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Aspects of discrete harmonic analysis

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Streszczenie (Summary in Polish)

Całki singularne i funkcje maksymalne należą do najważniejszych obiektów w analizie harmonicznej. Klasycznym przykładem całki singularnej jest transformata Hilberta zadana jako

$$\mathcal{H}f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \mathrm{d}y, \quad x \in \mathbb{R}. \tag{0.1}$$

Badanie tego typu operatorów wiąże się z wieloma trudnościami, ponieważ pojawiające się w ich definicjach „jądra całkowe" zwykle są niecałkowalne. Analizowanie całek singularnych wymaga użycia wyrafinowanych narzędzi, które biorą pod uwagę ich specyficzną naturę. Zbiór takich narzędzi został opracowany przez Calderóna i Zygmunda w ich przełomowej pracy [8], w której badane były operatory (zwane teraz operatorami Calderóna–Zygmunda) postaci

$$\mathcal{H}_{\mathrm{CZ}} f(x) := \mathrm{p.v.} \int_{\mathbb{R}^k} f(x-y) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^k,$$

gdzie $K\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{R}$ jest niecałkowalną funkcją spełniającą warunki Calderóna–Zygmunda (zobacz następny rozdział). Dla danego operatora $\mathcal{H}_{\mathrm{CZ}}$ można określić jego dyskretny analogon jako

$$H_{\mathrm{CZ}} f(x) := \sum_{n \in \mathbb{Z}^k \setminus \{0\}} f(x-n) K(n), \quad x \in \mathbb{Z}^k.$$

W swojej pracy Caderón i Zygmund zauważyli, że ograniczoność na $L^p(\mathbb{R}^k)$ operatorów $\mathcal{H}_{\mathrm{CZ}}$ implikuje ograniczoność na $\ell^p(\mathbb{Z}^k)$ ich dyskretnych odpowiedników.

Najbardziej znanym przykładem funkcji maksymalnej jest funkcja maksymalna Hardy'ego–Littlewooda, która jest zdana jako

$$\mathcal{M}_{\mathrm{HL}} f(x) := \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(x-y)| \mathrm{d}y, \quad x \in \mathbb{R}^k, \tag{0.2}$$

gdzie $B(0,t)$ jest standardową kulą euklidesową o promieniu t i o środku w punkcie 0. Analizowanie funkcji maksymalnych również nie jest łatwe, co spowodowane jest występowaniem w ich definicji normy supremum, której w znacznym stopniu ogranicza możliwości stosowania zwykłych narzędzi. Dzięki przełomowej pracy Hardy'ego i Littlewooda [22] ($k = 1$) oraz Wienera [63] ($k > 1$) wiadomo, że operator $\mathcal{M}_{\mathrm{HL}}$ jest ograniczony na przestrzeni $L^p(\mathbb{R}^k)$. Podobnie jak w przypadku operatorów Calderóna–Zygmunda, można określić dyskretną wersję funkcji maksymalnej Hardy'ego–Littlewooda jako

$$M_{\mathrm{HL}} f(x) := \sup_{t>0} \frac{1}{|B(0,t) \cap \mathbb{Z}^k|} \sum_{m \in B(0,t) \cap \mathbb{Z}^k} |f(x-m)|, \quad x \in \mathbb{Z}^k,$$

gdzie $|B(0,t) \cap \mathbb{Z}^k|$ oznacza liczbę punktów kratowych $m \in \mathbb{Z}^k$ zawartych w $B(0,t)$. Również w tym przypadku ograniczoność na $L^p(\mathbb{R}^k)$ ciągłej wersji funkcji maksymalnej implikuje ograniczoność M_{HL} na $\ell^p(\mathbb{Z}^k)$.

Operatory Calderóna–Zygmunda oraz funkcja maksymalna Hardy'ego–Littelwooda są obiektami dobrze zbadanymi, dlatego współcześnie rozpatruje się ich różnego rodzaju uogólnienia. W pracy doktorskiej skupimy się na tak zwanych operatorach typu Radona.

