). Suppose that* $\operatorname{Re}\psi \in \mathcal{R}_\alpha^0$ *with the parameter $\alpha \in (1,2)$. If $\gamma_1 = 0$, then the imaginary part* $\operatorname{Im}\psi$ *satisfies*

$$\operatorname{Im}\psi(\xi) \sim -\frac{C_u - C_d}{C_u + C_d}\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{Re}\psi(\xi), \quad \xi \to 0^+.$$

For $\gamma_1 \neq 0$ we have

$$\operatorname{Im}\psi(\xi) \sim \gamma_1\xi, \quad \xi \to 0.$$

Proof. By Proposition 5.2.1, the function $t \mapsto \nu_0([t,\infty))$ is regularly varying at infinity with the exponent $-\alpha$. More precisely,

$$\nu_0([t,\infty)) \sim \frac{1}{C_u + C_d}\frac{\Gamma(1+\alpha)}{\mathrm{B}\left(1-\frac{\alpha}{2}, 1+\frac{\alpha}{2}\right)}\operatorname{Re}\psi(1/t), \quad t \to \infty. \tag{5.2.3}$$

By Proposition 5.2.2, we have that $\int_{(-1,1)^c} |x|\,\nu(dx) < \infty$. We are therefore allowed to use the representation (5.2.1) of $\operatorname{Im}\psi$. For any $\xi > 0$ we have

$$\begin{aligned}
\operatorname{Im}\psi(\xi) =&\gamma_1\xi + \int_{\mathbb{R}}(\xi x - \sin\xi x)\,\nu(dx)\\
=&\gamma_1\xi + (C_u - C_d)\int_0^\infty\int_0^x(\xi t - \sin\xi t)'\,dt\,\nu_0(dx)\\
=&\gamma_1\xi + (C_u - C_d)\xi\int_0^\infty(1-\cos\xi t)\int_t^\infty \nu_0(dx)\,dt\\
=&\gamma_1\xi + (C_u - C_d)\xi\int_0^\infty(1-\cos\xi t)\,\nu_0([t,\infty))\,dt.
\end{aligned}$$

Let $C_u \neq C_d$. Observe that $f(s) = \nu([s,\infty))$ satisfies the assumptions of Lemma 5.2.3. Thus, the function $\xi \mapsto (C_u - C_d)\xi\int_0^\infty(1-\cos\xi t)\nu_0([t,\infty))\,dt$ is also regularly varying at the origin with the exponent α. Assume that $\gamma_1 \neq 0$. Then

$$\frac{\gamma_1\xi + \xi(C_u - C_d)\int_0^\infty(1-\cos\xi t)\nu_0([t,\infty))\,dt}{\gamma_1\xi} \to 1$$

as $\xi \to 0^+$, which follows from the Potter bounds for $(C_u - C_d)\xi\int_0^\infty(1-\cos\xi t)\nu_0([t,\infty))\,dt$. Then $\operatorname{Im}\psi$ is comparable at the origin with a linear function and the claim follows in this case. Now, assume that $\gamma_1 = 0$. Again by Lemma 5.2.3,

$$\xi\int_0^\infty(1-\cos\xi t)\nu_0([t,\infty))\,dt \sim \nu_0([1/\xi,\infty))\frac{\pi}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}, \quad \xi \to 0^+.$$

Using the identity $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin\pi z}$ and invoking (5.2.3), we obtain

$$\begin{aligned}
\operatorname{Im}\psi(\xi) \sim& -\frac{C_u - C_d}{C_u + C_d}\frac{\alpha\Gamma(\alpha)}{\Gamma(1-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}\frac{\pi}{2\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}\operatorname{Re}\psi(\xi)\\
\sim& -\frac{C_u - C_d}{C_u + C_d}\frac{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2})}\frac{\pi}{\cos\left(\frac{\pi\alpha}{2}\right)}\operatorname{Re}\psi(\xi)\\
\sim& -\frac{C_u - C_d}{C_u + C_d}\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{Re}\psi(\xi), \quad \xi \to 0^+.
\end{aligned}$$

□

5.3 Asymptotics

In this section, we aim first at providing some sufficient conditions for the existence of the compensated potential kernel S, and then at deriving asymptotics of the first hitting time of a compact set B. The reason for this particular order is the fact that the main result displayed by Theorem 5.3.7 describes the behaviour of $\mathbb{P}_x(T_B > t)$ precisely by means of kernel S. It is therefore sensible to discuss exclusively the existence of S before turning to the proof of the asymptotic behaviour. We note here that if the $\mathbf{X}$ is symmetric, then, by [105, Theorem 4.2], the kernel S is well defined. Furthermore, by [105, Proposition 6.1], for non-symmetric case the existence of first derivatives of $(\operatorname{Re}\psi(\xi))'$ and $(\operatorname{Im}\psi(\xi))'$ together with the following condition

$$\int_0^\infty \frac{\left(|(\operatorname{Re}\psi(\xi))'| + |(\operatorname{Im}\psi(\xi))'|\right)(\xi^2 \wedge 1)}{|\psi(\xi)|^2}\, d\xi < \infty \tag{5.3.1}$$

are also sufficient. We remark here that, in view of [93, Theorem 21.9] and discussion at the beginning of *Elements of fluctuation theory* in Chapter 2, the condition **L1'** in [105] always implies **L2**. However, (5.3.1) does not suit our case, and therefore, we first prove the existence of S in several cases. The first one requires the non-negativity of the imaginary part of the characteristic exponent on some positive neighbourhood of the origin.

Lemma 5.3.1. *Assume that* $1/(1+\operatorname{Re}\psi)$ *is integrable and* $\operatorname{Im}\psi \geq 0$ *on* $(0,\varepsilon)$ *for some* $\varepsilon > 0$. *Then the compensated potential kernel* S *exists and*

$$S(x) = \frac{1}{\pi}\int_0^\infty \operatorname{Re}\left[\frac{1}{\psi(s)}\left(1 - e^{-ixs}\right)\right] ds. \tag{5.3.2}$$

Proof. Since $e^{-x} \leq (1+x)^{-1}$ for $x \geq 0$, we get that $e^{-\psi} \in L^1(\mathbb{R})$. By the Riemann-Lebesgue Lemma we have $\operatorname{Re}\psi(\xi) \to \infty$ as $\xi \to \infty$. Since $\operatorname{Re}\psi(\xi) > 0$ for $\xi \neq 0$, this implies that $1/\operatorname{Re}\psi \in L^1([\delta,\infty))$ for any $\delta > 0$. Next, let us observe that $\operatorname{Re}\psi(\xi) \geq c\xi^2$, $|\xi| \leq 1$, for some $c > 0$. Hence,

$$\int_{\mathbb{R}} \frac{1-\cos(x\xi)}{\operatorname{Re}\psi(\xi)} d\xi < \infty.$$

By the dominated convergence theorem, for every $x \in \mathbb{R}$,

$$\lim_{\lambda\to 0^+}\int_{\mathbb{R}} \frac{(1-\cos(x\xi))(\lambda + \operatorname{Re}\psi(\xi))}{|\lambda+\psi(\xi)|^2}\, d\xi = \int_{\mathbb{R}} \frac{(1-\cos(x\xi))\operatorname{Re}\psi(\xi)}{|\psi(\xi)|^2}\, d\xi,$$

and

$$\lim_{\lambda\to 0^+}\int_{|\xi|\geq\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi(\xi)}{|\lambda+\psi(\xi)|^2}\, d\xi = \int_{|\xi|\geq\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi(\xi)}{|\psi(\xi)|^2}\, d\xi.$$

For $|\xi| < \varepsilon\wedge(\pi/|x|)$, the function $\xi \mapsto \sin(x\xi)\operatorname{Im}\psi(\xi)$ is non-negative; therefore, by the monotone convergence theorem,

$$\lim_{\lambda\to 0^+}\int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi(\xi)}{|\lambda+\psi(\xi)|^2}\, d\xi = \int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi(\xi)}{|\psi(\xi)|^2}\, d\xi. \tag{5.3.3}$$

Since $0 \leq S^\lambda(x) \leq H(x) < \infty$ for every $\lambda > 0$ and $x \in \mathbb{R}$, the above integral is finite. Finally, let us notice that the integrand is an even function which ends the proof. ☐

Corollary 5.3.2. *If* $1/(1+\operatorname{Re}\psi)$ *is integrable,* $\mathbb{E}X_1$ *exists and* $\mathbb{E}X_1 \neq 0$, *then the compensated potential kernel* S *is well defined and* (5.3.2) *holds.*

Proof. Since $\mathbb{E}|X_1| < \infty$, we have

$$\psi(\xi) = \sigma^2\xi^2 + i\gamma_1\xi + \int_{\mathbb{R}} (1 + i\xi z - e^{i\xi z})\,\nu(dz).$$

A consequence of the dominated convergence theorem is

$$\lim_{\xi\to 0^+} \frac{\operatorname{Im}\psi(\xi)}{\xi} = \gamma_1.$$

Hence, if $\gamma_1 = \mathbb{E}X_1 \neq 0$, then $\operatorname{Im}\psi$ has a constant sign on $(0,\varepsilon)$ for some $\varepsilon > 0$, which finishes the proof due to Lemma 5.3.1. □

In the next example we impose a certain growth condition on the tail of the first moment of the Lévy measure ν.

Proposition 5.3.3. *Assume that* $1/(1+\operatorname{Re}\psi)$ *is integrable and there is* $c > 0$ *such that*

$$\int_{|z|\geq r} |z|\,\nu(dz) \leq cr\operatorname{Re}\psi(1/r), \quad r > 1. \tag{5.3.4}$$

Then the compensated potential kernel S *exists and* (5.3.2) *holds.*

Remark 5.3.4. If $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$, then the assumptions of Proposition 5.3.3 are satisfied. Indeed, it is easy to see that the condition $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ implies $\psi^* \in \mathrm{WLSC}(\alpha,\chi)$. Hence, the claim follows by [43, Lemmas 2.3 and 2.10] and (2.2.6).

