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Mathematical Modelling of Investment Portfolio Management Strategies. Theory and Applications within Heston Market Model.

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1 Introduction

1.1 History of portfolio management research

The interest in selecting and holding investment assets, which we nowadays refer to as building and managing investment portfolio is an old and well know problem in the field of many disciplines including financial mathematics, financial engineering and econometrics. The foundation of the theory of selecting assets for an investment portfolio was laid by an American economist and Nobel Prize winner – Harry Markowitz. In his article from 1952 he postulated that an investor acting rationally should always try to maximise their expected returns and minimise the risk of the entire portfolio, taking into account possible correlations between the assets [1]. The approach presented in the article turned out to be extremely well-received and the strategies described there have been collectively named a Modern Portfolio Theory or MTP in short. The concept introduced by Markowitz was then widely developed and built up. In 1963 William Sharpe devised a very computationally efficient method of analysing available assets and creating a portfolio having desired properties [2] and in 1972 Robert C. Merton published a paper in which he shows how to analytically obtain the "efficient frontier" – a graphical illustration of a set of all portfolios optimal in Markowitz sense [3]. Notable extensions of Markowitz's model were made by Gerald A. Pogue who enhanced the model by taking into consideration transaction costs and short selling [4]. In 1992 Fisher Black and Robert Litterman built their own model based on the one of Markowitz, which did not require to associate assets with specific expected returns (which were difficult to estimate) [5] and in 1991 B.M. Rom and K. Ferguson extended that model even further by addressing the limitations of the original model regarding the distribution of returns and the usage of variance of returns as a measure of investment's risk [6]. What they accomplished is often called a Post Modern Portfolio Theory. More recent research is also still being conducted within the Markowitz framework. In 2013 Woodside-Oriakhi et.al. presented an extensive model of portfolio optimisation and solved the stated problem by mixed-

integer quadratic programming [7]. In 2014 Mittal and Mehlawat proposed a model including transaction costs and also solved an associated optimisation problem but by means of real-coded genetic algorithms [8].

There were also numerous publications on the topic of portfolio management where the fast developing theory of fuzzy sets was applied. One of the pioneering works in that field was the one of Hideo Tanaka et.al. from 1998 in which authors presented the first extension of Markovitz's idea to use the fuzzy set theory and exchanging the notion of probability for, what is called, fuzzy probability [9]. Research in the area was later continued by i.a. Christer Carlsson et.al. who were successful in finding an optimal portfolio in Markowitz sense assuming that returns are fuzzy variables (instead of classical random variables) [10]. Also, Yong Fang et.al. introduced the idea changing portfolio composition during portfolio's lifetime within the fuzzy set theory framework [11].

One of the methodologies which is becoming increasingly popular in recent years for studying portfolio management is machine learning. The abundance of financial data and the ease of access to most of it unleashes the great potential of machine learning methods in the field. One of the most promising methods originating from machine learning are deep learning and reinforcement learning. The deep learning methods utilise deep (multi-level) neural networks to effectively create the Markowitz frontier and have hence been cumulatively called the Deep Portfolio Theory [12, 13]. Reinforcement learning, on the other hand, creates an environment in which the program can take certain actions, analyse its results quantitatively in respect to what is called a reward function and adapt the actions taken in order to maximise the reward. The actions can be, for example, allocating cash in several financial assets (hence forming an investment portfolio) and the reward function could be the return form this portfolio after some time. This way of making the algorithms learn how to make investments decisions seems to be effective in many different contexts [14]. However, within both deep learning and reinforcement learning, the number of different approaches and research directions is really immense, hence — there are already books [15] and voluminous series of articles [16, 17] covering the variety of research used there.

Independently of the Markowitz framework, which mostly focuses on optimal selection of assets to the portfolio, the idea of altering an investment portfolio after it's created has also been studied. One of the first researchers who did that was Paul A. Samuelson. In his article from 1969 he was considering the problem of what changes should be done to a life-time portfolio at the number of discrete moments of time [18]. Similar problems were then studied by Robert C. Merton in his two famous articles, from 1969 and 1971, in which he discusses the topic of building a life-time

portfolio of stocks for the sake of calculating optimal consumption (the amount of money that can be withdrawn from the portfolio with no major detriment for this portfolio's performance) [19, 20]. Another worth-noticing piece of work in this stream was i.a. the paper of Morton and Pliska in which they introduced transaction costs to the Merton portfolio problem and eliminated the need of making transactions in continuous time, giving instead a sequence of discrete moments of time at which transactions are allowed to be made by an investor [21]. There was also a paper by Colin Atkinson et.al. in which the portfolio problem in question was translated into the language of differential equations and a numerical procedure to solve them was proposed [22]. On the other hand, Jakša Cvitanić and Ioannis Karatzas were studying a slightly different portfolio optimisation problem using martingale theory and advanced tools of stochastic calculus [23].

An entirely different approach to the topic of portfolio optimisation has been also proposed in regard to a famous paper of J.L. Kelly Jr [24]. The paper itself is not however related to portfolio management directly — actually all statements are formulated in the language of information theory and examples given are referencing gambling. The author argues that the best strategy for a gambler is to maximise the logarithm of wealth (which in this context simply meant the logarithm of the amount of money possessed by the gambler). Kelly proved that a strategy devised that way will outperform any other strategy in a long enough amount of time almost surely. This idea was called a "Kelly criterion". It was quickly implemented in the field of portfolio management and found a lot of attention but also a lot of controversy. One of the biggest criticisers of this approach were Paul Samuelson and Robert C. Merton, who were of the opinion that the research related to portfolio management should continue to be conducted in the language of utility functions (developed by them) calling the application of Kelly criterion a "fallacy" [25]. Despite this criticism, the idea of building and maintaining a Growth Optimal Portfolio (which evolved from the "Kelly criterion") is still being extensively studied. The first chapter of the book by Morten Mosegaard Christensen from 2012 constitutes a very broad overview of the history, mathematical and practical aspects and various discussions on the topic of the Growth Optimal Portfolio [26]. A study of a portfolio using Kelly strategy has also been presented in an article of Paolo Laureti et.al. [27].

The last set of methods in the field of investment portfolio management which we will name in this brief overview are Monte Carlo simulations. They were used by Riccardo Cesari and David Cremonini who provided a wide comparison between various portfolio management techniques in their article from 2003 [28]. Their main assumption was the normality and independence of returns. They were first testing those qualities in their data and later they were simulating synthetic data-sets

according to the results the tests they obtained. A slightly more theoretically-oriented article written by Ofer Alper et.al. appeared in 2017 [29] aiming to compare selected portfolio management strategies in presence of correlation between the assets and with inclusion of transaction fees. A significant part of the work in this dissertation can be seen as an extension of this particular article.

1.2 Overview of the content of this work and its goals

As it was indicated in section 1.1, the research in the field of investment portfolio management has a long and diverse history. Over the years, thousands of scientists published their works on this subject matter and they were using a variety of tools and methods originating from a lot of different fields of science — not only pure mathematics, but also statistics, physics and social sciences. This variety of research methodologies led to the situation in which it is difficult to acquire knowledge about investment portfolio management, as different approaches use different language for the description of the concepts of interest. Sometimes the same ideas are named or denoted completely differently by different authors. Some works, especially the ones focusing mostly on the practical and application aspects of how to manage the investment portfolio, lack mathematical precision and unambiguity, which makes it significantly more difficult to e.g. reproduce the results which they present. This dissertation can be seen as an attempt to address all of those problems. One of the main goals of this thesis is to present a rigorous, mathematically-rooted framework in which portfolio management subject area can be learnt, developed and researched. Moreover, we strongly believe that such framework is very much needed. As an analogy — we know it was possible to analyse various random experiments before the notion of probability space was introduced by Andrey Kolmogorov in 1933. But only after it happened, the theory of probability found its proper place in the world of mathematics as a branch of measure theory and only then it become one of the most prominent areas of mathematics as a whole. We hope this work can contribute to finding a proper place for the research over various aspects of investment portfolio management. The entire Chapter 2 has been devoted to present a proposal of a rigorous mathematical framework which can be used for mathematical analysis within portfolio management area. In this part of the dissertation we formally define the notion of a market, a portfolio and measures to assess its performance, including the most basic one — the portfolio wealth. We agree what exactly we will be calling a portfolio management strategy and introduce a number of them within a discrete-time

regime. The scope of this chapter has been cover in Ref. [30].

We do believe that a good mathematical foundation for a research topic is necessary for the results of this research to be meaningful. However, in come cases, designing mathematical concepts and ideas is not the goal in itself. More often then not, those concepts and ideas are created to model some real-life phenomena and studying them allows to draw incisive conclusions about the reality. This is another field which we think this work contributes to. Using the foundations set out in Chapter 2, in Chapter 3 we designed, performed and analysed the outcomes of a group of numerical experiments which allowed us to deepen our understanding of the behaviour of the portfolio management strategies, which we selected to study. We compared several strategies which utilise a scheme called portfolio rebalancing and examined the impact of a couple of factors on their performance. We also put to the test some strategies, the definitions of which we formalised ourselves, which use well-known trading indicators to make the buy-or-sell decision. All the numerical experiments were based on the Monte Carlo methodology and as such, the same experiment was always repeated usually at least several hundred times to draw reliable conclusions — this has been explained in section 3.1. The well-established model of Heston [31] has been chosen by us for simulating the trajectories meant to represent the prices of financial assets. The detailed specification of the model and its parameters have been described in section 3.3. In subsequent section, 3.4, the addition of discontinuities to the Heston model has been described. Since all our strategies have been defined to be discrete in time, section 3.5 describes the details of Heston model discretisation and simulation (as those topics are very much related to one another) both with and without jumps. The experiments themselves have been presented in section 3.6. Most of them have also been published in Ref. [32], although we added some new ones for completing this work in particular.

In many disciplines, computer simulation results provide a valuable insight into the character of the objects or phenomena studied, but the final step is to relate the outcomes obtained by simulations to real-life observations. Portfolio management is very much an example of where this should also be done. While in multiple fields of science the parameters of the model used for simulation are known up-front, since they represent some characteristics of the objects that are being studied, which can be directly measured, it is not the case for analysing investment strategies and moreover — it is usually not the case for most models used in mathematical finance. Those models often feature parameters or additional variables referred to as “drift” or “volatility” which most people in the realm of financial analysis are familiar with, but it is far from obvious how to get the exact values of those quantities for a given financial model. Usually, the only aspect of a financial instrument to which researchers and

practitioners have easy access is its price, changing over time. Everything else needs to be somehow extracted out of the price process. Addressing this problem has been another goal of this work. We managed to successfully assemble an effective estimation technique for the Heston model (without and with jumps), utilising both existing and new methods, devised by us. The tools which we used are based on the idea of Bayesian inference [33] and Monte Carlo Markov Chains [34]. It has been presented in Chapter 4. Not only did we provide an extensive, precise and easy-to-follow recipe for the estimation process (section 4.2), but we also showed the results of an exemplary estimation for a simulated asset, thoroughly analysed it and described all aspects of using the procedure which require additional caution (section 4.3). Results presented in this chapter have been published in Ref. [35]

Chapters 3 and 4 respectively provide results of simulation-based research and the estimation procedure, which allows obtaining the actual values of parameters the simulation experiments were based on. The final step, therefore, was to demonstrate the applicability of those synthetic results to the real data. This has been done in Chapter 5. By estimating the parameters of our model for three different financial assets we were able to place them on a spectrum of the applicability of portfolio management strategies which we studied and it indeed turned out that the choice of strategies advised by our research was optimal in terms of profits. We have demonstrated that in Ref. [36] as well.

Chapter 5 ties together all the research areas covered in this work — theoretical considerations, numerical experiments, the estimation scheme and a real-data experiment. It is followed by a short summary and outlook for further research in Chapter 6.

2 Formal framework for studying portfolio management

2.1 Fundamental concepts of portfolio dynamics

As a first step, we are going to formally define the notions used throughout the rest of the work. Some of those terms appeared in diverse forms in literature already but we provide our own definitions to systematise their usage, as mentioned in the Introduction. We start by defining the notion of *a market*.

Definition 1. A market $\{\mathcal{M}_i\}_{i\in I}$ is an indexed family with an index set $I = \{1, 2, \ldots, N\}, N \in \mathbb{N} \setminus \{0\}$ consisting of N stochastic processes $S_i \in I$, all defined on the same probability space $(\Omega, \mathcal{F}, P)$, the same time domain $[0, +\infty]$ and having the same state space — $\mathbb{R}_+$. Each stochastic processes S_i in $\mathcal{M}$ is called *an asset*.

While the elements of the market might theoretically represent various types of assets, throughout this thesis we will mostly use this term to represents shares of stocks. Hence, the terms: "asset" and "stock" will be used interchangeably.

We can now proceed with the definition of a *portfolio* on a market.

Definition 2. A portfolio $\mathcal{P}$ built on a market $\mathcal{M}$ is an ordered pair $(\mathcal{S}, \mathcal{Q})$. $\mathcal{S}$ is called an asset component and it is a partially stochastic, $(N+1)$-dimensional vector-valued function with coordinate functions of indices 1 to N being elements of $\mathcal{M}$,

$$\mathcal{S} : [0, +\infty] \to \mathbb{R}_+^{N+1}. \tag{2.1}$$

The coordinate function of the index 0 inside $\mathcal{S}$ is always equal 1.

$$S_0(t) \equiv 1 \text{ for all } t \in [0, +\infty]. \tag{2.2}$$

In other words, for a time moment $t \in [0, +\infty]$, $\mathcal{S}(t)$ is defined as

$$\mathcal{S}(t) = \Big(S_0(t), S_1(t), S_2(t), \ldots S_N(t)\Big) = \Big(1, S_1(t), S_2(t), \ldots S_N(t)\Big). \tag{2.3}$$

Likewise $\mathcal{Q}$, which is called a quantity component, is also a stochastic, $(N+1)$-dimensional vector-valued function with coordinate functions q_i being stochastic processes defined on the same probability space and time domain as all of the S_i processes, for $i \in \{1, 2, \ldots, N\}$, but having the entire real line as the state space

$$\mathcal{Q} : [0, +\infty] \to \mathbb{R}^{N+1}, \tag{2.4}$$

$$t \mapsto (q_0(t), q_1(t), q_2(t), \ldots q_N(t)) = \mathcal{Q}(t). \tag{2.5}$$

The S_0 asset represents the cash, which an investor may also wish to store in their portfolio, alongside the risky assets, being a part of the market $\mathcal{M}$. Cash does not change its value in time at all[1], hence it is modelled by a constant function, always equal to 1, since unlike stocks — this fact is reflected in Eq. (2.2). .

The elements of the quantity component at any given moment of time represent the amount of a given asset in a portfolio, e.g. number of shares if the type of assets we are considering is common stock. For the zero-th asset, it simply represents the amount of cash in the portfolio, i.e. q_0. It is highlighted in the definition of the quantity component that the coordinate functions have the entire real line as a state space, which means they can also take negative values. This would represent short-selling, i.e. selling an asset not physically owned by the owner of the portfolio[2].

We will usually only observe and work with portfolios for a predefined amount of time (e.g. one year, three years, ten years). We will call it *portfolio duration*, or *maturity* T and express it in years.

The notion of the portfolio allows us to define what we consider *an investment strategy*.

Definition 3. An investment strategy is a way of deciding upon a value of the portfolio's quantity components q_i at any given moment of time $t \in [0, T]$ based on previous values of this and other quantity components and the values of the assets. In other words, it is a function (possibly auto-recursive or delayed), which allows to establish the current value of the i-th quantity component of a given portfolio, i.e.

$$q_i(t) = f(t, q_j(u), S_j(u)), \tag{2.6}$$

where $i, j \in I$ and $u \in [0, t)$.

[1]This statement is debatable due to existence of various economic circumstances, e.g. periods of high inflation or a possibility of placing cash into a risk-free interest bearing deposit. We consider neither of those in this work and treat the constant value of portfolio cash as an assumption. This assumption could however be easily lifted by introducing another (possibly stochastic) model for cash.

[2]None of the management strategies studied by us allow for shot selling, but — similarly to the case of the variable value of cash — the framework is ready to support such models.

The definition mentions a very natural assumption related any portfolio management strategy which needs to be met in order for the strategy to be applicable in practice. At any point of time t the investor only knows prices of each of the N assets up to this moment and not further, i.e. for every $i \in \{1, \ldots, N\}$ values of $S_i(u)$ and $q_i(u)$ are known only for $u < t$. Violation of this assumption for any strategy would effectively mean a possibility to "predict the future", which is not possible in real-life.

An example of a very simple investment strategy is *a passive strategy*.

Definition 4. A passive strategy is a strategy for which values of the functions of the quantity component are constant in time, i.e. for each $i \in I$ we have $q_i(t) = q_i(0) = const$ for any $t \in [0, T]$.

A portfolio which is run by a passive strategy is often called a passive portfolio — it is very common that a portfolio is referred to using a name of a strategy according to which it is run. The passive portfolio is also sometimes referred to as *a buy-and-forget portfolio* or *a buy-and-hold portfolio*. It is admittedly very straightforward but it is very useful, as it can often be treated as a benchmark, i.e. a baseline to compare against other portfolios in which the owner actually makes some changes intended to make improvements in portfolio performance. To evaluate how an investment portfolio is performing, a number of measures can be introduced. The most basic one is its value changing over time. We call it the portfolio *wealth*.

Definition 5. Wealth of portfolio $\mathcal{P}$ at any moment of time $t \in [0, T]$ is a scalar product of the vector of the values of assets $\mathcal{S}$ and the vector of quantities of these assets $\mathcal{Q}$.

$$W_{\mathcal{P}}(t) = \langle \mathcal{S}(t), \mathcal{Q}(t) \rangle = \sum_{i=0}^{N} S_i(t) q_i(t), \tag{2.7}$$

In most cases, it is obvious from the context the wealth of which portfolio is being described, so we will drop the $\mathcal{P}$ subscript from wealth's denotement and write $W(t)$ instead of $W_{\mathcal{P}}(t)$ whenever possible.

2.2 Balanced portfolio, time discretisation and self-financing

The notion of portfolio wealth, presented in Definition 5 can be used to define a more complex method of portfolio management, than the passive one. It is based on undertaking specific actions when market prices change. It is called *a balanced portfolio*. To define it, we first need to introduce the concept of *fractions of wealth* [29].

Definition 6. An i-th fraction of wealth f_i (also called a wealth fraction or simply – a fraction), at any moment of time t, is a component of the portfolio wealth $W(t)$ associated with an i-th portfolio asset, i.e.

$$f_i(t) = \frac{S_i(t) \cdot q_i(t)}{W(t)} \quad \text{for all } i \in \{0, 1, 2 \dots N\}. \tag{2.8}$$

If we take into consideration the Definition 5 of portfolio wealth, is clear that $\sum_{i=0}^{N} f_i(t) = 1$ — hence the name: wealth fractions. Knowing what they are, we can define a balanced portfolio strategy [29].

Definition 7. A balanced strategy is a strategy, according to which quantity components of assets $q_i(t)$ are selected in such a way that all fractions of wealth f_i are constant in time, i.e. for all $i \in \{0, 1, 2 \dots N\}$ we have $f_i(t) = f_i(0) = const$ for any t.

Such a portfolio construct essentially guarantees that if the price of a particular asset increases, its amount in the portfolio is made smaller (hence — assets are sold when their prices are getting higher) and if the price drops — its amount is increased (thus — we buy new assets when their prices are diminishing). However, the above definition of the balanced portfolio is implicit — it does not provide any guidance on how to manipulate the quantities of the assets to keep portfolio in a balanced state. It is not even obvious if it is always possible to keep it in that state and if it is — what is the formula and can it be given in an explicit way? This is a problem for a lot of investment strategies when we try to describe them in a time-continuous context. This is, however, not actually necessary as in practice we hardly ever deal with continuous pricing of assets and even if we were — it is impossible to perform transactions in a continuous manner on a real market. Therefore, for the sake of practical application — it is sufficient to create an iterative procedure which captures the idea of building a balanced portfolio. To this end two things are necessary. One is the discretised time axis for the process. We can introduce it as a sequence of $(n + 1)$ discrete moments of time, equally distributed every Δt, i.e. $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, t_3 = 3\Delta t, \dots, t_n = n\Delta t = T$, where Δt is a small time increment, $\Delta t = \frac{T}{n}$. Once we have that, we also need a notion of what is called *temporary wealth* of the portfolio, i.e. the value of our portfolio which arises from the amount of assets and cash we had in the previous step.

Definition 8. Temporary wealth of portfolio $\mathcal{P}$ at any moment of time $t = k\Delta t, k \in \{1, 2, \dots, n\}$ is its value at that time if no changes to the amounts of assets in portfolio were made:

$$W^{temp}(k\Delta t) = \sum_{i=0}^{N} q_i^{temp}(k\Delta t) \cdot S_i(k\Delta t), \tag{2.9}$$

where $q_i^{temp}(t) = q_i\Big((k-1)\Delta t\Big)$.

Having both of those things defined, we propose a discretised version of the portfolio balancing strategy [29]

Definition 9. A discrete balanced strategy is a strategy for which the quantity update scheme has the following form:

$$q_i(k\Delta t) = f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}, \tag{2.10}$$

$$q_0(k\Delta t) = 0, \tag{2.11}$$

$$q_i(0) = f_i \cdot \frac{W(0)}{S_i(0)}, \tag{2.12}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $W(0)$ is a fixed, initial portfolio wealth and f_i is an i-th wealth fraction. A set of operations performed in a singe time step, $t = k\Delta t$ for some $k \in \{1, 2, \ldots, n\}$ is called portfolio rebalancing.

In the above definition, we can think of the initial wealth $W(0)$ as of an initial amount of money which will be invested in the portfolio. The temporary wealth of portfolio W^{temp} comes in useful here, as it allows to keep track of the wealth of portfolio in a given time step, before its rebalancing is performed, since before it is done the quantity of each asset is equal to the actual quantity of this asset from the previous step. It is also worth noting, that this portfolio strategy does not use any cash. Its amount is being kept 0, according to equation (2.11).

One of the characteristics of a good portfolio management strategy is being *self-financing*.

Definition 10. A strategy is called self-financing if it requires no additional cash inflow to be executed for any $t \in (0, T]$ and in which profits get immediately reinvested at any given moment $t \in [0, T)$. In terms of quantities and asset prices that means that for any $k \in \{1, 2, \ldots, n\}$, the following equality is preserved

$$\sum_{i=0}^{N} q_i\Big((k-1)\Delta t\Big) S_i(k\Delta t) = \sum_{i=0}^{N} q_i(k\Delta t) S_i(k\Delta t) \tag{2.13}$$

or, alternatively,

$$W^{temp}(k\Delta t) = W(k\Delta t) \tag{2.14}$$

It can be demonstrated that the above definition of a self-financing strategy is satisfied by the discretised balanced portfolio.

Proposition 1. The strategy of a discretised balanced portfolio, as described by Definition 9, is self-financing.

Proof. Let us get a closer look at the wealth of portfolio at an arbitrary point of time $k\Delta t$. Its value can be obtained by using formulas (2.7) and (2.10). We have

$$\begin{aligned} W(k\Delta t) = \sum_{i=0}^{N} S_i(k\Delta t) q_i(k\Delta t) &= \sum_{i=0}^{N} S_i(k\Delta t) f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)} \\ &= \sum_{i=0}^{N} f_i W^{temp}(k\Delta t) = W^{temp}(k\Delta t) \sum_{i=0}^{N} f_i. \end{aligned}$$

From the Definition 6 of wealth fractions we know that $\sum_{i=0}^{n} f_i = 1$. Henceforth, we have

$$W(k\Delta t) = W^{temp}(k\Delta t)$$

.

Thus, performing a rebalance operation does not require any additional money and all changes in portfolio wealth are results of fluctuating assets prices – there is neither any kind of withdrawal nor insertion of money from or into the portfolio. □

We should also prove that the discretised version of the balancing strategy is indeed a balancing strategy, according to its original definition.

Proposition 2. The discrete balancing strategy (from Definition 9), meets the general definition of the balancing strategy (Definition 7) in all the grid points $t = k\Delta t, k \in \{0, 1, 2, \ldots, N\}$.

Proof. According to the Definition 6, at $t = 0$, the value of the i-th wealth fraction is

$$f_i(0) = \frac{S_i(0) \cdot q_i(0)}{W(0)}.$$

Considering the definition of $q_i(0)$ given by Eq. (2.12), we have

$$f_i(0) = \frac{S_i(0) \cdot q_i(0)}{W(0)} = \frac{S_i(0)}{W(0)} \cdot f_i \cdot \frac{W(0)}{S_i(0)} = f_i.$$

At any other moment of time $t = k\Delta t$, we have

$$f_i(k\Delta t) = \frac{S_i(k\Delta t) \cdot q_i(k\Delta t)}{W(k\Delta t)} = \frac{S_i(k\Delta t)}{W(k\Delta t)} \cdot f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}.$$

We proved in Proposition 1, that for a discretised balancing strategy $W(k\Delta t) = W^{temp}(k\Delta t)$, hence

$$f_i(k\Delta t) = \frac{S_i(k\Delta t)}{W(k\Delta t)} \cdot f_i \frac{W(k\Delta t)}{S_i(k\Delta t)} = f_i.$$

We hence proved that the wealth fractions remain constant all the time, which means the discretised balancing strategy is indeed the balancing strategy for any $t = k\Delta t, k \in \{0, 1, 2, \ldots, N\}$. □

From now on, we will only be talking about discrete management strategies, hence the "discrete balanced strategy" will simply be called "balanced strategy".

2.3 Transactions costs and frugal balancing strategies

Most markets charge investors for making transactions, which is why keeping portfolio balanced becomes expensive if we take transaction costs into consideration. In most cases, the value of the fee is determined as a percentage of the value of assets being exchanged. This amount is normally deducted from the cash account related to the brokerage account. However, it is possible to introduce a (simplified) way of including transaction costs into portfolio strategy without the necessity of introducing the notion of portfolio cash. This has been presented for the balanced strategy in Ref. [29]. The idea is that the fee directly lowers the amounts of assets that would be necessary to reconstruct the portfolio and make it balanced again.

Definition 11. A balancing strategy with the inclusion of transaction fees is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = f_i \frac{W^{temp}(k\Delta t) - A(k\Delta t)}{S_i(k\Delta t)}, \tag{2.15}$$

$$q_0(k\Delta t) = 0, \tag{2.16}$$

$$q_i(0) = f_i \cdot \frac{W(0)}{S_i(0)}, \tag{2.17}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $W(0)$ is a fixed, initial portfolio wealth, f_i is an i-th wealth fraction. $A(t)$ is the total value of fees, calculated as

$$A(k\Delta t) = \sum_{i=1}^{N} \alpha |S_i(k\Delta t) \cdot (q_i^{temp}(k\Delta t) - q_i'(k\Delta t))|, \tag{2.18}$$

where α is a constant level of fees and q_i' is the desired quantity of each kind of asset, defined to be

$$q_i'(k\Delta t) = f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}. \tag{2.19}$$

A set of operations performed in a singe time step, $t = k\Delta t$ for some $k \in \{1, 2, \ldots, n\}$ is called portfolio rebalancing.

Note that if we take transaction costs into consideration, as described by formula (2.15), keeping portfolio in a balanced state becomes costly so that profits resulting from using this strategy might be overtaken by the expenses related to the fees. This is why two more frugal strategies have been devised. First one is based on the idea of rarefying the moments of rebalancing the portfolio, by some constant factor m, i.e. performing the rebalance m times less often. This strategy is called *periodically balanced* [29]. In such case the procedure of updating the quantities of assets takes place once every m time intervals of length Δt and between these moments portfolio acts like a passive one.

Definition 12. A periodically balanced strategy with the inclusion of transaction fees is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = \begin{cases} f_i \frac{W^{temp}(k\Delta t) - A(k\Delta t)}{S_i(k\Delta t)} & \text{if } k \bmod m = 0 \\ q_i^{temp}(k\Delta t) & \text{otherwise} \end{cases} \tag{2.20}$$

$$q_0(k\Delta t) = 0, \tag{2.21}$$

$$q_i(0) = f_i \cdot \frac{W(0)}{S_i(0)}; \tag{2.22}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $m \in \mathbb{N} \setminus \{0\}$ is the rebalance period, $W(0)$ is a fixed, initial portfolio wealth, f_i is an i-th wealth fraction. $A(t)$ is the total value of fees, calculated as

$$A(k\Delta t) = \sum_{i=1}^{N} \alpha |S_i(k\Delta t) \cdot (q_i^{temp}(k\Delta t) - q_i'(k\Delta t))|, \tag{2.23}$$

where α is a constant level of fees and q_i' is the desired quantity of each kind of asset, defined to be

$$q_i'(k\Delta t) = f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}. \tag{2.24}$$

A set of operations performed in a singe time step, $t = k\Delta t$ for k such that $k \bmod m = 0$ is called portfolio rebalancing.

Another method of reducing transaction costs while not leaving the idea of rebalancing the portfolio completely is to build a *partially balanced portfolio*. In this strategy, however, the goal is not to limit the number of transactions, but their value. Therefore, instead of making the portfolio 100% balanced in every iteration, each time transactions are made that bring the portfolio closer to the state of being fully balanced, but they do not make it entirely balanced. The parameter which will encode what part of portfolio is being rebalanced will be denoted by D. In each time step and for each asset, instead of buying or selling the amount of assets which would completely balance the portfolio, only D of it is actually exchanged. Consequently, the portfolio is never fully balanced, but the transaction costs are smaller by a factor of D.