Niech $d, k \geq 1$ będą ustalonymi liczbami naturalnymi. Niech

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d) \colon \mathbb{R}^k \to \mathbb{R}^d$$

będzie przekształceniem wielomianowym, takim że każda współrzędna $\mathcal{P}_j \colon \mathbb{R}^k \to \mathbb{R}^d$ jest wielomianem k zmiennych o współczynnikach całkowitych, spełniającym warunek $\mathcal{P}_j(0) = 0$. Dla dowolnej funkcji $f \in C_c^\infty(\mathbb{R}^d)$ określamy *ciągłą singularną transformatę Radona* jako

$$\mathcal{H}^{\mathcal{P}} f(x) := \text{p.v.} \int_{\mathbb{R}^k} f(x - \mathcal{P}(y)) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{0.3}$$

gdzie K jest jądrem typu Calderóna–Zygmunda. Łatwo widać, że $\mathcal{H}^{\mathcal{P}}$ jest uogólnieniem operatora $\mathcal{H}_{\mathrm{CZ}}$. Analogicznie, dla $f \colon \mathbb{Z}^d \to \mathbb{C}$ o zwartym nośniku, definiujemy *dyskretną singularną transformatę Radona* jako

$$H^{\mathcal{P}} f(x) := \sum_{m \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(m)) K(m), \quad x \in \mathbb{Z}^d. \tag{0.4}$$

Operatory postaci (0.3) oraz (0.4) były rozpatrywane po raz pierwszy przez Steina i współpracowników [15, 25, 59, 58] w kontekście parabolicznych równań różniczkowych. Wiadomo, że operator (0.3) jest ograniczony na $L^p(\mathbb{R}^d)$ i że jego ograniczoność jest konsekwencją ograniczoności standardowych operatorów Calderóna–Zygmunda $\mathcal{H}_{\mathrm{CZ}}$. Sytuacja ulega całkowitej zmianie, gdy rozważymy operator (0.4). Okazuje się, że ograniczoności na $\ell^p(\mathbb{Z}^d)$ operatora $H^{\mathcal{P}}$ nie można wywnioskować ani z ograniczoności jego ciągłego odpowiednika $\mathcal{H}^{\mathcal{P}}$, ani z oszacowań dla standardowych dyskretnych operatorów Calderóna–Zygmunda H_{CZ}. Ponadto klasyczne metody badania całek singularnych okazują się niewystarczające w tym przypadku.

W podobny sposób można uogólnić funkcję maksymalną Hardy'ego–Littewooda. W tym celu dla dowolnej funkcji $f \in C_c^\infty(\mathbb{R}^d)$ określamy *ciągłe średnie Radona* jako

$$\mathcal{M}_t^{\mathcal{P}} f(x) := \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(x - \mathcal{P}(y))| \mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{0.5}$$

Mozna zauważyć, że stowarzyszona funkcja maksymalna zadana jako $\mathcal{M}^{\mathcal{P}} f(x) := \sup_{t>0} \mathcal{M}_t^{\mathcal{P}} f(x)$ jest uogólnieniem funkcji maksymalnej Hardy'ego–Littlewooda. Tak jak w przypadku singularnej transformaty $\mathcal{H}^{\mathcal{P}}$, ograniczność $\mathcal{M}^{\mathcal{P}}$ na $L^p(\mathbb{R}^d)$ wynika z ograniczoności operatora $\mathcal{M}_{\mathrm{HL}}$. Dla $f \colon \mathbb{Z}^d \to \mathbb{C}$ o zwartym nośniku definiujemy *dyskretną średnią Radona* jako

$$M_t^{\mathcal{P}} f(x) := \frac{1}{|B(0,t) \cap \mathbb{Z}^k|} \sum_{m \in B(0,t) \cap \mathbb{Z}^k} |f(x - \mathcal{P}(m))|, \quad x \in \mathbb{Z}^d. \tag{0.6}$$

Podobnie jak w przypadku ciągłym, stowarzyszona funkcja maksymalna $M^{\mathcal{P}} f(x) := \sup_{t>0} M_t^{\mathcal{P}} f(x)$ jest uogólnieniem dyskretnej funkcji maksymalnej Hardy'ego–Littlewooda. Tutaj również pojawia się problem z pokazaniem ograniczoności na $\ell^p(\mathbb{Z}^d)$ operatora $M^{\mathcal{P}}$. Nie wynika ona ani z ograniczoności operatora $\mathcal{M}^{\mathcal{P}}$, ani z oszacowań dla dyskretnej funkcji Hardy'ego–Littlewooda.