Proof. By (5.3.4), we have $\mathbb{E}|X_1| < \infty$. If $\mathbb{E}X_1 \neq 0$, we apply Corollary 5.3.2 to get the claim of the proposition. Therefore, assume that $\mathbb{E}X_1 = 0$, and then

$$\psi(\xi) = \sigma^2\xi^2 + \int_{\mathbb{R}} (1 + i\xi z - e^{i\xi z})\,\nu(dz).$$

By the proof of Lemma 5.3.1, it is enough to prove that (5.3.3) holds. Let us consider a Lévy measure $\tilde{\nu}(dx) = 1_{(0,1)}(x)x^{-5/2}dx + 1_{[1,\infty)}\,\nu(dx)$ and a characteristic exponent

$$\tilde{\psi}(\xi) = \int_{(0,\infty)} (1 + i\xi z - e^{i\xi z})\,\tilde{\nu}(dz), \quad \xi \in \mathbb{R}.$$

Since $\operatorname{Re}\tilde{\psi}(\xi) \approx |\xi|^{3/2}$ for $|\xi| \geq 1$ and

$$\operatorname{Im}\tilde{\psi}(\xi) = \int_{(0,\infty)} (\xi z - \sin(\xi z))\,\tilde{\nu}(dz) \geq 0,$$

we can apply Lemma 5.3.1 and its proof to obtain the finiteness of

$$\int_0^\infty \frac{|\sin(x\xi)\operatorname{Im}\tilde{\psi}(\xi)|}{|\tilde{\psi}(\xi)|^2}\,d\xi, \quad x \in \mathbb{R}.$$

Now, let

$$\psi_1(\xi) = \int_{(0,\infty)} (1 + i\xi z - e^{i\xi z})\,\nu(dz),$$

and $\psi_2(\xi) = \psi(\xi) - \psi_1(\xi)$. Notice that $\operatorname{Re}\psi_1 \approx \operatorname{Re}\tilde{\psi}$, $\operatorname{Im}\psi_1 \approx \operatorname{Im}\tilde{\psi}$ on $(0,1)$ and $\operatorname{Im}\psi_1(\xi), \operatorname{Im}\psi_2(-\xi) \geq 0$ for $\xi \geq 0$. Hence,

$$\int_0^1 \frac{|\sin(x\xi)\operatorname{Im}\psi_1(\xi)|}{|\psi_1(\xi)|^2}\,d\xi < \infty.$$

But we also have that $|\operatorname{Im}\psi_1(\xi)| \leq c \operatorname{Re}\psi(\xi)$ for $|\xi| < 1$. Indeed, by Taylor's formula and (5.3.4),

$$\begin{aligned}
|\operatorname{Im}\psi_1(\xi)| &\lesssim \int_{(0,1/|\xi|)} (|\xi|z)^3\,\nu(dz) + |\xi| \int_{[1/|\xi|,\infty)} z\,\nu(dz) \\
&\lesssim |\xi|^2 \int_{(0,1/|\xi|)} z^2\,\nu(dz) + |\xi| \int_{[1/|\xi|,\infty)} z\,\nu(dz) \\
&\lesssim |\xi|^2 \int_{(0,1/|\xi|)} z^2\,\nu(dz) + \operatorname{Re}\psi(\xi) \\
&\lesssim \operatorname{Re}\psi(\xi)
\end{aligned}$$

where the last inequality follows by the fact that $1 - \cos \xi s \approx (\xi s)^2$ for $|\xi s| \leq 1$. These imply

$$\int_0^1 \frac{|\sin(x\xi)\operatorname{Im}\psi_1(\xi)|}{|\psi(\xi)|^2}\,d\xi < \infty.$$

Hence, by the monotone convergence theorem,

$$\lim_{\lambda\to 0^+} \int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi_1(\xi)}{|\lambda+\psi(\xi)|^2}\,d\xi = \int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi_1(\xi)}{|\psi(\xi)|^2}\,d\xi,$$

and the limit is finite. Again by the monotone convergence theorem,

$$\lim_{\lambda\to 0^+} \int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi_2(\xi)}{|\lambda+\psi(\xi)|^2}\,d\xi = \int_{|\xi|<\varepsilon\wedge(\pi/|x|)} \frac{\sin(x\xi)\operatorname{Im}\psi_2(\xi)}{|\psi(\xi)|^2}\,d\xi.$$

Combining the above limits together, we obtain (5.3.3), which ends the proof. □

Corollary 5.3.5. *Assume that* $\operatorname{Re}\psi \in \mathcal{R}^0_\alpha$ *with the exponent* $\alpha \in (1, 2]$. *Then the compensated potential kernel* S *is well defined and* (5.3.2) *holds.*

Now we turn our attention to the asymptotic behaviour of the tail of the distribution of the first hitting time. First, we concentrate on the asymptotic behaviour of $\lambda U^\lambda(0)$ as $\lambda \to 0^+$.

Lemma 5.3.6. *Assume that* $\operatorname{Re}\psi \in \mathcal{R}^0_\alpha$ *with* $\alpha \in (1, 2]$ *and*

$$\lim_{\xi\to 0^+} \frac{\operatorname{Im}\psi(\xi)}{\operatorname{Re}\psi(\xi)} = C_I$$

for some $C_I \in \mathbb{R}$. *Then*

$$\lambda U^\lambda(0) \sim (\operatorname{Re}\psi)^{-1}(\lambda) C(\alpha, C_I), \quad \lambda \to 0^+,$$

where

$$C(\alpha, C_I) = \frac{\cos(\arctan(C_I)/\alpha)}{\alpha(1 + C_I^2)^{1/(2/\alpha)} \sin(\pi/\alpha)}.$$

Proof. Denote for simplicity $\theta(\xi) = \operatorname{Re}\psi(\xi)$ and $\omega(\xi) = \operatorname{Im}\psi(\xi)$. We have

$$U^\lambda(0) = \frac{1}{\pi} \int_0^\infty \frac{\lambda + \theta(\xi)}{(\lambda + \theta(\xi))^2 + \omega(\xi)^2}\,d\xi.$$

Notice that, for any $\delta > 0$,

$$I_1(\lambda) = \int_\delta^\infty \frac{\lambda + \theta(\xi)}{(\lambda + \theta(\xi))^2 + \omega(\xi)^2}\,d\xi \leq \int_\delta^\infty \frac{1}{\theta(\xi)}\,d\xi < \infty.$$

Since $\alpha > 1$, we conclude that $\lim_{\lambda\to 0^+} \frac{\lambda}{\theta^{-1}(\lambda)} I_1(\lambda) = 0$, hence it does not have impact on the asymptotic behaviour.

Now set $I_2(\lambda) = \lambda U^\lambda(0) - I_1(\lambda)$. Since θ is a continuous function, we have $\theta(\theta^{-1}(s)) = s$. Hence,

$$\begin{aligned}
\frac{\lambda}{\theta^{-1}(\lambda)} I_2(\lambda) &= \int_0^{\frac{\delta}{\theta^{-1}(\lambda)}} \frac{1+\theta\left(\theta^{-1}(\lambda)w\right)/\lambda}{\left(1+\theta\left(\theta^{-1}(\lambda)w\right)/\lambda\right)^2 + \left(\omega\left(\theta^{-1}(\lambda)w\right)/\lambda\right)^2}\, dw \\
&= \int_0^{\frac{\delta}{\theta^{-1}(\lambda)}} \frac{1+\frac{\theta(\theta^{-1}(\lambda)w)}{\theta(\theta^{-1}(\lambda))}}{\left(1+\frac{\theta(\theta^{-1}(\lambda)w)}{\theta(\theta^{-1}(\lambda))}\right)^2 + \left(\frac{\omega(\theta^{-1}(\lambda)w)}{\theta(\theta^{-1}(\lambda)w)}\frac{\theta(\theta^{-1}(\lambda)w)}{\theta(\theta^{-1}(\lambda))}\right)^2}\, dw. \qquad (5.3.5)
\end{aligned}$$

Now we will choose δ. Set ρ such that $1 < \rho < \alpha$. By the Potter bounds, there exists $\delta > 0$ such that for $\lambda < \theta(\delta)$, $s > 1$ and $\theta^{-1}(\lambda)s < \delta$,

$$\frac{\theta\left(\theta^{-1}(\lambda)s\right)}{\theta\left(\theta^{-1}(\lambda)\right)} \geq \frac{1}{2}s^\rho.$$

The integrand in (5.3.5) is then dominated by

$$\frac{1}{1+\frac{\theta(\theta^{-1}(\lambda)w)}{\theta(\theta^{-1}(\lambda))}} \leq \frac{2}{1+w^\rho/2}, \qquad w \leq \delta/\theta(\lambda).$$

Thus, by the dominated convergence theorem,

$$\lim_{\lambda\to 0^+} \frac{\lambda}{\theta^{-1}(\lambda)} U^\lambda(0) = \frac{1}{\pi}\int_0^\infty \frac{1+w^\alpha}{(1+w^\alpha)^2 + (C_I w^\alpha)^2}\, dw,$$

which ends the proof, since the limit is equal to $u^1(0)$ for stable processes (see [85, p. 389]). □

Theorem 5.3.7. *Assume that* $\operatorname{Re}\psi \in \mathcal{R}^0_\alpha$ *with the exponent* $\alpha \in (1,2]$ *and suppose that* $\lim_{\xi\to 0^+} \operatorname{Im}\psi(\xi)/\operatorname{Re}\psi(\xi) = C_I$. *Let* B *be a compact set such that* $0 \in B$. *Then, for* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} t(\operatorname{Re}\psi)^{-1}(1/t)\mathbb{P}_x(T_B > t) = \frac{1}{C(\alpha, C_I)\Gamma(1/\alpha)}(S(-x) - \mathbb{E}_x S(-X_{T_B})).$$

Let us note here that Theorem 5.3.7 extends [85, Theorem 2], where general recurrent stable processes are treated.

Proof. We have

$$\mathcal{L}(\mathbb{P}_x(T_B > \cdot))(\lambda) = \frac{1}{\lambda}[1 - \mathbb{E}_x e^{-\lambda T_B}].$$

In view of Proposition 5.1.3 and the fact that $0 \leq G^\lambda_{B^c}(x,0) \leq G_{\{0\}^c}(x,0)$, we may conclude that $G^\lambda_{B^c}(x,0) = 0$. Hence,

$$\begin{aligned}
\lambda U^\lambda(0)\mathcal{L}(\mathbb{P}_x(T_B > \cdot))(\lambda) =& U^\lambda(0) - U^\lambda(-x) + U^\lambda(-x) \\
&- \mathbb{E}_x e^{-\lambda T_B}(U^\lambda(0) - U^\lambda(-X_{T_B})) - \mathbb{E}_x e^{-\lambda T_B}(U^\lambda(-X_{T_B})) \\
=& S^\lambda(-x) - \mathbb{E}_x e^{-\lambda T_B} S^\lambda(-X_{T_B}) + G^\lambda_{B^c}(x,0) \qquad (5.3.6)\\
=& S^\lambda(-x) - \mathbb{E}_x e^{-\lambda T_B} S^\lambda(-X_{T_B}).
\end{aligned}$$

Since S^λ is bounded by H and, by Proposition 5.1.2, H is bounded on B because of its compactness, using the dominated convergence theorem, Lemma 5.3.6 and Corollary 5.3.5, we infer that

$$\lim_{\lambda\to 0^+} (\operatorname{Re}\psi)^{-1}(\lambda)\mathcal{L}(\mathbb{P}_x(T_B > \,\cdot\,))(\lambda) = (S(-x) - \mathbb{E}_x S(-X_{T_B}))/C(\alpha, C_I).$$

Let $V(s) = \int_0^s \mathbb{P}_x(T_B > t)dt$. We have

$$\mathcal{L}V(\lambda) = \frac{1}{\lambda}\mathcal{L}(\mathbb{P}_x(T_B > \,\cdot\,))(\lambda).$$

Hence,

$$\lim_{\lambda\to 0^+} \lambda(\operatorname{Re}\psi)^{-1}(\lambda)\mathcal{L}V(\lambda) = (S(-x) - \mathbb{E}_x S(-X_{T_B}))/C(\alpha, C_I).$$

Notice that $(\operatorname{Re}\psi)^{-1} \in \mathcal{R}^0_{1/\alpha}$ (see e.g. [5, the proof of Theorem 1.5.12]); thus, by the Tauberian theorem [5, Theorem 1.7.1], we can observe that

$$\lim_{t\to\infty} (\operatorname{Re}\psi)^{-1}(1/t)V(t) = \frac{1}{C(\alpha, C_I)\Gamma(1+1/\alpha)}(S(-x) - \mathbb{E}_x S(-X_{T_B})).$$

Eventually, by the monotone density theorem [5, Theorem 1.7.2],

$$\lim_{t\to\infty} t(\operatorname{Re}\psi)^{-1}(1/t)\mathbb{P}_x(T_B > t) = \frac{1}{\alpha C(\alpha, C_I)\Gamma(1+1/\alpha)}(S(-x) - \mathbb{E}_x S(-X_{T_B})).$$

□

Since $\mathbb{E}_x S(X_{T_0}) = 0$, we immediately obtain the following Corollary.