Definition 13. A partially balanced strategy with the inclusion of transaction fees is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = \frac{W_i^{temp}(k\Delta t) + D \cdot (f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t))}{S_i(k\Delta t)}, \tag{2.25}$$

$$q_0(k\Delta t) = 0, \tag{2.26}$$

$$q_i(0) = f_i \cdot \frac{W(0)}{S_i(0)}, \tag{2.27}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $D, D \in [0, 1]$ is the partial rebalancing coefficient, $W(0)$ is a fixed, initial portfolio wealth, f_i is an i-th wealth fraction, W_i^{temp} is an i-th temporary wealth fraction

$$W_i^{temp} = S_i(k\Delta t) \cdot q_i((k-1)\Delta t). \tag{2.28}$$

$A(t)$ is the total value of fees

$$A(k\Delta t) = \sum_{i=1}^{N} \alpha |S_i(k\Delta t) \cdot (q_i^{temp}(k\Delta t) - q_i'(k\Delta t))|, \tag{2.29}$$

where α is a constant level of fees and q_i' is the desired quantity of each kind of asset,

$$q_i'(k\Delta t) = f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}. \tag{2.30}$$

A set of operations performed in a singe time step, $t = k\Delta t$ for some $k \in \{1, 2, \ldots, n\}$ is called partial portfolio rebalancing.

An interesting fact about the partially balanced strategy is that for specific values of parameter D, it can turn into other strategies, which have already been discussed.

Fact 1. The partially balanced portfolio management strategy reduces to the passive strategy for $D = 0$.

Proof. To prove that, one needs to demonstrate that the actual quantities of assets never change i.e. $q_i(k\Delta t) = q_i((k-1)\Delta t)$. For an arbitrary choice of $k > 0$, we have

$$\begin{aligned} q_i(k\Delta t) &= \frac{W_i^{temp}(k\Delta t) + D \cdot (f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t))}{S_i(k\Delta t)} \\ &= \frac{W_i^{temp}(k\Delta t) + 0 \cdot (f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t))}{S_i(k\Delta t)} \\ &= \frac{W_i^{temp}(k\Delta t)}{S_i(k\Delta t)} = \frac{S_i(k\Delta t) \cdot q_i((k-1)\Delta t)}{S_i(k\Delta t)} = q_i((k-1)\Delta t). \end{aligned}$$

□

Fact 2. The partially balanced portfolio management strategy reduces to the regular balanced strategy for $D = 1$.

Proof. To prove that, it suffices to demonstrate that Eq. (2.25) reduces to Eq. (2.15) for $D = 1$, as the rest of the updating scheme is the same. We have

$$\begin{aligned} q_i(k\Delta t) &= \frac{W_i^{temp}(k\Delta t) + D \cdot (f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t))}{S_i(k\Delta t)} \\ &= \frac{W_i^{temp}(k\Delta t) + 1 \cdot (f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t))}{S_i(k\Delta t)} \\ &= \frac{W_i^{temp}(k\Delta t) + f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t)}{S_i(k\Delta t)} \\ &= \frac{f_i \cdot W^{temp}(k\Delta t) - f_i \cdot A(k\Delta t)}{S_i(k\Delta t)} = f_i \frac{W^{temp}(k\Delta t) - A(k\Delta t)}{S_i(k\Delta t)}. \end{aligned}$$

□

Hence, for a partially balanced portfolio one could think of D as of a "slider" between passive and fully balanced portfolio which can be considered two extremes.

The approaches of rebalancing the portfolio periodically and partially can also be merged together to jointly form a composite periodically and partially balanced strategy [29].

Definition 14. A periodically and partially balanced strategy with the inclusion of transaction fees is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = \begin{cases} \frac{1}{S_i(k\Delta t)}\Big(W_i^{temp}(k\Delta t) + D\times \\ \Big(f_i \cdot W^{temp}(k\Delta t) - W_i^{temp}(k\Delta t) - f_i \cdot A(k\Delta t)\Big)\Big) & \text{if } k \bmod m = 0 \\ q_i^{temp}(k\Delta t) & \text{otherwise} \end{cases} \tag{2.31}$$

$$q_0(k\Delta t) = 0, \tag{2.32}$$

$$q_i(0) = f_i \cdot \frac{W(0)}{S_i(0)}; \tag{2.33}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $m, m \in \mathbb{N} \setminus \{0\}$ is the rebalance period, $D \in [0, 1]$ is the partial rebalancing coefficient, $W(0)$ is a fixed, initial portfolio wealth, f_i is an i-th wealth fraction, W_i^{temp} is an i-th temporary wealth fraction. $A(t)$ is the total value of fees

$$A(k\Delta t) = \sum_{i=1}^{N} \alpha |S_i(k\Delta t) \cdot (q_i^{temp}(k\Delta t) - q_i'(k\Delta t))|, \tag{2.34}$$

where α is a constant level of fees and q_i' is the desired quantity of each kind of asset

$$q_i'(k\Delta t) = f_i \frac{W^{temp}(k\Delta t)}{S_i(k\Delta t)}. \tag{2.35}$$

A set of operations performed in a singe time step, $t = k\Delta t$ for k such that $k \bmod m = 0$ is called partial portfolio rebalancing.

In the periodically and partially balanced strategy we have two parameters — m and D — which we can use to optimise portfolio performance in given market conditions.

2.4 Strategies based on trading indicators

Keeping the wealth fractions constant is one of the ways of altering an investment portfolio in order to increase its performance. Throughout the years, investors have been attempting to find various mathematical tools which would allow them to find the correct moment to buy or sell a stock. This resulted in creation of a wide set of indicators and markers designed for that particular purpose. A branch of trading activities which studies usefulness of those markers is called *technical analysis* [37]. In this section we will have a closer look at two of the most well-known trading indicators and we will build investment strategies utilising them in order to systematically examine their usefulness in particular market conditions.

2.4.1 MACD trading indicator

One of the most widespread indicators, well known among all investors using technical analysis is MACD — *Moving Average Convergence Divergence* [38].

MACD was invented by Gerard Appel in 1979. It is based on a couple of time series which are derived from the asset price process by means of a transformation very commonly used in technical analysis — EMA — the *Exponential Moving Average*. As the name itself suggests it is a kind of a moving average, but with exponentially decreasing weights of factors more distant in time from the current one.

Definition 15. Exponential Moving Average is a transformation of a discrete stochastic process $\{X(k\Delta t)\}_{k=0}^{n}$ described with the following recursive formula

$$\mathrm{EMA}_{X,p}(k\Delta t) = \begin{cases} X(k\Delta t), & \text{for } t = 0 \\ \alpha X(k\Delta t) + (1-\alpha)\,\mathrm{EMA}_{X,p}\Big((k-1)\Delta t\Big), & \text{for } t > 0 \end{cases} \tag{2.36}$$

where $\alpha = \frac{2}{p+1}$ and $p \in \mathbb{N} \setminus \{0\}$ is a parameter of the transformation, called *the lag*, .

As one can see, EMA itself can be interpreted as a time series. Besides the time itself, two additional inputs are required for the EMA value to be computed. As mentioned earlier, one of them is the base process X. As far as the applications in portfolio management are concerned, this process is almost always the stock price of one of the portfolio assets S_i, for some $i \in \{1, 2, \ldots, N\}$. Hence, to simplify the notation, instead of writing $\mathrm{EMA}_{S_i,p}$, we will just put $\mathrm{EMA}_{i,p}$. The second parameter of EMA, p, is largely responsible for the weight of the past values taken to the average. The bigger p, the bigger is the weight of older values of the underlying process in the final result, which has an effect in a smoother EMA curve.

MACD indicator makes its predictions based on the difference between two time series. The first is the MACD line, i.e. a line obtained by subtracting two EMAs of different lags.

Definition 16. The difference between two EMA time series with different lags p and q, with $p, q \in \mathbb{N} \setminus \{0\}, p < q$, is called the MACD line.

$$\mathrm{MACD}_{i,p,q}(k\Delta t) = \mathrm{EMA}_{i,p}(t) - \mathrm{EMA}_{i,q}(k\Delta t) \tag{2.37}$$

EMA related to the lag parameter p is called the *fast line* whereas the one related to the parameter q is called the *slow line*.

Buy and sell signals are generated in places where the MACD line crosses what is called the *signal line* — another EMA, with a new lag parameter $s < p$ — see the example plots in Figs. 2.1a and 2.1b. Therefore, we can introduce what we will call the indicator line. Whenever this line changes from positive to negative (or the other way around) will be interpreted by the strategy as a buy or sell signal.

Definition 17. The MACD indicator line is the difference between an EMA with lag s (called the signal line) and the MACD line with fast line lag p and slow line lag q :

$$\mathrm{Ind}_{i,p,q,s}(k\Delta t) = \mathrm{EMA}_{i,s}(k\Delta t) - \mathrm{MACD}_{i,p,q}(k\Delta t), \tag{2.38}$$

where $s \in \mathbb{N} \setminus \{0\}, s < p$.

Whenever the indicator line changes its value from negative to positive — MACD gives a "buy" signal. Contrarily, when this line drops from positive values to the negative ones — we obtain a "sell" signal. Hence we, can introduce the buying and selling indicators:

Definition 18. The MACD buy indicator is a time series of 1-s and 0-s defined in the following way :

$$\mathbb{1}^{+}_{i,p,q,s}(k\Delta t) = \begin{cases} 1, & \text{if } t > 0 \wedge \mathrm{Ind}_{i,p,q,s}\Big((k-1)\Delta t\Big) < 0 \wedge \mathrm{Ind}_{i,p,q,s}(k\Delta t) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.39}$$

Similarly, the MACD sell indicator is the following time series

$$\mathbb{1}^{-}_{i,p,q,s}(k\Delta t) = \begin{cases} 1, & \text{if } t > 0 \wedge \mathrm{Ind}_{i,p,q,s}\Big((k-1)\Delta t\Big) > 0 \wedge \mathrm{Ind}_{i,p,q,s}(k\Delta t) < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.40}$$

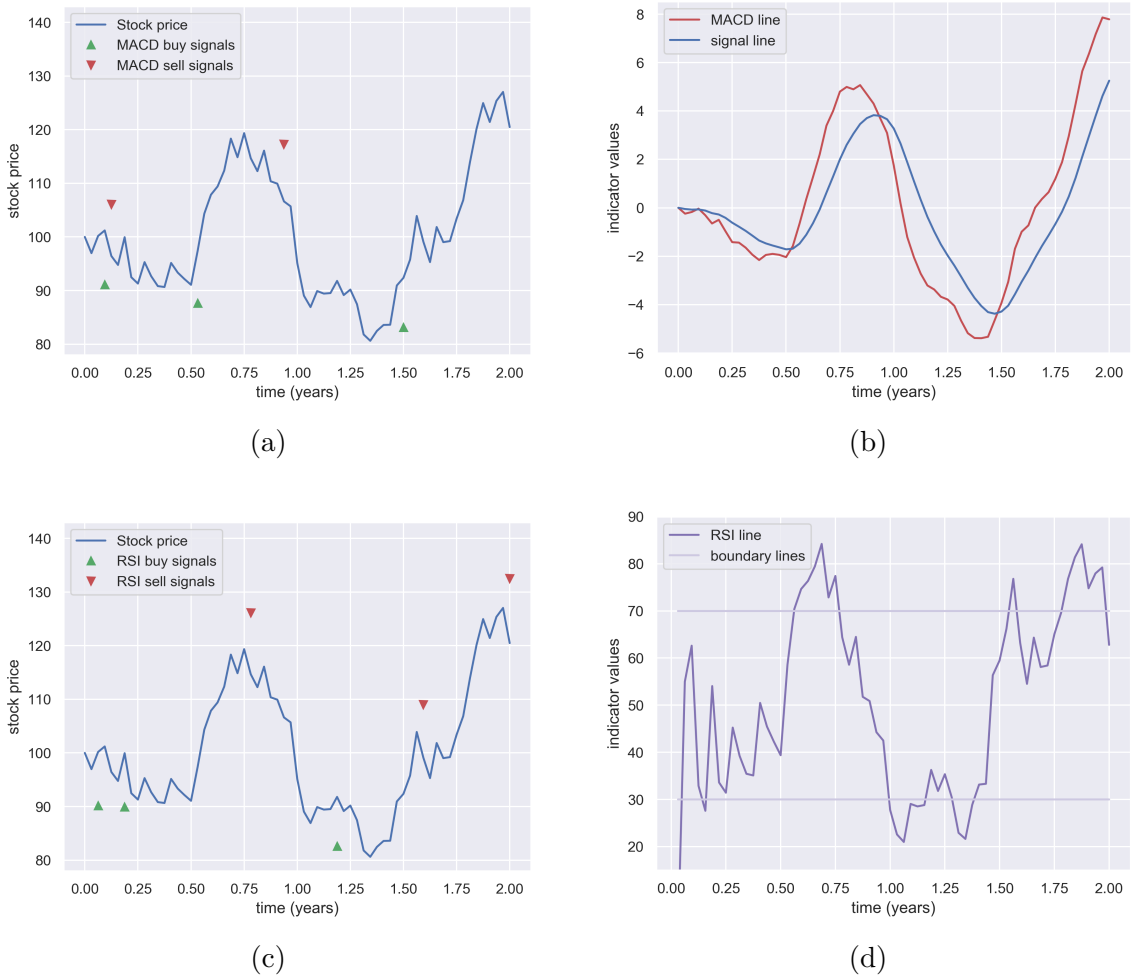


Figure 2.1: Illustration of buy and sell signals for MACD and RSI strategies: (a) and (b) — buy signals are generated when the MACD line crosses the signal line from the bottom, sell signals are generated when the MACD line crosses the signal line from the top; (c) and (d) — buy signals are generated when the RSI line leaves the oversold (below 30) area, sell signals are generated when it leaves the overbought (above 70) area.

Until that moment, we were only using mathematical notation to write down ideas and concepts known previously, related to the MACD indicator. However, the MACD only tells *when* to buy or sell a particular asset, but not *how much* and, as we know already, for that we need what is called a strategy. Hence, we propose our own way of formalising the usage of MACD in form of a proper investment strategy, as defined by the Definition 3 for that purpose.

Definition 19. An MACD-based strategy (or "MACD strategy", in short) is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = q'_i(k\Delta t) + \frac{c(k\Delta t)}{S_i(k\Delta t)} \left(\sum_{i=1}^{N} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t) \right)^{-1} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t), \tag{2.41}$$

$$q_0(k\Delta t) = q'_0(k\Delta t) - c(k\Delta t), \tag{2.42}$$

$$q_i(0) \geqslant 0, \tag{2.43}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $q_i(0)$ are initial amounts of assets (for $i > 1$) and cash (for $i = 0$), $c(k\Delta t)$ is an amount of cash used to buy new assets, defined as

$$c(k\Delta t) = \phi q'_0(k\Delta t). \tag{2.44}$$

$q'_i(k\Delta t)$ is the temporary quantity of $i-th$ assets after (potentially) selling some of them but before buying new ones at time $t = k\Delta t$ and $q'_0(k\Delta t)$ is the temporary amount of cash at that moment:

$$q'_i(k\Delta t) = q_i\Big((k-1)\Delta t\Big)(1 - \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t)), \tag{2.45}$$

$$q'_0(k\Delta t) = q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t). \tag{2.46}$$

$\phi, \psi \in [0, 1]$ are called a buy and a sell factor respectively.

Let us explain more descriptively how the strategy works. The factors ψ and ϕ decide upon the actual amount of assets bought and sold when the signal from MACD comes. They can be picked arbitrarily by any investor, as they are meant to represent their trust in selling and buying through signals generated by the MACD indicator. The sell factor ψ controls the assets' disposal process. Whenever MACD generates a sell signal for a given stock, a ψ part of the amount of this asset is sold (see Eq. (2.45)) and the money from selling is converted into portfolio cash (see Eq. (2.46)). Similarly, the buy factor ϕ drives the purchase procedure. This time however, it is used to

decide, what part of available cash (including new portion obtained from selling some stocks) will be used to buy new stocks. The fraction of portfolio cash $c(k\Delta t)$ is then used to buy assets indicated by a buy signal (see Eqs. (2.44), (2.42) and (2.41)).

It can be proven that the strategy described in Definition 19 is self-financing.

Theorem 1. The strategy of an MACD portfolio, as described by Definition 19, is self-financing.

Proof. Let us now look into the movements which happen at an arbitrary point of time $t = k\Delta t$. On one hand, we know that the wealth of it is expressed by formula (2.7). This we can write as

$$W(k\Delta t) = q_0(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t)q_i(k\Delta t). \tag{2.47}$$

On the other hand however, we know that before we do any buy or sell transaction at time t, the value of our portfolio arises from the amount of assets we had in the previous step, so the temporary wealth, as defined in Definition 8 can be written as

$$W^{temp}(k\Delta t) = q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t)q_i\Big((k-1)\Delta t\Big). \tag{2.48}$$

Using equations (2.41) and (2.42) we have

$$
\begin{aligned}
W(k\Delta t) =& q_0(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i(k\Delta t) = \\
=& q_0'(k\Delta t) - c(k\Delta t) + \\
& \sum_{i=1}^{N} S_i(k\Delta t) \left(q_i'(k\Delta t) + \frac{c(k\Delta t)}{S_i(k\Delta t)} \left(\sum_{j=1}^{N} \mathbb{1}^{+}_{j,p,q,s}(k\Delta t) \right)^{-1} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t) \right) = \\
=& q_0'(k\Delta t) - c(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i'(k\Delta t) + \\
& \sum_{i=1}^{N} S_i(k\Delta t) \frac{c(k\Delta t)}{S_i(k\Delta t)} \left(\sum_{j=1}^{N} \mathbb{1}^{+}_{j,p,q,s}(k\Delta t) \right)^{-1} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t) = \\
=& q_0'(k\Delta t) - c(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i'(k\Delta t) + \\
& \sum_{i=1}^{N} c(k\Delta t) \mathbb{1}^{+}_{i,p,q,s}(k\Delta t) \left(\sum_{j=1}^{N} \mathbb{1}^{+}_{j,p,q,s}(k\Delta t) \right)^{-1} = \\
=& q_0'(k\Delta t) - c(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i'(k\Delta t) + \\
& c(k\Delta t) \left(\sum_{j=1}^{N} \mathbb{1}^{+}_{j,p,q,s}(k\Delta t) \right)^{-1} \sum_{i=1}^{N} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t) = \\
=& q_0'(k\Delta t) - c(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i'(k\Delta t) + c(k\Delta t) = \\
=& q_0'(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q_i'(k\Delta t).
\end{aligned}
$$

Now, using equations (2.45) and (2.46) we obtain

$$
\begin{aligned}
W(k\Delta t) =& q'_0(k\Delta t) + \sum_{i=1}^{N} S_i(k\Delta t) q'_i(k\Delta t) = \\
=& q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t) + \\
&+ \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big)(1 - \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t)) = \\
=& q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t) + \\
&+ \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) - \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t) = \\
=& q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big) = W^{temp}(k\Delta t).
\end{aligned}
$$

Thus, since $W(k\Delta t) = W^{temp}(k\Delta t)$, the strategy is indeed self-financing. □

It can also be proven that the quantities of assets and the amount of cash in the MACD portfolio will never be negative.

Theorem 2. The value of $q_i(k\Delta t)$ in the MACD strategy stays non-negative for all $k \in \{0, 1, \ldots, n\}$ and for all $i \in \{0, 1, \ldots, N\}$.

Proof. We will prove that if an amount a given asset (including cash) brought from the previous time step is non-negative, it will remain such in the next step. That, together with an assumption that the initial amount of each asset at time $t = 0$ is non-negative (given as an inequality (2.43) in Definition 19) proves it will always be that way.

Therefore, let us assume the quantities in the previous step were all non-negative $q_i\Big((k-1)\Delta t\Big) \geqslant 0$ for all $i \in \{0, 1, \ldots, N\}$. We will start by proving non-negativity for cash. At time t, the ultimate amount of cash is given by equation (2.42),

$$q_0(k\Delta t) = q'_0(k\Delta t) - c(k\Delta t).$$

Plugging equation (2.44) into the above one we get

$$q_0(k\Delta t) = q'_0(k\Delta t) - \phi q'_0(k\Delta t) = q'_0(k\Delta t)(1 - \phi).$$

Using equation (2.46) yields

$$q_0(k\Delta t) = \Bigg(q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big)\psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t)\Bigg)(1-\phi) =$$
$$= \Bigg(q_0\Big((k-1)\Delta t\Big) + \psi \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big)\mathbb{1}^{-}_{i,p,q,s}(k\Delta t)\Bigg)(1-\phi).$$

Now, $q_0\Big((k-1)\Delta t\Big) \geqslant 0$ and $q_i\Big((k-1)\Delta t\Big) > 0$ (by assumption), as well as $\psi \in [0,1]$, $S_i(k\Delta t) > 0$, and $\mathbb{1}^{-}_{i,p,q,s}(k\Delta t) \geqslant 0$ (by their respective definitions). Therefore, the entire expression $q_0\Big((k-1)\Delta t\Big) + \psi \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big)\mathbb{1}^{-}_{i,p,q,s}(k\Delta t)$ must be non-negative. Also, since $\phi \in [0,1]$, the expression $(1-\phi) \geqslant 0$. Hence, multiplication of those two expressions must also be non-negative and thus

$$q_0(k\Delta t) \geqslant 0.$$

for any $t \in [0,T]$.

The ultimate quantity of each of the risky assets is given by (2.41).

$$q_i(k\Delta t) = q'_i(k\Delta t) + \frac{c(k\Delta t)}{S_i(k\Delta t)} \left(\sum_{i=1}^{N} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t)\right)^{-1} \mathbb{1}^{+}_{i,p,q,s}(k\Delta t).$$

Now, $q'_i(k\Delta t)$ itself is defined by (2.45)

$$q'_i(k\Delta t) = q_i\Big((k-1)\Delta t\Big)(1 - \psi \mathbb{1}^{-}_{i,p,q,s}(k\Delta t)).$$

In the above expression, we see that $q_i\Big((k-1)\Delta t\Big) \geqslant 0$ (by assumption) as well as $\psi \in [0,1]$ and $\mathbb{1}^{-}_{i,p,q,s}(k\Delta t) \geqslant 0$ (by their respective definitions). Hence, we see that $q'_i(k\Delta t) \geqslant 0$.

Looking at the other term and plugging equations (2.44) and (2.46) into it we have

$$\frac{c(k\Delta t)}{S_i(k\Delta t)}\left(\sum_{i=1}^{N}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)\right)^{-1}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)=$$
$$=\frac{\phi q_0'(k\Delta t)}{S_i(k\Delta t)}\left(\sum_{i=1}^{N}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)\right)^{-1}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)=$$
$$=\phi\frac{q_0\Big((k-1)\Delta t\Big)+\sum_{i=1}^{N}S_i(k\Delta t)q_i\Big((k-1)\Delta t\Big)\psi\mathbb{1}^{-}_{i,p,q,s}(k\Delta t)}{S_i(k\Delta t)}\times$$
$$\left(\sum_{i=1}^{N}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)\right)^{-1}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t).$$

The obtained expression consists of four factors. In the first one we have $\phi \in [0,1]$, so it is non-negative. In the second factor, we have a fraction in which the numerator is non-negative (as $q_0\Big((k-1)\Delta t\Big)$ is non-negative by assumption and all terms in the sum $\sum_{i=1}^{N} S_i(k\Delta t)q_i\Big((k-1)\Delta t\Big)\psi\mathbb{1}^{-}_{i,p,q,s}(k\Delta t)$ are non-negative — either by definition or by assumption) and the denominator is positive ($S_i(k\Delta t) > 0$, by definition for any i and k). The third part is the inverse of the sum of indicators $\left(\sum_{i=1}^{N}\mathbb{1}^{+}_{i,p,q,s}(k\Delta t)\right)^{-1}$, all having non-negative values of either 0 or 1, sum of which will also be non-negative and the inverse of it too. The fourth term is one of the indicators which were summed in term three, which is also non-negative, as $\mathbb{1}^{+}_{i,p,q,s}(k\Delta t) \in \{0,1\}$. Hence, the expression is a multiplication of four non-negative terms, therefore — it must be non-negative itself.

Thus, $q_i(k\Delta t)$, as a sum of two non-negative terms, must be non-negative too.

□

2.4.2 RSI trading indicator

Besides MACD, another, commonly used trading indicator is called RSI — the *Relative Strength Index*. It was proposed by J. Welles Wilder Jr. in 1978 [39]. Its values can only be in the interval $[0, 100]$ and the marker can be used to identify when an instrument is oversold (index' value below certain level, usually 30 — a signal for buying) or when it is overbought (index' value above certain level, usually 70 — a signal for selling). In order to calculate the value of RSI, the difference sequence will be used

Definition 20. The difference (or increments) process of a process $\{X(k\Delta t)\}_{k=0}^{n}$ is defined to be

$$D_X(k\Delta t) = \begin{cases} 0, & \text{for } t = 0, \\ X_i(k\Delta t) - X_i\Big((k-1)\Delta t\Big), & \text{for } t > 0. \end{cases} \tag{2.49}$$

Similarly to what we have done for EMA, we simplify the notation by stating that $D_{S_i}(k\Delta t) = D_i(k\Delta t)$, since also in this case the only process we will apply this transformation to will be the process of prices.

To calculate the RSI index, positive and negative increments need to be sorted out from each other

Definition 21. The positive differences (or increments) process is a process defined as

$$D_i^+(k\Delta t) = \begin{cases} D_i(k\Delta t), & \text{where } D_i(k\Delta t) > 0 \\ 0, & \text{where } D_i(k\Delta t) \leqslant 0. \end{cases} \tag{2.50}$$

Similarly, the negative differences (or increments) process is:

$$D_i^-(k\Delta t) = \begin{cases} 0, & \text{where } D_i(k\Delta t) \geqslant 0 \\ D_i(k\Delta t), & \text{where } D_i(k\Delta t) < 0. \end{cases} \tag{2.51}$$

The RSI index relies on the notion of asset's *relative strength*. This quantity can be defined in various different ways, we will follow the original one, given by Wilder [39].

Definition 22. Relative strength of the i-th asset of a portfolio $\mathcal{P}$ is defined to be

$$\mathrm{RS}_i(k\Delta t) = \frac{\mathrm{EMA}_{D_i^+,s}(k\Delta t)}{\mathrm{EMA}_{D_i^-,s}(k\Delta t)}. \tag{2.52}$$

where s is EMA's lag parameter, which, in the context of RS_i, is often called *the period*.

RSI is an index obtained by normalising relative strength, so that it's value is always between 0 and 100

Definition 23. The RSI Index is a process defined as

$$\mathrm{RSI}_i(k\Delta t) = 100 - \frac{100}{1 + RS_i(k\Delta t)}. \tag{2.53}$$

In order to construct the actual strategy out of the values of RSI, we can construct dedicated indicators, similar to the ones that have been proposed for the MACD in Eqs. (2.39) and (2.40). This time however arbitrary levels of overselling and overbuying need to be fixed additionally. Let us denote them by d^+ and d^- respectively. As mentioned above, practitioners usually stick to $d^+ = 30$ and $d^- = 70$ [40]. Having that fixed, we can define the indicators for RSI strategy

Definition 24. The RSI buy indicator is a time series of 1-s and 0-s defined in the following way:

$$\mathbb{1}^+_{i,d^+}(k\Delta t) = \begin{cases} 1, & \text{if } \mathrm{RSI}_i\Big((k-1)\Delta t\Big) < d^+ \wedge \mathrm{RSI}_i(k\Delta t) > d^+, \\ 0, & \text{otherwise,} \end{cases} \tag{2.54}$$

Similarly, the RSI sell indicator is the following time series

$$\mathbb{1}^-_{i,d^-}(k\Delta t) = \begin{cases} 1, & \text{if } \mathrm{RSI}_i\Big((k-1)\Delta t\Big) > d^- \wedge \mathrm{RSI}_i(k\Delta t) < d^-, \\ 0, & \text{otherwise.} \end{cases} \tag{2.55}$$

With those indicators, we can construct the exact same strategy as in case of MACD (reacting for signals, as described by Eqs. (2.41)–(2.46)) simply by replacing MACD-related buy and sell indicators by the newly defined RSI-based ones.

