**Pierre PATIE**
School of ORIE
Cornell University
USA

Report on the thesis of **Łukasz Leżaj** entitled
*Non-symmetric Lévy processes on the real line*

Ithaca, August 25th 2021

The thesis of Łukasz Leżaj deals with several substantial and delicate problems arising in the theory of Lévy processes and related processes. The class of Lévy processes is central in probability theory and has ramifications in several branches of mathematics, including functional analysis, potential theory, differential geometry and mathematical physics. They are also fundamental in many areas of applied mathematics such as mathematical finance, risk theory, biology, astrophysics...

More specifically, the work of Łukasz Leżaj provides an in-depth and very accurate asymptotic analysis as well as sharp two-sided estimates of the heat kernel and the distribution of the first hitting times of several important classes of non-symmetric Lévy processes. There is a rich and fascinating literature on these topics in the case of symmetric Lévy processes, or more generally symmetric Markov processes, which mostly relies on theories which have been successively developed for diffusions processes. However, any attempt to extend these techniques to the non-symmetric case have met with resistance so far. To overcome this difficulty, one definitely needs novel ideas and original combinations of various mathematical techniques.

The thesis is rigorously and well-written mostly in English except a Polish version of the introduction, that, unfortunately, I am not be able to comment. It is composed of 5 chapters, an introduction, one which are preliminaries and the 3 others cover different mathematical problems. These last three chapters correspond to 3 research papers, one written as a single author. They all have been published or accepted in excellent journals in the area of probability theory or pure mathematics.

The **Introduction** motivates nicely the main topics of the work by describing thoroughly the impressive span of applications of Lévy processes in many areas of sciences since their characterization by Paul Lévy in the 1930's. It also contains a clear explanation of the challenges that one faces when studying problems related to heat kernel estimates for Lévy processes with non-symmetric jump structures compared to the symmetric case that has been only addressed, beside some isolated cases, in the literature up to date. Although each subsequent chapter of the thesis contains a comprehensive list of references, the additions of a few references in this part of the work would have made the introduction more self-contained, this is of course a minor comment. The rest of the introduction is devoted to a concise description of the results along with a broad and useful explanation of the original ideas and techniques that Łukasz Leżaj has implemented to prove them.

**Chapter 2** sets up the notation used throughout the thesis and provides a very well-written review on properties of Lévy processes and other more specific mathematical tools needed by Łukasz Leżaj to develop his theory. This includes measure theoretical and path properties, essential results on their potential theory, some elements of their fluctuation theory and some functional analytical results such as Harnack inequality. It also contains detailed information about the class of regular variation functions and its substantial generalization which enjoy the so-called weak scaling properties. The latter class which is roughly the set of measurable functions that are bounded from below or/and above by power functions of possible different indexes is the standing assumption in this work.

**Chapter 3** is concerned with the study of the transition densities (aka the heat kernel) of an important subclass of Lévy processes, namely the subordinator, which have non-decreasing sample paths. Their law is fully characterized by their so-called Laplace exponent which are Bernstein functions. Beyond their own interests, subordinators and Bernstein functions are ubiquitous in a variety of mathematical contexts, such as Bochner subordination, harmonic analysis, complex analysis and operator theory, revealing the purpose of substantial contribution in this area.

In order to illustrate the relevance of the findings, this chapter also presents intriguing applications of the two-sided estimates of the transition densities of subordinators to subordination in a very general context and also to Green functions of subordinators.

After motivating and reviewing the main results of this chapter, a very interesting and original analysis of the weak lower and upper scaling properties (WLSC and WLUC for short) of Bernstein functions and completely monotone functions is carried out. In particular, closure properties under differentiation and integration operation is proved, comparison results between Bernstein functions and their derivatives, and, also with complete Bernstein functions are also provided.

The first main result Theorem 3.1.1 presents, under the mere assumption that the second derivative of the corresponding Bernstein function satisfies the WLSC condition, an explicit Gaussian type asymptotic for large time. This generalizes a recent result by Chen, Kim, Kumagai and Wang (2020) which was obtained under a global stable-like assumption, which is much more restrictive. With the additional WUSC condition but with a possible different index, this result is improved to be valid uniformly in a time-space domain. The proof is based on a saddle point approximation, a powerful complex analytical technique used in asymptotic analysis. The implementation of such an approach in the framework stems on fine and delicate estimates on the representation of the inverse Laplace transform of the transition density as a

Bromwich contour integral. The existence of the latter as well as its estimates require bounds on the holomorphic extension of the Bernstein functions which explain the need of the WLSC assumption. As a side remark, there is an unspoken reference from Doney and Rivero (*Asymptotic behaviour of first passage time distributions for subordinators,* Electron. J. Probab. 20 (2015), no. 91) where the authors obtain uniform local estimates of Gaussian type for the one dimensional distribution of subordinator that belongs to the so-called Feller class. Also this latter includes the regularly varying class, it is not obvious to compare the two type of assumptions. Moreover, the conditions and arguments of proofs, which are based on known uniform estimates of the density of some random walks, are of completely different nature than the one developed by Łukasz Leżaj, which seems, a priori, more comprehensive.