Corollary 5.3.8. *Assume that* $\operatorname{Re}\psi \in \mathcal{R}^0_\alpha$ *with the exponent* $\alpha \in (1, 2]$ *and suppose that* $\lim_{\xi\to 0^+} \operatorname{Im}\psi(\xi)/\operatorname{Re}\psi(\xi) = C_I$. *Then, for* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} t(\operatorname{Re}\psi)^{-1}(1/t)\mathbb{P}_x(T_0 > t) = \frac{1}{C(\alpha, C_I)\Gamma(1/\alpha)}S(-x). \tag{5.3.7}$$

Using Theorem 5.2.4, we conclude the asymptotic behaviour for Lévy measures of specific type (5.2.2).

Corollary 5.3.9. *Suppose that* $\nu(dx)$ *is of the form* (5.2.2). *Assume that* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathcal{R}^0_\alpha$ *with parameter* $\alpha \in (1, 2)$. *Then* (5.3.7) *holds true. In particular, this is the case for spectrally one-sided Lévy processes.*

Proposition 5.3.10. *Suppose that* $1/(1 + \operatorname{Re}\psi)$ *is integrable and* $\mathbb{E}X_1$ *exists. If* $\mathbb{E}X_1 \neq 0$, *then*

$$\mathbb{P}_x(T_B > t) \sim (S(-x) - \mathbb{E}_x S(-X_{T_B}))\kappa, \quad t \to \infty,$$

where κ *is as in* (5.1.3).

Proof. Let us observe that, by [93, Theorem 36.7], we have $\kappa > 0$. Hence, by (5.3.6), Corollary 5.3.2 and the Tauberian theorem [5, Theorem 1.7.1], we obtain the claim. □

Corollary 5.3.11. *Under the assumptions of the above proposition, the compensated kernel exists and it is coharmonic on* $\mathbb{R}$.

5.4 Harnack inequality and boundary behaviour

This Section is devoted to the proof of the Harnack inequality and a discussion on its consequences. The main result here is Theorem 5.4.4 which will then allow us to deduce some useful properties, including boundary behaviour of harmonic functions. We follow the approach from Grzywny and Ryznar [40], which, in turn, is based on the work of Bass and Levin [3]. We first provide an uniform interior lower bound on the Green function of an interval.

Lemma 5.4.1. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha, \chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0, 1]$. *Then there are* $\delta_1 \in (0, 1]$ *and* $c > 0$*, depending only on the scalings, such that for any* $R > 0$,

$$G_{(-R,R)}(x, y) \geq cH(R), \quad |x|, |y| \leq \delta_1 R.$$

Proof. By the sweeping formula (2.2.16), for any $\lambda > 0$ and any $x, y \in \mathbb{R}$ we have

$$G_{(-R,R)}(x, y) \geq G^{\lambda}_{(-R,R)}(x, y) = U^{\lambda}(y - x) - \mathbb{E}_x \left[e^{-\lambda\tau_{(-R,R)}} U^{\lambda} \left(y - X_{\tau_{(-R,R)}} \right) \right].$$

Since $\mathbb{E}X_1 = 0$, we have $b_r = -\int_{|z| \geq r} z \, \nu(dz)$, thus, by [43, Lemma 2.10], there is $c_1 \in (0, 1]$ such that $t|b_{h^{-1}(1/t)}| \leq c_1 h^{-1}(1/t)$ for all $t > 0$. Hence, by [43, Theorem 5.4] with $\theta = (2 + c_1)h^{-1}(\lambda)$, there is $c_2 \in (0, 1]$ such that for all $|x|, |y| < h^{-1}(\lambda)$,

$$U^{\lambda}(y - x) \geq \int_{1/\lambda}^{\infty} e^{-\lambda t} p(t, y - x) \, dt \geq c_2 \int_{1/\lambda}^{\infty} e^{-\lambda t} \frac{dt}{h^{-1}(1/t)}.$$

By [43, Lemma 2.3], there is $c_3 \in (0, 1]$ such that

$$U^{\lambda}(y - x) \geq \frac{c_2 c_3}{\lambda h^{-1}(\lambda)} \int_1^{\infty} e^{-s} \frac{ds}{s^{1/\alpha}} = c_4 \frac{1}{\lambda h^{-1}(\lambda)},$$

with $c_4 = c_2 c_3 / \int_1^{\infty} e^{-s} s^{-1/\alpha} \, ds$.

Next, using the estimate of the supremum of the density $p(t, \cdot)$ (see [43, Theorem 3.1]), we infer that

$$\mathbb{E}_x e^{-\lambda\tau_{(-R,R)}} U^{\lambda} \left(y - X_{\tau_{(-R,R)}} \right) \leq \mathbb{E}_x e^{-\lambda\tau_{(-R,R)}} \int_0^{\infty} e^{-\lambda t} \frac{dt}{h^{-1}(1/t)}.$$

By the scaling property of h^{-1},

$$\int_0^{1/\lambda} e^{-\lambda t} \frac{dt}{h^{-1}(1/t)} \lesssim \frac{1}{\lambda^{1/\alpha} h^{-1}(\lambda)} \int_0^{1/\lambda} \frac{dt}{t^{1/\alpha}} = c_\alpha \frac{1}{\lambda h^{-1}(\lambda)}.$$

Moreover, by monotonicity of h^{-1},

$$\int_{1/\lambda}^{\infty} e^{-\lambda t} \frac{dt}{h^{-1}(1/t)} \leq \frac{1}{h^{-1}(\lambda)} \int_{1/\lambda}^{\infty} e^{-\lambda t} \, dt = e^{-1} \frac{1}{\lambda h^{-1}(\lambda)}.$$

Now let $t_0 > 0$. By Pruitt's estimates, there is $c_5 > 0$ such that

$$\mathbb{E}_x \Big[\tau_{(-R,R)} \leq t_0; \, e^{-\lambda\tau_{(-R,R)}} \Big] \leq c_5 t_0 \big(h(R) + R^{-1}|b_R| \big).$$

Furthermore,

$$\mathbb{E}_x \Big[\tau_{(-R,R)} \geq t_0; \, e^{-\lambda\tau_{(-R,R)}} \Big] \leq \frac{c_5 e^{-\lambda t_0}}{t_0 \big(h(R) + R^{-1}|b_R| \big)}.$$

Thus, if we set $t_0 = c_6/(h(R) + R^{-1}b_R)$ and $\lambda = c_7\big(h(R) + R^{-1}b_R\big)$, where c_6 and c_7 are such that $c_6 \le c_4/(4c_5(c_\alpha + e^{-1}))$ and $c_7 \ge c_6^{-1} \ln \frac{4c_5(c_\alpha+e^{-1})}{c_4 c_6}$, then putting everything together yields

$$G_{(-R,R)}(x,y) \ge \frac{c_4}{2} \frac{1}{\lambda h^{-1}(\lambda)}.$$

Since, by [43, Lemma 2.10], we have $\lambda \approx h(R)$, using scaling properties of h^{-1} we get that

$$G_{(-R,R)}(x,y) \gtrsim \frac{1}{Rh(R)}, \quad |x|, |y| \le \delta_1 R,$$

with some $\delta_1 \in (0,1]$, and the claim follows by Proposition 5.1.2. □

Proposition 5.4.2. *Suppose* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *There is* $\delta_2 \le \delta_1$ *dependent only on the scalings such that for any* $R > 0$ *any non-empty* $A \subset (-\delta_2 R, \delta_2 R)$,

$$\mathbb{P}_x\Big(T_A < \tau_{(-R,R)}\Big) \ge \frac{1}{2}, \quad |x| \le \delta_2 R.$$

Proof. Let $|a| \le R/4$ and $D = (-R/2,0) \cup (0,R/2)$. By [34, Lemma 3] and Proposition 5.1.4, there is $C_1 > 0$ such that for $|x-a| \le R/4$,

$$\mathbb{P}_x\Big(T_a > \tau_{(-R,R)}\Big) \le \mathbb{P}_{x-a}\Big(T_0 > \tau_{(-R/2,R/2)}\Big) \le C_1 h(R/2)\mathbb{E}_{x-a}\tau_D \le 8C_1 Rh(R)H(x-a).$$

In view of Proposition 5.1.2, there is $C_2 > 0$ dependent only on the scalings such that

$$\mathbb{P}_x\Big(T_a > \tau_{(-R,R)}\Big) \le C_2 \frac{H(x-a)}{H(R)}, \quad |x-a| < R/4.$$

Since, by Proposition 5.1.2, $H \in \mathrm{WLSC}(\alpha-1,\tilde{\chi})$ for some $\tilde{\chi} \in (0,1]$, we can pick $\delta_2 < 1/2$ such that

$$\mathbb{P}_x\Big(T_a > \tau_{(-R,R)}\Big) \le \frac{1}{2}, \quad |x-a| < 2\delta_2 R.$$

It follows that if $x \in A \subset (-\delta_2 R, \delta_2 R)$ and $a \in A$, then

$$\mathbb{P}_x\Big(T_A \ge \tau_{(-R,R)}\Big) \le \mathbb{P}_x\Big(T_a > \tau_{(-R,R)}\Big) \le \frac{1}{2},$$

and the proof is completed. □

Denote $R_0 = \delta_2 R$, where δ_2 is taken from Proposition 5.4.2.

Proposition 5.4.3. *Suppose* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *Then for any* $R > 0$ *and any non-negative function* F *such that* $\operatorname{supp} F \subset (-R,R)^c$,

$$\mathbb{E}_x F\Big(X_{\tau_{(-R_0,R_0)}}\Big) \lesssim \frac{1}{c}\mathbb{E}_y F\Big(X_{\tau_{(-R,R)}}\Big), \quad |x|, |y| \le R_0,$$

where c is taken from Lemma 5.4.1. The implied comparability depends only on the scalings.