Definition 25. An RSI-based strategy (or simply an "RSI strategy") is a strategy for which the quantity update scheme has the following form

$$q_i(k\Delta t) = q'_i(k\Delta t) + \frac{c(k\Delta t)}{S_i(k\Delta t)} \left(\sum_{i=1}^{N} \mathbb{1}^+_{i,d^+}(k\Delta t)\right)^{-1} \mathbb{1}^+_{i,d^+}(k\Delta t), \tag{2.56}$$

$$q_0(k\Delta t) = q'_0(k\Delta t) - c(k\Delta t), \tag{2.57}$$

$$q_i(0) \geqslant 0, \tag{2.58}$$

for consecutive $k \in \{1, 2, \ldots, n\}$, and within each k — for consecutive $i \in \{1, 2, \ldots, N\}$, where $q_i(0)$ are initial amount of assets (for $i > 1$) and cash (for $i = 0$), $c(k\Delta t)$ is an amount of cash used to buy new assets, defined as

$$c(k\Delta t) = \phi q'_0(k\Delta t), \tag{2.59}$$

$q'_i(k\Delta t)$ is the temporary quantity of $i-th$ assets after (potentially) selling some of them but before buying new ones at time $t = k\Delta t$ and $q'_0(k\Delta t)$ is the temporary

amount of cash at that moment:

$$q'_i(k\Delta t) = q_i\Big((k-1)\Delta t\Big)(1-\psi \mathbb{1}^-_{i,d^-}(k\Delta t)), \tag{2.60}$$

$$q'_0(k\Delta t) = q_0\Big((k-1)\Delta t\Big) + \sum_{i=1}^{N} S_i(k\Delta t) q_i\Big((k-1)\Delta t\Big)\psi \mathbb{1}^-_{i,d^-}(k\Delta t). \tag{2.61}$$

$\phi, \psi \in [0,1]$ are called buy and sell factors respectively.

One can observe how RSI indicator generates buy and sell signals by looking at the example in Figs. 2.1c and 2.1d.

2.5 Portfolio performance measures

The most basic measure of portfolio performance is its wealth, defined already in Definition 5. It simply represents the value of the portfolio, measured in units of cash. Although it is the most intuitive indicator of how the value of the portfolio changes, it has been noticed in numerous works that portfolio wealth is not a reliable measure of performance [41]. It suffers from few weaknesses. First of all, it is not invariant to the initial values of the assets the portfolio is composed of, whereas a good portfolio performance measure should be. Second — it does not properly reflect the *relative character* of investment's value in general. This relativity is associated with a fact that the measure of portfolio's performance should be dependent on its current value. For example – if the value of some portfolio grows by \$1, it is considered to be a much bigger change if it was a \$10 portfolio than if it would be a \$1 000 000 one, although the absolute gain is the same in both cases. Finally, the third issue is that a good portfolio performance measure should take into consideration the passage of time. The more time we give to our investment strategy, the better results it should present. All these three factors are addressed in a function that we call a *portfolio growth*, or *logarithmic growth* [29].

Definition 26. Growth of portfolio $\mathcal{P}$ (or "the GoP", in short), denoted by $g_{\mathcal{P}}(k\Delta t)$, is a measure of portfolio performance given by

$$g_{\mathcal{P}}(t) = \frac{1}{t}\log\left(\frac{W_{\mathcal{P}}(t)}{W_{\mathcal{P}}(0)}\right) \tag{2.62}$$

This measure does not exhibit any of the weaknesses mentioned above. Since wealth of portfolio gets divided by its initial value — the measure is ultimately not

dependent on it. Imposing the logarithm on the scaled portfolio value "flattens" the measure in its right end. Hence, more significant movements of portfolio wealth are needed to change the value of the measure by the same amount for big portfolios than for the smaller ones. Time dependence of an investment has also been captured. As the time passes and the value of t in the denominator gets bigger, the logarithm of the normalised portfolio wealth must increase with the similar rate in order not to lower the value of the whole measure.

The growth of wealth is a measure focusing on portfolio evolution in time. But, although the time is a natural variable along which we track changes in general, it is not the only one possible. Let us imagine an indexed set of M investment portfolios $\{\mathcal{P}\}_{i=1}^{M}$, which are similar to one another except for one characteristic feature which is different for each of those portfolios — let's call it τ. If this characteristic feature is quantitative (possible to be expressed by a number) and if there is sufficiently many different values of it for different portfolios (ideally — each one of all M portfolios has a different value of τ) — it may provide another "dimension" along which we can observe the changes of values of portfolios involved and it could give us insights different to the ones obtained through time-oriented observations. Having a suitable set of investment portfolios $\{\mathcal{P}\}_{i=1}^{M}$, we can design a measure like that. We will call it a *characteristic measure*, as it shows the dynamics of a set of portfolios along some characteristic property τ.

Definition 27. A characteristic measure $\varphi(t, \tau)$ of a set of M portfolios $\{\mathcal{P}\}_{i=1}^{M}$, each characterised by a unique value of the portfolio's characteristic property τ, i.e. $\mathcal{P}_i = \mathcal{P}(\tau_i)$, for some value of $\tau_i \in \{1, 2, \ldots, M\}$, is given by

$$\varphi_{\mathcal{P}}(t, \tau) = g_{\mathcal{P}(\tau)}(t). \tag{2.63}$$

As we can see, the characteristic measure is exactly equivalent to growth of wealth, except it highlights the possibility of comparing several different portfolios (varying in the value of τ) at the same moment of time t. Since we prefer our measures to be one-dimensional (for easier plotting and analysis), we will often only be interested in the changes of the characteristic measure along with the second parameter τ and just one, fixed point of time t. It would make sense for this point of time to be the maturity of the portfolio $t = T$, as the end of an investment is a natural moment for making comparisons. Hence, we state that $\varphi_{\mathcal{P}}(\tau) = \varphi_{\mathcal{P}}(T, \tau) = g_{\mathcal{P}(\tau)}(T)$.

Since there is a lot of features and qualities by which investment portfolios can differ from one another, it can also be the case that one will want to compare two sets of portfolios — $\{\mathcal{P}\}_{i=1}^{M}$ and $\{\mathcal{R}\}_{i=1}^{M}$. In such case, we can formalise a measure of differences between two sets of portfolios — *a benchmark measure*.

Definition 28. A benchmark measure of a portfolio set $\{\mathcal{P}\}_{i=1}^{M}$ against another portfolio set $\{\mathcal{R}\}_{i=1}^{M}$, denoted by $\delta(t,\tau)$, is given by

$$\delta_{\mathcal{P},\mathcal{R}}(t,\tau) = \varphi_{\mathcal{P}}(t,\tau) - \varphi_{\mathcal{R}}(t,\tau). \tag{2.64}$$

Similarly to the characteristic measure, dropping the time dimension in the notation simply means that involved measures are evaluated at $t = T$, i.e. $\delta_{\mathcal{P},\mathcal{R}}(\tau) = \delta_{\mathcal{P},\mathcal{R}}(T,\tau) = \varphi_{\mathcal{P}}(\tau) - \varphi_{\mathcal{R}}(\tau)$.

Also, similarly to portfolio wealth, if it is clear from the context which portfolios are measured, the subscripts denoting them can be dropped for clarity as well, giving us $g(t)$ for the growth of wealth, $\varphi(t,\tau)$ or $\varphi(\tau)$ for the characteristic measure and $\delta(t,\tau)$ or $\delta(\tau)$ for the benchmark measure.

3 Empirical results under Heston market model

3.1 Importance of synthetic data and Monte-Carlo-style experiments

A big part of modern research related to methods of portfolio management focuses on testing obtained results based on real market data. Although it may seem justified and legitimate, studying financial markets only that way has some drawbacks as well. First of all, research which focuses on one particular market, stock, or even basket of stocks cannot be treated as fully universal. This is because any conclusions of such research are only fully applicable to this one particular class of assets which have been used to prove authors' claims. For that reason, in our work, we chose to focus on a different research methodology. Namely, we conducted a number of experiments in which we used synthetic data, i.e. asset price trajectories are neither prices of any real-life stocks nor any other financial instruments. Instead, they were simulated using a dedicated mathematical model. Thanks to that we were able to draw more general conclusions which can be applied to any market and any sets of assets to which our base model can be applied. Using the Monte Carlo framework [42] enabled us to generate arbitrary number of trajectories and average the results on. We eradicated the bias related to picking some particular assets from existing markets. Therefore, we consider our results to be more generic since they represent an average scenario of what may happen at any possible market in the world.

3.2 Basic terms assumptions for numerical experiments

The most important assumption that we make within all our numerical experiments is that the prices of assets do not depend on decisions a single investor makes. That means the character of trajectories S_i for any $i \in \{1, \ldots, N\}$ is fully random and modelled by a dedicated stochastic process, described in details in the next section. Therefore, the only way an investor can influence performance of their portfolio is by altering quantities q_i of the assets held. The other assumption, related to the first one, is that an investor is not able to alter the price of an asset itself by performing any market transaction. Admittedly, this is not always true in case of real markets, especially if we consider transactions performed by big market players like banks, mutual funds or hedge funds. However, for a single, individual investor, this does not need to be treated as a limitation as most of market participants operate within a range of financial means way too small to be able to significantly influence a typical stock market.

3.3 Heston model

Like we mentioned in section 3.2, we assume that a single investor can only watch the prices of assets randomly changing on the market, not being able to influence them. As a model describing the behaviour of the assets, we chose the *Heston model* [31].

Definition 29. Heston model is a mathematical model used for modelling the dynamics of prices of financial assets in time. It comprises of three stochastic differential equations

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dB^S(t), \tag{3.1}$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}S(t)dB^v(t), \tag{3.2}$$

$$dB^S(t)dB^v(t) = \rho dt, \tag{3.3}$$

and two initial conditions

$$S(0) = S_0 > 0, \tag{3.4}$$

$$v(0) = v_0 > 0, \tag{3.5}$$

where S is the price of the asset, v is the volatility process, B^S and B^v are Wiener processes and $\mu, \kappa, \theta, \sigma, \rho$ are the parameters of the model, such that $\mu \in \mathbb{R}, \kappa > 0, \theta > 0, \sigma > 0, \rho \in [-1, 1]$.

This model is often used as a description of the movement of an underlying assets while pricing derivative instruments, particularly options[43]. Heston model as well as some of its variants, are heavily studied not only in mathematics, but also in physics — and are often referred to as *diffusive diffusivity* models [44]. One can find examples of its applications in multiple branches of physics, including econophysics, where it often serves as a basic model e.g. for pricing [45] and forecasting [46] as well as in sociophysics, in combination with agent-based models [47].

As one can see, within Heston model, the price of a financial asset $S(t)$ is considered to be a stochastic process which solves a system of stochastic differential equations, given by Eqs. (3.1)–(3.3) and initial conditions (3.4)–(3.5). The price process is built upon the value of μ which is constant and the value of $v(t)$ which is a stochastic process in and of itself (hence — the diffusive diffusivity term, often used by econophisicists). μ is called a drift and it represents a general tendency of an asset to grow (if $\mu > 0$) or fall (if $\mu < 0$). On the other hand, $v(t)$ represents the volatility of the asset and is actually modelled by a process known as CIR, originally introduced by Cox, Ingersoll and Ross to model the movement of interest rates [48]. One of the defining features of the CIR model is the so called *mean-reversion*. The value of the process generally oscillates around a long-term average θ, randomly converging to and diverging from it with the rate of κ. A random factor of severity of those oscillations is reflected in the value of σ, which hence can be called "a volatility of a volatility". Thus, the Heston model, featuring volatility of an asset changing in time, appears to reflect the behaviour of the real-life markets well. In reality, we indeed observe periods of time when prices of assets do not move significantly (practitioners often refer to such behaviour as a *consolidation*), but another times prices of those very same assets fluctuate strongly, achieving e.g. daily return rates, which can be orders of magnitude greater than during the peaceful times [49]. Other, simpler models, like Geometric Brownian Motion, which is also frequently used for the purpose of describing prices of financial assets changing in time [43], are often not able to simulate such behaviour.

The actual randomness of the prices of assets and their volatility is achieved by including Wiener processes (also known as Brownian motion processes) [50], denoted by $B^S(t)$ and $B^v(t)$ respectively. The model also allows for the possibility that those two are correlated with instantaneous correlation ρ. The need for this feature in the model can also explained from the practical point of view as here seems to be an actual correlation between prices of assets and their volatility. It is usually observed to be negative — i.e. an increased volatility of a market usually occurs when prices drop, especially as a consequence of some kind of a market event, often related to some critical political or economical news. On the other hand, when the prices casually grow, lower market volatility can be observed [51].

Similarly as in case of classical (deterministic) differential equations, initial conditions for both $S(t)$ and $v(t)$ are required to complete the set-up of the model. For the Heston model, s_0 represents the initial price of an asset at time $t = 0$, and v_0 is the value of the market volatility at that point of time. Since, as mentioned earlier, the volatility process oscillates around the value of parameter θ, for all our numerical experiments we assumed $v_0 = \theta$.

3.4 Merton-style jumps

Although the trajectories coming from the Heston model are continuous, the model itself can easily be extended to include discontinuities. The most common type of jumps which can be incorporated into the model is called *Merton log-normal jump* [52]. To add it, one needs to augment the equation (3.1) with an additional term.

Definition 30. Heston model with Merton-style jumps is a mathematical model used for modelling the dynamics of prices of financial assets in time. It comprises of three stochastic differential equations

$$dS(t) = \mu S(t)dt + \sqrt{v(t)}S(t)dB^S(t) + (e^{Z(t)} - 1)S(t)dq(t) \tag{3.6}$$

$$dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}S(t)dB^v(t), \tag{3.7}$$

$$dB^S(t)dB^v(t) = \rho dt, \tag{3.8}$$

and two initial conditions

$$S(0) = S_0 > 0, \tag{3.9}$$

$$v(0) = v_0 > 0, \tag{3.10}$$

where S is the price of the asset, v is the volatility process, B^S and B^v are Wiener processes and $\mu \in , \kappa, \theta, \sigma, \rho$ are the parameters of the model, such that $\mu \in \mathbb{R}, \kappa \in \mathbb{R}_+, \theta \in \mathbb{R}_+, \sigma \in \mathbb{R}_+, \rho \in [-1, 1]$, $Z(t)$ is a series of i.i.d. normally distributed random variables with mean μ^J and standard deviation σ^J, whereas $q(t)$ is a Poisson counting process with constant intensity λ.

Comparing to the Definition 29 of the original Heston model, the only change in the updated definition is one added term in Eq. (3.6). It is responsible for the discontinuities in the price process $S(t)$, called Merton-style jumps. They turn the Heston model into what is sometimes called a Bates model [53]. Although there is multitude of ways jumps can be incorporated into a continuous model, the one

described by Merton is used particularly often and it has an easy real-life interpretation. Namely, $e^{Z(t)}$ is the actual (absolute) rate of the difference between the price before the jump at time t and right after it, i.e. $S(t-) \cdot e^{Z(t)} = S(t+)$. So if for example, for a given t, $e^{Z(t)} \approx 0.85$, that means the stock experienced $\sim 15\%$ drop in value at that moment.

3.5 Model discretisation and simulation

As mentioned in section 2.2, in real-life we almost always work with the data in form of discrete series of values. Thus, in order to make the model applicable in practice, especially for the sake of computer simulations, one needs to discretise it, that is — to rewrite the continuous (theoretical) equations in such a way that the values of the process are given in specific, equidistant points of time. The time discretisation should be consistent with the one we performed to define our portfolio management strategies. Thus, we split the time domain $[0, T]$ into n short intervals, each of length Δt. Thus $n \cdot \Delta t = T$. To properly transform the SDEs of the model into this new time domain, a discretisation scheme is necessary. We used Euler-Maruyama discretisation for that purpose [54]. The stock price equation (3.1) of the regular Heston model can be discretised as

$$S(k\Delta t) - S\Big((k-1)\Delta t\Big) = \mu S\Big((k-1)\Delta t\Big)\Delta t + S\Big((k-1)\Delta t\Big)\sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^S(k\Delta t)\sqrt{\Delta t}, \quad (3.11)$$

where $k \in \{1, \cdots, n\}$ and ε^S is a series of n i.i.d. standard normal random variables.

To highlight the ratio between two consecutive values of the stock price, Eq. (3.11) is often re-written as

$$\frac{S(k\Delta t)}{S\Big((k-1)\Delta t\Big)} = \mu\Delta t + 1 + \sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^S(k\Delta t)\sqrt{\Delta t}. \quad (3.12)$$

The same discretisation scheme can be applied to the Eq. (3.2), to obtain:

$$v(k\Delta t) - v\Big((k-1)\Delta t\Big) = \kappa\left(\theta - v\Big((k-1)\Delta t\Big)\right)\Delta t + \sigma\sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^v(k\Delta t)\sqrt{\Delta t}. \quad (3.13)$$

If $\rho = 0$, then ε^v in the above formula is also a series of n i.i.d. standard normal random variables. However, if $\rho \neq 0$, then — to ensure the proper dependency between S and v — we take

$$\varepsilon^v(k\Delta t) = \rho\varepsilon^S(k\Delta t) + \sqrt{1-\rho^2}\varepsilon^{add}(k\Delta t) \quad (3.14)$$

where ε^{add} is an additional series of n i.i.d. standard normal random variables, which are "mixed" with the ones from ε^S and hence — become dependent on them.

The equations of the discretised model can be directly used to simulate trajectories from the Heston model. To do that, one simply needs to fix $s_0 = S(0)$ and $v_0 = v(0)$, generate two (possibly correlated) normally distributed series of random variables ε^S and ε^v and proceed with the generation of first $v(k\Delta t)$ (as indicated by Eq. (3.13)) and finally — of $S(k\Delta t)$ (as indicated by Eq. (3.11)) for $k \in \{1, 2, \ldots n\}$.

If needed, Merton style jumps can be added after simulating the regular Heston model. To this end, one should generate a random variable from the Poisson distribution with intensity λT

$$\nu \sim Poiss(\lambda T). \quad (3.15)$$

Then, a series of ν random normal variables $Z_1, Z_2, \ldots, Z_\nu$ should be generated, having mean $\mu^J J$ and standard variation $\sigma^J J$. To incorporate them as jumps, one should also find positions for them, hence, another ν random variables will be needed — this time from the uniform distribution on the interval $[0, T]$.

$$u_i \sim \mathcal{U}(0, T), \text{ for } i \in \{1, 2, \ldots, \nu\} \quad (3.16)$$

However, these jump positions will not, with probability 1, land on our discrete grid. Hence, they need to be "snapped" to the nearest grid point, so that the jumps occur at the dicretised points of time $k\Delta t, k \in \{1, 2, \ldots n\}$ too. This can be done, for example by calculating a series of k_i, which will indicate the ordinals of grid points where the jumps occur.

$$K(u_i) = k_i = \left[n \cdot \frac{u_i}{T}\right], \text{ for } i \in \{1, 2, \ldots, \nu\} \quad (3.17)$$

where $[\cdot]$ is a function of rounding the number to the nearest integer (a "floor" function $\lfloor\cdot\rfloor$ or a "ceiling" function $\lceil\cdot\rceil$ can also be used). Having calculated the actual jump positions on the grid, they should be placed on it

$$Z(k\Delta t) = \begin{cases} Z_i & \text{if there is } i \in \{1, 2, \ldots, \nu\} \text{ such that } k = k_i, \\ 1 & \text{otherwise.} \end{cases} \tag{3.18}$$

This is in order to make the "enumeration" of jumps and prices consistent. Hence, although values of $Z(k\Delta t)$ for where $k \neq k_i$ are set to 1, they could actually be anything as they will not be used anyway.

To incorporate the jumps into the trajectory, one puts

$$S^J(k\Delta t) = S(k\Delta t) \cdot \prod_{\{i:k_i \leqslant k\}} \exp\Big(Z(k_i\Delta t)\Big) \tag{3.19}$$

for $k \in \{1, 2, \ldots n\}$ and $i \in \{1, 2, \ldots, \nu\}$.

One of the assumptions of the discretised Merton-style jumps model is that there should be not more than one jump in one time step. In case this assumption is violated (i.e. for $\{u_i, u_j\}, i \neq j$ there is $K(u_i) = K(u_j)$), it is best to simply repeat the generation of all u_i-s, as this procedure is not very computationally expensive.

3.6 Numerical experiments and their results

This section is devoted to present the actual numerical experiments which have been conducted for the purpose of this work. Experiments have been split into groups and for each group a premise for the experiments in the group has been outlined. Each experiment has also been precisely described, including the actual reason of conducting it and what it meant to have provided (**Aim**), a step-by-step procedural description of how it was executed, for easier reproduction (**Procedure**), a conclusion or a list of conclusions that can be drawn from the experiment (**Observations**), a list of the values of parameters used to synthesise the data required for the experiment (**Simulation parameters**) and identifications of figures, on which the experiment results were visualised (**Figure references**). This style of describing numerical experiments has been designed by us to make it as easy as possible to follow the procedure and if necessary, to reproduce the results. Outcomes of all of the presented experiments have been published in our research articles [30, 32, 36].

3.6.1 Comparison of balancing strategies

The first experiment was performed to compare various strategies related to portfolio rebalancing with each other and with the passive strategy. It also allowed us to compare our results against the ones presented in the work of Alper et al. [29], in which authors proved that the growth of wealth of the passive portfolio decays over time if the assets are modelled by the Geometric Brownian Motion. We were able to reproduce those results for the Heston model (in simulations), although we also note that the decay of the growth of wealth is a rather theoretical result, as it is very slow and the passive portfolio actually falls beneath the balanced ones only after $\sim 20-30$ years (see Ref. [30] for further details).

Experiment 1.

Aim: Comparing balancing portfolios to the passive one, examining the impact of the length of investment.

Procedure:

1. Define the list of the values of portfolio maturities T to be examined.
2. Create a synthetic market by simulating asset price trajectories.
3. Run five different portfolio management strategies on the market:
   - fully balanced strategy with no fees involved,
   - fully balanced strategy with fees involved,
   - periodically balanced strategy with fees involved,
   - partially balanced strategy with fees involved,
   - passive strategy.
4. Repeat steps 2-3 independently and collect the results after each Monte Carlo trial.
5. Average the results across all Monte Carlo trials for each t.
6. Repeat steps 2-5 for subsequent values of portfolio maturity.

Observations:

- Among the strategies utilising rebalancing, the fully-balanced portfolio performs best in the absence of the exchange fees and worst when they are involved. Partially- and periodically-rebalanced strategies both find themselves between those two extremes, regardless of the time of the investment.

- For short-term investments ($T = 1$, i.e. one-year portfolio, presented in Fig. 3.1), the passive portfolio acts almost identically as the fully-balanced portfolio in the absence of fees.
- For extremely long-term investments ($T = 100$, i.e. 100-years portfolio, presented in Fig. 3.3), the growth of wealth of of the passive portfolio decreases visibly. After ~ 30 years from the portfolio set-up, the buy-and-hold strategy performs worse than all rebalance-related portfolios.

Simulation parameters: $T \in \{1, 10, 100\}$, $\Delta t = 0.1, N = 8, s_0 = 1, \mu = 0.2, \kappa = 1.5, \theta = 0.5, \sigma = 0.1, \rho = -0.66, \alpha \in \{0, 0.03\}$, $m = 5$, $D = 0.2$, Monte Carlo trials: 1000.
Figure references: 3.1 — 3.3.

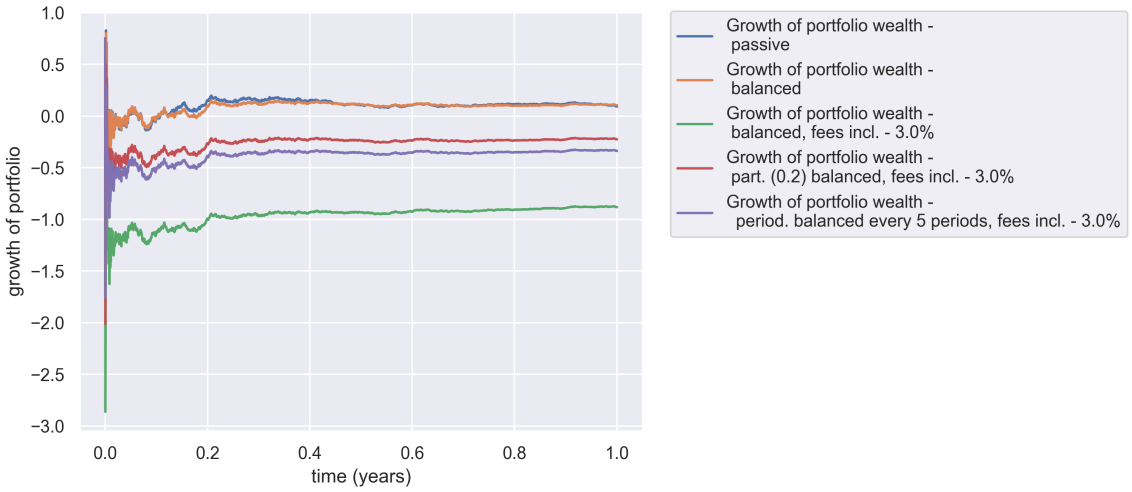


Figure 3.1: Growth of various types of 1-year portfolios, with and without the exchange fees.

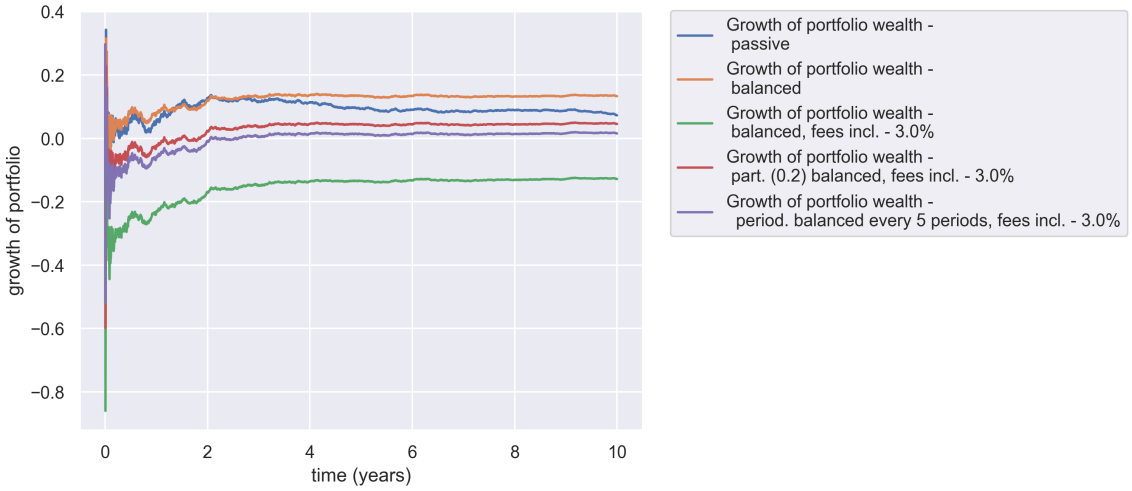


Figure 3.2: Growth of various types of 10-year portfolios, with and without the exchange fees.

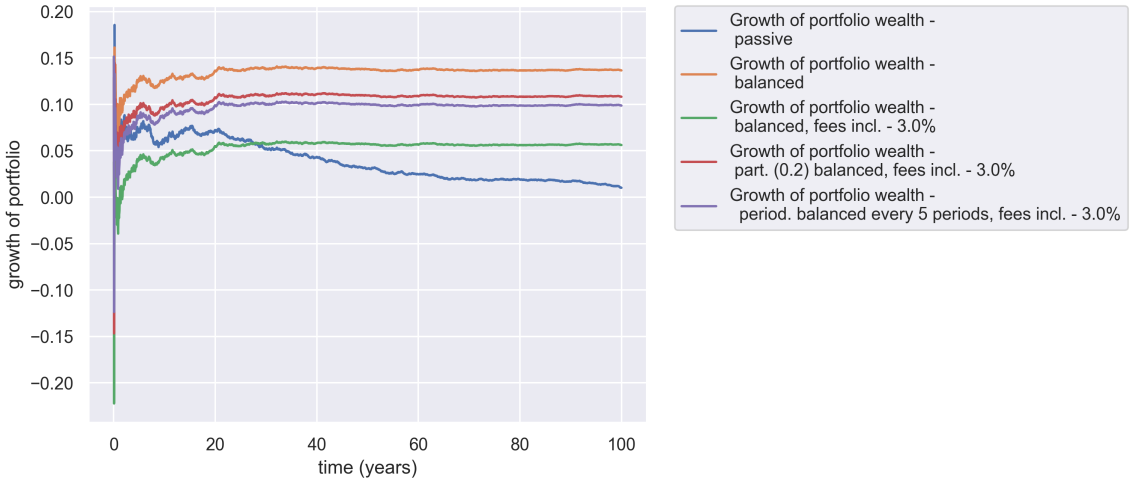


Figure 3.3: Growth of various types of 100-year portfolios, with and without the exchange fees.

3.6.2 Optimal rebalance period and optimal rebalance coefficient

As demonstrated by Experiment 1, partially and periodically balanced portfolios seem to perform best in presence of transaction fees, especially for longer investment times. We decided to study what parameters of these portfolios maximise the growth of wealth of portfolio at the point where simulation terminates (i.e. at portfolio's maturity time T). It was natural to use the characteristic measure for that, as defined in Definition 27. For the partially balanced portfolio, the characteristic feature was the partial rebalance coefficient, i.e. $\tau = D$, for the periodically balanced one — the rebalance period, $\tau = m$. In both cases the level of transactions fees α turned out to have crucial meaning when determining the optimal parameters.

Experiment 2.

Aim: Finding optimal parameters for partially balanced and periodically balanced portfolios in presence of transaction fees.

Procedure:

1. Define the list of values of the strategy parameter (partial rebalance coefficient D for partially balanced portfolios, period of rebalance m for periodically balanced ones) to be examined.

2. Create a synthetic market by simulating asset price trajectories.

3. For a given strategy parameter, run three different portfolios on the market for three different levels of fees.

4. Repeat steps 2-3 independently and collect $\varphi(D)$ (for partially balanced portfolios) or $\varphi(m)$ for periodically balanced portfolios) for each fee level after each Monte Carlo trial.