Bourgain w przełomowej serii prac [4, 5, 6] o zbieżności średnich ergodycznych opracował zestaw narzędzi, który pozwala analizować dyskretne operatory związane z trajektoriami wielomianowymi.
W szczególności przedstawił on pierwszy dowód ograniczoności na $\ell^p(\mathbb{Z})$ jednowymiarowej funkcji maksymalnej $M^{\mathcal{P}}$. Mianowicie pokazał on, że dla $d = k = 1$, każdego wielomianu $\mathcal{P}$ i każdego $p \in (1, \infty)$ istnieje stała $C_{p,\mathcal{P}}$ taka, że

$$\|M^{\mathcal{P}} f\|_{\ell^p(\mathbb{Z})} \leq C_{p,\mathcal{P}} \|f\|_{\ell^p(\mathbb{Z})}, \quad f \in \ell^p(\mathbb{Z}). \tag{0.7}$$

Podejście zaproponowane przez Bourgaina było rozwijane przez wielu innych autorów [58, 60, 37, 39, 43]. Tutaj należy wspomnieć o pracy Ionescu i Waingera [26], którzy w znaczący sposób udoskonalili podejście Bourgaina i pokazali, że dyskretna singularna transformata Radona jest ograniczona, tj. istnieje stała $C_{p,k,d,\mathcal{P}} > 0$ taka, że

$$\big\|H^{\mathcal{P}} f\|_{\ell^p(\mathbb{Z}^d)} \le C_{p,k,d,\mathcal{P}} \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d).$$

Bourgain w swojej serii prac [4, 5, 6] oprócz udowodnienia oszacowania (0.7) wprowadził również cały zestaw narzędzi niezbędnych do badania zbieżności punktowej prawie wszędzie. Jednym z takich narzędzi jest półnorma oscylacyjna. Niech $I = (I_j : j \in \mathbb{N}) \subset \mathbb{R}_+$ będzie dowolnym ściśle rosnącym ciągiem o wartościach dodatnich. Dla funkcji $f\colon (0,\infty) \to \mathbb{C}$ oraz $N \in \mathbb{N}$ określamy *półnormę oscylacyjną* jako

$$O^2_{I,N}(f(t) : t > 0) := \Big(\sum_{j=1}^{N} \sup_{\substack{I_j \le t < I_{j+1} \\ t>0}} |f(t) - f(I_j)|^2 \Big)^{1/2}. \tag{0.8}$$

Półnorma $O^2_{I,N}$ jest bardziej wymagającym, z punktu widzenia późniejszej analizy, obiektem od normy supremum, ponieważ dla $f\colon X \times (0,\infty) \to \mathbb{C}$ mamy

$$\big\| \sup_{t>0} |f(\cdot,t)| \big\|_{L^p(X)} \le \sup_{t>0} \big\| f(\cdot,t) \big\|_{L^p(X)} + \sup_{N\in\mathbb{N}} \sup_{I \subset \mathbb{R}_+} \big\| O^2_{I,N}(f(\cdot,t) : t > 0) \big\|_{L^p(X)},$$

gdzie ostatnie supremum po I jest brane po wszystkich rosnących ciągach o wartościach dodatnich. Zatem w przypadku operatorów o jednostajnie ograniczonej normie L^p oszacowania oscylacyjne implikują oszacowania maksymalne.

Celem niniejszej rozprawy jest badanie oszacowań typu L^p dla różnego rodzaju półnorm, w tym oscylacyjnej $O^2_{I,N}$, dla średnich Radona (0.5), (0.6) oraz dla przyciętych całek singularnych postaci

$$\mathcal{H}^{\mathcal{P}}_t f(x) := \text{p.v.} \int_{B(0,t)} f(x - \mathcal{P}(y)) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{0.9}$$

$$H^{\mathcal{P}}_t f(x) := \sum_{m \in B(0,t) \cap \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(m)) K(m), \quad x \in \mathbb{Z}^d. \tag{0.10}$$

W tym celu korzystamy z metod opracowanych przez Bourgaina [4, 5, 6] , Ionescu–Waingera [26] oraz przez Mirka, Steina, Trojana i Zorin-Kranicha [40, 43], które zostały użyte w kontekście oszacowań wariacyjnych i skokowych.