Proof. Let us denote, for any $w \in \mathbb{R}$ and a Borel set A, $\nu(w,A) = \nu(A-w)$. By the Ikeda-Watanabe formula (2.2.19) and Lemma 5.4.1,

$$\begin{aligned}\mathbb{E}_y F\Big(X_{\tau_{(-R,R)}}\Big) &\ge \int_{(-R,R)^c} \int_{-R_0}^{R_0} F(z) G_{(-R,R)}(y,w)\, \nu(w,dz)\, dw \\ &\ge \tilde{c} H(R) \int_{(-R,R)^c} \int_{-R_0}^{R_0} F(z)\, \nu(w,dz)\, dw.\end{aligned}$$

On the other hand, by the Ikeda-Watanabe formula, Proposition 5.1.3, subadditivity and almost monotonicity of H,

$$\mathbb{E}_x F\Big(X_{\tau_{(-R_0.R_0)}}\Big) \le \int_{(-R,R)^c} \int_{-R_0}^{R_0} F(z) G_{\{0\}}(x+R_0, w+R_0)\, \nu(w,dz)\, dw$$
$$\lesssim H(R_0) \int_{(-R,R)^c} \int_{-R_0}^{R_0} F(z)\, \nu(w,dz)\, dw.$$

Hence,

$$\mathbb{E}_x F\Big(X_{\tau_{(-R_0,R_0)}}\Big) \le \frac{1}{c} \mathbb{E}_y F\Big(X_{\tau_{(-R,R)}}\Big).$$

□

With these tools at our disposal, we are now able to prove the global scale invariant Harnack inequality.

Theorem 5.4.4. *Suppose* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *Then the global scale invariant Harnack inequality holds, i.e. there is a constant* C_H *dependent only on the scalings such that for any* $R > 0$ *and any non-negative harmonic function on* $(-R,R)$ *we have*

$$\sup_{x\in(-R/2,R/2)} h(x) \le C_H \inf_{x\in(-R/2,R/2)} h(x).$$

Proof. Suppose first that h is bounded. Then, using the approach of [3], we infer that there exist constants $c_1 = c_1(\alpha,\chi)$ and $a = a(\alpha,\chi) \in (0,1]$ such that for any non-negative, bounded and harmonic function on $(-R,R)$,

$$\sup_{x\in(-aR,aR)} h(x) \le c_1 \inf_{x\in(-aR,aR)} h(x). \tag{5.4.1}$$

For the justification of this claim we observe that Lemma 5.4.1 is an analogue of [3, Lemma 3.2], Proposition 5.4.2 corresponds to [3, Proposition 3.4], and Proposition 5.4.3 mirrors [3, Proposition 3.5]. We also observe that [3, Lemma 3.3] in our setting follows immediately from Pruitt's estimates and Proposition 5.1.2. With these tools at our disposal, one can follow the proof of [3, Theorem 3.6] almost directly.

Now, we apply a standard chain argument to get

$$\sup_{x\in(-R/2,R/2)} h(x) \le C_H \inf_{x\in(-R/2,R/2)} h(x).$$

Indeed, observe that h is harmonic in $(aR-(1-a)R, aR+(1-a)R)$. Thus, after applying (5.4.1) to the function $\tilde{h}(x) = h(x-aR)$, we conclude that (5.4.1) holds true also for $x \in (-(a+a(1-a))R, (a+a(1-a))R)$ with the constant c_1^2. By verbatim repetition of this argument, we get that (5.4.1) holds true for $x \in \Big(-aR\sum_{k=0}^{n-1}(1-a)^k, aR\sum_{k=0}^{n-1}(1-a)^k\Big) = \Big(-R(1-(1-a)^n), R(1-(1-a)^n)\Big)$ with a constant c_1^n. It is clear now that we get the claim with $C_H = c_1^n$ for sufficiently large n which depends only on a. It remains to observe that the boundedness assumption on h may be removed in the similar way as in the proof of [98, Theorem 2.4]. □

Thanks to the Harnack property, we are able to prove a relation between renewal functions and their derivatives, and provide a sharp estimate of the Green function of the positive half-line.

Corollary 5.4.5. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *Then there is* $c \geq 1$ *such that for all* $x > 0$,

$$c^{-1}\frac{V(x)}{x} \leq V'(x) \leq c\frac{V(x)}{x}$$

and

$$c^{-1}\frac{\widehat{V}(x)}{x} \leq \widehat{V}'(x) \leq c\frac{\widehat{V}(x)}{x}.$$

In particular, $V', \widehat{V}' \in \mathrm{WLSC}(\alpha-2,\tilde{\gamma})$ *for some* $\tilde{\gamma} \in (0,1]$.

Proof. First, let us consider the second part of the claim. Let $x > 0$. Recall that $\widehat{V}'$ is harmonic on $(0,\infty)$. Thus, by Theorem 5.4.4,

$$\widehat{V}(x) \geq \int_{x/2}^{x} \widehat{V}'(s)\,ds \geq \frac{1}{2C_H} x\widehat{V}'(x).$$

On the other hand, since $\mathrm{Re}\,\psi$ is the same for $\mathbf{X}$ and $\widehat{\mathbf{X}}$, we may apply [35, Lemma 8] for $\widehat{V}$. Let c_1 be taken from [35, Lemma 8] and $\delta \in \left(0, (c_1/2)^{1/(\alpha-1)}\right]$. Then, again by Theorem 5.4.4,

$$C_{\mathrm{H}}(1-\delta)x\widehat{V}'(x) \geq \int_{\delta x}^{x} \widehat{V}'(s)\,ds = \widehat{V}(x) - \widehat{V}(\delta x) \geq (1 - c_1^{-1}\delta^{\alpha-1})\widehat{V}(x) \geq \frac{1}{2}\widehat{V}(x).$$

Now, the lower scaling property follows immediately by [35, Lemma 8]. For the proof of the first part it remains to observe that, by the previous remark on the real part of the characteristic exponent, V also satisfies the Harnack inequality (with the same constant), and one can repeat the reasoning above to finish the proof. □

Corollary 5.4.6. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$. *Then*

$$G_{(0,\infty)}(x,y) \approx \begin{cases} \widehat{V}(x)V'(y), & 0 < x \leq y, \\ \widehat{V}'(x)V(y), & 0 < y < x. \end{cases}$$

The comparability constant depends only on the scaling characteristics.

Proof. First assume that $0 < x \leq y$. Recall that

$$G_{(0,\infty)}(x,y) = \int_0^x \widehat{V}'(u)V'(y-x+u)\,du, \quad 0 < x \leq y,$$

see (2.2.23). Since V is monotone and subadditive, for any $\lambda \geq 1$ and $x > 0$ we have

$$V(\lambda x) \leq 2\lambda V(x).$$

That, in view of Corollary 5.4.5, implies that V' is almost decreasing, and consequently,

$$G_{(0,\infty)}(x,y) \gtrsim \int_0^x \widehat{V}'(u)V'(y)\,du = \widehat{V}(x)V'(y).$$

Next, let $x < y < 2x$. By Corollary 5.4.5, [35, Corollary 5], and almost monotonicity of V',

$$G_{(0,\infty)}(x,y) \lesssim \int_0^x \widehat{V}'(u)V'(u)\,du \approx \int_0^x \frac{du}{u^2 h(u)}.$$

Using scaling property of h, [35, Corollary 5] and Corollary 5.4.5, we conclude that

$$G_{(0,\infty)}(x,y) \lesssim \frac{1}{xh(x)} \approx \frac{\widehat{V}(x)V(x)}{x} \lesssim \frac{\widehat{V}(x)V(y)}{y} \lesssim \widehat{V}(x)V'(y)$$

where the third inequality follows from monotonicity of V. Finally, for $y \geq 2x$, we use scaling property of V' with index $\alpha - 2$ (Corollary 5.4.5) to obtain

$$V'(y-x+u) \lesssim V'(y)\left(\frac{y}{y-x+u}\right)^{2-\alpha} \leq 2^{2-\alpha}V'(y),$$

and the first part follows.

If $0 < y \leq x$, we use the Green function for the dual process to get the claim. □

We note one important observation. Assume that $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. By [95, Theorem 1], V' is coharmonic on $(0,\infty)$, that is, harmonic on $(0,\infty)$ for the dual process $\widehat{\mathbf{X}}$. Since $\operatorname{Re}\psi$ is symmetric, the Harnack inequality for $\widehat{\mathbf{X}}$ holds as well. Thus, by Theorem 5.4.4, for any $0 < \delta \leq w \leq u \leq w + 2\delta$,

$$V'(u) \leq C_{\mathrm{H}} V'(w).$$

With that property at hand, proofs of the remaining lemmas in this section follow directly results obtained in [40, Subsection 4.2], and therefore, they are omitted.

Lemma 5.4.7. *Suppose that* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$*. Let* $F(z)$ *be non-negative,* $F(x) \leq F_1(x)$ *on* $\mathbb{R}$*, and* $F(x+y) \leq F_1(x) + F_1(y)$ *for* $x, y \in \mathbb{R}$*. Suppose that* $\mathbb{E}_x F(X_{\tau_{(0,\infty)}}) \leq F(x)$ *and* $\mathbb{E}_x F_1(X_{\tau_{(0,\infty)}}) \leq F_1(x)$ *for* $x > 0$*. Then there is* $c > 0$ *such that for any* $0 < x < 1$*,*

$$\mathbb{E}_x\left[X_{\tau_{(0,\infty)}} \leq -2;\, F\left(X_{\tau_{(0,\infty)}}\right)\right] \leq cC_{\mathrm{H}}^2 F_1^*(1)\frac{\widehat{V}(x)}{\widehat{V}(1)}.$$

The constant c *depends only on the scalings.*

Proof. Follows directly by proof of [40, Lemma 4.7] with applications of Lemma 4.6 and Lemma 2.9 replaced by Corollary 5.4.6 and [35, Corollary 5], respectively, and using a function F_1 instead of subadditivity of F. □

Lemma 5.4.8. *Suppose* $\mathbb{E}X_1 = 0$ *and* $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0,1]$*. Let* F *be a non-negative harmonic function on* $(0,2R)$ *for some* $R > 0$*. Suppose that* $r > 0$ *is such that* $\widehat{V}(R) \geq 2\widehat{V}(r)/c$*, where* c *is taken from Proposition 5.1.5. Then, for* $0 < x < r$*,*

$$\frac{F(x)}{F(r)} \geq \frac{c}{4} C_{\mathrm{H}}^{-1-R/r} \frac{\widehat{V}(x)}{\widehat{V}(r)}.$$

Proof. Follows directly the proof of [40, Lemma 4.8] with applications of Theorem 4.5 and Lemma 2.11 replaced by Theorem 5.4.4 and Proposition 5.1.5, respectively. □

5.5 Estimates

In the last section of this chapter we prove sharp two-sided estimates of the tail of the first hitting time of the interval. Our main result here is Theorem 5.5.11. We also provide an analogous estimate for the specific case of spectrally negative Lévy processes. Again, we generalise ideas developed in [40] and, in particular, if $\mathbf{X}$ is symmetric, then our results boil down to the ones obtained in [40]. As already mentioned, the third assumption of Theorem 5.5.11 may not be straightforward; therefore, in Subsection 5.5.1 we point out a large class of admissible non-symmetric Lévy processes.

We begin with two auxiliary results; the former is the sharp estimate of $U^\lambda(0)$, while the latter provides some lower bound on S^λ.