5. Average obtained values of $\varphi(D)$ or $\varphi(m)$ across all Monte Carlo trials for each fee level.

6. Repeat steps 3-5 for subsequent values of strategy parameters and collect the averaged value $\varphi(D)$ or $\varphi(m)$ for each value of D or m.

7. Repeat steps 2-6 for the other strategy parameter.

Observations:

- For periodically balanced portfolios (Fig. 3.4), the bigger the fees the larger is the value of m for which $\varphi(m)$ attains maximum. This means that for bigger level of the fees, rebalancancing should be performed less frequently.
- For partially balanced portfolios (Fig. 3.5), the bigger the fees the smaller is the value of D for which $\varphi(D)$ attains maximum. This means that for bigger level of the fees, the level of portfolio rebalancing should be smaller.
- The variability of $\varphi(D)$ is much smaller across all levels of D than the variability of $\varphi(m)$ across m, which results in much smoother curves for $\varphi(D)$ than for $\varphi(m)$.

Simulation parameters: $T = 10$, $\Delta t = 0.03, N = 2, s_0 = 1, \mu = 0.125, \kappa = 1.5, \theta = 0.5, \sigma = 0.1, \rho = -0.66, \alpha \in \{0.01, 0.03, 0.05\}$, $m \in \{1, 2, \ldots, 60\}$ or $m = 1$, $D \in \{0, 0.01, 0.02, \ldots, 1\}$ or $D = 1$, Monte Carlo trials: 1000.
Figure references: 3.4, 3.5.

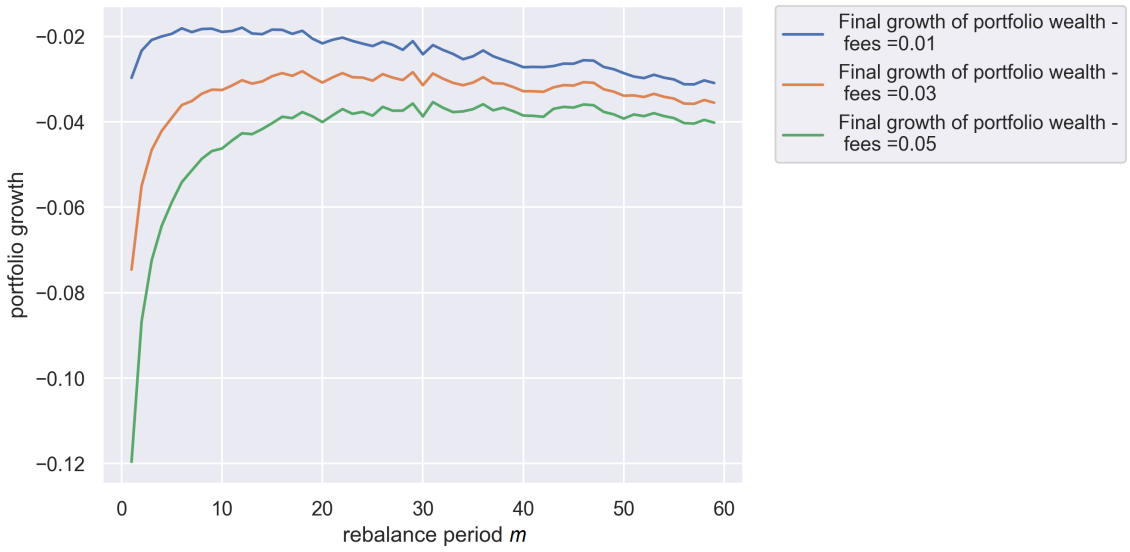


Figure 3.4: Final growth of portfolio (for $t = T$) in dependence of the period of rebalance.

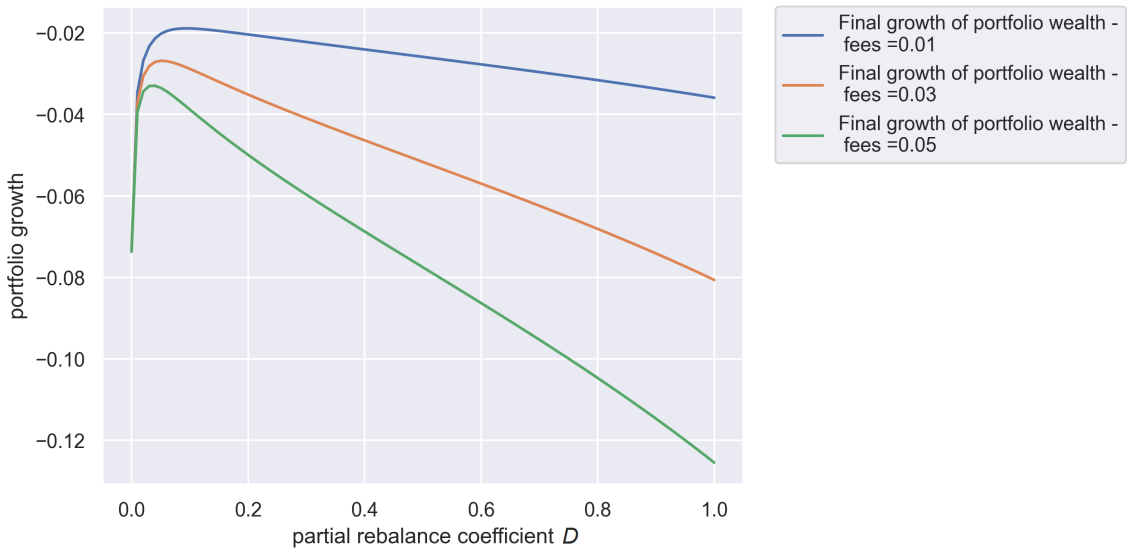


Figure 3.5: Final growth of portfolio (for $t = T$) in dependence of the partial rebalance coefficient

In Experiment 2 we were analysing periodically balanced and partially balanced strategies separately, but both of those approaches can be merged into one strategy, according to Definition 14. Such strategy is then dependent on two equally important parameters — period of rebalance m and partial rebalance coefficient D. Hence, to best visualise it, a measure similar the characteristic measure would be useful, but generalised to be multi-dimensional. For this purpose specifically we introduce a *multi-dimensional characteristic measure*.

Definition 31. A multi-dimensional characteristic measure $\varphi(t, \boldsymbol{\tau})$ of a set of M portfolios $\{\mathcal{P}\}_{i=1}^{M}$, each characterised by a unique L-dimensional vector of values of the portfolio's characteristic properties $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_L)$, i.e. $\mathcal{P}_i = \mathcal{P}(\boldsymbol{\tau}_i)$, for a particular vector of values $\boldsymbol{\tau}_i \in \{1, 2, \ldots, M\}$, is given by

$$\varphi(t, \boldsymbol{\tau}) = g_{\mathcal{P}(\boldsymbol{\tau})}(t). \tag{3.20}$$

By definition $\varphi(\boldsymbol{\tau}) = \varphi(T, \boldsymbol{\tau})$.

Using $\boldsymbol{\tau} = (m, D)$ as a vector of characteristic properties for periodically and partially balanced portfolio, we get $\varphi(m, D)$ as a multi-dimensional characteristic measure for this strategy and we can use it for visualising the performance of this strategy in two dimensions.

Experiment 3.

Aim: Analysis performance of the periodically and partially balanced portfolio.
Procedure:

1. Define the list of values of the period of rebalance m and partial rebalance coefficient D to be examined.
2. Create a synthetic market by simulating asset price trajectories.
3. For a given combination of the values of m and D, run the periodically and partially balanced portfolio on a simulated market.
4. Repeat steps 2-3 independently and collect $\varphi(m, D)$ after each Monte Carlo trial.
5. Average obtained values of $\varphi(m, D)$ across all Monte Carlo trials.
6. Repeat steps 2-5 for subsequent combination of (m, D) and collect the averaged value $\varphi(m, D)$.

Observations:

- The bigger the rebalance period, the bigger the optimal partial rebalance coefficient for which the optimal portfolio growth is attained. This means that if one rebalances a portfolio less frequently, he should rebalance in bigger part.
- For a fixed period of rebalance, there is usually a whole spectrum of partial rebalance coefficients for which portfolios present similar performance.
- For all periods of rebalance, portfolios with partial rebalance coefficients close to 0 perform worse than the ones close to 1. This effect is getting stronger for bigger periods of rebalance.
- The passive portfolio, when the partial rebalance coefficient is 0, performs the worst across the board.
- Variability across periods of rebalance is bigger than across partial rebalance coefficients, which can be seen as horizontal "stripes" on the heat map.

Simulation parameters: $T = 10$, $\Delta t = 0.02, N = 2, s_0 = 1, \mu = 0.125, \kappa = 1.5, \theta = 0.5, \sigma = 0.1, \rho = -0.66, \alpha = 1, m \in \{1, 2, \ldots, 50\}$, $D \in \{0, 0.02, 0.04, \ldots, 1\}$, Monte Carlo trials: 1000.

Figure references: 3.6.

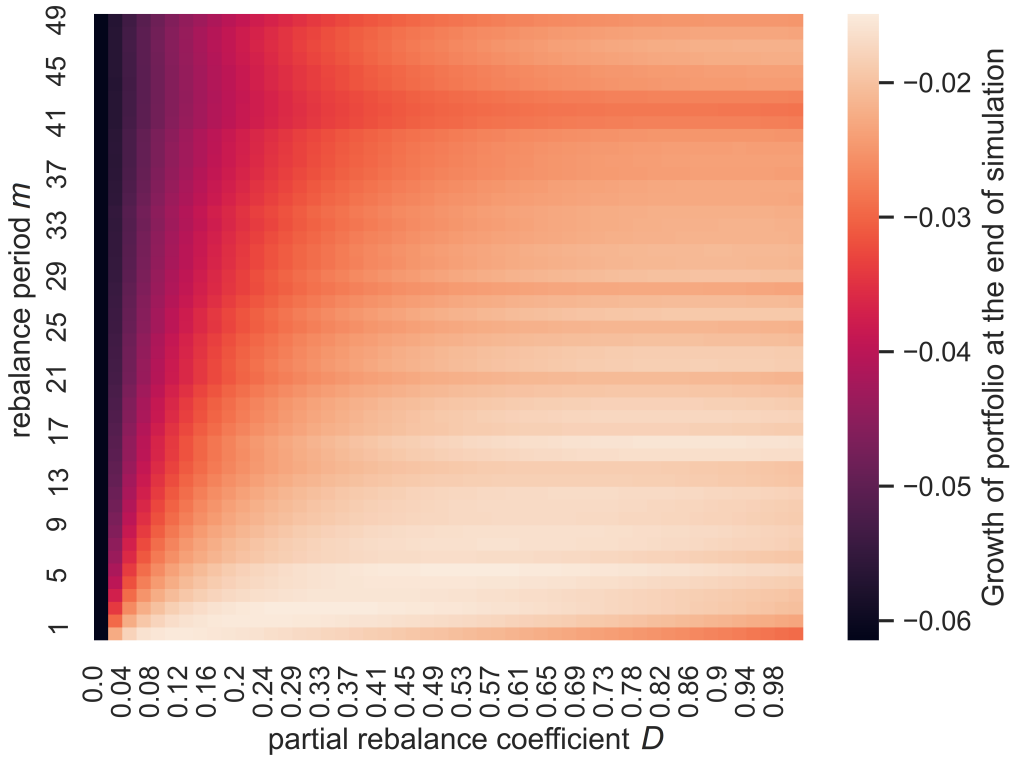


Figure 3.6: Final growth (for $t = T$) of partially and periodically balanced portfolios in dependence of the period of rebalance and the partial rebalance coefficient.

3.6.3 Impact of cash on portfolio performance

An important question investors often ask themselves is whether they should leave some money aside in form of cash for future investments and — if so — how much should they leave. Not investing all possessed money into stocks at the very first moment has some immediate advantages. It reduces the risk, as the more money remains inside a portfolio, the more stable it becomes. This is because cash, unlike stocks, does not change its value in time in our model. Moreover, leaving some cash aside allows an investor to react when an opportunity on the market appears, without the need to make any changes in the existing portfolio stock arrangement. On the

other hand however, holding cash in a portfolio diminishes the amount of money earned from an actual investment, which seems to be especially dissatisfying in case of a very successful initial choice of the risky assets.

Experiment 4 illustrates the impact of storing cash in an investment portfolio. The intuition here would be that a portfolio with left-away money will perform worse compared to an analogous portfolio with all money resources invested in stocks. Rather interestingly however, it turns out it might not always be the case. We found out that stocks' drift is the decisive parameter here.

Experiment 4.

Aim: Analysing the impact of cash in a passive investment portfolio in two special cases.

Procedure:

1. Define two separate values of μ to be examined.
2. For a given value of μ create a synthetic market by simulating asset price trajectories.
3. Run two different portfolio management strategies on the market, one with cash and one without cash.
4. Repeat steps 2-3 independently and collect the results in form of the growth of wealth for both portfolios after each Monte Carlo trial.
5. Average the results across all Monte Carlo trials for both portfolios for each t.
6. Repeat steps 2-5 for the other value of μ.

Observations:

- Portfolio without cash demonstrated better performance for small value of μ (Fig. 3.8), but it performed worse if μ was bigger (Fig. 3.7).

Simulation parameters: $T = 2, \Delta t = 2^{-8}, N = 8, s_0 = 100, \mu \in \{0.001, 0.1\}$, $\kappa = 1.22, \theta = 0.06, \sigma = 0.56, \rho = -0.66, \alpha = 0, q_0(0) \in \{0, 200\}$, Monte Carlo trials: 1000.

Figure references: 3.7, 3.8.

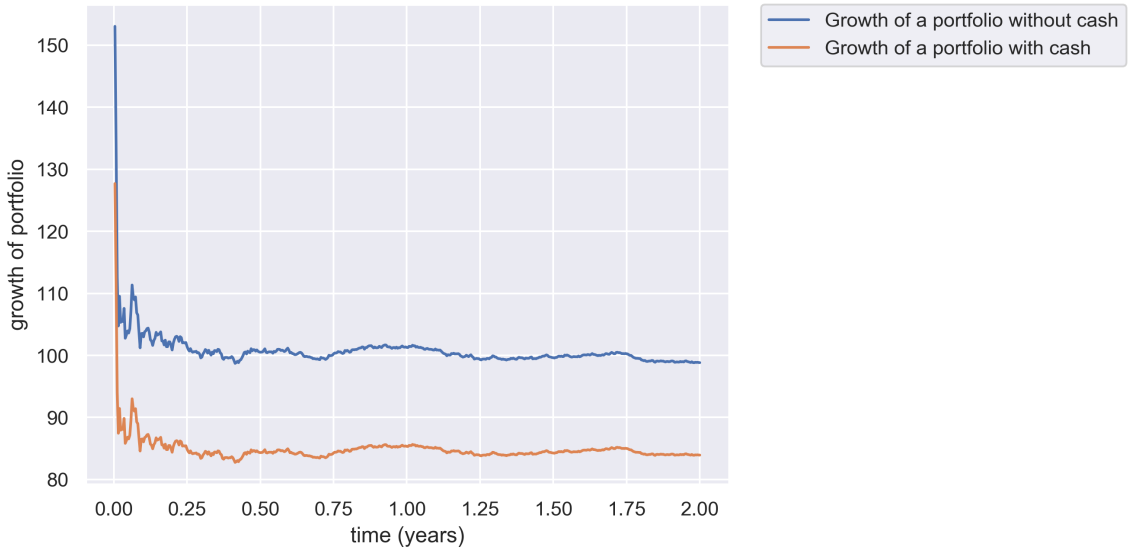


Figure 3.7: Portfolio growth in time for a passive portfolio and a portfolio with a cash contribution (approx. 28.5%) for $\mu = 0.1$.

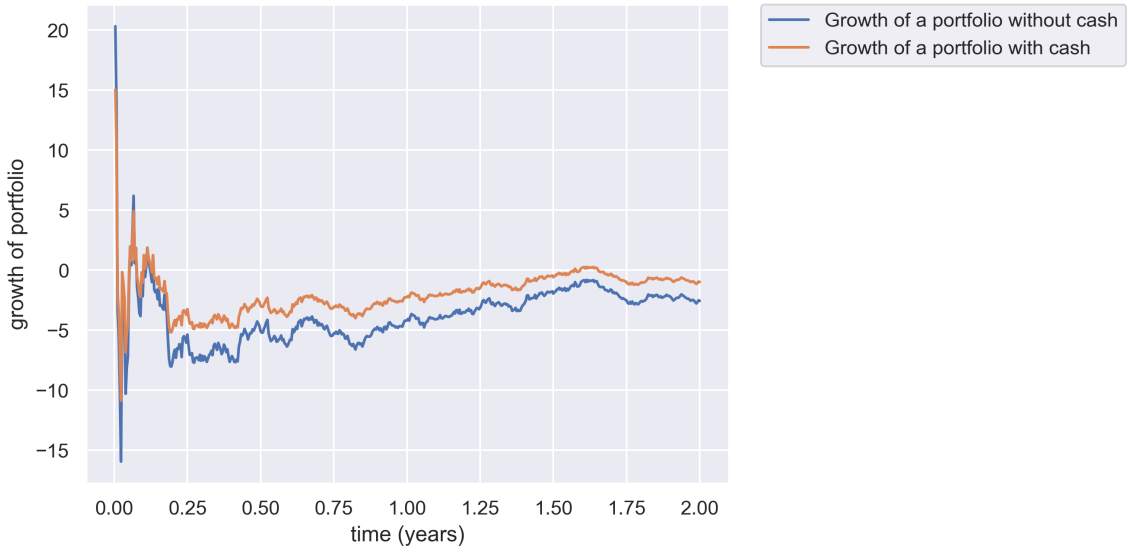


Figure 3.8: Portfolio growth in time for a passive portfolio and a portfolio with a cash contribution (approx. 28.5%) for $\mu = 0.001$.

Results of Experiment 4 are special cases of a more general analysis, presented in Experiment 5. It uses the benchmark measure, with the characteristic property being the drift parameter μ. The measure is set to calculate the difference between a passive portfolio with some amount of cash in it and a classical passive portfolio, in which all money is invested in the risky assets (also called an *all-in portfolio*).

Experiment 5.

Aim: General analysis of the impact of cash in a passive investment portfolio using benchmark measure.

Procedure:

1. Define the list of values of the drift parameter μ and five non-zero portfolio cash levels to be examined.

2. For a given drift parameter μ create a synthetic market by simulating asset price trajectories.

3. For a simulated market, run six different portfolios — one cashless and five containing various amounts of cash, as defined in step 1.

4. Repeat steps 2-3 independently and collect $\delta(\mu)$ for each combination of cash- and cashless portfolio after each Monte Carlo trial.

5. Average obtained values of $\delta(\mu)$ across all Monte Carlo trials for each portfolio pair.

6. Repeat steps 2-5 for subsequent values of μ and collect the averaged value $\delta(\mu)$ for each portfolio pair.

Observations:

- Portfolios consisting mostly of cash perform worse if assets have a strong tendency to grow (big values of μ), but they give the best results in case of lower growth potential (small values of μ).

Simulation parameters: $T = 2, \Delta t = 2^{-8}, N = 5, s_0 = 100, \mu \in \{0.001, 0.002, \ldots, 0.05\}$, $\kappa = 1.22, \theta = 0.06, \sigma = 0.56, \rho = -0.66, \alpha = 0, q_0(0) \in \{0, 100, 250, 500, 750, 1000\}$, Monte Carlo trials: 1000.

Figure references: 3.9.

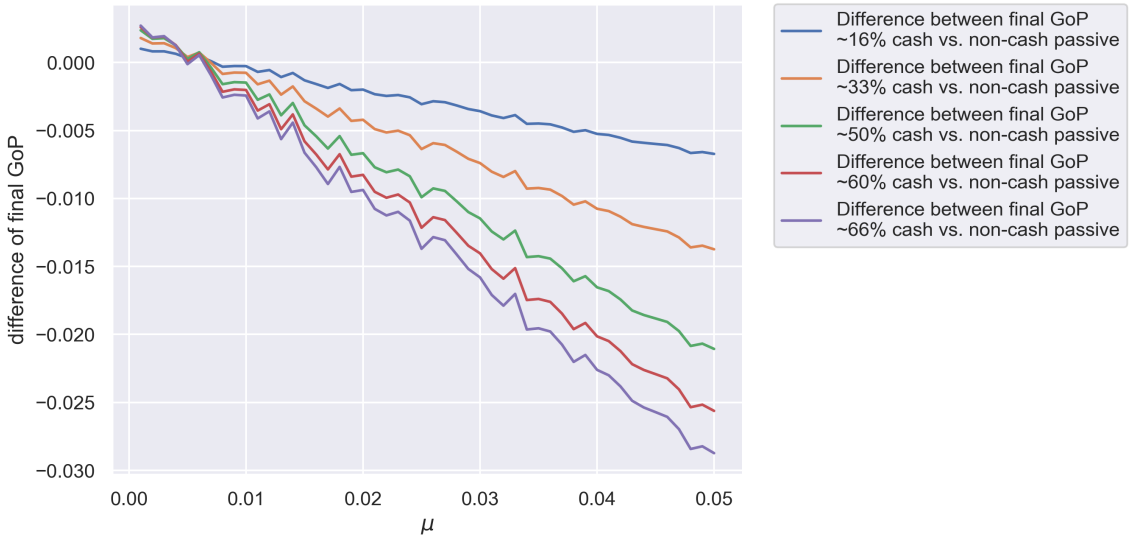


Figure 3.9: Impact of the amount of cash stored in the portfolio for various values of the drift parameter μ.

3.6.4 Performance of MACD- and RSI-driven portfolios

In the previous section we have demonstrated how parameter μ of the Heston model is related to the impact of cash on portfolio performance. It turns out that the drift is a critical parameter not only when it comes to the cash inclusion, but it also has a huge impact on the effectiveness of strategies based on using technical analysis indicators like MACD or RSI. In Experiment 6, the behaviour of passive and MACD-driven portfolios has been compared for different values of the drift μ.

Experiment 6.

Aim: Analysis the MACD-driven portfolio in two special cases.
Procedure:

1. Define two separate values of μ to be examined.
2. For a given value of μ, create a synthetic market by simulating asset price trajectories.
3. Run two portfolios on a simulated market — one MACD-driven and one passive.
4. Repeat steps 2-3 independently and collect the results in form of the growth of wealth for both portfolios after each Monte Carlo trial.

5. Average the results across all Monte Carlo trials for both portfolios, for each t.
6. Repeat steps 2-5 for the other value of μ.

Observations:

- For a small value of μ (Fig. 3.10), both strategies perform similarly throughout the lifetime of the portfolio and at the end they give very similar results.
- For a bigger value of μ (Fig. 3.11), it becomes clear that the portfolio managed by an MACD strategy outperforms a simple buy-and-hold strategy.

Simulation parameters: $T = 2, \Delta t = 2^{-8}, N = 10, s_0 = 100, \mu \in \{0.005, 0.1\}$ $\kappa = 1.2, \theta = 0.05, \sigma = 0.5, \rho = -0.66, \alpha = 0, q_0(0) = 200, p = 12, q = 26, s = 9, \psi = 0.5, \phi = 0.5$, Monte Carlo trials: 1000.
Figure references: 3.10, 3.11.

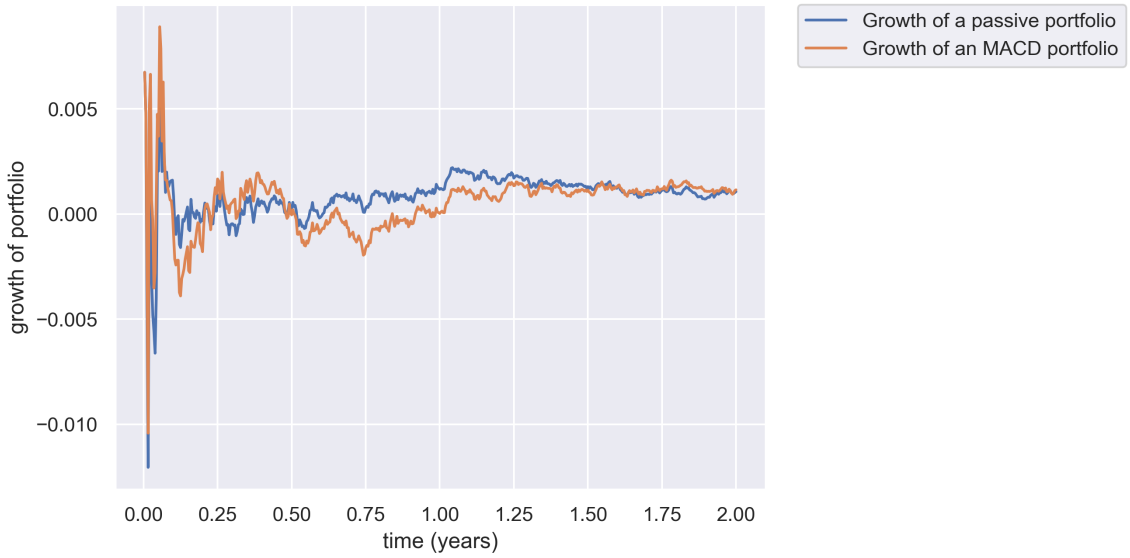


Figure 3.10: Portfolio growth in time for a passive portfolio and a MACD portfolio for $\mu = 0.005$.

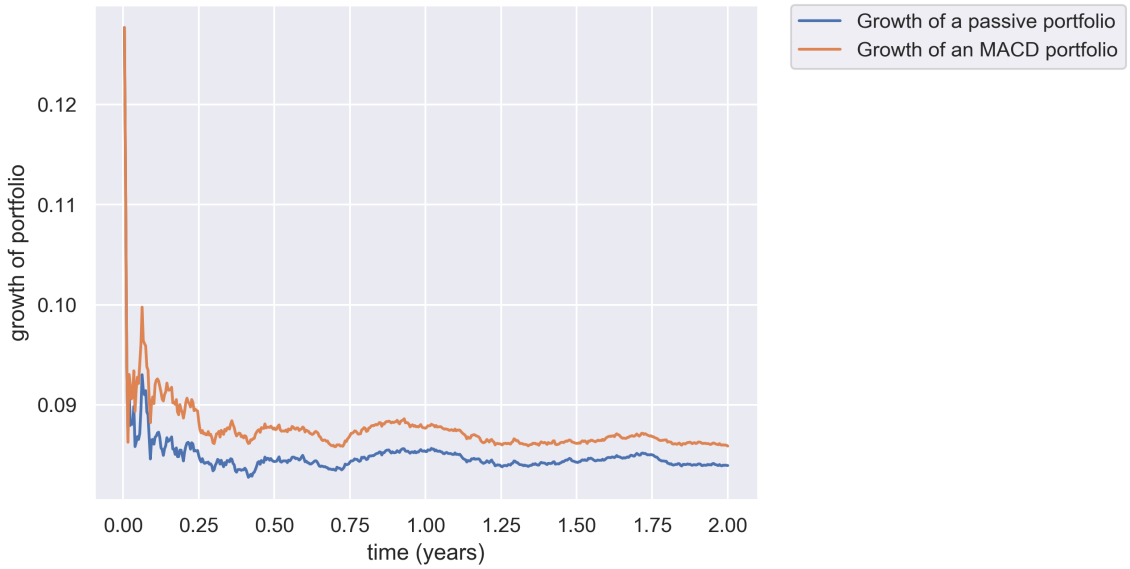


Figure 3.11: Portfolio growth in time for a passive portfolio and a MACD portfolio for $\mu = 0.1$.

One could expect that it would be the case for strategies utilising various indicators, that the bigger the drift, the better the results. It turns out, that the behaviour can be different depending on the indicator. Experiment 7 allows to see the results for RSI.

Experiment 7.

Aim: Analysis the RSI-driven portfolio in two special cases.

Procedure:

1. Define two separate values of μ to be examined.
2. For a given value of μ create a synthetic market by simulating asset price trajectories.
3. Run two portfolios on a simulated market — one RSI-driven and one passive.
4. Repeat steps 2-3 independently and collect the results in form of the growth of wealth for both portfolios after each Monte Carlo trial.
5. Average the results across all Monte Carlo trials for both portfolios, for each t.
6. Repeat steps 2-5 for the other value of μ.

Observations:

- For the smaller value of μ (Fig. 3.12), both strategies perform similarly throughout the lifetime of the portfolio, however, at the end, the passive portfolio achieves better results.
- For the bigger value of μ (Fig. 3.13) — portfolio managed by the RSI strategy performs drastically worse than the buy-and-hold strategy.

Simulation parameters: $T = 2, \Delta t = 2^{-8}, N = 10, s_0 = 100, \mu \in \{0.1, 0.5\}$, $\kappa = 1.2, \theta = 0.05, \sigma = 0.5, \rho = -0.66, \alpha = 0, q_0(0) = 200, s = 14, d^- = 30, d^+ = 70, \psi = 0.5, \phi = 0.5$, Monte Carlo trials: 1000.
Figure references: 3.12, 3.13.