Pierwszy rozdział pracy stanowi wstęp. Przedstawiamy w nim zarys historyczny oraz formułujemy główne wyniki pracy.

W rozdziale drugim przedstawiamy podstawowe narzędzia i własności, z których będziemy korzystali w pozostałej części pracy. Prezentujemy w nim również dowód zasady transfererecji Calderóna, dzięki której nasze wyniki mają zastosowanie w teorii ergodycznej.

Rozdział trzeci jest poświęcony udowodnieniu jednostajnej nierówności oscylacyjnej postaci

$$\sup_{N\in\mathbb{N}} \sup_{I \subset \mathbb{R}_+} \big\| O^2_{I,N}(N_t f : t > 0) \big\|_{L^p(X)} \lesssim_{p,d,k,\deg\mathcal{P}} \|f\|_{L^p(X)}, \quad f \in L^p(X),$$

gdzie N_t jest jednym z operatorów $\mathcal{M}^{\mathcal{P}}_t$, $\mathcal{H}^{\mathcal{P}}_t$ (dla $X = \mathbb{R}^d$) lub $M^{\mathcal{P}}_t$, $H^{\mathcal{P}}_t$ (dla $X = \mathbb{Z}^d$).

W czwartym rozdziale zajmujemy się tzw. bootstrapowym podejściem do badania oszacowań ℓ^p dla różnego rodzaju półnorm, w tym półnormy oscylacyjnej $O^2_{I,N}$, dla dyskretnych operatorów typu Radona. Metoda bootstrapowa dowodzenia zadanej nierówności polega na oszacowaniu lewej strony nierówności, nazywanej umownie L, poprzez wyrażenie postaci CL^θ dla $\theta \in [0,1)$, przy czym $C > 0$ jest niezależne od L. Prowadzi to do następującej zależności

$$L \le CL^\theta.$$

Dzieląc obie strony przez L^θ otrzymujemy $L^{1-\theta} \leq C$, a ponieważ $\theta \in [0,1)$, to otrzymujemy

$$L \leq C^{\frac{1}{1-\theta}}$$

co daje nietrywialne oszacowanie wielkości L. Określenie „bootstrap" dla tej procedury odnosi się do operowania tylko wielkością L, która jest podana na początku. W 2018 roku Mirek, Stein i Zorin-Kranich [42] rozwinęli podejście bootstrapowe w celu otrzymania oszacowań typu L^p dla półnormy wariacyjnej i skokowej dla ciągłych operatorów typu Radona. W pracy [D3] udało się rozwinąć analogiczne podejście w przypadku dyskretnych operatorów. Dzięki temu udało nam się podać nowy, krótszy dowód głównych wyników uzyskanych w pracach [6, 40, 43, D1].

Wszystkie nowe wyniki przedstawione w rozdziałach trzecim i czwartym można znaleźć w artykułach:

[D1] Mirek, M., Słomian, W., Szarek, T.Z. Some remarks on oscillation inequalities. *Ergodic Theory and Dynamical Systems*, 1–30 (2022), doi:10.1017/etds.2022.77,

[D2] Słomian, W. Oscillation Estimates for Truncated Singular Radon Operators. *J. Fourier Anal. Appl.* **29**, 4 (2023),

[D3] Słomian, W. Bootstrap methods in bounding discrete Radon operators. *J. Funct. Anal.* **283**, 9 (2022).

Opisane w doktoracie wyniki oraz metody w znacznej części opierają się na wyżej wymienionych pracach. W większości przypadków treść rozprawy została poszerzona o dodatkowe szczegóły, które nie były przedstawione w artykułach.