This chapter proceeds with several one and two-sided estimates for the transition density for small time. First, Lukasz Leżaj manages to derive upper estimates on the transition density valid for small time, under the WLSC condition, which is refined under the additional condition that the Lévy measure admits a decreasing density. This is achieved by very clever and non-trivial manipulations of a general upper bound obtained on transition densities of Lévy processes by Kaleta and Sztonik (2015). The study of the lower bounds for small time, which is split into two space domains, is more demanding and is developed under the WUSC condition. It is inspired by some ideas that were proposed in Picard (1995) and developed further by Grzywny and Szczypkowski (2020). For the one associated to say small values of the space variable, Lukasz Leżaj uses an original combination of analytical and measure theoretical techniques based on intriguing limiting argument. Eventually, after showing, which is a non-trivial exercise, that both the lower and upper estimates coincide on the time-space domain, sharp two-sided estimates are obtained under the combination of the WLSC, WUSC conditions with the almost decreasing property of the Lévy measure.

This chapter closes with two interesting and substantial applications of these two-sided estimates. The first one is two-sided estimates for the transitions densities of Bochner subordinated semigroups of a Markov process valued in a locally compact separable metric space for which it is assumed the existence and estimates of the transition densities defined with respect to a reference measure. The second application provides, under both the WLSC and WUSC conditions, another two-sided estimates for the density of the potential measure of a subordinator. Throughout the chapter, examples are provided to illustrate the novelty of the approach.

**Chapter 4** contains an in-depth study of the transitions densities, including their asymptotic and two-sided estimates, of spectrally negative Lévy processes. These are real-valued Lévy processes experiencing only downward jumps. This property implies that their corresponding Lévy measures have support included in the negative half-line. Several substantial examples of stochastic processes fall into this class such as the Brownian motion with drift, spectrally negative stable processes, and, also the classical Cràmer-Lundberg risk processes which are canonical models in insurance mathematics. The distribution of spectrally negative Lévy processes is fully characterized by their so-called Laplace exponent which extends to a function holomorphic in the right-half plane. This property enables Lukasz Leżaj to carry out a similar program than the one developed in Chapter 3 to obtain, under the same type of conditions, large asymptotic estimates of the transition densities (heat kernel) of these processes as well as sharp two-sided estimates.

After describing accurately the motivation and challenges underlying its study, the two main claims of this chapter are presented. The first one is an asymptotic result for large time of the transition densities under the condition that the second derivative of the Laplace exponents belong to the WLSC class. As a companion result of Lukasz Leżaj's asymptotic, there is Corollary 2.1 in Patie and Vaidyanathan (*Non-classical Tauberian and Abelian type criteria for the moment problem,* arxiv, 2018, to appear in Mathematische Nachrichten) that contains similar large asymptotic estimate under the presence of a Gaussian component, a case excluded in Lukasz Leżaj's claim. The second one consists on sharp two-sided estimates of the transition densities of spectrally negative Lévy processes of unbounded variation but without Gaussian components.

The subsequent section develops an interesting and useful analysis, including bounds and closure properties, of the class of Laplace exponents that satisfy the WSC conditions. A complementary analysis is exposed in the Appendix A at the end of the thesis. The remaining part of this chapter is devoted to the proof of the main results. Although the strategy is similar to the one developed in Chapter 3, the detailed justification of the different steps is non-trivial and differs substantially from the previous case.

This thesis ends with **Chapter 5** which focusses on sharp two-sided estimates of the distribution of the first hitting time of a compact set by a (non-symmetric) Lévy process. These estimates are expressed in terms of two different versions of the compensated potential kernel. Its proof requires several steps of different nature. First, under a global WLS condition, a Harnack inequality is derived inspired by a reasoning due to Bass and Levin (2002). As a byproduct, the sharp two-sided estimate is derived under a technical condition which is difficult to check. However, relying on a recent preprint by Grzywny (2019), this condition is shown to be satisfied by some important classes of Lévy processes, e.g. the symmetric and the spectrally negative ones.

It is worth pointing out that the section 2 of this chapter discusses the regular variation property for the characteristic exponent of Lévy processes. More precisely, it contains, Proposition 5.2.1, an original and very interesting result which relates the regular variation of the real part of the characteristic exponent, either at 0 or $\infty$ to the asymptotic behaviour of the tails of the Lévy measure at $\infty$ and 0 respectively. This is definitely a substantial add-in to the very developed theory of regularly varying functions in the context of Lévy processes. The proof relies on clever combination of Mellin convolution, approximation procedures and asymptotic analysis techniques.

Overall, the work of Lukasz Leżaj is a fundamental and original contribution to the study of the substantial class of Lévy processes which have been intensively studied over the last three decades. The diverse asymptotic as well sharp two-sided estimates of several heat kernels and distribution functions presented in this thesis reveal that Lukasz Leżaj not only master some deep techniques in a variety of mathematical areas including probability and potential theory, complex and asymptotic analysis and to some extend in functional analysis, but also has the ability to develop original and rigorous ideas.

For all these reasons, I recommend distinguishing the thesis as excellent.

Pierre Patie