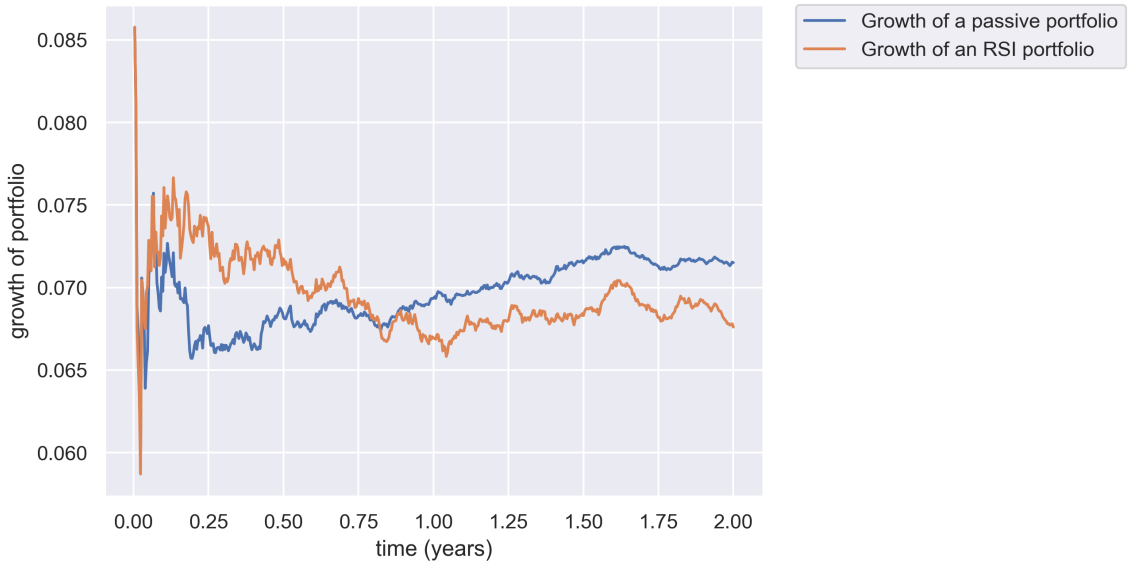


Figure 3.12: Portfolio growth in time for a passive portfolio and a RSI portfolio for $\mu = 0.1$.

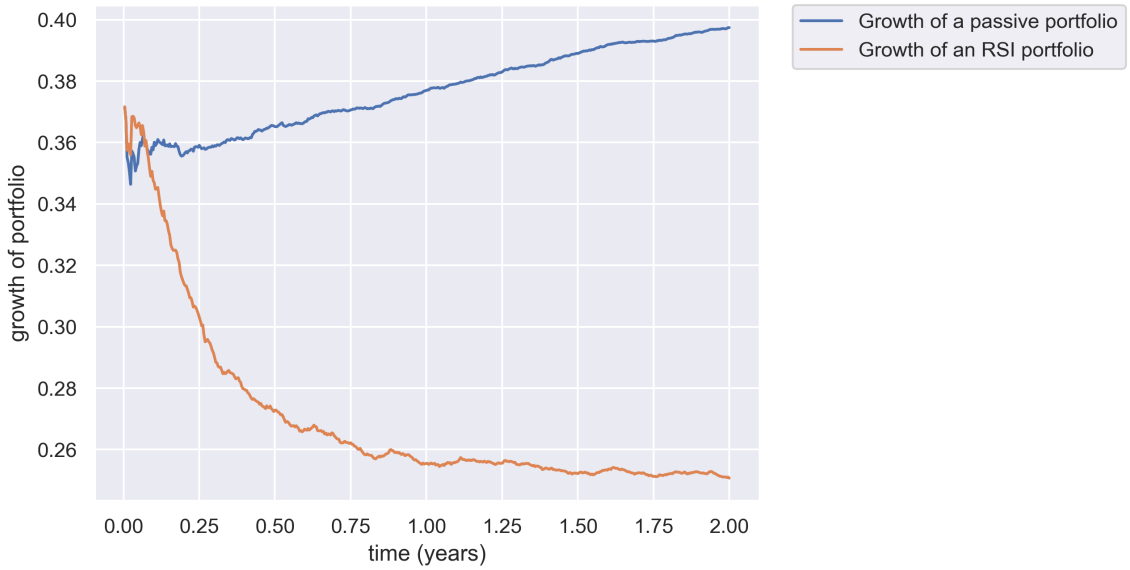


Figure 3.13: Portfolio growth in time for a passive portfolio and a RSI portfolio for $\mu = 0.5$.

Similarly to the cash inclusion experiment, also here we have performed an experiment with a benchmark measure, using differences between actively-managed portfolios and the passive one in dependence of the drift parameter μ. Also, for the previous experiment we used MACD and RSI strategies in their basic form, but one should keep in mind that their definitions (Definition 19 and 25, respectively) allow for customising the way they work. One of the possible customisations is changing the values of buy and sell factors (ϕ and ψ, respectively). The default values of those indicators are both 0.5. Changing them may result in changes of portfolio performance. This effect has been studied in Experiment 8.

Experiment 8.

Aim: General analysis of trading-indicator-driven strategies using benchmark measure.

Procedure:

1. Define the list of values of the drift parameter μ and two sets of values for the buy and sell factors (ϕ and ψ respectively) to be examined.

2. For a given value of μ create a synthetic market by simulating asset price trajectories.

3. For a simulated market, run three different portfolios — passive, MACD-driven and RSI-driven for a given selection of ϕ and ψ.

4. Repeat steps 2-3 independently and collect $\delta(\mu)$ for each combination of an indicator-driven and passive portfolio after each Monte Carlo trial.

5. Average obtained values of $\delta(\mu)$ across all Monte Carlo trials for both portfolio pairs.

6. Repeat steps 2-5 for subsequent values of μ and collect the averaged value $\delta(\mu)$ for each portfolio pair.

7. Repeat steps 2-6 for the other set of values for ϕ and ψ.

Observations:

- For the default values of $\phi = 0.5$ and $\psi = 0.5$ (Fig. 3.14), MACD performs best for portfolios where assets are characterised by a big value of the drift μ, whereas RSI turns out to be useful in the interval of values of μ between 0 and 0.1. However, even then it does not really outperform a passive portfolio.

- For the altered values of $\phi = 0.8$ and $\psi = 0.1$ (Fig. 3.15), the MACD strategy performed better for bigger values of μ and worse for lower values of μ, compared to standard values of buy and sell indicators for this type of strategy. The RSI strategy exacerbated slightly for smaller values of μ and improved considerably for bigger μ, compared to standard values of ϕ and ψ for this type of strategy.

- MACD and RSI present similar results for the altered values of ϕ and ψ (lines on the plot in Fig. 3.15 are very close to one another).

Simulation parameters: $T = 2, \Delta t = 2^{-8}, N = 5, s_0 = 100, \mu \in \{-0.2, -0.198, \ldots, 0.2\}$, $\kappa = 1.2, \theta = 0.05, \sigma = 0.5, \rho = -0.66, \alpha = 0, q_0(0) = 200, p = 12, q = 26, s_{MACD} = 9, s_{RSI} = 14, d^- = 30, d^+ = 70, \psi \in \{0.1, 0.5\}$, $\phi \in \{0.5, 0.8\}$, Monte Carlo trials: 1000.

Figure references: 3.14, 3.15.

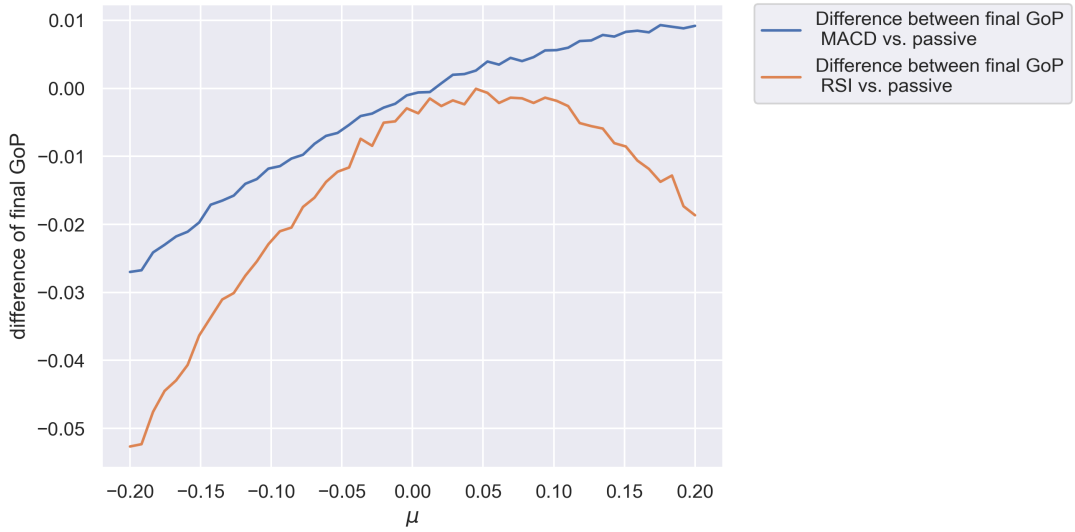


Figure 3.14: Difference between MACD and RSI portfolios and a passive portfolio for various values of the drift parameter μ and buy and sell factors equal to $\phi = 0.5$ and $\psi = 0.5$, respectively.

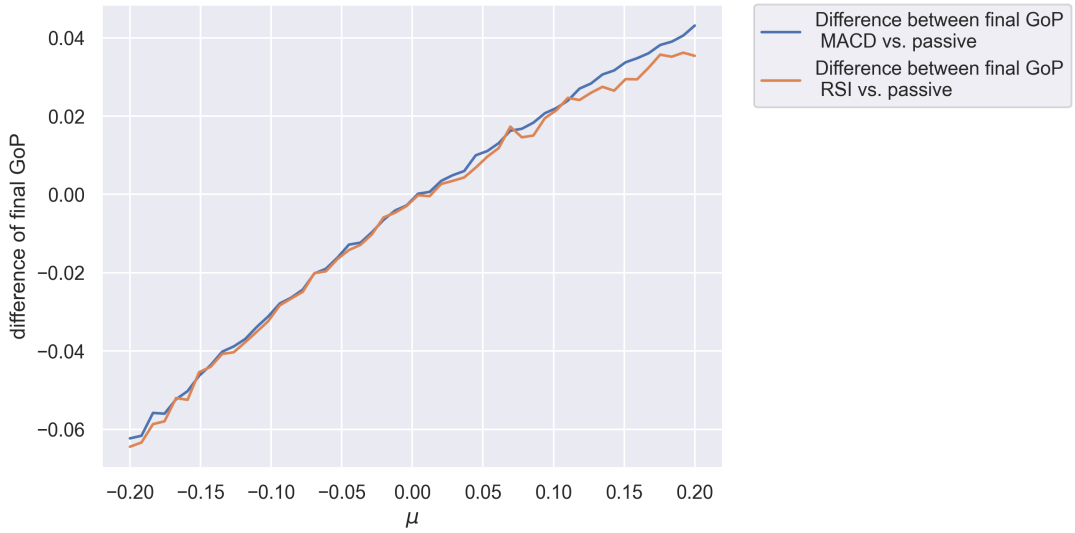


Figure 3.15: Difference between MACD and RSI portfolios and a passive portfolio for various values of the drift parameter μ and buy and sell factors equal to $\phi = 0.8$ and $\psi = 0.1$, respectively.

Experiments 6–8 demonstrate clearly, that performance of MACD- and RSI-based management strategies depends a lot on the growth potential of the assets making up the portfolio, reflected in the value of the drift parameter μ. Indeed, the drift is the main factor to be considered when the original Heston model is used for simulating the assets. If however jumps are included in the prices, two more parameters will affect the growth potential of the assets involved, i.e. the intensity of jumps λ, and the mean size of the jumps μ^J. Their impact on portfolio performance is the subject of Experiment 9.

Experiment 9.

Aim: Analysis of trading-indicator-driven strategies in presence of price jumps, using benchmark measure.

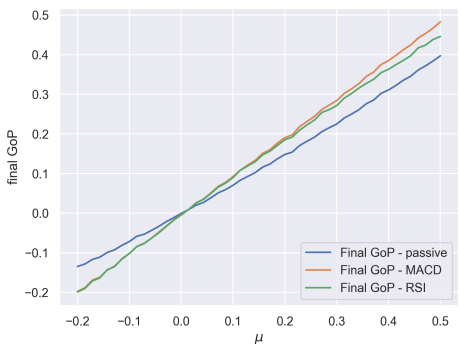
Procedure:

1. Define the list of values of the drift parameter μ, three values of the jump intensity parameter λ and three values of the average jump size μ^J to be examined.
2. For a given value of μ, λ and μ^J, create a synthetic market by simulating asset price trajectories.
3. For a simulated market, run three different portfolios — passive, MACD-driven and RSI-driven.
4. Repeat steps 2-3 independently and collect $\varphi(\mu)$ for each of the three portfolios after each Monte Carlo trial.
5. Average obtained values of $\varphi(\mu)$ for each of the three portfolios across all Monte Carlo trials.
6. Repeat steps 2-5 for subsequent value of λ or μ^J.

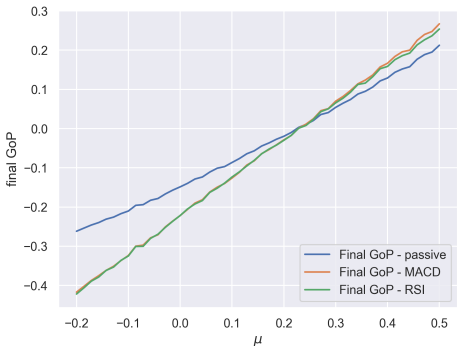
Observations:

- For a fixed μ and μ^J, the bigger the value of λ, the worse is the performance of indicator-based strategies compared to the passive one (Fig. 3.16).
- Similar effect observed for fixed μ and λ and increasing μ^J (Fig. 3.17).
- The three parameters together — μ, λ and μ^J collectively decide upon the "growth potential" of the asset and hence — have direct impact on performance of all studied portfolio management strategies.

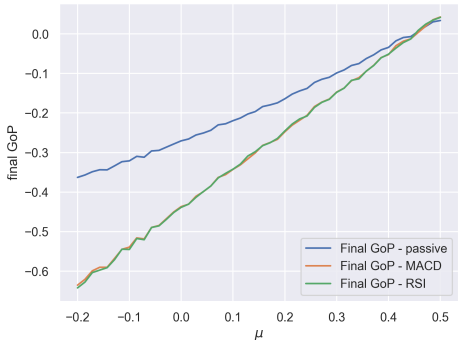
Simulation parameters: $T = 10, \Delta t = 0.02, N = 2, s_0 = 1, \mu \in \{-0.2, -0.198, \ldots, 0.5\}$, $\kappa = 1.2, \theta = 0.05, \sigma = 0.5, \rho = -0.66$, $\lambda \in \{0, 1, 2\}$ or $\lambda = 1$, $\mu^J = -0.25$ or $\mu^J \in -0.2, -0.6, -1$, $\sigma^J = 0.1$ $\alpha = 0, q_0(0) = 200, p = 12, q = 26, s_{MACD} = 9, s_{RSI} = 14, d^- = 30, d^+ = 70, \psi = 0.1, \phi = 0.8$, Monte Carlo trials: 1000.
Figure references: 3.16, 3.17.



(a) $\lambda = 0$

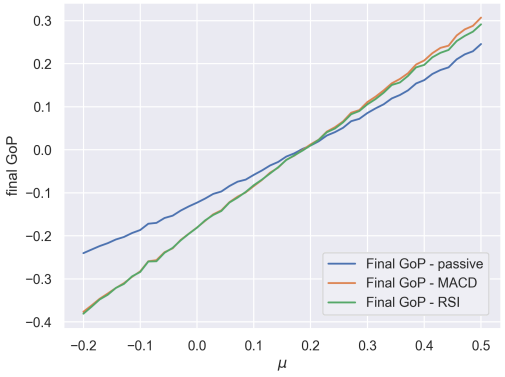


(b) $\lambda = 1$

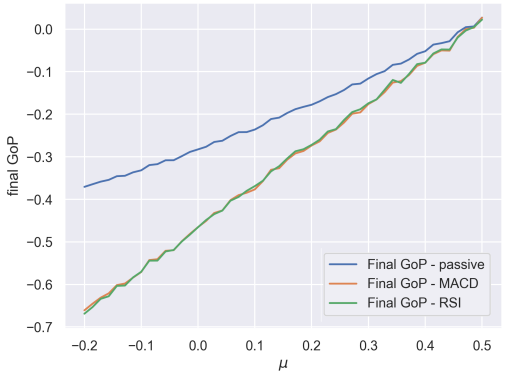


(c) $\lambda = 2$

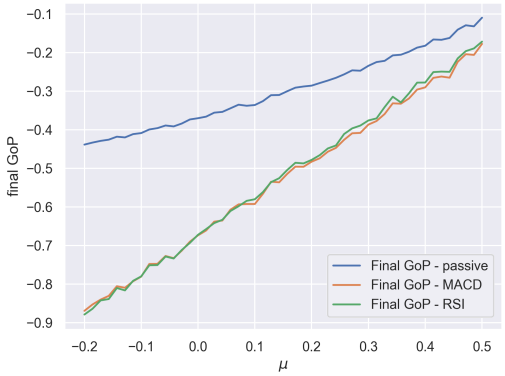
Figure 3.16: Growth of portfolio for different investment strategies and different values of λ.



(a) $\mu^J = -0.2$



(b) $\mu^J = -0.6$



(c) $\mu^J = -1$

Figure 3.17: Growth of portfolio for different investment strategies and different values of μ^J.

By now, we have demonstrated through several experiments that the choice of the optimal portfolio management strategy depends on the character of assets which we want to invest in. Thus, we decided to create a map of applicability of active and passive portfolio management strategies. In order for the map to have a normalised values, we introduce another measure for comparing portfolios to one another, called ASPI — *Active Strategy Performance Indicator*. It can be thought of as a variation of the benchmark measure. For a pair of two portfolios consisting of the assets of a similar character, one actively managed and one passive, ASPI simply indicates whether the difference between their terminal growth of wealth is positive.

Definition 32. For a set of M actively-managed portfolios $\{\mathcal{P}\}_{i=1}^{M}$, each characterised by a unique L-dimensional vector of values of the portfolio's characteristic

properties $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_L)$, i.e. $\mathcal{P}_i = \mathcal{P}(\boldsymbol{\tau}_i)$, for a particular vector of values $\boldsymbol{\tau}_i \in \{1, 2, \ldots, M\}$, and another set of M passive portfolios $\{\mathcal{R}\}_{i=1}^{M}$ with the same characteristic properties, the ASPI measure is an indicator given by

$$\mathrm{ASPI}_{\mathcal{P},\mathcal{R}}(\boldsymbol{\tau}) = \begin{cases} 1 & \text{if } \delta_{\mathcal{P},\mathcal{R}}(\boldsymbol{\tau}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.21}$$

Of course, trajectories of the asset prices are random so it is possible, that one pair of portfolios will result in ASPI equal to 1, and another one, generated for the same vector of characteristic properties $\boldsymbol{\tau}$, will give us the ASPI measure of 0. Hence, to get a full picture, as always, averaging is needed. Experiment 10 provides more details about how to do it, taking $\boldsymbol{\tau} = (\mu, \lambda)$. We used MACD strategy as a model of an actively managed portfolio, since, as Fig. 3.15 suggests, both active strategies perform similarly in some configurations.

Experiment 10.

Aim: General analysis of the trading indicator-driven strategies in presence of price jumps using ASPI measure.

Procedure:

1. Define the list of values of the drift parameter μ and the jump intensity parameter λ.
2. For a given combination of the values of μ and λ, create a synthetic market by simulating asset price trajectories.
3. For a simulated market, run two portfolios — MACD-driven (as actively-managed) and passive.
4. Repeat steps 2-3 independently and collect $\mathrm{ASPI}(\mu, \lambda)$ after each Monte Carlo trial.
5. Average obtained values of $\mathrm{ASPI}(\mu, \lambda)$ across all Monte Carlo trials.
6. Repeat steps 2-5 for all combinations of (μ, λ).

Observations:

- Darker colours on the heat map (Fig. 3.18), associated with higher frequency of jumps and smaller drift, are an indication that it is better to apply a passive strategy. On the other hand, bright hues represent the regions where active strategies should be preferable. Hence, the map can be used to select proper strategy for assets of certain (μ, λ) specification.

- A significant portion of the map is occupied by colours from the red-to-purple spectrum, which indicate assets where the results where not equivocal. Hence — the division between when each of the strategies should be used is not clear-cut and for some assets active and passive strategies perform (on average) similarly.

Simulation parameters: $T = 10$, $\Delta t = 0.02, N = 2, s_0 = 1, \mu \in \{0.0, 0.02, \ldots, 0.5\}$, $\kappa = 1.2, \theta = 0.05, \sigma = 0.5, \rho = -0.66, \lambda \in \{0, 1, 2, \ldots, 10\}$, $\mu^J = 0.05, \alpha = 0, q_0(0) = 200, p = 12, q = 26, s_{MACD} = 9, \psi = 0.1, \phi = 0.8$, Monte Carlo trials: 1000.
Figure references: 3.18.

Having the map allows for selecting the optimal strategy, but the problem which persists is identifying the values of parameters μ, λ and μ^J for an asset[1] which an investor wants to invest in. We address this problem in Chapter 4.

[1]It only makes sense to use the map if the identified value of the mean jump size parameter is similar to the one which we used in simulation. If it differs significantly, another map should be crafted, with the proper value of this parameter.

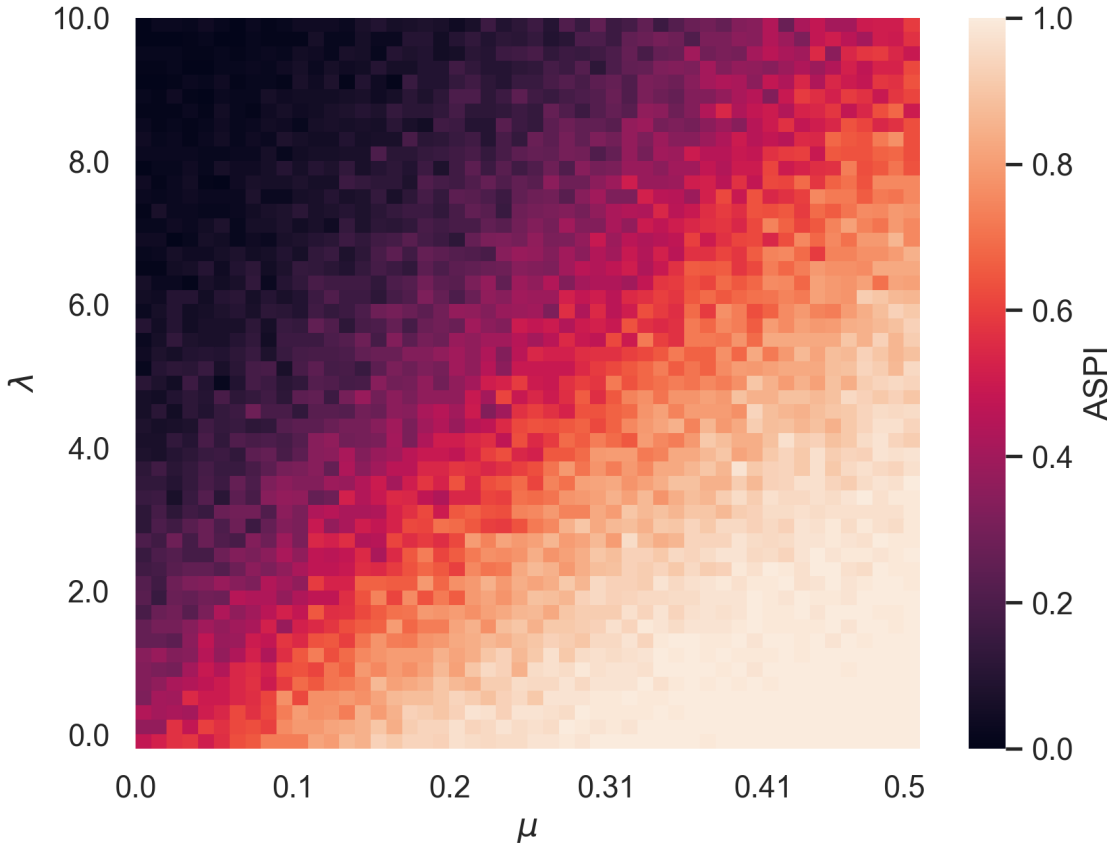


Figure 3.18: A heat map presenting the average value of the ASPI measure. Darker colours (values closer to 0) are an indication that it is better to use a passive strategy, brighter colours (values closer to 1) — that the active strategy is preferable.

4 Estimation scheme for the Heston model

4.1 Estimation techniques

The problem of parameter estimation of mathematical models applied in the fields of economy and finance is of critical importance. In order to use most of the models, like the ones for pricing financial instruments or finding an optimal investment portfolio, one needs to provide values of the model parameters which are often not easily available.

Over the last decades, mathematical models describing the behaviour of observed market quantities (e.g. prices of assets, interest rates etc.) became more and more complicated. Those complications were introduced in order to better reflect the dynamics of those quantities. The problem is that more complicated models typically use more parameters, which need to be estimated and moreover, standard estimation techniques, like Maximum Likelihood Estimators (MLE) or Generalised Method of Moments (GMM), fail very often for them [34]. Fortunately, modern research in the field of financial mathematics and econometrics revealed an approach which seems to be especially effective where others are not. It is the Bayesian approach [33]. Among methods based on Bayesian inference, the ones using Monte Carlo Markov Chains (MCMC) are the most prominent for complex financial models [34]. In this group of methods one assumes some distribution for the value of each of the parameters of a model (called the *prior distribution*) and uses it, along with the data, to produce what is called the *posterior distribution* — samples from which we can treat as possible values of our parameters (Ref. [34] provides a great overview of MCMC methods used for financial mathematics).

The MCMC concept can be applied in multiple ways and by utilising various algorithms, including Gibbs sampling or the Metropolis-Hastings algorithm [55]. Both are generally very useful for effective estimation of "scalar" parameters, i.e. those

parameters which only have one, constant number as their value. However, some models assume that the directly observable dynamic quantities (e.g. prices) are dependent on other dynamically changing properties of the model. The latter are often called latent variables or state variables. In case of the Heston model for example, the volatility process is a state variable. Estimation of state variables is inherently more complicated than of the regular parameters, as each value which was observed directly was partly determined by value of the state variable at that particular point of time. A very elegant solution to this complication is a methodology called particle filtering. It is based on the idea of creating a collection of values (called particles) which are meant to represent the distribution of the latent variable at a given point of time. Each particle then has a probability assigned to it, which serves as a measure of how likely it is that a given value of the state variable generated the outcome observed at a given moment of time. For an overview of particle filtering methods, we recommend Refs. [56] and [57].

Methods outlined above have been studied quite thoroughly for the past years. However, the research articles and literature focuses on the theoretical aspect of the estimation process and is often lacking precision and concreteness. In this part of the dissertation we address this nuisance by presenting a complete set-up for parameter estimation of the Heston model, using only the prices of the basic instrument one wants to study using the model (an index, a stock, a commodity[1] etc.). We provide the estimation process for both the pure Heston model and its extended version, with the inclusion of Merton-style jumps (discontinuities), which is known as the Bates model. In the following section, 4.2, we describe in detail how to obtain the posterior distributions to get the values of the parameters of the Heston model. We also provide a detailed description of the particle filtering scheme needed to reconstruct the volatility process. The whole procedure is summarised in an easy-to-follow pseudo-code algorithm. An exemplary estimation as well as the analysis of the factors that impact the quality of the estimation in general is presented in section 4.3.

The entirety of content of this chapter has been adapted from our research article about the estimation process of the Heston model [35].

[1]We point out explicitly that the estimation scheme only uses the data about the basic instrument, as we consider this a significant value added to the presented method. This is because the Heston model is often used in the context of derivative instruments and prices of those derivatives, often difficult to obtain, are a necessary input to the estimation procedure. This is not the case however for the scheme that we compiled.

4.2 Estimation framework

Estimation of the Heston model consists of two major parts. First one is estimating scalar parameters, i.e. μ, κ, θ, σ and ρ, for the basic version of the model and additionally λ, μ^J and σ^J after inclusion of jumps. The second part is estimating the state variable — volatility $v(t)$. For all the estimation procedures we used the Bayesian inference methodology, in particular Monte Carlo Markov Chains (for parameter estimation within the base model) and particle filtering (for estimation of the volatility as well as jump-related parameters).

4.2.1 Regular Heston model

In order to estimate the Heston model with no jumps, we will mainly be using the principles of Bayesian inference, and in particular — Bayesian regression [58].