Lemma 5.5.1. *Assume that there exist constants $a > 0$, $b \geq 0$ such that $|\operatorname{Im}\psi(\xi)| \leq b \operatorname{Re}\psi(\xi)$, $\xi \in \mathbb{R}$ and $a\psi^*(x) \leq \operatorname{Re}\psi(x)$, $x \geq 0$. Then we have*

$$\frac{a}{4(1+b^2)} H\left(\frac{1}{(\operatorname{Re}\psi)^{-1}(\lambda)}\right) \leq U^\lambda(0) \leq \frac{3\pi^2(1+b^2)}{2a} H\left(\frac{1}{(\operatorname{Re}\psi)^{-1}(\lambda)}\right).$$

Proof. Since $|\operatorname{Im}\psi(\xi)| \leq b \operatorname{Re}\psi(\xi)$, we have

$$\frac{1}{\pi(1+b^2)} \int_0^\infty \frac{d\xi}{\lambda + \operatorname{Re}\psi(\xi)} \leq U^\lambda(0) \leq \frac{1}{\pi} \int_0^\infty \frac{d\xi}{\lambda + \operatorname{Re}\psi(\xi)}.$$

Hence, by [40, Lemma 2.15], for $\lambda > 0$,

$$U^\lambda(0) \geq \frac{a}{4\pi(1+b^2)} \int_0^\infty (1 - \cos(s/(\operatorname{Re}\psi)^{-1}(\lambda))) \frac{ds}{\operatorname{Re}\psi(s)}$$

and

$$U^\lambda(0) \leq \frac{3\pi^2(1+b^2)}{2a} \int_0^\infty (1 - \cos(s/(\operatorname{Re}\psi)^{-1}(\lambda))) \frac{ds}{\operatorname{Re}\psi(s)}.$$

Using $|\operatorname{Im}\psi(\xi)| \leq b \operatorname{Re}\psi(\xi)$, we infer that

$$\pi H(x) \leq \int_0^\infty (1 - \cos(xs)) \frac{ds}{\operatorname{Re}\psi(s)} \leq (1+b^2)\pi H(x),$$

which ends the proof. □

Lemma 5.5.2. *Suppose that $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \operatorname{WLSC}(\alpha, \chi)$ for some $\alpha > 1$ and $\chi \in (0, 1]$. Then there exists $c = c(\alpha, \chi)$ such that, for any $a, x > 0$,*

$$S^\lambda(x) \geq c(1 - e^{-a}) U^\lambda(0), \quad \lambda \geq a h(x).$$

Proof. Let us define

$$f(t) = \mathbb{P}_x(T_0 > t), \quad t > 0.$$

Since

$$S^\lambda(x) = \lambda U^\lambda(0) \mathcal{L}f(\lambda),$$

it is enough to prove that $\mathcal{L}f(\lambda) \geq c/\lambda$ if $\lambda \geq ah(x)$. Using estimates of the tail distribution of the first exit time from the positive half-line [35, Theorem 6], we conclude that there is c_1 such that

$$\mathcal{L}f(\lambda) \geq c_1 \int_0^\infty \left(1 \wedge \frac{\hat{V}(x)}{\hat{V}(h^{-1}(1/s))}\right) e^{-\lambda s}\, ds \geq c_1 \int_0^{1/h(x)} e^{-\lambda s}\, ds \geq c_1(1 - e^{-a})\lambda^{-1}.$$

□

Now, we turn our attention to various upper and lower estimates of tails of the first hitting time of the origin and of the unit interval.

Proposition 5.5.3. *Assume that there exist constants $a > 0$, $b \geq 0$ such that $|\operatorname{Im}\psi(\xi)| \leq b\operatorname{Re}\psi(\xi)$, $\xi \in \mathbb{R}$ and $a\psi^*(x) \leq \operatorname{Re}\psi(x)$ for $x \geq 0$. Then*

$$\mathbb{P}_x(T_0 > t) \leq \frac{4(e-1)}{e}\frac{(1+b^2)}{a}\frac{H(x)}{H\big(1/(\operatorname{Re}\psi)^{-1}(1/t)\big)} \wedge 1.$$

Proof. Let us define

$$f(t) = \mathbb{P}_x(T_0 > t), \quad t > 0.$$

Recall that

$$\lambda\mathcal{L}f(\lambda) = 1 - \mathbb{E}_x e^{-\lambda T_0} = \frac{U^\lambda(0) - U^\lambda(-x)}{U^\lambda(0)} = \frac{S^\lambda(-x)}{U^\lambda(0)}.$$

By Lemma 5.5.1,

$$\mathcal{L}f(\lambda) \leq \frac{4(1+b^2)}{a}\frac{H(x)}{H\big(1/(\operatorname{Re}\psi)^{-1}(\lambda)\big)}.$$

Therefore, using [10, Lemma 5], we conclude that

$$\mathbb{P}_x(T_0 > t) \leq \frac{e}{e-1}\frac{4(1+b^2)}{a}\frac{H(x)}{H\big(1/(\operatorname{Re}\psi)^{-1}(1/t)\big)}.$$

□

Corollary 5.5.4. *Assume that $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \operatorname{WLSC}(\alpha, \chi)$ for some $\alpha > 1$ and $\chi \in (0, 1]$. Then there is $c > 0$ such that for all $t > 0$,*

$$\mathbb{P}_x(T_0 > t) \leq c\frac{H(x)}{H(h^{-1}(1/t))} \wedge 1.$$

The constant c depends only on the scalings.

Proof. Using [35, Lemma 12] and [43, Remark 3.2], we see that the assumptions of Proposition 5.5.3 are satisfied. Now it remains to apply comparability of $1/(\operatorname{Re}\psi)^{-1}$ and h^{-1} together with Proposition 5.1.2. □

Lemma 5.5.5. *Suppose $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \operatorname{WLSC}(\alpha, \chi)$ for some $\alpha > 1$ and $\chi \in (0, 1]$. If $x > 1$ and $t < 1/h(1)$, then*

$$\mathbb{P}_x(T_{B_1} > t) \approx \frac{\hat{V}(x-1)}{\hat{V}(h^{-1}(1/t))} \wedge 1.$$

The comparability constant depends only on the scalings.

Proof. Of course, the lower bound is a consequence of the estimates of the tail for the first exit time from a half-line, that is [35, Theorem 6]. By subadditivity of $\hat{V}$, it is enough to consider $1 < x < 1 + h^{-1}(1/t)/2$, because if x is larger, by the lower bound the probability is comparable to 1.

To prove the estimate from the above, let us denote $r = h^{-1}(1/t)$. Notice that $r < 1$ and we have

$$\mathbb{P}_x(T_{B_1} > t) \leq \mathbb{P}_x(\tau_{(1,1+r)} > t) + \mathbb{P}_x(|X_{\tau_{(1,1+r)}} - 1| > r).$$

Combining [34, Lemma 3] and [35, the proof of Proposition 4], we obtain

$$\mathbb{P}_x(|X_{\tau_{(1,1+r)}} - 1| > r) \leq c\mathbb{E}_x\tau_{(1,1+r)}h(r) \leq c\hat{V}(x-1)V(r)h(r)$$

for some $c > 0$. Finally, by [35, Corollary 5],

$$\mathbb{P}_x(|X_{\tau_{(1,1+r)}} - 1| > r) \leq c\frac{\hat{V}(x-1)}{\hat{V}(r)}.$$

This together with [35, Theorem 6] imply

$$\mathbb{P}_x(T_{B_1} > t) \leq c\frac{\hat{V}(x-1)}{\hat{V}(r)}.$$

□

Lemma 5.5.6. *Assume that* $\mathbb{E}X_1 = 0$ *and* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha, \chi)$ *for some* $\alpha > 1$ *and* $\chi \in (0, 1]$. *If* $x > 1$ *and* $t \geq 1/h(1)$, *then*

$$\mathbb{P}_x(T_{B_1} > t) \leq c\frac{\hat{V}(x-1)}{\hat{V}(x)}\frac{H(x)}{H(h^{-1}(1/t))} \wedge 1 \approx \frac{\hat{V}(x-1)}{\hat{V}(x)}\frac{H(x)}{t/h^{-1}(1/t)} \wedge 1.$$

The constant c depends only on the scalings.

Proof. If $x \geq 2$, we have, by subadditivity and monotonicity of $\hat{V}$, $\frac{\hat{V}(x-1)}{\hat{V}(x)} \geq \frac{1}{2}$, hence the claim follows from [35, Lemma 12] and Corollary 5.5.4.

Let $1 < x < 2$. By [35, Theorem 6],

$$\mathbb{P}_x(\tau_{(1,\infty)} > t) \approx 1 \wedge \frac{\hat{V}(x-1)}{\hat{V}\big(h^{-1}(1/t)\big)}.$$

Since $t > 1/h(1)$, using subadditivity of $\hat{V}$ and [43, Lemma 2.1], we obtain

$$\begin{aligned}\mathbb{P}_x(\tau_{(1,\infty)} > t/2) &\leq c_1\frac{\hat{V}(x-1)}{\hat{V}(1)}\frac{\hat{V}(1)}{\hat{V}(h^{-1}(1/t))} \leq c_2\frac{\hat{V}(x-1)}{\hat{V}(1)}\mathbb{P}_2(\tau_{(1,\infty)} > t)\\ &\leq c_3\frac{\hat{V}(x-1)}{\hat{V}(x)}\mathbb{P}_1(T_0 > t).\end{aligned}$$

Since

$$\mathbb{P}_x(T_{B_1} > t) \leq \mathbb{P}_x(\tau_{(1,\infty)} > t/2) + \mathbb{E}_x\mathbb{P}_{X_{\tau_{(1,\infty)}}}(T_{B_1} > t/2),$$

due to Proposition 5.1.2 and Corollary 5.5.4, it is enough to estimate the second term. We have

$$\begin{aligned}\mathbb{E}_x\mathbb{P}_{X_{\tau_{(1,\infty)}}}(T_{B_1} > t/2) &\leq \mathbb{E}_x[X_{\tau_{(1,\infty)}} \leq -1; \mathbb{P}_{X_{\tau_{(1,\infty)}}}(T_1 > t/2)]\\ &= \mathbb{E}_{x-1}[X_{\tau_{(0,\infty)}} \leq -2; \mathbb{P}_{X_{\tau_{(0,\infty)}}}(T_0 > t/2)].\end{aligned}$$

Let $F(z) = \mathbb{P}_z(T_0 > t/2)$. Observe that

$$\begin{aligned} F(z) &= \mathbb{P}_z(\tau_{(0,\infty)} > t/2) + \mathbb{E}_z\Big[\tau_{(0,\infty)} \le t/2; \mathbb{P}_{X_{\tau_{(0,\infty)}}}(T_0 > t/2 - \tau_{(0,\infty)})\Big] \\ &\ge \mathbb{E}_z\Big[\mathbb{P}_{X_{\tau_{(0,\infty)}}}(T_0 > t/2)\Big] \\ &= \mathbb{E}_z F(X_{\tau_{(0,\infty)}}). \end{aligned}$$

Furthermore,

$$F(x+y) \le \mathbb{P}_{x+y}(T_x > t/4) + \mathbb{E}_x\Big[T_x \le t/4; \mathbb{P}_{X_{T_x}}(T_0 > t/4)\Big] \le \mathbb{P}_y(T_0 > t/4) + \mathbb{P}_x(T_0 > t/4).$$

Hence, F and $F_1(z) = \mathbb{P}_z(T_0 > t/4)$ satisfy the assumptions of Lemma 5.4.7. Therefore, the conclusion follows from Lemma 5.4.7 and Proposition 5.5.3. ☐

Lemma 5.5.7. *Assume $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. If $x_0 > 1$, $1 < x \le x_0$ and $t > 1/h(1)$, then there is $c = c(x_0,\alpha,\chi) > 0$ such that*

$$\mathbb{P}_x(T_{B_1} > t) \ge c\frac{\hat{V}(x-1)}{\hat{V}(x_0)}\mathbb{P}_{x_0}(T_0 > 2t).$$

Proof. With Lemma 5.4.8 and [35, Theorem 6] at hand, the proof is the same as the first part of the proof of [40, Lemma 5.4], and therefore, it is omitted. ☐

Lemma 5.5.8. *Assume that $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. Suppose that there exist constants $c > 0$ and $a > 0$ such that, for $x > 0$, we have $S^\lambda(x) \ge cH(x)$ if $\lambda \le ah(x)$. Then there is $\tilde{c} = \tilde{c}(a,\alpha,c) > 0$ such that*

$$\mathbb{P}_x(T_0 > t) \ge \tilde{c}\Big(1 \wedge \frac{H(x)}{H(h^{-1}(1/t))}\Big), \quad x,t > 0.$$

Remark 5.5.9. In case of symmetric Lévy processes, the last assumption follows from [40, Lemma 2.15]. The same remark applies to Proposition 5.5.10 and Theorem 5.5.11.