Chapter 1

Introduction

1.1 Discrete analogues in harmonic analysis

Classical examples

The discrete analogues are present in harmonic analysis since the very beginning. The one of the most famous operators in the "continuous" harmonic analysis is the *Hilbert transform* defined by

$$\mathcal{H}f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \mathrm{d}y, \quad x \in \mathbb{R}. \tag{1.1}$$

The operator $\mathcal{H}$ arose in Hilbert's 1904 work on a problem Riemann posed concerning analytic functions [23]. At this time it was unknown whether the operator $\mathcal{H}$ is bounded on $L^p(\mathbb{R})$. A positive answer to this question was given by M. Riesz [52] in 1928 who showed that for $p > 1$ there is a positive constant C_p such that

$$\|\mathcal{H}f\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R}). \tag{1.2}$$

In the same paper Riesz made an observation that this result implies the boundedness on $\ell^p(\mathbb{Z})$, with $p > 1$, of the *discrete Hilbert transform* given by

$$Hf(x) := \frac{1}{\pi} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{f(x-n)}{n}, \quad x \in \mathbb{Z}. \tag{1.3}$$

Riesz approach relied heavily on some properties of analytic functions and it was not possible to use it in higher dimensions. In 1952 Calderón and Zygmund in their groundbreaking paper [8] developed a real-variable method which allowed them to study singular integrals in higher dimensions and resulted in introducing *Calderón–Zygmund operators* of the form

$$\mathcal{H}_{\mathrm{CZ}} f(x) := \mathrm{p.v.} \int_{\mathbb{R}^k} f(x-y) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^k,$$

where $K \colon \mathbb{R}^k \setminus \{0\} \to \mathbb{R}$ is a *Calderón–Zygmund kernel* which satisfies the following conditions[1]:

(1) The size condition. For every $x \in \mathbb{R}^k \setminus \{0\}$, we have

$$|K(x)| \leq |x|^{-k}. \tag{1.4}$$

(2) The cancellation condition. For every $0 < r < R < \infty$, we have

$$\int_{\Omega_R \setminus \Omega_r} K(y) \mathrm{d}y = 0. \tag{1.5}$$

[1] The conditions given here are not the weakest possible, see [20] for more details.

(3) The Hölder continuity condition. For some $\sigma \in (0,1]$ and every $x, y \in \mathbb{R}^k \setminus \{0\}$ with $2|y| \leq |x|$, we have

$$|K(x-y) - K(x)| \leq |y|^{\sigma}|x|^{-k-\sigma}. \tag{1.6}$$

Calderón and Zygmund proved that if $\mathcal{H}_{\mathrm{CZ}}$ is a operator associated with the kernel K which satisfies the above conditions then for $p > 1$ there is a positive constant $C_{p,k}$ for which the inequality

$$\|\mathcal{H}_{\mathrm{CZ}} f\|_{L^p(\mathbb{R}^k)} \leq C_{p,k}\|f\|_{L^p(\mathbb{R}^k)} \tag{1.7}$$

holds for any $f \in L^p(\mathbb{R}^k)$. It was noted by Calderón and Zygmund (see Proposition 1.15) that, as in the case of the Hilbert transform, the estimate (1.7) implies the boundedness on $\ell^p(\mathbb{Z}^k)$ of the *discrete Calderón–Zygmund operators* given by

$$H_{\mathrm{CZ}} f(x) := \sum_{n \in \mathbb{Z}^k \setminus \{0\}} f(x-n)K(n), \quad x \in \mathbb{Z}^k.$$

Another objects of great importance in harmonic analysis are maximal functions. The best known example is the *Hardy–Littlewood maximal function* which is given

$$\mathcal{M} f(x) := \sup_{t>0} \frac{1}{2r} \int_{-t}^{t} |f(x-y)| \mathrm{d}y, \quad x \in \mathbb{R}.$$

The operator $\mathcal{M}$ was introduced in 1930 by Hardy and Littlewood [22]. They proved that for any $p > 1$ there is a positive constant C_p such that

$$\|\mathcal{M} f\| \leq C_p\|f\|_{L^p(\mathbb{R})}. \tag{1.8}$$

Here the story is somewhat the opposite of the Hilbert transform one because in their work Hardy and Littlewood first considered the *discrete Hardy–Littlewood maximal function* given by

$$M(x) := \sup_{t>0} \frac{1}{|(-t,t) \cap \mathbb{Z}|} \sum_{m \in (-t,t) \cap \mathbb{Z}} |f(x-m)|, \quad x \in \mathbb{Z}. \tag{1.9}$$

They showed that for any $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\|Mf\|_{\ell^p(\mathbb{Z})} \leq C_p\|f\|_{\ell^p(\mathbb{Z})}, \quad f \in \ell^p(\mathbb{Z}),$$

and then they argued that the above inequality implies (1.8).