Estimation of μ

We will start by finding a way to estimate the drift parameter μ. First, Eq. (3.12) will be transformed to a regression form. To this end, we will introduce several additional variables. The first, η, is defined as

$$\eta = \mu\Delta t + 1. \tag{4.1}$$

Let $R(t)$ be a series of ratios between consecutive prices of assets,

$$R(k\Delta t) = \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)}, \tag{4.2}$$

for $k \in \{1, 2, \ldots, n\}$. Taking the above definitions into consideration, Eq. (3.12) can be rewritten as:

$$R(k\Delta t) = \eta + \sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^S(k\Delta t)\sqrt{\Delta t}. \tag{4.3}$$

Now, let us divide both sides of this equation by $\sqrt{v\Big((k-1)\Delta t\Big)}\sqrt{\Delta t}$, as Δt is known and, at this stage, we consider $v(t)$ to be known too. Let us now introduce another two new variables: $y^S(t)$,

$$y^S(k\Delta t) = \frac{1}{\sqrt{v\Big((k-1)\Delta t\Big)\sqrt{\Delta t}}} R(k\Delta t) \tag{4.4}$$

and $x^S(t)$ as

$$x^S(k\Delta t) = \frac{1}{\sqrt{v\Big((k-1)\Delta t\Big)\sqrt{\Delta t}}}. \tag{4.5}$$

Inserting them into Eq. (4.3) gives

$$y^S(k\Delta t) = \eta x^S(k\Delta t) + \varepsilon^S(k\Delta t). \tag{4.6}$$

The last expression has the form of a linear regression with $y^S(t)$ explained by $x^S(t)$. We want to treat it with the Bayesian regression framework. To this end we first collect all discretised values of $y^S(t)$ and $x^S(t)$ into n-element column vectors — $\mathbf{y}^S$ and $\mathbf{x}^S$ respectively,

$$\mathbf{y}^S = \frac{1}{\sqrt{\Delta t}} \left[\frac{R(\Delta t)}{\sqrt{v(0)}} \quad \frac{R(2\Delta t)}{\sqrt{v(\Delta t)}} \quad \cdots \quad \frac{R(n\Delta t)}{\sqrt{v\Big((n-1)\Delta t\Big)}} \right]', \tag{4.7}$$

$$\mathbf{x}^S = \frac{1}{\sqrt{\Delta t}} \left[\frac{1}{\sqrt{v(0)}} \quad \frac{1}{\sqrt{v(\Delta t)}} \quad \cdots \quad \frac{1}{\sqrt{v\Big((n-1)\Delta t\Big)}} \right]', \tag{4.8}$$

where the prime symbol is used for the transpose.

Assuming a prior distribution for η to be normal with mean μ_0^η and standard deviation σ_0^η, it follows from the Bayesian regression general results [58] that the posterior distribution for η will also be normal with precision (inverse of variance) τ^η, which can be calculated as

$$\tau^\eta = \Big(\boldsymbol{x}^S\Big)' \cdot \boldsymbol{x}^S + \tau_0^\eta. \tag{4.9}$$

Here, τ_0^η is precision of the prior distribution, i.e. $\tau_0^\eta = \frac{1}{\left(\sigma_0^\eta\right)^2}$. Mean μ^η of the posterior distribution is of the following form

$$\mu^\eta = \frac{1}{\tau^\eta} \left(\tau_0^\eta \mu_0^\eta + \Big(\boldsymbol{x}^S\Big)' \cdot \boldsymbol{x}^S \hat{\eta} \right), \tag{4.10}$$

where $\hat{\eta}$ is a classical, ordinary-lest-squares (OLS) estimator of η, i.e.

$$\hat{\eta} = \left(\Big(\boldsymbol{x}^S\Big)' \cdot \boldsymbol{x}^S \right)^{-1} \Big(\boldsymbol{x}^S\Big)' \boldsymbol{y}^S. \tag{4.11}$$

Having a realisation of η, we can quickly turn it into a realisation of the μ parameter itself by a simple transform, inverse to Eq. (4.1)

Corollary 1. In the i-th iteration of the MCMC estimation procedure, the estimate of parameter μ is obtained via

$$\mu_i = \frac{\eta_i - 1}{\Delta t}, \tag{4.12}$$

where η_i is a sample from the posterior distribution, generated as

$$\eta_i \sim \mathcal{N}\left(\mu^\eta, \frac{1}{\sqrt{\tau^\eta}}\right). \tag{4.13}$$

Estimation of κ, θ and σ

In order to estimate parameters related to the volatility process, i.e. κ, θ and σ, we will do a similar exercise but this time using the volatility process. Let us first rewrite Eq. (3.13) as

$$v(k\Delta t) = \kappa\theta\Delta t + (1 - \kappa\Delta t)v\Big((k-1)\Delta t\Big) + \sigma\sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^v(k\Delta t)\sqrt{\Delta t}. \tag{4.14}$$

Now let us introduce two new parameters,

$$\beta_1 = \kappa\theta\Delta t \tag{4.15}$$

and

$$\beta_2 = 1 - \kappa\Delta t. \tag{4.16}$$

From Eqs. (4.14)–(4.16) we get

$$v(k\Delta t) = \beta_1 + \beta_2 v\Big((k-1)\Delta t\Big) + \sigma\sqrt{v\Big((k-1)\Delta t\Big)}\varepsilon^v(k\Delta t)\sqrt{\Delta t}. \tag{4.17}$$

In a fashion similar to the equation for the stock price, we can rewrite this last expression as

$$\frac{v(k\Delta t)}{\sqrt{\Delta t}\sqrt{v\Big((k-1)\Delta t\Big)}} = \frac{\beta_1}{\sqrt{\Delta t}\sqrt{v\Big((k-1)\Delta t\Big)}} + \frac{\beta_2 v\Big((k-1)\Delta t\Big)}{\sqrt{\Delta t}\sqrt{v\Big((k-1)\Delta t\Big)}} + \sigma\varepsilon^v(k\Delta t). \tag{4.18}$$

Introducing vectors

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \tag{4.19}$$

$$\mathbf{y}^v = \frac{1}{\sqrt{\Delta t}} \left[\frac{v(2\Delta t)}{\sqrt{v(\Delta t)}} \quad \frac{v(3\Delta t)}{\sqrt{v(2\Delta t)}} \quad \dots \quad \frac{v(n\Delta t)}{\sqrt{v\Big((n-1)\Delta t\Big)}} \right]', \tag{4.20}$$

$$\mathbf{x}_1^v = \frac{1}{\sqrt{\Delta t}} \left[\frac{1}{\sqrt{v(\Delta t)}} \quad \frac{1}{\sqrt{v(2\Delta t)}} \quad \dots \quad \frac{1}{\sqrt{v\Big((n-1)\Delta t\Big)}} \right]', \tag{4.21}$$

$$\mathbf{x}_2^v = \frac{1}{\sqrt{\Delta t}} \left[\frac{v(\Delta t)}{\sqrt{v(\Delta t)}} \quad \frac{v(2\Delta t)}{\sqrt{v(2\Delta t)}} \quad \dots \quad \frac{v\Big((n-1)\Delta t\Big)}{\sqrt{v\Big((n-1)\Delta t\Big)}} \right]' = \frac{1}{\sqrt{\Delta t}} \left[\sqrt{v(\Delta t)} \quad \sqrt{v(2\Delta t)} \quad \dots \quad \sqrt{v\Big((n-1)\Delta t\Big)} \right]', \tag{4.22}$$

allows us to rewrite the original volatility equation in form of a linear regression

$$\mathbf{y}^v = \mathbf{X}^v \boldsymbol{\beta} + \sigma \boldsymbol{\varepsilon}^v, \tag{4.23}$$

where

$$\mathbf{X}^v = \begin{bmatrix} \mathbf{x}_1^v & \mathbf{x}_2^v \end{bmatrix} \tag{4.24}$$

and

$$\boldsymbol{\varepsilon}^v = \begin{bmatrix} \varepsilon^v(\Delta t) & \varepsilon^v(2\Delta t) & \dots & \varepsilon^v\Big((n-1)\Delta t\Big) \end{bmatrix} \tag{4.25}$$

Using the formulas for Bayesian regression and assuming multivariate (2-dimensional) normal prior for $\boldsymbol{\beta}$ with mean vector $\boldsymbol{\mu}_0^\beta$ and precision matrix $\boldsymbol{\Lambda}_0^\beta$, we get the conjugate posterior distribution being also multivariate normal with precision matrix given by

$$\boldsymbol{\Lambda}^\beta = (\boldsymbol{X}^v)' \cdot \boldsymbol{X}^v + \boldsymbol{\Lambda}_0^\beta \tag{4.26}$$

and mean vector given by

$$\boldsymbol{\mu}^\beta = \Big(\boldsymbol{\Lambda}^\beta\Big)^{-1} \Big(\boldsymbol{\Lambda}_0^\beta \boldsymbol{\mu}_0^\beta + (\boldsymbol{X}^v)' \cdot \boldsymbol{X}^v \hat{\boldsymbol{\beta}}\Big) \tag{4.27}$$

where, again, $\hat{\boldsymbol{\beta}}$ is a standard OLS estimator of $\boldsymbol{\beta}$,

$$\hat{\boldsymbol{\beta}} = \Big((\boldsymbol{X}^v)' \cdot \boldsymbol{X}^v\Big)^{-1} (\boldsymbol{X}^v)' \, \boldsymbol{y}^v. \tag{4.28}$$

Obtaining realisations of the actual parameters is very easy — one simply needs to inverse the equations defining β_1 and β_2 (Eqs. (4.15) and (4.16), respectively).

Corollary 2. In the i-th iteration of the MCMC estimation procedure, the estimate of the parameter κ is obtained via

$$\kappa_i = \frac{1 - \boldsymbol{\beta}_i[2]}{\Delta t}, \tag{4.29}$$

and the estimate of the parameter θ — via

$$\theta_i = \frac{\boldsymbol{\beta}_i[1]}{\kappa_i \Delta t}, \tag{4.30}$$

where $\boldsymbol{\beta}_i[1]$ and $\boldsymbol{\beta}_i[2]$ are respectively the first and the second component of the $\boldsymbol{\beta}_i$ vector, which is a sample from the multi-variate posterior distribution

$$\boldsymbol{\beta}_i \sim \mathcal{N}(\boldsymbol{\mu}^\beta, (\sigma_i)^2 (\boldsymbol{\Lambda}^\beta)^{-1}). \tag{4.31}$$

It is worth noting that the realisation of σ appears in the equation (4.31), but we have not showed how to calculate it yet. This can be done, however, using prior and posterior parameters for $\boldsymbol{\beta}$, which we are also using for κ and θ. The most common approach for estimating σ is assuming the inverse-gamma prior distribution for σ^2. If the parameters of the prior distribution are a_0^σ and b_0^σ, then the conjugate posterior distribution is also inverse gamma, with new parameters a^σ and b^σ, where

$$a^\sigma = a_0^\sigma + \frac{n}{2} \tag{4.32}$$

and

$$b^\sigma = b_0^\sigma + \frac{1}{2}\left((\boldsymbol{y}^v)' \cdot \boldsymbol{y}^v + \left(\boldsymbol{\mu}_0^\beta\right)' \boldsymbol{\Lambda}_0^\beta \boldsymbol{\mu}_0^\beta - \left(\boldsymbol{\mu}^\beta\right)' \boldsymbol{\Lambda}^\beta \boldsymbol{\mu}^\beta \right). \tag{4.33}$$

Corollary 3. In the i-th iteration of the MCMC estimation procedure, the estimate of parameter σ^2 is obtained via

$$(\sigma_i)^2 \sim \mathcal{IG}\left(a^\sigma, b^\sigma\right) \tag{4.34}$$

Estimation of ρ

For the estimation of ρ we follow the approach presented in Ref. [59]. We first define the residuals for the stock price equation

$$e_1^\rho(k\Delta t) = \frac{R(k\Delta t) - \mu_i \Delta t - 1}{\sqrt{\Delta t}\sqrt{v\Big((k-1)\Delta t\Big)}} \tag{4.35}$$

and for the volatility equation,

$$e_2^\rho(k\Delta t) = \frac{v(k\Delta t) - v\Big((k-1)\Delta t\Big) - \kappa_i\left(\theta_i - v\Big((k-1)\Delta t\Big)\right)\Delta t}{\sqrt{\Delta t}\sqrt{v\Big((k-1)\Delta t\Big)}}. \tag{4.36}$$

By calculating those residuals we try to retrieve the error terms from the Eqs. (3.1) and (3.2) — $\varepsilon^S(t)$ and $\sigma\varepsilon^v(t)$ respectively, as we know they are tied with each other by a relationship given by equation (3.14). Taking this fact into consideration, we end up with the following equation

$$e_2^\rho(k\Delta t) = \sigma\rho e_1^\rho(k\Delta t) + \sigma\sqrt{1-\rho^2}\varepsilon^{add}(k\Delta t). \tag{4.37}$$

We now introduce two new variables, traditionally called $\psi = \sigma\rho$ and $\omega = \sigma^2(1-\rho^2)$. It is not difficult to deduce that the relationship between ρ and a newly-introduced variables is

$$\rho = \frac{\psi}{\sqrt{\psi^2 + \omega}} \tag{4.38}$$

Then, Eq. (4.37) becomes

$$e_2^\rho(k\Delta t) = \psi e_1^\rho(k\Delta t) + \sqrt{\omega}\varepsilon^{add}(k\Delta t), \tag{4.39}$$

which is again, a linear regression of $e_2^\rho(t)$ on e_1^ρ. Thus, we can use the exact same estimation scheme as in the previous case[2]. We first collect the values of e_1^ρ and e_2^ρ in two n-element vectors:

$$\mathbf{e}_1^\rho = \begin{bmatrix} e_1^\rho(\Delta t) & e_1^\rho(2\Delta t) & \dots & e_1^\rho(n\Delta t) \end{bmatrix}', \tag{4.40}$$

$$\mathbf{e}_2^\rho = \begin{bmatrix} e_2^\rho(\Delta t) & e_2^\rho(2\Delta t) & \dots & e_2^\rho(n\Delta t) \end{bmatrix}'. \tag{4.41}$$

Then we appose both vectors, forming into an n-by-2 matrix:

$$\mathbf{e}^\rho = \begin{bmatrix} \mathbf{e}_1^\rho & \mathbf{e}_2^\rho \end{bmatrix} \tag{4.42}$$

[2]Although the way of obtaining the coefficients through Bayesian regression is exactly the same, the notation differs slightly compared to the ones presented above, as we wanted to stay consistent with our main reference related to the estimation of ρ — Ref. [59]

Next, we define a 2-by-2 matrix $\mathbf{A}^\rho$ as

$$\mathbf{A}^\rho = (\mathbf{e}^\rho)' \cdot \mathbf{e}^\rho \tag{4.43}$$

If we assume a normal prior for ψ with mean μ_0^ψ and precision τ_0^ψ, the posterior distribution for ψ is going to also be normal with mean μ^ψ given by

$$\mu^\psi = \frac{\mathbf{A}_{12}^\rho + \mu_0^\psi \tau_0^\psi}{\mathbf{A}_{11}^\rho + \tau_0^\psi} \tag{4.44}$$

and precision τ^ψ equal to

$$\tau^\psi = \mathbf{A}_{11}^\rho + \tau_0^\psi, \tag{4.45}$$

where $\mathbf{A}_{11}^\rho$, $\mathbf{A}_{12}^\rho$ and $\mathbf{A}_{22}^\rho$ are the elements of the matrix $\mathbf{A}^\rho$ on positions $(1,1)$, $(1,2)$ and $(2,2)$ respectively.

Also, if we assume an inverse-gamma prior for ω, with parameters a_0^ω and b_0^ω, the posterior distribution for ω is also going to be inverse-gamma with parameters a^ω and b^ω, calculated as

$$a^\omega = a_0^\omega + \frac{n}{2} \tag{4.46}$$

and

$$b^\omega = b_0^\omega + \frac{1}{2}\left(\mathbf{A}_{22}^\rho - \frac{(\mathbf{A}_{12}^\rho)^2}{\mathbf{A}_{11}^\rho}\right). \tag{4.47}$$

Corollary 4. In the i-th iteration of the MCMC estimation procedure, the estimate of parameter ρ is obtained via

$$\rho_i = \frac{\psi_i}{\sqrt{\psi_i^2 + \omega_i}} \tag{4.48}$$

where ω_i is a sample from the inverse-gamma posterior distribution, generated as:

$$\omega_i \sim \mathcal{IG}\left(a^\omega, b^\omega\right), \tag{4.49}$$

whereas ψ_i is a sample from the normal posterior distribution, generated as

$$\psi_i \sim \mathcal{N}\left(\mu^\psi, \frac{\sqrt{\omega_i}}{\sqrt{\tau^\psi}}\right). \tag{4.50}$$

Estimation of $v(t)$ – particle filtering

For all estimation procedures shown in the previous sections, we assumed $v(t)$ to be known. However, in practice, the volatility is not a directly observable quantity, it is "hidden" in the process of prices, which we have access to. Hence — we need a way to extract volatility from the price process and particle filtering methodology is extremely useful for that purpose. We will therefore outline the particle filtering logic, namely the SIR algorithm, which we will utilise to get the volatility estimator designed for Heston model specifically. More in-depth review of particle filtering can be found in Refs. [56] and [57]. Here we follow a procedure similar to the one presented in Ref [60].

We start by fixing the number of particles G. In each moment of time $t = k\Delta t$, we will produce G particles, which are going to represent various possible values of the volatility at that point of time. By averaging out all of those particles we will get an estimate of the true volatility $v(t)$. The process of creating the particles is as follows: at the time $t = 0$, we create G initial particles, all with the initial value of the volatility, which we assume to be the long term average θ. Denoting each of the particles by V_j, for $j \in \{1, 2, \dots G\}$, we have

$$V_j(0) = \theta_i. \tag{4.51}$$

For any subsequent moment of time except the last one, i.e $t = k\Delta t, k \notin \{0, n\}$, we define three sequences of size G. ε_j will be a series of independent, standard normal random variables

$$\varepsilon_j(k\Delta t) \sim \mathcal{N}(0, 1). \tag{4.52}$$

The series z_j contains residuals from the stock price process, where the past values of volatility are replaced by the values of the particles from the previous time step

$$z_j(k\Delta t) = \frac{R(k\Delta t) - \mu_i \Delta t - 1}{\sqrt{\Delta t}\sqrt{V_j\Big((k-1)\Delta t\Big)}}. \tag{4.53}$$

And finally the series w_j, which incorporates the possible dependency between the stock process and the volatility particles

$$w_j(k\Delta t) = z_j(k\Delta t)\rho_i + \varepsilon_j(k\Delta t)\sqrt{1 - (\rho_i)^2}. \tag{4.54}$$

Having all that, the candidates for the new particles $\widetilde{V}_j$ are created as follows

$$\widetilde{V}_j(k\Delta t) = V_j\Big((k-1)\Delta t\Big) + \kappa_i\left(\theta_i - V_j\Big((k-1)\Delta t\Big)\right)\Delta t + \sigma_i\sqrt{\Delta t}\sqrt{V_j\Big((k-1)\Delta t\Big)}w_j. \quad (4.55)$$

Those candidates are also often called *raw particles*, to differentiate them from *refined particles*, which will be used for the actual volatility estimation. In order for the raw particle to become a refined one, it must meet certain conditions. First of all, since it is meant to represent the volatility, which is a positive quantity, each candidate for a particle must be positive too. Once it is ascertained that all the generated raw particles are positive, they are assigned a value of a measure based on how probable it is that such value of the volatility would generate the return that has actually been observed. This measure — $\widetilde{W}_j$ — is defined to be a value of a normal distribution PDF function designed specifically for this purpose[3]

$$\widetilde{W}_j(k\Delta t) = \frac{1}{\sqrt{2\pi\widetilde{V}_j(k\Delta t)\Delta t}}\exp\left(-\frac{1}{2}\frac{\Big(R\Big((k+1)\Delta t\Big) - \mu_i\Delta t - 1\Big)^2}{\widetilde{V}_j(k\Delta t)\Delta t}\right). \quad (4.56)$$

To be able to treat the values of the proposed measure along with the values of particles as a proper probability distribution on its own, we normalise them, so that their sum is equal to 1,

$$\breve{W}_j(k\Delta t) = \widetilde{W}_j(k\Delta t)\left(\sum_{j=1}^{G}\widetilde{W}_j(k\Delta t)\right)^{-1}. \quad (4.57)$$

Now, we combine the particles and their respective probabilities, forming two-element vectors $\mathbf{U}_j$

$$\mathbf{U}_j(k\Delta t) = \Big(\widetilde{V}_j(k\Delta t), \breve{W}_j(k\Delta t)\Big). \quad (4.58)$$

In order to get the refined particles we should now sample from the probability distribution described by the event-probability pairs outlined in Eq. (4.58). One of

[3]Equation (4.56) is the reason we cannot run this procedure for $k = n$, as we would not be able to obtain $R\big((n+1)\Delta t\big)$, since the last available value is $R(n\Delta t)$.

the easiest method to do it is through inverse transform sampling. To use this method — the values of particles need to be sorted in an ascending order. Mathematically speaking, we create another sequence and call it $\widetilde{V}_j^{sort}$, ensuring that the following conditions are all met:

1. the particle with the smallest value will be the first in the new sequence, i.e.

$$\widetilde{V}_1^{sort}(k\Delta t) = \min_{j\in\{1,2,\ldots,G\}} \{\widetilde{V}_j(k\Delta t)\}, \tag{4.59}$$

2. the particle with the biggest value will be the last in the new sequence, i.e.

$$\widetilde{V}_G^{sort}(k\Delta t) = \max_{j\in\{1,2,\ldots,G\}} \{\widetilde{V}_j(k\Delta t)\}, \tag{4.60}$$

3. for any $j \in \{2, 3, \ldots G-1\}$ we will have

$$\widetilde{V}_{j-1}^{sort}(k\Delta t) < \widetilde{V}_j^{sort}(k\Delta t) < \widetilde{V}_{j+1}^{sort}(k\Delta t). \tag{4.61}$$

We also want to keep track of the probabilities of our sorted particles, so we order the probabilities in the very same way (apply the same permutation as we applied to $\widetilde{V}_j^{sort}$). We can mathematically define this new order as another probability sequence $\breve{W}_j^{sort}$,

$$\breve{W}_j^{sort}(k\Delta t) = \breve{W}_m(k\Delta t) \text{ for } m \text{ such that } \widetilde{V}_j^{sort}(k\Delta t) \in \mathbf{U}_m(k\Delta t) \tag{4.62}$$

Sequences $\widetilde{V}_j(k\Delta t)$ and $\breve{W}_j(k\Delta t)$ can now be used to define the CDF function of the distribution described by Eq. (4.58):

$$F_{\mathbf{U}}(v) = \begin{cases} 0 & \text{if } v \leqslant \widetilde{V}_1^{sort}, \\ \sum_{m=1}^{j} \breve{W}_m^{sort}(k\Delta t) & \text{if } v \in (\widetilde{V}_j^{sort}, \widetilde{V}_{j+1}^{sort}] \\ & \text{for } j \in \{1, 2, 3, \ldots, G-1\}, \\ 1 & \text{if } v > \widetilde{V}_G^{sort}. \end{cases}. \tag{4.63}$$

This function indeed can be used for the inverse transform sampling. However, since the distribution is discrete, all the refined particles will have the same values as the raw ones — just the proportions will be changed (the exact same raw particle can be drawn several times, if it has probability bigger than others). To address this problem, we propose a different distribution to sample from. Similarly to the

original one the marginal particles (i.e. extreme in terms of value), $\widetilde{V}_1^{sort}$ and $\widetilde{V}_G^{sort}$, will be the edges of the support of our new distribution. The CDF function of this distribution is given by the formula below (time labels have been dropped for the sake of legibility, as all variables are evaluated at $t = k\Delta t$):

$$F^c_{\mathbf{U}}(v) = \begin{cases} 0 & \text{if } v \leqslant \widetilde{V}_1^{sort} \\ \frac{v-\widetilde{V}_1^{sort}}{\widetilde{V}_2^{sort}-\widetilde{V}_1^{sort}} \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right) & \text{if } v \in (\widetilde{V}_1^{sort}, \widetilde{V}_2^{sort}] \\ \left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right) + & \\ \frac{v-\widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort}-\widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) & \text{if } v \in (\widetilde{V}_j^{sort}, \widetilde{V}_{j+1}^{sort}] \\ & \text{for } j \in \{2, 3, \dots G-2\} \\ \left(\sum_{m=1}^{G-2} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right) + & \\ \frac{v-\widetilde{V}_{G-1}^{sort}}{\widetilde{V}_G^{sort}-\widetilde{V}_{G-1}^{sort}} \left(\frac{1}{2}\breve{W}_{G-1}^{sort} + \breve{W}_G^{sort} \right) & \text{if } v \in (\widetilde{V}_{G-1}^{sort}, \widetilde{V}_G^{sort}] \\ 1 & \text{if } v > \widetilde{V}_G^{sort} \end{cases}. \tag{4.64}$$

The advantage of the CDF function outlined in the Eq. (4.64), compared to the one in Eq. (4.63) is that the former one is continuous.

Proposition 3. The CDF function, presented in Eq. (4.64) is continuous.

Proof. The function is piece-wise linear and hence — continuous in the inside of every interval formed by the neighbouring sorted particles, i.e. for every $v \in (\widetilde{V}_j^{sort}, \widetilde{V}_{j+1}^{sort})$ for $j \in \{1, 2, 3, \dots, G-1\}$, as well as for $v < \widetilde{V}_1^{sort}$ (where it's 0) and $v > \widetilde{V}_G^{sort}$ (where it's 1). Thus, the only places when continuity needs to be verified are the exact values of particles, where the one line segment ends and another begins, i.e. $v \in \{\widetilde{V}_G^{sort} : j \in \{1, 2, 3, \dots, G\}\}$. Therefore

- for $v = \widetilde{V}_1^{sort}$

  $F^c_{\mathbf{U}}(v-) = 0,$

  $F^c_{\mathbf{U}}(v+) = \frac{\widetilde{V}_1^{sort}-\widetilde{V}_1^{sort}}{\widetilde{V}_2^{sort}-\widetilde{V}_1^{sort}} \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right) = \frac{0}{\widetilde{V}_2^{sort}-\widetilde{V}_1^{sort}} \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right) = 0,$

  hence $F^c_{\mathbf{U}}(v-) = F^c_{\mathbf{U}}(v+) = F^c_{\mathbf{U}}(v)$;

- for $v = \widetilde{V}_2^{sort}$

$$F_{\mathbf{U}}^c(v-) = \frac{\widetilde{V}_2^{sort} - \widetilde{V}_1^{sort}}{\widetilde{V}_2^{sort} - \widetilde{V}_1^{sort}} \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right) = \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right),$$

$$F_{\mathbf{U}}^c(v+) = \left(\sum_{m=1}^{1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right) + \frac{\widetilde{V}_2^{sort} - \widetilde{V}_2^{sort}}{\widetilde{V}_3^{sort} - \widetilde{V}_2^{sort}} \left(\frac{1}{2}\breve{W}_2^{sort} + \frac{1}{2}\breve{W}_3^{sort} \right) = \left(\breve{W}_1^{sort} + \frac{1}{2}\breve{W}_2^{sort} \right),$$

  hence $F_{\mathbf{U}}^c(v-) = F_{\mathbf{U}}^c(v+) = F_{\mathbf{U}}^c(v)$;

- for $v = \widetilde{V}_j^{sort}, j \in \{2, 3, \ldots, G-2\}$

$$F_{\mathbf{U}}^c(v-) = \left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right) + \frac{\widetilde{V}_j^{sort} - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) = \left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right),$$

$$F_{\mathbf{U}}^c(v+) = \left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right) + \frac{\widetilde{V}_j^{sort} - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) = \left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right),$$

  hence $F_{\mathbf{U}}^c(v-) = F_{\mathbf{U}}^c(v+) = F_{\mathbf{U}}^c(v)$;

- for $v = \widetilde{V}_{G-1}^{sort}$

$$F_{\mathbf{U}}^c(v-) = \left(\sum_{m=1}^{G-1-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right) + \frac{\widetilde{V}_{G-1}^{sort} - \widetilde{V}_{G-1}^{sort}}{\widetilde{V}_{G-1+1}^{sort} - \widetilde{V}_{G-1}^{sort}} \left(\frac{1}{2}\breve{W}_{G-1}^{sort} + \frac{1}{2}\breve{W}_{G-1+1}^{sort} \right) = \left(\sum_{m=1}^{G-2} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right) + \frac{0}{\widetilde{V}_G^{sort} - \widetilde{V}_{G-1}^{sort}} \left(\frac{1}{2}\breve{W}_{G-1}^{sort} + \frac{1}{2}\breve{W}_G^{sort} \right) = \left(\sum_{m=1}^{G-2} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right)$$

$$F_{\mathbf{U}}^c(v+) = \left(\sum_{m=1}^{G-2} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right) + \frac{\widetilde{V}_{G-1}^{sort} - \widetilde{V}_{G-1}^{sort}}{\widetilde{V}_G^{sort} - \widetilde{V}_{G-1}^{sort}} \left(\frac{1}{2}\breve{W}_{G-1}^{sort} + \breve{W}_G^{sort} \right) = \left(\sum_{m=1}^{G-2} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_{G-1}^{sort} \right)$$

hence $F^c_{\mathbf{U}}(v-) = F^c_{\mathbf{U}}(v+) = F^c_{\mathbf{U}}(v)$;

- for $v = \widetilde{V}^{sort}_G$

$$F^c_{\mathbf{U}}(v-) = \left(\sum_{m=1}^{G-2} \breve{W}^{sort}_m + \frac{1}{2}\breve{W}^{sort}_{G-1}\right) + \frac{\widetilde{V}^{sort}_G - \widetilde{V}^{sort}_{G-1}}{\widetilde{V}^{sort}_G - \widetilde{V}^{sort}_{G-1}} \left(\frac{1}{2}\breve{W}^{sort}_{G-1} + \breve{W}^{sort}_G\right) =$$
$$\left(\sum_{m=1}^{G-2} \breve{W}^{sort}_m + \frac{1}{2}\breve{W}^{sort}_{G-1}\right) + \left(\frac{1}{2}\breve{W}^{sort}_{G-1} + \breve{W}^{sort}_G\right) =$$
$$\left(\sum_{m=1}^{G-2} \breve{W}^{sort}_m + \breve{W}^{sort}_{G-1} + \breve{W}^{sort}_G\right) = \sum_{m=1}^{G} \breve{W}^{sort}_m = 1,$$
$$F^c_{\mathbf{U}}(v-) = 1,$$

hence $F^c_{\mathbf{U}}(v-) = F^c_{\mathbf{U}}(v+) = F^c_{\mathbf{U}}(v)$;

As demonstrated above, the left-side and right-side limits have the same values in all the points in which the function could potentially not be continuous. Thus, we conclude it is continuous. □

A visual comparison between the CDF functions of the discrete and continuous distributions can be seen in Fig. 4.1. On the plot, one can observe how the ends of the linear parts get connected to each other.