Proof. Let us define

$$f(t) = \mathbb{P}_x(T_0 > t), \quad t > 0.$$

By Lemma 5.5.1, comparability of h and ψ, and Proposition 5.1.2,

$$\lambda \mathcal{L}f(\lambda) \approx \frac{S^\lambda(x)}{H\left(h^{-1}(\lambda)\right)}.$$

Let $x > 0$, $\lambda > 0$ and $s > 1$. Combining Lemma 5.5.2 with the assumption on S^λ, we obtain, for $\lambda \ge ah(x)$ or $\lambda s \le ah(x)$,

$$\frac{S^{\lambda s}(x)}{S^\lambda(x)} \lesssim 1.$$

If $\lambda s \ge ah(x) \ge \lambda$ we have, by Lemma 5.5.1 and almost monotonicity of H,

$$\frac{S^{\lambda s}(x)}{S^\lambda(x)} \approx \frac{U^\lambda(0)}{H(x)} \approx \frac{H\left(h^{-1}(\lambda s)\right)}{H(x)} \lesssim \frac{H\left(h^{-1}(ah(x))\right)}{H(x)} \lesssim 1.$$

Thus,

$$\frac{S^{\lambda s}(x)}{S^\lambda(x)} \le c,$$

and consequently,

$$\frac{\mathcal{L}f(\lambda s)}{\mathcal{L}f(\lambda)} \le c\frac{\lambda}{\lambda s}\frac{S^{\lambda s}(x)}{S^{\lambda}(x)}\frac{H\left(h^{-1}(\lambda)\right)}{H\left(h^{-1}(\lambda s)\right)} \le c\frac{h^{-1}(\lambda s)}{h^{-1}(\lambda)} \le cs^{-1/2},$$

where c depends only on the scalings and a. Hence, by [10, Lemma 13], there exists a constant c_1 that depends only on the scalings such that

$$\mathbb{P}_x(T_0 > t) \ge c_1 \frac{S^{1/t}(x)}{H\left(\frac{1}{\psi^{-1}(1/t)}\right)}.$$

For $t > 1/ah(x)$, we get the claim by the comparability $S^{1/t}$ with S, and for $t \le 1/(ah(x))$, we use estimates for the positive half-line [35, Theorem 6]. □

Proposition 5.5.10. *Assume $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. Suppose that there exist constants $c > 0$ and $a > 0$ such that for $x > 0$ we have $S^{\lambda}(x) \ge cH(x)$ if $\lambda \le ah(x)$. Then there is $x_0 \ge 2$ which depends only on the scaling characteristics and a, such that for $x \ge x_0$ and $t > 1/h(1)$, we have*

$$\mathbb{P}_x(T_{B_1} > t) \ge \tilde{c}\left(\frac{H(|x|)}{H(h^{-1}(1/t))} \wedge 1\right) \approx \left(\frac{H(|x|)}{t/h^{-1}(1/t)} \wedge 1\right).$$

The constant $\tilde{c}$ depends only on the scalings and a.

The proof is very similar to the proof of [40, Proposition 5.3] with modifications like in the proof above; therefore, it is omitted.

We now proceed to the proof of the main result of this Section.

Theorem 5.5.11. *Suppose that $\mathbb{E}X_1 = 0$ and $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. Then for any $R > 0$ and $x > R$,*

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{\widehat{V}(x-R)}{\widehat{V}(h^{-1}(1/t))} \wedge 1, \quad t < 1/h(R).$$

Furthermore, if we additionally assume that there exist constants $c_1 > 0$ and $a > 0$ such that for $x > 0$ we have $S^{\lambda}(x) \ge c_1 H(x)$ if $\lambda \le ah(x)$, then

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{\widehat{V}(x-1)}{\widehat{V}(x)}\frac{H(x)}{H(h^{-1}(1/t))} \wedge 1 \approx \frac{\widehat{V}(x-1)}{\widehat{V}(x)}\frac{H(x)}{t/h^{-1}(1/t)} \wedge 1, \quad t \ge 1/h(R).$$

Proof. The case $R = 1$ follows from Lemmas 5.5.5 and 5.5.6, and Proposition 5.5.10. Now we may proceed as in the proof of [40, Theorem 5.5] to obtain the claim for any $R > 0$. □

Let us now turn our attention to spectrally negative Lévy processes and prove the analogue of Theorem 5.5.11. This part is possible thanks to the recent preprint by Grzywny [35] which together with our former results allows us to treat this specific class as well.

Lemma 5.5.12. *Suppose $\mathbb{E}X_1 = 0$, $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$, $\chi \in (0,1]$, and let $R \in [0,\infty)$. Assume that*

$$\int_0^1 \frac{\nu(y,\infty)}{h(y)}\frac{dy}{y} < \infty.$$

Then

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{H(x-R)}{H(h^{-1}(1/t))} \wedge 1, \quad R < x < R+1,\ 0 < t < 1/h(1).$$

Proof. A consequence of Proposition 14 and Corollary 5 in [35] is $\tilde{V}(x) \approx \frac{1}{xh(x)}$, $0 < x \le 1$. This together with Proposition 5.1.2 imply $\tilde{V}(x) \approx H(x)$, $0 \le x \le 1$. Hence, the claim holds due to [35, Theorem 6] and Corollary 5.5.4. □

Similarly, the consequence of [35, Proposition 15 and Section 3] is the following.

Lemma 5.5.13. *Suppose* $\mathbb{E}X_1 = 0$ *and* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha, \chi)$ *with* $\alpha > 1$, $\chi \in (0, 1]$, *and let* $R \in [0, \infty)$. *Assume that any of the following holds true:*

(i) $\mathbb{E}X_1^2 < \infty$,

(ii) there are $C, r > 0$ *such that* $\nu(x, \infty) \le C\nu(-\infty, -x)$, $x > r$, *and*

$$\int_1^\infty \frac{\nu(y, \infty)}{h(y)} \frac{dy}{y} < \infty.$$

Then

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{H(x - R)}{H(h^{-1}(1/t))} \wedge 1, \quad x \ge R + 1, t > 0.$$

Finally, let us observe that the previous two lemmas may be applied to the case of spectrally negative Lévy processes.

Corollary 5.5.14. *Suppose* $\mathbb{E}X_1 = 0$ *and* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha, \chi)$ *with* $\alpha > 1$, $\chi \in (0, 1]$, *and let* $R \in [0, \infty)$. *Assume that* $\mathbf{X}$ *is spectrally negative, i.e.* $\nu(0, \infty) = 0$. *Then*

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{H(x - R)}{H(h^{-1}(1/t))} \wedge 1, \quad x > R, t > 0,$$

and

$$\mathbb{P}_x(T_{B_R} > t) \approx \frac{|x + R|}{h^{-1}(1/t)} \wedge 1, \quad x < -R, t > 0.$$

Proof. The proof mirrors the proofs of the previous two lemmas but we provide it for the convenience of the reader. Since $\mathbf{X}$ is spectrally negative, it trivially satisfies both integral conditions in Lemmas 5.5.12 and 5.5.13. Thus, [35, Corollary 5 and Propositions 14 and 15] yield that $V(x) \approx x$ and $\widehat{V}(x) \approx \frac{1}{xh(x)}$ for $x > 0$, which together with Proposition 5.1.2 imply that $\widehat{V}(x) \approx H(x)$ for $x > 0$. Now it remains to observe that the first part of the claim follows by [35, Theorem 6] and Corollary 5.5.4, while the second is a consequence of [35, Theorem 6] combined with the fact that for spectrally negative Lévy processes $T_{[0,\infty)} = T_0$ if the process starts from the negative half-line. □

5.5.1 A class of processes that satisfy the assumptions of Theorem 5.5.11

At the end of this chapter, let us provide an aforementioned example of a class of non-symmetric Lévy processes that satisfy the assumptions of Theorem 5.5.11. As one can suspect, the main difficulty here is the lower estimate of S^λ for small λ, which is far from obvious for general non-symmetric processes even if the remaining two assumptions are satisfied. Let us repeat one more time that if process is symmetric, then the third assumption follows from [40, Lemma 2.15].

Let ν be of the form (5.2.2) and assume that $\operatorname{Re}\psi \in \mathrm{WLSC}(\alpha,\chi)$ for some $\alpha > 1$ and $\chi \in (0,1]$. Since $\int_{|z|\geq 1} |z|\nu(dz) < \infty$, the characteristic exponent ψ is differentiable and

$$(\operatorname{Im}\psi)'(\xi) = \int_{\mathbb{R}} z(1-\cos\xi z)\,\nu(dz) = (C_u - C_d)\int_0^\infty z(1-\cos\xi z)\,\nu_0(dz), \quad \xi \in \mathbb{R}.$$

Now we specify ν_0. Assume that $0 < \beta_1 \leq \beta_2 < 1$ and $0 < a_2 \leq 1 \leq a_1$. Let $\nu_0(dz) = \frac{f(z)}{z^2}dz$, where f is non-negative, non-increasing and satisfies

$$a_2\lambda^{-\beta_2} f(z) \leq f(\lambda z) \leq a_1 \lambda^{-\beta_1} f(z), \quad \lambda > 1,\ z > 0.$$

For such ν_0 it is easy to verify that $\operatorname{Re}\psi$ is non-decreasing on $[0,\infty)$, and, by [10, Proposition 28], there is $c_1 = c_1(\beta_1,\beta_2,a_1,a_2)$ such that

$$\big|(\operatorname{Im}\psi)'(\xi)\big| \leq c|C_u - C_d| f(1/\xi) \leq c_1 \frac{|C_u - C_d|}{C_u + C_d}\frac{\operatorname{Re}\psi(\xi)}{\xi}, \quad \xi \in \mathbb{R}.$$

