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REVIEW OF THE DOCTORAL DISSERTATION OF MGR ŁUKASZ LEŻAJ

The main objects of investigation in this dissertation are the asymptotic behavior and bounds on the transition density $p(t, x, y)$ of a Lévy process X_t when it moves from a point x to a point y during time $t > 0$. Such density is implicitly defined by the formula

$$\mathbb{E}_x f(X_t) = \int_{\mathbb{R}} p(t, x, y) f(y)\, dy, \quad t > 0,\ x \in \mathbb{R}$$

for a suitable class of functions f. By an analogy to the case when X_t is a Brownian motion, $p(t, x, y)$ is also called a (generalized) heat kernel. One should mention that while the behavior and estimates on the transition density are important for Lévy processes in $\mathbb{R}^d$, for any $d \geq 1$, the case $d = 1$ for non-symmetric Lévy processes has not been fully understood, even for such simple processes as the subordinators. The work presented in mgr Leżaj's dissertation significantly changes this situation by making an important contribution to this area of research. Moreover, the case $d = 1$ exemplifies the difficulties of dealing with the non-symmetry in general. It is plausible that generalizations of some methods and tools presented in this dissertation may eventually lead to progress in the multidimensional case.

Three papers of mgr Leżaj form the core of his dissertation. These are a single authored paper [69] and two joint papers [37] and [38]. Papers [37] and [69] have already appeared, while [38] has recently been accepted for publication in the *Journal of the Institute of Mathematics of Jussieu.* These citation numbers follow the Bibliography provided at the end of this dissertation.

The dissertation of mgr Leżaj consists of five chapters, an appendix, and the bibliography. Chapter 1 is an introduction. Chapter 2 contains extensive yet essential preliminary material on Lévy processes, including their Fourier and Laplace transforms, infinitesimal generators, concentration functions and Pruitt's estimates. The preliminaries also include some elements of a potential theory for Lévy processes, pertinent parts of fluctuation theory, regular variation, and weak scaling properties. Chapters 3, 4, and 5 are based on the material from papers [38], [69], and [37], respectively, which will be discussed below. The Appendix includes a number of useful relations between various characteristics of Lévy processes. An extensive bibliography citing 108 books and articles concludes this work. The dissertation is well written, having the material carefully introduced and properly explained. The pertinent references are provided. If the candidate would decide to publish this dissertation, it will become an excellent source of knowledge for researchers and students who want to get familiar with this area of research and its results.

Chapter 3 is concerned with the existence, estimates, and asymptotic behavior of transition densities of subordinators. The Laplace exponent ϕ of a subordinator T_t, $t \geq 0$ is the main characteristic of $(T_t)_{t\geq 0}$, in terms of which other characteristics of T_t are built. The crucial notions are the weak lower scaling (WLSC) and the weak upper scaling (WUSC) properties of functions, which lead to the robustness of the results. Theorems 3.4.10 and 3.4.13 have elegant conclusions and are robust.

Two applications are considered. The first one is heat kernel estimates of a subordinated Markov process. Namely, let $(X_t)_{t\geq 0}$ be a Markov process in a locally compact Polish space with a full support reference Radon measure μ and transition densities $h(t,\cdot,\cdot)$ (with respect to μ) so that

$$\mathbb{P}(X_t \in B | X_0 = x) = \int_B h(t,x,y)\,\mu(dy) \quad B \text{ Borel}. \tag{1}$$

Let T_t, $t \geq 0$ be a subordinator which is independent of $(X_t)_{t\geq 0}$. Then the subordinated Markov process $(X_{T_t})_{t\geq 0}$ has the heat kernel of the form

$$H(t,x,y) = \int_0^\infty h(s,x,y)\,G(t,ds),$$

where

$$G(t,s) = \mathbb{P}(T_t \geq s).$$

It is shown that using the results of this Chapter one can obtain sharp estimates on the heat kernel $H(t,x,y)$. In another application, sharp estimates on the Green function U of $(T_t)_{t\geq 0}$ are derived.

In Chapter 4 transition densities of spectrally positive Lévy processes are investigated. Such processes have their Lévy measures concentrated on the positive half-line, similarly like subordinators. However, unlike subordinators, spectrally positive Lévy processes may have locally unbounded variation. It appears that the approach and techniques used in the previous Chapter 3 can be adapted to spectrally positive Lévy processes as well. Theorem 3.4.10 has its counterpart in Theorem 4.5.1 while Theorem 3.4.13 is the counterpart in Theorem 4.5.3 (notice that there are three cases in the last theorem).

The final Chapter 5 concentrates on the study of the first hitting time of a compact set located on the real line by a one-dimensional Lévy process. The approach in this chapter is based on the potential theory methods. Let's discuss Theorem 5.3.7 from this chapter, which brings a significant progress on a long standing open problem. Namely, the theorem gives the asymptotic behavior for the tail of the distribution of the first hitting time of a compact set that contains the origin in the case when the Lévy process is non-symmetric. There are some assumptions in the theorem on the regular variation of the real and imaginary parts of the characteristic exponent at 0. All stable Lévy processes satisfy conditions of Theorem 5.3.7, which makes it an important contribution to the program of developing methods for dealing with the asymmetry.

In conclusion, this dissertation is full of new original solutions to research problems that will find applications in various areas of sciences and economy. It also opens new directions and the methodology to follow. The candidate demonstrated deep understanding of the problems, available methods, and techniques. He is fully capable to conduct his own independent scientific work. I would judge the present dissertation as excellent and deserving the award.

Sincerely yours,

Jan Rosinski
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Elected Fellow of the IMS
Elected Member of the ISI