Continuous CDF function and hence — continuous distribution of the refined particles — is surely an improvement, as the possibility that any two of the refined particles, sampled from this distribution, will have the same value is equal to 0. However, it is critical that the improved distribution is "close" to the original one. The following theorem demonstrates that the impact of the change we introduced diminishes with the increasing number of particles.

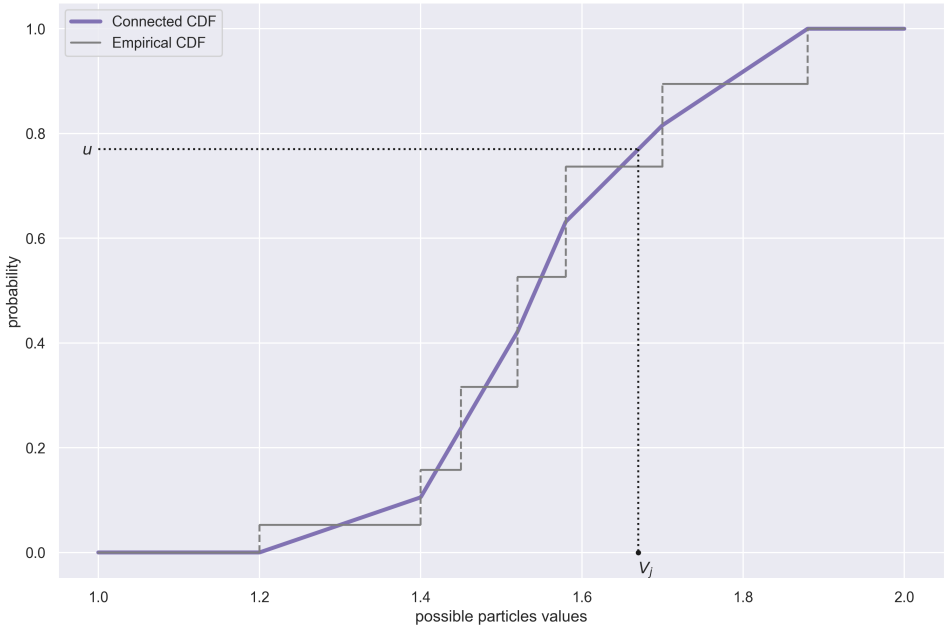


Figure 4.1: Visualisation of the process of resampling particles according to their probabilities. Values of raw particles are the places, where the empirical cumulative ditribution function (ECDF in short) jumps and each jump size represents probability of a respective raw particle. The Connected CDF is a continuous modification of the ECDF, build according to the formula (4.64). In order to resample, a uniform random variable u is generated and then its inverse through the Connected CDF function becomes a new, resampled particle — V_j.

Theorem 3. The difference between the distribution characterised by Eq. (4.63) and Eq. (4.64) approaches zero as the number of particles increases to infinity, i.e.

$$\lim_{G\to\infty} F^c_{\mathbf{U}}(v) - F_{\mathbf{U}}(v) = 0, \tag{4.65}$$

for any value of $v > 0$.

Proof. Looking at the way of sourcing raw particles (Eq. (4.55)) we see that, conditionally on the value of $V_j\Big((k-1)\Delta t\Big), \kappa_i, \theta_i, \sigma_i$ and $R(k\Delta t)$, the distribution of $\widetilde{V}_j$ is normal and considering a raw particle will only be accepted if it is positive —-

we conclude that the support of the distribution of $\widetilde{V}_j^{sort}$ is the entire positive ray of the real line. Hence — for sufficiently large G we can assume, with no loss of generality, that for any choice of $v > 0$, it will fall into an interval $(\widetilde{V}_j^{sort}, \widetilde{V}_{j+1}^{sort})$ for $j \in \{2, 3, \ldots, G-1\}$ (i.e. not into the first one $(\widetilde{V}_1^{sort}, \widetilde{V}_2^{sort})$ and not into the last one $(\widetilde{V}_{G-1}^{sort}, \widetilde{V}_G^{sort})$). Knowing that, we have

$$\begin{aligned} F_{\mathbf{U}}^{c}(v) - F_{\mathbf{U}}(v) = &\left(\sum_{m=1}^{j-1} \breve{W}_m^{sort} + \frac{1}{2}\breve{W}_j^{sort} \right) + \\ &\frac{v - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \sum_{m=1}^{j} \breve{W}_m^{sort} = \\ &\frac{v - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \frac{1}{2}\breve{W}_j^{sort}. \end{aligned}$$

Now, let us look at the factor $\frac{v - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}}$. It is linear in v and for $v \in (\widetilde{V}_j^{sort}, \widetilde{V}_{j+1}^{sort})$, its value is bounded by 0 (when $v = \widetilde{V}_j^{sort}$) and 1 (when $v = \widetilde{V}_{j+1}^{sort}$). Thus, $F_{\mathbf{U}}^{c}(v) - F_{\mathbf{U}}(v)$, is bounded between

$$\begin{aligned} F_{\mathbf{U}}^{c}(\widetilde{V}_j^{sort}) - F_{\mathbf{U}}(\widetilde{V}_j^{sort}) = &\frac{\widetilde{V}_j^{sort} - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \frac{1}{2}\breve{W}_j^{sort} = \\ &\frac{0}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \frac{1}{2}\breve{W}_j^{sort} = \\ &-\frac{1}{2}\breve{W}_j^{sort} \end{aligned}$$

and

$$\begin{aligned} F_{\mathbf{U}}^{c}(\widetilde{V}_{j+1}^{sort}) - F_{\mathbf{U}}(\widetilde{V}_{j+1}^{sort}) = &\frac{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}}{\widetilde{V}_{j+1}^{sort} - \widetilde{V}_j^{sort}} \left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \frac{1}{2}\breve{W}_j^{sort} = \\ &\left(\frac{1}{2}\breve{W}_j^{sort} + \frac{1}{2}\breve{W}_{j+1}^{sort} \right) - \frac{1}{2}\breve{W}_j^{sort} = \\ &\frac{1}{2}\breve{W}_{j+1}^{sort}. \end{aligned}$$

According to Eq. (4.57), $\breve{W}_j$ is a ratio of a value of the function given by Eq. (4.56) evaluated at a particular $\widetilde{V}_j$ and the sum of the values of this function evaluated at all the $\widetilde{V}_j$-s. Therefore, it becomes clear that that as long as the function given

in Eq. (4.56) is bounded, increasing number of particles lead to diminishing the value of the ratio denoted by $\breve{W}_j$. We thus conclude that for any $j \in \{1, 2, \ldots, G\}$, $\lim_{G\to\infty} \breve{W}_j = 0$ and $\lim_{G\to\infty} \breve{W}_j^{sort} = 0$ (as this sequence has the same values, just ordered differently).

Therefore, since we have

$$F_{\mathbf{U}}^c(\widetilde{V}_j^{sort}) - F_{\mathbf{U}}(\widetilde{V}_j^{sort}) \leqslant F_{\mathbf{U}}^c(v) - F_{\mathbf{U}}(v) \leqslant F_{\mathbf{U}}^c(\widetilde{V}_{j+1}^{sort}) - F_{\mathbf{U}}(\widetilde{V}_{j+1}^{sort})$$

and

$$\lim_{G\to\infty} F_{\mathbf{U}}^c(\widetilde{V}_j^{sort}) - F_{\mathbf{U}}(\widetilde{V}_j^{sort}) = 0,$$
$$\lim_{G\to\infty} F_{\mathbf{U}}^c(\widetilde{V}_{j+1}^{sort}) - F_{\mathbf{U}}(\widetilde{V}_{j+1}^{sort}) = 0,$$

by the squeeze theorem, it must hold that

$$\lim_{G\to\infty} F_{\mathbf{U}}^c(v) - F_{\mathbf{U}}(v) = 0$$

□

The refined particles can be generated by drawing from the distribution given by $F_{\mathbf{U}}^c$. As mentioned earlier, this can be achieved easily using the inverse transform method.

$$V_j(k\Delta t) \sim F_{\mathbf{U}}^c. \tag{4.66}$$

After following the described procedure for each $k \in \{1, 2, \ldots, n-1\}$, we can specify the actual estimate of the volatility process as the mean of "refined" particles.

$$v(k\Delta t) = \frac{1}{G}\sum_{j=1}^{G} V_j(k\Delta t). \tag{4.67}$$

For $k = n$, we can simply assume $v(n\Delta t) = v\Big((n-1)\Delta t\Big)$, which should not have any tangible negative impact on any procedures using the $v(t)$ estimate for sufficiently dense time discretisation grid.

4.2.2 Heston model with jumps

The above estimation framework can be used with minor changes to also estimate Heston model with jumps. The model's SDE is defined in Eq. (3.6). Below we present the changes to the procedure allowing for the estimation of jumps. This novel approach to jumps was presented for the first time in Ref. [35].

After incorporation of jumps, changes are needed particularly in the particle filtering part of the estimation procedure. The particles need to be created not only for various possible values of volatility $V_j(t)$, but also for a possibility of a jump in that particular moment of time — $J_j(t)$ — and size of that jump — $Z_j(t)$. So one can now think of a particle as of a triple (V_j, J_j, Z_j). Generating raw values for J_j and Z_j is easy — for each $j \in \{0, 1, \ldots, G\}$, J_j is simply a random variable from a Bernoulli distribution with parameter λ^{th},

$$\widetilde{J}_j(k\Delta t) \sim \mathcal{B}(\lambda^{th}). \tag{4.68}$$

Parameter $\lambda^{th} \in [0, 1)$ can be thought of as a "threshold" value i.e. a proportion of the number of particles which encode the occurrence of a jump to all the particles. If the number of jumps is expected to be significant, it is good to increase the value of λ^{th}, hence increasing the number of particles suggesting the jump in each step.

Raw particles for Z_j are simply independent normal random variables with mean μ_0^J and standard deviation σ_0^J, which depict our prior beliefs about the size and variance of the jumps

$$\widetilde{Z}_j(k\Delta t) \sim \mathcal{N}(\mu_0^J, \sigma_0^J). \tag{4.69}$$

Assigning probabilities to the particles is different as well, since the normal PDF function which we use is different when there is a jump. Hence, equation (4.56) needs to be updated to

$$\widetilde{W}_j(k\Delta t) = \begin{cases} \dfrac{1}{\sqrt{2\pi\widetilde{V}_j\Big((k-1)\Delta t\Big)\Delta t}}\times \\ \exp\left(-\dfrac{1}{2}\dfrac{\Big(R(k\Delta t)-\mu_i\Delta t-1\Big)^2}{\widetilde{V}_j\Big((k-1)\Delta t\Big)\Delta t}\right) & \text{if } \widetilde{J}_j = 0 \\ \dfrac{1}{\exp\Big(\widetilde{Z}_j(k\Delta t)\Big)\sqrt{2\pi\widetilde{V}_j\Big((k-1)\Delta t\Big)\Delta t}}\times \\ \exp\left(-\dfrac{1}{2}\dfrac{\Big(R(k\Delta t)-\exp\Big(\widetilde{Z}_j(k\Delta t)\Big)(\mu_i\Delta t+1)\Big)^2}{\exp\Big(2\widetilde{Z}_j(k\Delta t)\Big)\widetilde{V}_j\Big((k-1)\Delta t\Big)\Delta t}\right) & \text{if } \widetilde{J}_j = 1 \end{cases} \tag{4.70}$$

We then normalise $\widetilde{W}_j$ so that it sums to 1, thus obtaining $\breve{W}_j$, and resample $\widetilde{V}_j$ according to this normalised probability, exactly as in the case with no jumps. However, this measure needs to also be used to resample $\widetilde{Z}_j$, as now raw particles do not only consist of a single value , $\widetilde{V}_j$, but the 3-element vector $(\widetilde{V}_j, \widetilde{Z}_j, \widetilde{J}_j)$. Resampling is exactly analogous — we need to sort the values of $\widetilde{Z}_j$ and reorder $\breve{W}_j$ in the exact same way, put the elements of those two reordered series into two-element vectors $\mathbf{U}'_j$, build the connected CDF function $F^c_{\mathbf{U}'}$ and sample refined particles from it

$$Z_j(k\Delta t) \sim F^c_{\mathbf{U}'}. \tag{4.71}$$

Finally, for the estimate of λ, for each $k \in \{1, 2, \dots n\}$ one needs to sum the cumulative value of all particles declaring a jump. That way we will get a probability that a jump took place at the time $t = k\Delta t$,

$$\lambda(k\Delta t) = \sum_{j=1}^{G} J_j(k\Delta t)\breve{W}_j(k\Delta t). \tag{4.72}$$

To get the actual estimate of λ one needs to average $\lambda(t)$ across all time points obtained for different values of k,

$$\lambda_i = \frac{1}{T}\sum_{k=1}^{n}\lambda(k\Delta t). \tag{4.73}$$

Similarly, to obtain the estimate of μ^J and σ^J, for each k one needs to first calculate the average size of a jump from the refined particles

$$Z(k\Delta t) = \frac{1}{G}\sum_{j=1}^{G} Z_j(k\Delta t) \tag{4.74}$$

and then calculate the mean and standard deviation of the results, weighed by the probability of a jump at time moment t indicated by $\lambda(t)$. For the weighted mean of the jumps we get

$$\mu_i^J = \left(\sum_{k=1}^{n} Z(k\Delta t)\lambda(k\Delta t)\right)\left(\sum_{k=1}^{n}\lambda(k\Delta t)\right)^{-1} \tag{4.75}$$

and for the standard deviation:

$$\sigma_i^J = \sqrt{\left(\sum_{k=1}^{n}\lambda(k\Delta t)\Big(Z(k\Delta t) - \mu_i^J\Big)^2\right)\left(\frac{n-1}{n}\sum_{k=1}^{n}\lambda(k\Delta t)\right)^{-1}}. \tag{4.76}$$

The presence of jumps also influences the estimation of other parameters — some of the procedures presented in the previous subsection are not fully correct, as jumps added to the stock price will additionally increase or — more likely — decrease the returns. To improve that, a correction of the definitions of $R(t)$ is needed in order to neutralise the impact of jumps on the parameters. In other words, Eq. (4.2) should be replaced with

$$r(k\Delta t) = \frac{S^J(k\Delta t)}{S^J\Big((k-1)\Delta t\Big)}\left(1 - \lambda(k\Delta t)\Big(1 - \exp\big(-Z(k\Delta t)\big)\Big)\right) \tag{4.77}$$

Proposition 4. Assume that jumps appearing in the price trajectory are Merton-style (as described in sections 3.4 and 3.5), there is ν of them. Moreover, assume the estimate $\lambda(k\Delta t)$ provides fully accurate information about an occurrence of a jump at time $t = k\Delta t$ and the estimate $Z(k\Delta t)$ — about the size of a jump at that point of time (if there was one). Then returns defined by equation (4.77) are equivalent to the ones defined by (4.2), i.e

$$r(k\Delta t) = R(k\Delta t)$$

for $k \in \{1, 2, \ldots, n\}$.

Proof. From Eq. (3.19) we know how jumps appear in the price trajectory

$$S^J(k\Delta t) = S(k\Delta t)\cdot \prod_{\{i:k_i\leqslant k\}} \exp\Big(Z(k_i\Delta t)\Big).$$

Therefore, the ratio between two neighbouring values of S^J is

$$\frac{S^J(k\Delta t)}{S^J\Big((k-1)\Delta t)\Big)} = \begin{cases} \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} \cdot \exp\Big(Z(k\Delta t)\Big) & \text{if there is } i \in \{1,2,\ldots,\nu\} \\ & \text{such that } k = k_i, \\ \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} & \text{otherwise.} \end{cases} \tag{4.78}$$

The condition in the upper variant of the function in Eq. (4.78) simply means that here was a jump at $t = k\Delta t$. If there is a jump, the accurate $\lambda(k\Delta t)$ should be equal to 1. In such case, plugging Eq. (4.78) into Eq. (4.77), we get

$$\begin{aligned} r(k\Delta t) =& \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} \cdot \exp\Big(Z(k\Delta t)\Big) \cdot \Big(1 - 1 \cdot \Big(1 - \exp\big(-Z(k\Delta t)\big)\Big)\Big) = \\ & \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} \cdot \exp\Big(Z(k\Delta t)\Big) \cdot \Big(1 - 1 + \exp\big(-Z(k\Delta t)\big)\Big) = \\ & \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} \cdot \exp\Big(Z(k\Delta t)\Big) \cdot \exp\Big(-Z(k\Delta t)\Big) = \\ & \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} = R(k\Delta t) \end{aligned}$$

If, however, there was no jump at $t = k\Delta t$, we have

$$\begin{aligned} r(k\Delta t) =& \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} \Big(1 - 0 \cdot \Big(1 - \exp\big(-Z(k\Delta t)\big)\Big)\Big) = \\ & \frac{S(k\Delta t)}{S\Big((k-1)\Delta t)\Big)} = R(k\Delta t) \end{aligned}$$

Therefore, we have proved that in both cases (for k where there was a jump and where there was not) it holds that

$$r(k\Delta t) = R(k\Delta t).$$

□

For trajectories for which we expect presence of jumps, using $r(t)$ instead of $R(t)$ makes the jumps disappear (see Fig. 4.2) and the estimation of the parameters of the model can be carried out as shown before.

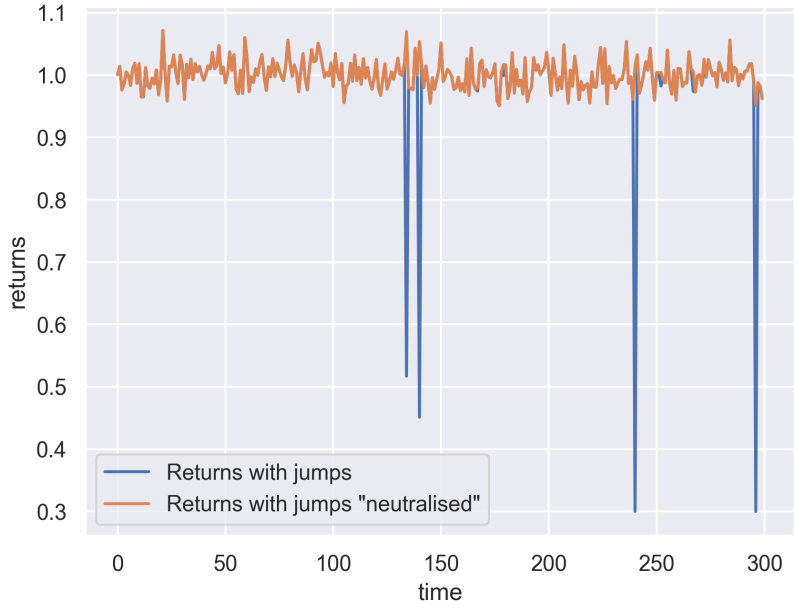


Figure 4.2: Comparison of returns for a process with jumps calculated based on formula (4.2) (blue line) and (4.77) (orange line). It can be clearly seen that the jumps have been "neutralised" in the latter case.

4.2.3 Estimation procedure

The Bayesian estimation framework presented above relies of several parameters for the prior distributions that cannot be calculated within the procedure itself. They are often referred to as metaparameters. For example, for the estimation of the μ parameter, values of two metaparameters are required — μ_0^η and τ_0^η (see Eqs. (4.9) and (4.10)). They should reflect our preexisting beliefs regarding the parameter, which we are trying to estimate — μ in this case. Let us say that for a given trajectory of the Heston model process, we assume the value of μ to be around 0.5. What values should we than choose for the metaparameters? First of all, we need to note that μ_0^η and τ_0^η are not the parameters of the prior distribution for μ directly. They are

parameters of another random variable, which we introduced to utilise the Bayesian framework — namely η. The connection between μ and η is known and given by the equation (4.1). Hence, if we assume μ to be around certain value, then, using this relationship, we can deduce the value of η. And since the prior distribution of η is normal, with mean μ_0^η and variance $\sigma_0^\eta = \frac{1}{\tau_0^\eta}$, we can propose the value of the mean of this distribution to be whatever η is for the supposed value of μ. Selecting a value for σ_0^η (and thus — τ_0^η) is even more equivocal — it should reflect the level of confidence that we have for picking a mean parameter. That is, if we feel that value we chose for μ_0^η would bring us close to the true μ, we should put a smaller value for σ_0^η. However, if we are not so sure about it, bigger value of σ_0^η should be used. Similar analysis can be repeated for picking the values of other metaparameters. We need to be aware that prior which is used will always in some way influence the final estimate of a given parameter. More detailed analysis on this topic is given in section 4.3.2.

Another problem which emerges when applying Bayesian inference (especially for more complex models) is that estimating one parameter often requires knowing a value of some of the others and vice-versa. Hence — there is not an obvious way of how to start the whole procedure. One way to address this problem is to come up with the initial guesses for the values of all the parameters (as described in the paragraph above) and use them in the first round of samplings. A well designed MCMC estimation algorithm should bring us closer to the true values of parameters with each new round of samplings. In Algorithms 1 and 2 our procedure for the Heston model without and with jumps is shown, respectively.

Algorithm 1 Estimation of the Heston model

Require:
number of samples: n
time step Δt
maturity T
prices $S(k\Delta t)$ for $k \in \{0, 1, \ldots, n\}$, so that $n\Delta t = T$
number of particles G
initial value of μ: μ_0
initial value of κ: κ_0
initial value of θ: θ_0
initial value of σ: σ_0
initial value of ρ: ρ_0
prior distribution parameters for η: μ_0^η and τ_0^η
prior distribution parameters for $\boldsymbol{\beta}$: $\boldsymbol{\mu}_0^\beta$ and $\boldsymbol{\Lambda}_0^\beta$
prior distribution parameters for σ^2: a_0^σ and b_0^σ
prior distribution parameters for ψ: μ_0^ψ and τ_0^ψ
prior distribution parameters for ω: a_0^ω and b_0^ω

Ensure:
estimate of parameter μ: $\hat{\mu}$
estimate of parameter κ: $\hat{\kappa}$
estimate of parameter θ: $\hat{\theta}$
estimate of parameter σ: $\hat{\sigma}$
estimate of parameter ρ: $\hat{\rho}$
estimate of the volatility process: $v(t)$

for $k = 1 \to n$ **do**
  set $R(k\Delta t)$ as shown in eq.(4.2)
end for
for $i = 0 \to n_s$ **do**
  for $k = 1 \to n - 1$ **do** ▷ particle filtering procedure
    for $j = 1 \to G$ **do**
      obtain $V_j(k\Delta t)$ as shown in eq. (4.51) – (4.66)
    end for
    obtain $v(k\Delta t)$ as shown in eq. (4.67)
  end for
  obtain μ_i as shown in eq. (4.3) – (4.12)

obtain κ_i, θ_i and σ_i as shown in eq. (4.15) – (4.33)
obtain ρ_i as shown in eq. (4.35) – (4.50)
end for
set $\hat{\mu} = \frac{1}{n_s} \sum_{i=1}^{n_s} \mu_i$
set $\hat{\kappa} = \frac{1}{n_s} \sum_{i=1}^{n_s} \kappa_i$
set $\hat{\theta} = \frac{1}{n_s} \sum_{i=1}^{n_s} \theta_i$
set $\hat{\sigma} = \frac{1}{n_s} \sum_{i=1}^{n_s} \sigma_i$
set $\hat{\rho} = \frac{1}{n_s} \sum_{i=1}^{n_s} \rho_i$

Algorithm 2 Estimation of the Heston model with jumps

Require:
number of samples: n
time step Δt
maturity T
prices $S(k\Delta t)$ for $k \in \{0, 1, \ldots, n\}$, so that $n\Delta t = T$
number of particles G initial value of μ: μ_0
initial value of κ: κ_0
initial value of θ: θ_0
initial value of σ: σ_0
initial value of ρ: ρ_0
prior distribution parameters for η: μ_0^η and τ_0^η
prior distribution parameters for $\boldsymbol{\beta}$: $\boldsymbol{\mu}_0^\beta$ and $\boldsymbol{\Lambda}_0^\beta$
prior distribution parameters for σ^2: a_0^σ and b_0^σ
prior distribution parameters for ψ: μ_0^ψ and τ_0^ψ
prior distribution parameters for ω: a_0^ω and b_0^ω
ratio of particle indicating jumps: λ^{th}
prior distribution parameters for Z: μ_0^J and σ_0^J
Ensure:
estimate of parameter μ: $\hat{\mu}$
estimate of parameter κ: $\hat{\kappa}$
estimate of parameter θ: $\hat{\theta}$
estimate of parameter σ: $\hat{\sigma}$
estimate of parameter ρ: $\hat{\rho}$
estimate of the volatility process: $v(t)$

for $k = 1 \to n$ **do**
set $R(k\Delta t)$ as shown in eq.(4.2)
end for
for $i = 0 \to n_s$ **do**
for $k = 1 \to n$ **do** ▷ particle filtering procedure
for $j = 1 \to G$ **do**
generate $\widetilde{J}_j(k\Delta t)$ as shown in eq. (4.68)
generate $\widetilde{Z}_j(k\Delta t)$ as shown in eq. (4.69)
obtain $V_j\Big((k-1)\Delta t\Big)$ as shown in eq. (4.52) – (4.66) and (4.70)
end for
obtain $v(k\Delta t)$ as shown in eq. (4.67)

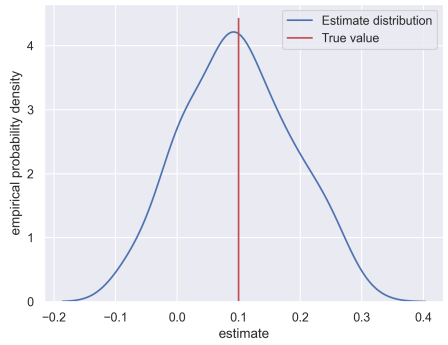
obtain $Z(k\Delta t)$ and $\lambda(k\Delta t)$ as shown in eq. (4.71) – (4.72) and (4.74)
end for
for $k = 1 \rightarrow n$ **do**
update $R(k\Delta t)$ as shown in eq.(4.77)
end for
obtain μ_i as shown in eq. (4.3) – (4.12)
obtain κ_i, θ_i and σ_i as shown in eq. (4.15) – (4.33)
obtain ρ_i as shown in eq. (4.35) – (4.50)
obtain λ_i as shown in eq. (4.73)
obtain μ_i^J as shown in eq. (4.75)
obtain σ_i^J as shown in eq. (4.76)
end for
set $\hat{\mu} = \frac{1}{n_s} \sum_{i=1}^{n_s} \mu_i$
set $\hat{\kappa} = \frac{1}{n_s} \sum_{i=1}^{n_s} \kappa_i$
set $\hat{\theta} = \frac{1}{n_s} \sum_{i=1}^{n_s} \theta_i$
set $\hat{\sigma} = \frac{1}{n_s} \sum_{i=1}^{n_s} \sigma_i$
set $\hat{\rho} = \frac{1}{n_s} \sum_{i=1}^{n_s} \rho_i$
set $\hat{\lambda} = \frac{1}{n_s} \sum_{i=1}^{n_s} \lambda_i$
set $\hat{\mu^J} = \frac{1}{n_s} \sum_{i=1}^{n_s} \mu_i^J$
set $\hat{\sigma^J} = \frac{1}{n_s} \sum_{i=1}^{n_s} \sigma_i^J$

4.3 Analysis of the estimation results

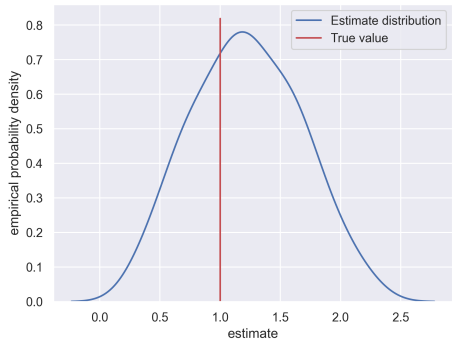
4.3.1 Exemplary estimation

We present here an exemplary estimation of the Heston model with jumps, to show the outcomes of the entire procedure. We assumed relatively non-informative prior distributions, with expected values shifted from the true parameters to make the task more challenging for the algorithm and to better reflect the real-life situation in which the used priors are most of the time not matching true parameters exactly, but should be rather close to them. Table 4.1 lists all the values of priors which we used. Table 4.2 summarises the results obtained and Fig. 4.3 elaborates on those results by showing empirical distributions of samples for all parameters of the model.