Next, we obtain the lower bound for S^λ for small λ. We have, for $x > 0$,

$$\begin{aligned} S^\lambda(x) &= \frac{1}{\pi}\int_0^\infty (1-\cos x\xi)\frac{\lambda + \operatorname{Re}\psi(\xi)}{(\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi))^2}\,d\xi \\ &+ \frac{1}{\pi}\int_0^\infty \sin x\xi \frac{\operatorname{Im}\psi(\xi)}{(\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi))^2}\,d\xi \\ &= \frac{1}{2}H^\lambda(x) + I_\lambda(x). \end{aligned}$$

Since $|\operatorname{Im}\psi(\xi)| \leq c_2 \operatorname{Re}\psi(\xi)$, $\xi \in \mathbb{R}$, where $c_2 = c_2(\beta_1, a_1)$, by [40, Lemma 2.15], for $x > 0$ and $\lambda \leq h(x)$,

$$H^\lambda(x) \geq \frac{1}{\pi(1+c_2^2)}\int_0^\infty (1-\cos x\xi)\frac{d\xi}{\lambda + \operatorname{Re}\psi(\xi)} \geq c_3 \frac{1}{xh(x)}$$

where c_3 depends only on β_1 and a_1. The integration by parts implies, for $x > 0$,

$$\pi x I_\lambda(x) = \int_0^\infty (1-\cos x\xi)\frac{g(\xi)}{((\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi)^2)^2}\,d\xi$$

where

$$\begin{aligned} g(\xi) &= 2\operatorname{Im}\psi(\xi)\big((\operatorname{Re}\psi)'(\xi)(\lambda + \operatorname{Re}\psi(\xi)) + (\operatorname{Im}\psi)'(\xi)\operatorname{Im}\psi(\xi)\big) \\ &\quad - (\operatorname{Im}\psi)'(\xi)\left((\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi)^2\right). \end{aligned}$$

Assume that $C_u \geq C_d$; then $\operatorname{Im}\psi$, $(\operatorname{Im}\psi)'$ and $(\operatorname{Re}\psi)'$ are non-negative on the positive half-line. Therefore,

$$\begin{aligned} \pi x I_\lambda(x) &\geq -\int_0^\infty (1-\cos x\xi)\frac{(\operatorname{Im}\psi)'(\xi)}{(\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi)^2}\,d\xi \\ &\geq -c_1\frac{C_u - C_d}{C_u + C_d}\int_0^\infty (1-\cos x\xi)\frac{\operatorname{Re}\psi(\xi)}{\xi((\lambda + \operatorname{Re}\psi(\xi))^2 + (\operatorname{Im}\psi(\xi)^2)}\,d\xi, \\ &\geq -c_1\frac{C_u - C_d}{C_u + C_d}\int_0^\infty (1-\cos x\xi)\frac{1}{\xi\operatorname{Re}\psi(\xi)}\,d\xi \\ &\geq -c_4\frac{C_u - C_d}{C_u + C_d}\frac{1}{\operatorname{Re}\psi(1/x)}, \end{aligned}$$

where in the last inequality we used [10, Corollary 22] and c_4 depends only on β_1, β_2, a_1 and a_2. Finally, we obtain

$$S^\lambda(x) \geq \frac{1}{xh(x)} \left(c_3 - \frac{c_4}{\pi} \frac{C_u - C_d}{C_u + C_d} \right), \quad \lambda \leq h(x).$$

Hence, for small $\frac{C_u - C_d}{C_u + C_d}$, we have, for $x > 0$,

$$S^\lambda(x) \approx \frac{1}{xh(x)}, \quad \lambda \leq h(x).$$

For $x < 0$, additional assumptions on $f(s) - sf'(s)$ are needed in order to provide similar calculations.

Appendix A

A.1 Further comparability properties of characteristics of the Lévy process

The aim of this Appendix is to deliver a number of comparability and scaling results concerning main characteristics of the process (i.e. the Lévy-Khintchine exponent ψ, the related function Φ and their inverses) and concentration functions K and h, as well as to provide useful sufficient conditions for the weak lower scaling property of the (minus) second derivative of the Laplace exponent. The content of this part will serve as a set of basic properties which will be in constant usage throughout Chapters 3 and 4.

Let $\mathbf{X} = (X_t \colon t \geq 0)$ be a one-dimensional Lévy process with the Lévy measure ν supported on the positive half-line and with the Lévy-Khintchine exponent of the form

$$\psi(\xi) = \sigma^2 \xi^2 - i\gamma\xi - \int_{(0,\infty)} \left(e^{i\xi x} - 1 - x\mathbb{1}_{(-1,1)}(x) \right) \nu(dx), \quad \xi \in \mathbb{R}.$$

Recall that the concentration functions K and h are defined as

$$K(r) = \frac{\sigma^2}{r^2} + r^{-2} \int_{(0,r)} |x|^2 \, \nu(dx), \quad r > 0,$$

and

$$h(r) = \frac{\sigma^2}{r^2} + \int_{(0,\infty)} \left(1 \wedge \frac{|x|^2}{r^2} \right) \nu(dx), \quad r > 0.$$

Next, let us define the auxiliary function

$$\Upsilon(x) = 2\sigma^2 + \int_0^\infty s^2 e^{-xs} \, \nu(ds), \quad x > 0.$$

It is straightforward after lecture of Chapters 3 and 4 that the function Υ corresponds to (minus) second derivative of the Laplace exponent of $\mathbf{X}$. Let us define

$$\Phi(x) = x^2 \Upsilon(x), \quad x > 0.$$

In this appendix, under the assumption of weak lower scaling property of Υ with the scaling index $\alpha > 0$, we eventually prove the following chain of comparabilities:

$$\psi^*(x) \approx h(1/x) \approx K(1/x) \approx \Phi(x) \approx \Phi^*(x), \quad x > x_0.$$

At the end, we also provide two sufficient conditions that imply the scaling property of Υ.

A.1.1 Various comparability properties

In this part we always assume that $\Upsilon \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. It follows immediately that $\Phi \in \mathrm{WLSC}(\alpha, c, x_0)$. Recall that the generalised (right-sided) inverse function Φ^{-1} is defined as

$$\Phi^{-1}(s) = \sup\{r > 0\colon \Phi^*(r) = s\},$$

where

$$\Phi^*(r) = \sup_{0<s\leq r} \Phi(s).$$

Clearly, Φ^{-1} is non-decreasing. Similarly to ψ^{-1}, we have (see the Notation subsection at the beginning of Chapter 3) that

$$\Phi^*(\Phi^{-1}(s)) = s, \qquad \Phi^{-1}(\Phi^*(s)) \geq s. \tag{A.1.1}$$

Observe that, since, by monotonicity of Υ, for all $x > 0$ and $\lambda \geq 1$,

$$\Phi(\lambda x) \leq \lambda^2 \Phi(x), \tag{A.1.2}$$

by taking a supremum, we conclude that

$$\Phi^*(\lambda x) \leq \lambda^2 \Phi^*(x). \tag{A.1.3}$$

Furthermore, for any $r > 0$, let u be such that $\Phi^{-1}(r) = u$. Then, by (A.1.3) and (A.1.1), for any $\lambda \geq 1$,

$$\Phi^{-1}(\lambda r) = \Phi^{-1}(\lambda \Phi^*(u)) \geq \Phi^{-1}(\Phi^*(\sqrt{\lambda} u)) \geq \sqrt{\lambda} u.$$

Thus, for any $r > 0$ and $\lambda \geq 1$,

$$\Phi^{-1}(\lambda r) \geq \sqrt{\lambda}\Phi^{-1}(r). \tag{A.1.4}$$

Lemma A.1.1. *Suppose $\Upsilon \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. There is a constant $C > 0$ such that for all $x > x_0$,*

$$C\Upsilon(x) \leq \sigma^2 + \int_{(0,1/x)} s^2\, \nu(ds).$$

Proof. First assume that $\sigma = 0$; the extension to any σ is immediate. Let $f\colon (0,\infty) \mapsto \mathbb{R}$ be a function defined as

$$f(t) = \int_{(0,t)} s^2\, \nu(ds).$$

Observe that, by the Fubini-Tonelli theorem, for $x > 0$, we have

$$\mathcal{L}f(x) = \int_0^\infty e^{-xt} \int_{(0,t)} s^2\, \nu(ds)\, dt = \int_{(0,\infty)} s^2 \int_s^\infty e^{-xt}\, dt\, \nu(ds) = x^{-1}\Upsilon(x).$$

Since f is non-decreasing, for any $s > 0$,

$$\Upsilon(x) = x\mathcal{L}f(x) \geq \int_s^\infty e^{-t} f(t/x)\, dt \geq e^{-s} f(s/x).$$

Hence, for any $u > 2$,

$$\Upsilon(x) = \int_0^u e^{-s} f(x/s)\, ds + \int_u^\infty e^{-s} f(x/s)\, ds \leq f(u/x) + \int_u^\infty e^{-s/2}\Upsilon(x/2)\, ds.$$

Therefore, setting $x = \lambda u > 2x_0$, by the weak scaling property of Υ,

$$f(1/\lambda) \geq \Upsilon(u\lambda) - 2e^{-u/2}\Upsilon(u\lambda/2) \geq (2^{\alpha-2}c - 2e^{-u/2})\Upsilon(u\lambda/2).$$

At this stage, we select $u > 2$ such that

$$2^{\alpha-2}c - 2e^{-u/2} \geq 2^{-2}c.$$

Then again, by the weak scaling property of Υ, for $\lambda > x_0$,

$$f(1/\lambda) \geq c2^{-2}\Upsilon(u\lambda/2) \geq (c/u)^2\Upsilon(\lambda),$$

which ends the proof. □

Since

$$K(1/x) \leq e\Phi(x),$$

by Lemma A.1.1 we immediately obtain the following corollary.

Corollary A.1.2. *Suppose $\Upsilon \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$ and $\alpha > 0$. Then there is $C \geq 1$ such that for all $x > x_0$,*

$$C\Phi(x) \leq K(1/x) \leq e\Phi(x).$$

Proposition A.1.3. *Suppose that $\Upsilon \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0, 1]$, $x_0 \geq 0$, and $\alpha > 0$. Then there is $C \geq 1$ such that for all $0 < r < 1/x_0$,*

$$K(r) \leq h(r) \leq CK(r).$$

Proof. Since $h(r) \geq K(r)$, it is enough to show that for some $C \geq 1$ and $0 < r < 1/x_0$,

$$h(r) \leq CK(r).$$

In view of (2.2.4), we have

$$h(r) = 2\int_r^\infty K(s)\frac{ds}{s} = 2\int_r^{1/x_0} K(s)\frac{ds}{s} + 2\int_{1/x_0}^\infty K(s)\frac{ds}{s}. \tag{A.1.5}$$

Let us consider the first term on the right-hand side of (A.1.5). By Corollary A.1.2, we have $K(r) \approx \Phi(1/r)$ for $0 < r < 1/x_0$, which implies

$$\int_r^{1/x_0} K(s)\frac{ds}{s} \lesssim K(r), \quad 0 < r < 1/x_0.$$

This finishes the proof in the case $x_0 = 0$. If $x_0 > 0$, then, for $1/(2x_0) \leq r < 1/x_0$, we have

$$K(r) \gtrsim \Phi(1/r) \gtrsim \Phi(x_0) > 0.$$

Hence, $K(r) \gtrsim 1$ for all $0 < r < 1/x_0$. Since the second term on the right-hand side of (A.1.5) is constant, the proof is completed. □

Let us notice that, by (2.2.6), Proposition A.1.3 and Corollary A.1.2, we have

$$\psi^*(x) \approx h(1/x) \approx K(1/x) \approx \Phi(x) \tag{A.1.6}$$

for all $x > x_0$. In particular, there is $c_1 \in (0,1]$ such that $\psi^* \in \mathrm{WLSC}(\alpha, c_1, x_0)$. Moreover,

$$\begin{aligned}\psi^*(x) \lesssim K(1/x) &= x^2 \left(\sigma^2 + \int_{(0,1/x)} s^2 \,\nu(ds)\right) \\ &\lesssim \sigma^2 x^2 + \int_{(0,1/x)} (1 - \cos sx)\,\nu(ds),\end{aligned}$$

thus, for all $x > x_0$,

$$\psi^*(x) \lesssim \operatorname{Re}\psi(x). \tag{A.1.7}$$

Proposition A.1.4. *Suppose that $\Upsilon \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha > 0$. Then, for all $r > 2h(1/x_0)$,*

$$\frac{1}{h^{-1}(r)} \approx \psi^{-1}(r). \tag{A.1.8}$$

Furthermore, there is $C \geq 1$ such that for all $\lambda \geq 1$ and $r > 2h(1/x_0)$,

$$\psi^{-1}(\lambda r) \leq C\lambda^{1/\alpha}\psi^{-1}(r).$$

Proof. Using (2.2.6), we immediately get

$$\frac{1}{h^{-1}(r/2)} \leq \psi^{-1}(r) \leq \frac{1}{h^{-1}(24r)}$$

for all $r > 0$. On the other hand, by Proposition A.1.3 and [43, Lemma 2.3], there is $C \geq 1$ such that for all $\lambda \geq 1$ and $r > h(1/x_0)$,

$$\frac{1}{h^{-1}(\lambda r)} \leq C\lambda^{1/\alpha}\frac{1}{h^{-1}(r)}. \tag{A.1.9}$$

Hence, for $r > 2h(1/x_0)$,

$$C^{-1}2^{-1/\alpha}\frac{1}{h^{-1}(r)} \leq \psi^{-1}(r) \leq C(24)^{1/\alpha}\frac{1}{h^{-1}(r)}, \tag{A.1.10}$$

proving (A.1.8). The weak upper scaling property of ψ^{-1} is a consequence of (A.1.9) and (A.1.10). □

Proposition A.1.5. *Suppose that $\Upsilon \in \mathrm{WLSC}(\alpha - 2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$, and $\alpha > 0$. Then, for all $x > x_0$,*

$$\psi^*(x) \approx \Phi^*(x), \tag{A.1.11}$$

and for all $r > \Phi(x_0)$,

$$\psi^{-1}(r) \approx \Phi^{-1}(r). \tag{A.1.12}$$

Furthermore, there is $C \geq 1$ such that for all $\lambda \geq 1$ and $r > \Phi(x_0)$,

$$\Phi^{-1}(\lambda r) \leq C\lambda^{1/\alpha}\Phi^{-1}(r). \tag{A.1.13}$$

Proof. We start by showing that there is $C \geq 1$ such that for all $x > x_0$,

$$C^{-1}\psi^*(x) \leq \Phi^*(x) \leq C\psi^*(x). \tag{A.1.14}$$

The first inequality in (A.1.14) immediately follows from (A.1.6). If $x_0 = 0$, then the second inequality is also the consequence of (A.1.6). In the case $x_0 > 0$ we observe that, for $x > x_0$, we have

$$\begin{aligned} \Phi^*(x) &= \max\Big\{ \sup_{0<y\leq x_0} \Phi(y), \sup_{x_0\leq y\leq x} \Phi(y) \Big\} \\ &\lesssim \max\left\{\Phi^*(x_0), \psi^*(x)\right\} \\ &\leq \left(1 + \frac{\Phi^*(x_0)}{\psi^*(x_0)}\right)\psi^*(x), \end{aligned}$$

proving (A.1.14).

Now, using (A.1.14), we easily get

$$\psi^{-1}(C^{-1}r) \leq \Phi^{-1}(r) \leq \psi^{-1}(Cr)$$

for all $r > C\psi^*(x_0)$. Hence, by Proposition A.1.4,

$$\Phi^{-1}(r) \approx \psi^{-1}(r)$$

for $r > C\max\left\{\psi^*(x_0), 2h(1/x_0)\right\}$. Finally, since both ψ^{-1} and Φ^{-1} are positive and continuous, at the possible expense of the constant we can extend the area of comparability to conclude (A.1.12). Now, the scaling property of Φ^{-1} follows by (A.1.12) and Proposition A.1.4. □

Since $\Phi \leq \Phi^*$, by Proposition A.1.5 and (A.1.6) we immediately obtain the following.

Remark A.1.6. Suppose $\Upsilon \in \mathrm{WLSC}(\alpha-2, c, x_0)$ for some $c \in (0,1]$, $x_0 \geq 0$ and $\alpha > 0$. There is $c_1 \in (0,1]$ such that for all $x > x_0$,

$$c_1\Phi^*(x) \leq \Phi(x) \leq \Phi^*(x).$$

Remark A.1.7. Note that, alternatively, one can define the (left-sided) generalized inverse

$$\Phi_{-1}(x) = \inf\{r > 0\colon \Phi_*(r) = x\},$$

where

$$\Phi_*(r) = \inf_{r\leq x} \Phi(x).$$

In such case we have

$$\Phi_*(\Phi_{-1}(s)) = s \qquad \text{and} \qquad \Phi_{-1}(\Phi_*(s)) \leq s.$$

Clearly, for all $x > 0$,

$$\Phi_*(x) \leq \Phi(x) \leq \Phi^*(x).$$

Let $u > x_0$ and set

$$r_0 = \inf\{r > 0\colon \Phi^*(r) = u\}.$$

By Proposition A.1.5, $\Phi^* \in \mathrm{WLSC}(\alpha, c, x_0)$ for some $c \in (0,1]$ and $x_0 \geq 0$. Thus, for $\lambda > c^{-1/\alpha}$, we get $\Phi^*(\lambda r_0) > \Phi^*(r_0)$. It follows that, for all $u > x_0$,

$$\begin{aligned} \sup\{r > 0\colon \Phi^*(r) = u\} &\leq \lambda \inf\{r > 0\colon \Phi^*(r) = u\} \\ &\leq \lambda \inf\{r > 0\colon \Phi_*(r) = u\}. \end{aligned}$$

Thus, for all $r > x_0$,

$$\Phi^{-1}(\Phi^*(r)) \lesssim r.$$

A.1.2 Sufficient conditions

Let us end this section with two useful sufficient conditions that entail the weak lower scaling property of the (minus) second derivative of the Laplace exponent and allow us to apply results from Chapters 3 and 4. First, we present the equivalence between scaling property of Υ and of the real part of the characteristic exponent.

Proposition A.1.8. *We have* $\Upsilon \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$, $x_0 \geq 0$ *and* $\alpha > 0$ *if and only if* $\mathrm{Re}\,\psi \in \mathrm{WLSC}(\alpha, \tilde{c}, x_0)$ *for some* $\tilde{c} \in (0,1]$.

Proof. In view of (A.1.6) and (A.1.7), it remains to prove the second implication in the corollary. We first prove that $\psi^* \in \mathrm{WLSC}(\alpha, c_1, x_0)$ for some $c_1 \in (0,1]$. Let $x \geq x_0$ and $\lambda \geq 1$. By monotonicity of ψ^* and scaling property of $\mathrm{Re}\,\psi$,

$$\begin{aligned}\psi^*(\lambda x) &= \max\big\{\psi^*(\lambda x_0), \sup_{\lambda x_0 < x \leq \lambda x} \mathrm{Re}\,\psi(r)\big\} \\ &\gtrsim \max\big\{\psi^*(x_0), \lambda^\alpha \sup_{x_0 < r \leq x} \mathrm{Re}\,\psi(r)\big\}.\end{aligned}$$

Now observe that, since $\lim_{r\to\infty} \mathrm{Re}\,\psi(r) = \infty$, there is $x_1 \geq x_0$ such that $\mathrm{Re}\,\psi(x) \geq \psi^*(x_0)$ for all $x \geq x_1$, and consequently, for all $\lambda \geq 1$ and $x \geq x_1$,

$$\psi^*(\lambda x) \gtrsim \max\big\{\psi^*(x_0), \lambda^\alpha \sup_{r \leq x} \mathrm{Re}\,\psi(r)\big\} = \lambda^\alpha \psi^*(x),$$

and by standard extension argument we get scaling property of ψ^* as desired.

Now, it remains to notice that, by the integral representation of Υ,

$$x^{-2}K(1/x) \lesssim \Upsilon(x) \lesssim x^{-2}h(1/x).$$

Thus, [43, Lemma 2.3] yields the claim. □

Next, although our starting object to operate with is the Laplace exponent, in some cases it is more convenient to work with Lévy measure instead. The following proposition states that an analogous condition on the tail of the Lévy measure is enough in our setting.

Proposition A.1.9. *Suppose that there are* $x_0 \geq 0$, $C \geq 1$ *and* $\alpha \in (0,2]$ *such that for all* $0 < r < 1/x_0$ *and* $0 < \lambda \leq 1$,

$$\nu((r,\infty)) \leq C\lambda^\alpha \nu((\lambda r, \infty)). \tag{A.1.15}$$

Then $\Upsilon \in \mathrm{WLSC}(\alpha-2, c, x_0)$ *for some* $c \in (0,1]$.

Proof. Let us first notice that, by the Fubini–Tonelli theorem,

$$\begin{aligned}h(r) &= r^{-2}\left(\sigma^2 + \int_{(0,\infty)} \min\{r^2, s^2\}\nu(ds)\right) \\ &= r^{-2}\left(\sigma^2 + 2\int_0^r t\nu((t,\infty))\,dt\right).\end{aligned}$$

Thus, by (A.1.15), for all $0 < r < 1/x_0$ and $0 < \lambda \leq 1$,

$$\begin{aligned}C\lambda^\alpha h(\lambda r) &= \frac{C}{\lambda^{2-\alpha} r^2}\sigma^2 + \frac{2C\lambda^\alpha}{r^2}\int_0^r t\nu((\lambda t,\infty))\,dt \\ &\geq \frac{\sigma^2}{r^2} + \frac{2}{r^2}\int_0^r t\nu((t,\infty))\,dt \\ &= h(r).\end{aligned} \tag{A.1.16}$$

Hence, by [43, Lemma 2.3], there is $C' \geq 1$ such that, for all $0 < r < 1/x_0$,

$$K(r) \leq h(r) \leq C'K(r). \tag{A.1.17}$$

The integral representation of Υ entails that

$$e^{-1}x^{-2}K(1/x) \leq \Upsilon(x) \leq e^2 2^{-2}x^{-2}h(1/x), \quad x > 0,$$

thus, by (A.1.17), we obtain

$$\Upsilon(x) \approx x^{-2}h(1/x)$$

for all $x > x_0$. Now, the weak lower scaling property of Υ is a consequence of (A.1.16). $\square$

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