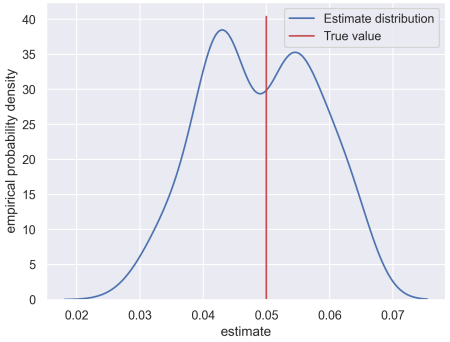
Analysing estimate samples for each of the parameters (presented in Fig. 4.3) one can observe that for most of them (Figs. 4.3a–4.3e and 4.3h) the true value of the parameter is within the support of the distribution of all samples. However, in case of two parameters — λ and μ^J (Figs. 4.3f and 4.3g respectively) — the scope of samples generated by the estimation procedure seems not even to include the parameter's true value. This is due to the fact that those parameters are related to intensity and size of jumps and for the simulation parameters which we picked jumps do not happen frequently (same as in case of real-life price falls). Hence, despite the procedure correctly identifies the moments of jumps and estimates their sizes, those estimates are relatively far from the true values simply because there was very little source material for the estimation in the first place. To be precise — the stock price simulated for our exemplary estimation experienced four jumps, and times of those jumps have been easily identified by our procedure with almost 100% certainty. Thus, since the length of time of the price observation (in years) was $T = 3$, the most probable value of the jump intensity λ was around $\frac{4}{3}$ (compare to the actual result in Table 4.2), although, obviously, other values of λ (slightly smaller or bigger) could have also lead to four jumps and this is exactly what happened in our case, as our true intensity was $\lambda = 1$ (again, see Table 4.2). Similarly, in case of μ^J, the reason for the estimated average jump to be bigger (in terms of magnitude) than the actual one was that the four jumps which were simulated all happened to be more severe than the true value of μ^J would suggest (by pure chance) and this pushed the procedure towards overestimating the (absolute) size of the jump.



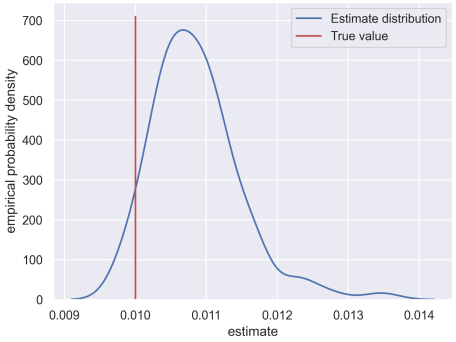
(a) parameter μ



(b) parameter κ

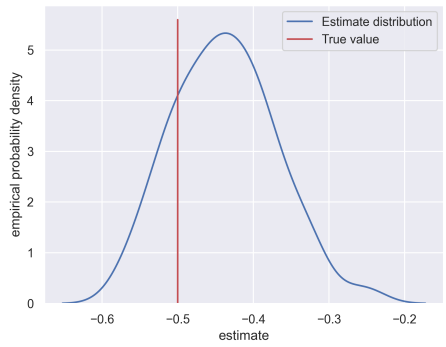


(c) parameter θ

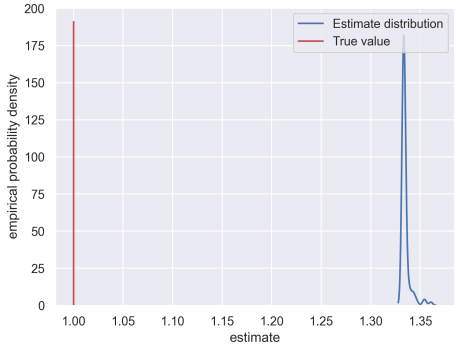


(d) parameter σ

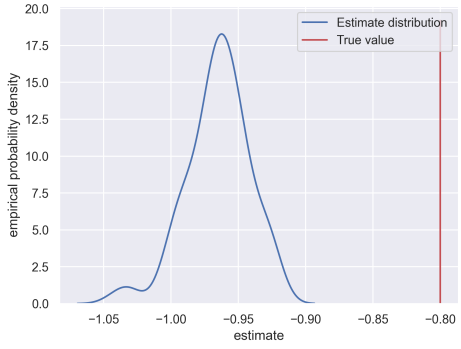
Figure 4.3: Empirical PDFs made of exemplary Heston parameter estimates.



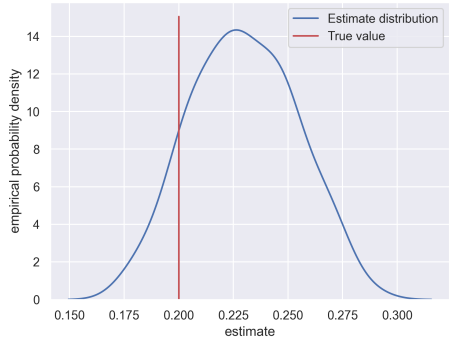
(e) parameter ρ



(f) parameter λ



(g) parameter μ^J



(h) parameter σ^J

Figure 4.3: (continued from the previous page) Empirical PDFs made of exemplary Heston parameter estimates.

Prior parameter	**Value**
μ_0^η	1.00125
σ_0^η	0.001
$\boldsymbol{\Lambda}_0$	$\begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$
$\boldsymbol{\mu}_0$	$\begin{bmatrix} 35 \cdot 10^{-6} \\ 0.988 \end{bmatrix}$
a_0^σ	149
b_0^σ	0.025
μ_0^ψ	-0.45
σ_0^ψ	0.3
a_0^ω	1.03
b_0^ω	0.05
λ^{th}	0.15
μ_0^J	-0.96
σ_0^J	0.3

Table 4.1: Priors for the exemplary estimation procedure

Parameter	**True value**	**Estimated value**	**Relative Error** [%]
μ	0.1	0.09829	1.77
κ	1	1.2190	21.90
θ	0.05	0.0493	1.92
σ	0.01	0.0108	8.55
ρ	-0.5	0.4379	12.40
λ	1	1.3349	33.49
μ^J	-0.8	-0.9651	20.64
σ^J	0.2	0.2298	14.88

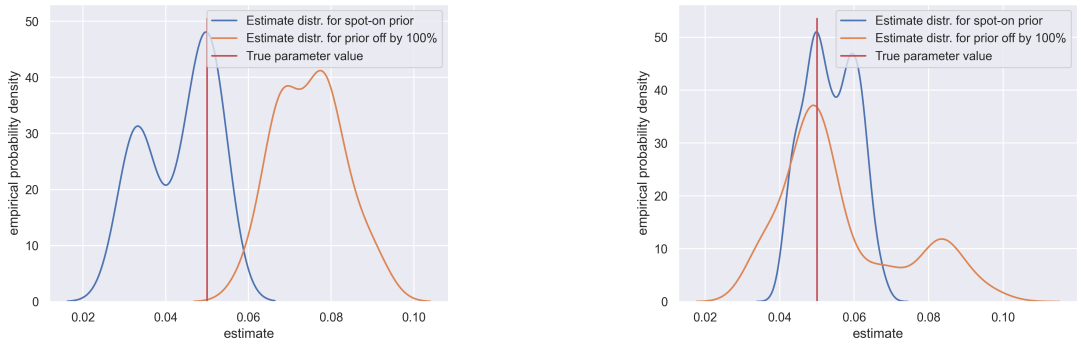
Table 4.2: Results of the exemplary estimation procedure.

4.3.2 Important findings

Although estimation through the joint forces of Bayesian inference, Monte Carlo Markov Chains and particle filtering is generally considered very effective [34], there are several areas the user needs to be aware of while using this estimation scheme. One of the issues worth considering is the impact of the prior parameters. Bayesian estimator of any kind needs to be fed with parameters of the prior distribution which should reflect our preexisting beliefs of what the value of the actual estimated parameter could be. The amount of information conveyed by a prior can be different, depending on several factors. One of them are the values of the parameters of the prior distribution itself. Consider μ_0^η and σ_0^η, mentioned already in the previous section. They are the prior parameters for η — the predecessor for the μ estimates. The bigger σ_0^η we take, the more volatile the estimates of η — and hence μ — are going to be. This is a pretty intuitive fact, being a direct consequence of the Bayesian approach itself. A more subtle influence of priors is hidden in the alternation between the MCMC sampling and particle filtering procedures.

As mentioned in the previous section MCMC and particle filtering depend on one another. As can be seen in the Algorithms 1 and 2, we have taken the approach that the particle filtering procedure should be done first and singular parameters which it needs in the first iteration should be the expected values of the prior distributions which we assume. Having volatility estimated that way, we can estimate the parameters, then based on them re-estimate the volatility process and so on. Although we can keep alternating that way as many times as we want, till the planned end of the estimation procedure, one might be tempted to perform the particle filtering procedure fewer times, as it is much more computationally expensive than the MCMC draws. The premise for that would be that after several trials, the volatility estimate becomes "good enough" and from that point onward, one can only generate more MCMC samples. A critical observation that we have made is that the quality of the initial volatility estimates depends very highly on the prior parameters which were used to initiate it. With little number of particle filtering procedures followed by multiple MCMC draws, the entire scheme does not have enough time to properly calibrate and results tend to stick to the priors which have been used. That means — for a prior leading exactly to the true value of the parameter — the estimator returns almost error-less results, however, if one uses a prior leading to value of the true parameter, e.g. 20% bigger than it really is — the estimate will probably be off by roughly 20%, which does not make the estimator very useful. A counter-proposal can then be made, to perform particle filtering as long as possible. This however, is not an ideal solution either. Firstly, as we said, it is very computationally expensive, and

secondly — a very long chain of samples increases the probability that the estimation procedure would at some point return an outlier, i.e. an estimate really far away from the true value of the parameter, which is especially likely if we use meta-parameters responsible for such parameter's variance (like e.g. σ_0^η for η) bigger. Appearance of such outliers is especially unfavourable in case of the MCMC methods, since its nature is that each sample is directly dependent on the previous one, so the whole procedure is likely to stay in the given region of the parameter space for a some number of subsequent simulations, thus impacting the final estimate of the parameter (which is the mean of all observed samples). Therefore, a clear trade-off appears. If one believes strongly that the prior is rather correct and only needs some minor correction to adjust it to the particular data-set — a modest number of particle filtering can be applied[4], followed by an arbitrary number of MCMC draws. If however we do not know much about our data-set and do not want to convey too much information through the prior — even at the cost of a bit worse final results — they should run particle filtering bigger number of times. The visual interpretation of this rule has been presented in Figs.4.4 and 4.5.



(a) (b)

Figure 4.4: Empirical distributions of the estimate samples for parameter θ in case when the mean of the prior distribution matches exactly the true parameter and when it is twice bigger. Distributions in figure 4.4a was based on 10 sampling cycles, and the one in figure 4.4b — 500 cycles. One can observe that in the first case the distribution with spot-on prior gives very good results, much better then the shifted one. In the second figure, both distributions are comparable.

[4]For applications in finance, this task is sometimes easier than for some other fields of science, as numerous works have been published already, presenting the results of the estimates of well-known stocks or market indices within various models — see e.g. Ref [61]

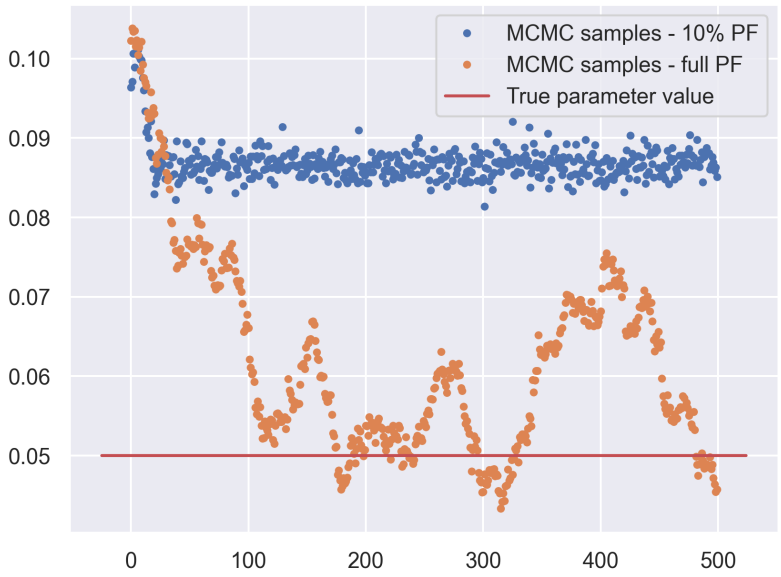


Figure 4.5: Sequences of estimate samples for a procedure in which particle filtering was done only for the first 5% samplings and another one, in which particle filtering was done for all the samplings. In both cases the mean of the prior distribution was shifted by 100% compared to its true value. It can be observed that the samples of the first procedure get stuck around the value close to the one dictated by the prior, whereas samples of the other procedure converge to the true value of the parameter, which leads to the better final result, less dependent on the prior parameters.

Another factor which should be taken into consideration is that the quality of results depends highly on the very parameters we try to estimate. The σ parameter seems to play a critical role for the Heston model in particular. This can be observed in Fig. 4.6. To produce it, an identical estimation procedure has been performed for two sample paths (which we can think of as of two different stocks). They have been simulated with the very same parameters, besides σ. Path no. 1 has been simulated with $\sigma = 0.01$ and path no. 2 with $\sigma = 0.1$, i.a. ten times bigger. The histograms present the distribution of the estimated values of the κ parameter, true κ was $\kappa = 1$ and the red vertical line illustrates this true value. It is clearly visible, that for the value of $\sigma = 0.01$ the samples were much more concentrated around the true value, while for a bigger value $\sigma = 0.1$ — they are more dispersed and the variance of the distribution is significantly bigger. This incommodity cannot be easily resolved, as

the true values of parameters of the trajectories are idiosyncratic — they cannot be influenced by the estimation procedure itself. However, we wanted to highlight the fact that the bigger the value of σ, the less trustworthy the results of the estimation of the other parameters might be.

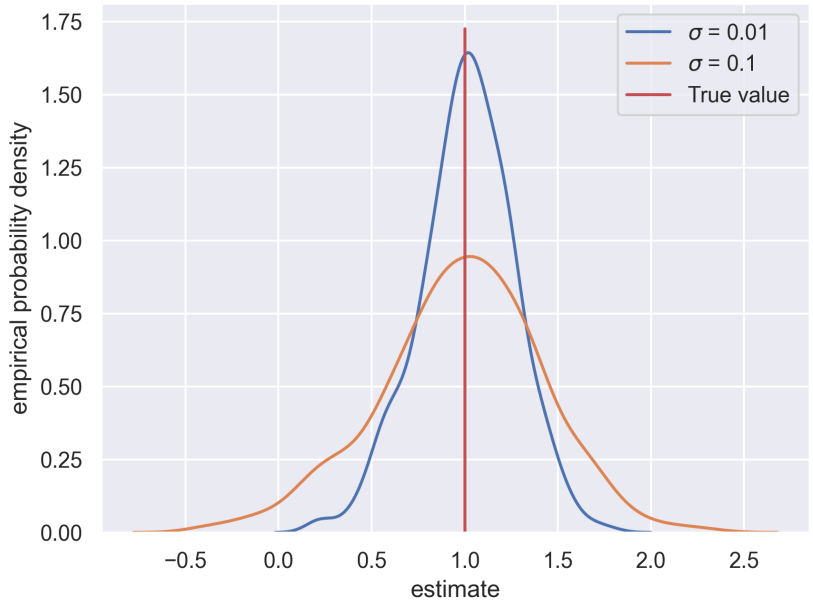


Figure 4.6: Empirical distributions of the estimate samples for the κ parameter of two different trajectories of the Heston model - one with $\sigma = 0.01$ and the other for $\sigma = 0.1$. The distribution of the estimate samples of the trajectory with smaller value of σ is narrower and more concentrated, hence — is likely to give less variable final estimates.

5 Real-life data application

5.1 Using synthetic results for investment decisions

In Chapter 3 we have presented a number of numerical experiments unravelling various characteristics of portfolio management strategies and dependencies between them. All results were obtained using trajectories simulated from the Heston model and have been synthesised in Monte-Carlo experiments. Results of certain experiments were often differing from one another depending on the simulation parameters which were used. Critically, in section 3.6.4 we demonstrated that depending on the character of the asset, encoded in the simulation parameteres of the price trajectory, some portfolio management strategies perform better than others. That could theoretically be an indication of the fact that it is possible for an investor to analyse assets they plan to invest in and then, based on the obtained results, choose an optimal portfolio management strategy. What we mean here by "analysing" the asset is actually performing the estimation process of the price trajectory (as explained in Chapter 4) and checking what set of parameters of the Heston model best reflects the actual dynamics of this particular investment. Then, the obtained estimation results can be used to identify the corresponding numerical findings, which in turn could be helpful in making the actual investment strategy selection. In this chapter we demonstrate that the decision-making process described above can indeed be performed and investments made that way lead to significant increase of profits compared to investing against the premises resulting from our analysis. The entire content of this chapter is also an adaptation of one of our research articles — namely our most recent paper [36].

5.2 Real-data experiment

We decided to estimate the parameter values of the Heston model with jumps for three well-known market indices — American S&P500, German DAX and Polish WIG20. We took into consideration the daily closing values of each of the indices, over the time period between the beginning of 2018 and the middle of 2022. The values of all three of the indices have been presented in Fig. 5.1.

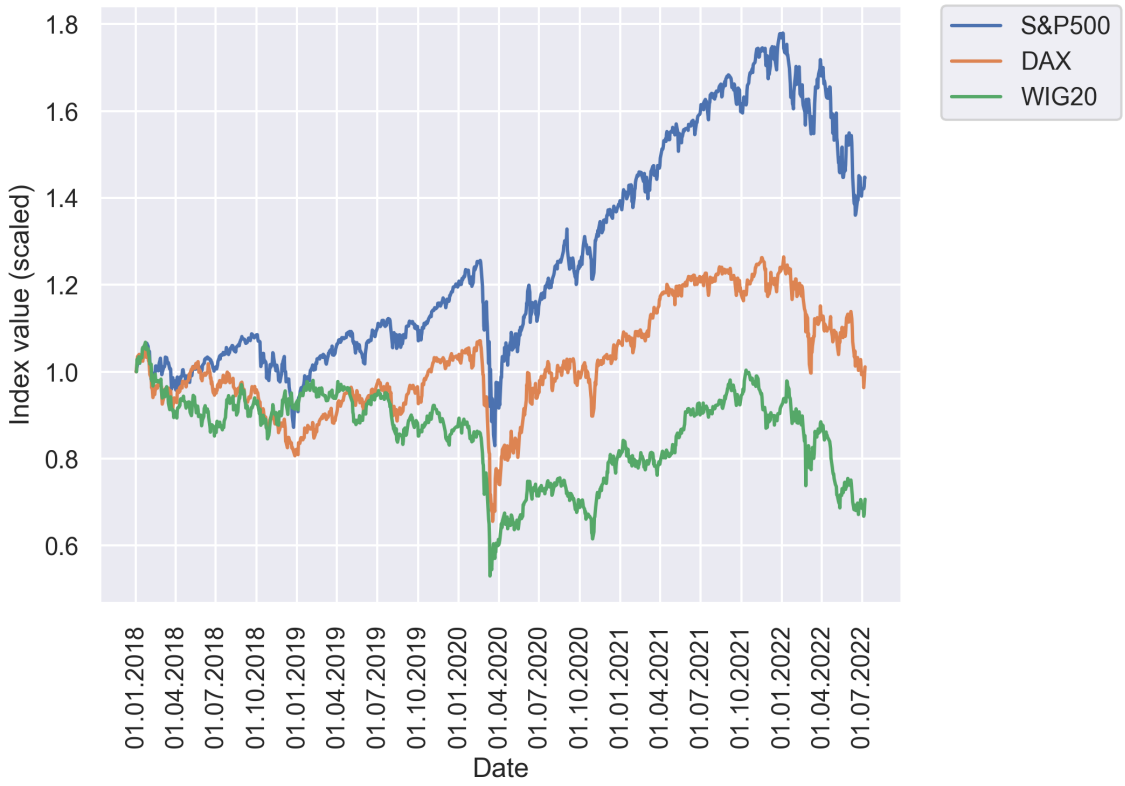


Figure 5.1: A visualisation of the dynamics of three stock market indices — S&P500, DAX and WIG20, over the time span of approximately 4.5 years. The values have been scaled down, so that the initial value of each index is 1, for easier comparison.

Before we analyse the estimation results, let us quickly recall that experiments in Chapter 3 revealed there are three major parameters responsible for performance of assets under all portfolio management strategies — the drift μ, the jump intensity λ and the average jump size μ^J. When it comes to the general growth potential, it can be noticed quite easily, just by looking at Fig. 5.1, that the American index features

the strongest drive upwards, while WIG20 values increase the most slowly among all three indices. However, the intensity and average size of jumps cannot really be assessed as easily just by looking at the graph.

In Table 5.1, the results of the parameter estimation for all three of the indices are presented. While the drift parameter μ turns out to indeed have the greatest value for S&P500, this index also has the biggest intensity of jumps λ. Since the average size of a jump is similar for all three indices, the question of what strategy should be used boils down to what are the values of μ and λ. Hence, we can use the heat-map in Fig. 3.18 as a reference for choosing proper portfolio management strategies for investing in our indices. Based on the values estimated for S&P500, this index can be positioned close to the bright area of the heat map, indicating that active strategies would be more suitable for it. On the other hand, the WIG20 index would be located in the darker area of the map which suggests the choice of the passive strategy. The exact numerical values of the ASPI measure (see Definition 32) were presented in Table 5.2. One can see that indeed, the ASPI score for WIG20 is below 0.5 (and hence — closer to zero), while fo S&P500 it is above 0.5 (and thus — closer to one). It should be noted however that the differences in the average ASPI value for the analysed indices are relatively small. Bigger differences are expected to occur between instruments with high tendency to grow combined with very infrequent jumps in prices and the ones with little growth potential and frequent jumps. However, finding such instruments in the real financial markets is rather unlikely, as the big growth potential of an instrument is often related to more significant price jumps. But, as we can see from the results of the experiments presented below, even those small differences in the value of ASPI allow us to aptly choose a proper investment strategy for a given asset.

Parameter	S&P500	DAX	WIG20
μ	0.44	0.30	0.19
κ	1.17	0.93	0.95
θ	0.06	0.06	0.06
σ	0.006	0.005	0.006
ρ	-0.41	-0.48	-0.40
λ	8.45	6.45	4.81
μ^J	-0.05	-0.05	-0.05
σ^J	0.001	0.001	0.001

Table 5.1: Estimation results for selected three stock market indices, under the assumption they follow the Heston model with Merton-style jumps (see Chapter 4 for the details of the estimation procedure).

In our experiment, we applied all three strategies to all three of our indices. The results are presented in Table 5.3 and they seem to confirm our findings. The MACD and RSI strategies indeed perform better than the passive portfolio when the asset has bigger growth potential, like in the case of S&P500. It is also true that for assets performing worse — like WIG20 — passive strategies allow if not to earn money, then to lose less than the active ones.

Parameter	S&P500	DAX	WIG20
ASPI score	0.68	0.43	0.36

Table 5.2: ASPI score for selected three stock market indices, based on the estimation of parameters μ and λ, as presented in Table 5.1.

Strategy	S&P500	DAX	WIG20
passive	9.48%	0.63%	−5.8%
MACD-driven	36.32%	0.79%	−12.51%
RSI-driven	32.66%	3.00%	−27.30%

Table 5.3: Returns on investment obtained by executing portfolio management strategies based on investing in one of three selected stock market indices. For the index with the weakest growth potential (WIG20) the passive strategy was the best, allowing to minimise the incurred loss. Investment in the index with the biggest growth potential (S&P500) earned more when active strategies (MACD- and RSI-driven) were utilised.

6 Conclusions

6.1 Summary of the work

This dissertation is a comprehensive analysis of investment portfolio management strategies in discrete time context. It contains numerous results obtained by mathematical simulations under the Heston market model, which have been supported by real-data evidence. We made a lot of effort so our results are both mathematically precise and unambiguous, but also expressed and in a straightforward way and possible to be reproduced.

We believe that the work contained in this dissertation can be utilised by a relatively wide audience. The theoretical framework, outlined in Chapter 2, provides a mathematically rigorous way of working with portfolio management strategies. It can be thought of as a foundation — a system of related concepts providing a common-ground for studying portfolio management strategies. This common-ground presents a natural environment for scientists and — in general — people who are interested in modelling portfolio management strategies to describe how they create them and study their properties. Ourselves, not only did we prove some features of portfolio management strategies, which were already known, but we also created ones of our own and successfully demonstrated some of their properties (like e.g. self-financing) too.

Chapter 3 can also provide significant value, especially for those who try to execute portfolio management strategies in practice, on the market. Thanks to the fact that we used simulated assets of various character, we were able to get the results that are general and can be treated as an "average" behaviour for a wide range of real-life assets. Those results can be used to better understand the evolution of financial portfolios in general, in various market conditions and circumstances under which they were examined (e.g. presence of trading fees), the assets' intrinsic properties encoded in the parameters of the model which we selected for simulating the data (e.g. growth potential — the 'drift') or the parameters of the investment strategy

(e.g. amount of portfolio cash).

In order to use the general results from Chapter 3, one needs to know the character of assets he is working with. Chapter 4 makes it possible by introducing the estimation procedure of the Heston model to real assets. Estimating the model using the real data allows for localising our actual investment reality in an appropriate region of the outcomes that the general, simulated solutions had given us. Using the estimation procedure one is able to know what kind of assets they are really working with and hence — to be better prepared for what kind of performance they can expect from them in certain investment scenarios.

Chapter 5, demonstrates that it is possible (within known limitations) to apply the knowledge obtained by means of simulations to the real data and get similar outcomes. It is an empirical evidence of the practical applicability of tools and methods provided in the earlier parts of the work and hence, constitutes a natural way to sum up all the considerations in this dissertation.

6.2 Future research

Results presented in this work open a plethora of ways in which the research in the area of managing investment portfolio can be continued. In the first part of Chapter 2 we formulated a foundation for a mathematical framework that we think is general enough to encapsulate formal description of a wide variety of different portfolio management strategies. In the experiments presented in Chapter 3 we mainly focused on management strategies related to portfolio rebalancing and the ones using trading indicators from technical analysis like MACD or RSI, but there is an abundance of other factors and markers which can be included in the buy-sell decisions investors make. New portfolio management strategies could e.g. be dependent on the current situation of the company, in stocks of which one wants to invest. Those conditions can be quantitatively captured by indicators which are studied via a branch competitive to technical analysis, called *fundamental analysis* [62]. Examples of such indicators are *price-to-book ratio* (also called *P/B ratio*) [62] or *price-to-earnings ratio* (also called *P/E ratio*) [63] which measure the ratio of the stock price to the company's market capitalisation and earnings per share, respectively. New management strategies could also make use of the newest algorithmic and computational methods, which are based on genetic algorithms, machine learning or big data analysis, as most of those methods perform especially well when they have access to large volumes of data [64, 65], which is, in most cases, very easily available in the field of finance [66]. It is also known now that despite most economical models assume members of any

market act rationally, humans do not always do that. Hence — an additional aspect of behavioural finance can be taken into account. It could be incorporated into any other model of a well-defined portfolio management strategy e.g. as an additional disturbance in the transactions performed by the investors, dependent, for instance, on the current general market trends (in a "behaviourised" strategy one would sell less assets after a sell signal compared to the base strategy if the market seemed to be in an upward tendency because selling at that point is counter-intuitive from a human point of view). Another aspect of researching portfolio management strategies, besides just how they work is how their performance should be measured. We mostly used measures based on the logarithmic growth of portfolio wealth but other measures can be introduced, according to which the results of comparing strategies could be totally different. Besides the sheer performance, measures of other aspects of portfolio dynamics should probably be considered too, one of the most important of such aspects being the risk. An investment strategy might be very profitable in a long term but might also lead to frequent and volatile changes in portfolio value which can be seen as a significant disadvantage for some risk-averse investors.

Another area of this work which provides a multitude of possible directions for the future research is estimation of the models used in financial mathematics. In Chapter 4 we presented a full estimation framework for Heston model, along with the analysis of the estimation process and pointing out possible difficulties and aspects in which the MCMC sampling algorithm performs better and worse. It is however of the utmost importance to continue the research of the applications of Bayesian methods in finance, as they seem to give solutions to the problems which were not possible to be solved otherwise, especially in the field of financial models' parameter estimation. While it is critical to develop new estimation methods and introduce further optimisations, making all of the procedures simpler and faster, it is also critical to study the existing methods, their accuracy and applicability.

Finally, the content of Chapter 5 also leaves some unanswered questions. It would be interesting to study if there are any indicators of the applicability of certain investment strategies other than the parameters of the Heston model. Possibly those new indicators could be simpler to obtain than trough the stochastic model estimation process, which, as we already mentioned, is often a challenge both from the perspective of mathematical complexity and computational performance. The reason behind studying investment portfolios is getting better understanding of the way financial markets behave and using their dynamics to our advantage. Making investment decisions is a notoriously challenging topic, so it is crucial to develop proper tools and conduct useful analyses in order to make this process easier but at the same time still backed by legitimate scientific research.

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