In 1930 Wiener [63] generalized the Hardy–Littlewood result to the higher dimensional setting. Namely, let $B(0,t)$ be the Euclidean ball centered at 0 with radius $t > 0$. The higher dimensional Hardy–Littlewood maximal function is defined as

$$\mathcal{M}_{\mathrm{HL}} f(x) := \sup_{t>0} \frac{1}{|B(0,t)|} \int_{B(0,t)} |f(x-y)| \mathrm{d}y, \quad x \in \mathbb{R}^k. \tag{1.10}$$

Wiener showed that for any $p \in (1, \infty)$ and any $k \in \mathbb{N}$ there is a constant $C_{p,k} > 0$ such that

$$\|\mathcal{M}_{\mathrm{HL}} f\|_{L^p(\mathbb{R}^k)} \leq C_{p,k}\|f\|_{L^p(\mathbb{R}^k)}, \quad f \in L^p(\mathbb{R}^k).$$

As in the case of the Calderón–Zygmund operators, the above inequality implies the boundedness on $\ell^p(\mathbb{Z}^k)$ of the discrete higher dimensional Hardy–Littlewood maximal function defined as

$$M_{\mathrm{HL}} f(x) := \sup_{t>0} \frac{1}{|B(0,t) \cap \mathbb{Z}^k|} \sum_{m \in B(0,t) \cap \mathbb{Z}^k} |f(x-m)|, \quad x \in \mathbb{Z}^k.$$

Operators of Radon type

There are many ways to generalize the theory of Calderón–Zygmund and Hardy–Littlewood operators. One type of such generalization concerns Radon type operators. Let $d, k \in \mathbb{N}$ be fixed natural numbers. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d) \colon \mathbb{R}^k \to \mathbb{R}^d \tag{1.11}$$

be a polynomial mapping, where each $\mathcal{P}_j \colon \mathbb{R}^k \to \mathbb{R}^d$ is a polynomial of k variables with integer coefficients such that $\mathcal{P}_j(0) = 0$. For any $f \in C_c^\infty(\mathbb{R}^d)$ we define the *continuous singular Radon transform* as

$$\mathcal{H}^{\mathcal{P}} f(x) := \mathrm{p.v.} \int_{\mathbb{R}^k} f(x - \mathcal{P}(y)) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{1.12}$$

where $K \colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a Calderón–Zygmund kernel. It can be easily seen that the above definition generalizes $\mathcal{H}_{\mathrm{CZ}}$. The operators $\mathcal{H}^{\mathcal{P}}$ originate in some problems related to curvatures and parabolic differential equations, see [15, 25, 59, 58]. It is well known that the operator $\mathcal{H}^{\mathcal{P}}$ is bounded on $L^p(\mathbb{R}^d)$ with $p \in (1, \infty)$. Roughly speaking, this is due to the L^p-boundedness of classical Calderón–Zygmund operators. We illustrate this with a particular example. Let $d = k = 1$, $\mathcal{P}(y) = y^3$ and let $K(y) = y^{-1}$. Then

$$\mathcal{H}^{\mathcal{P}} f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x - y^3)}{y} \mathrm{d}y, \quad x \in \mathbb{R}. \tag{1.13}$$

By making a substitution $y^3 = t$ we get

$$\int_{|y| > \varepsilon} \frac{f(x - y^3)}{y} \mathrm{d}y = \frac{1}{3} \int_{|t| > \varepsilon^3} \frac{f(x - t)}{t} \mathrm{d}t.$$

Consequently, we get that $\mathcal{H}^{\mathcal{P}} f = \frac{1}{3} \mathcal{H} f$, where $\mathcal{H}$ is the standard Hilbert transform (1.1). Therefore, we see that the boundedness of $\mathcal{H}^{\mathcal{P}}$ follows from (1.2). Obviously, in general case a much more work is required but the core of the proof is the boundedness of the Calderón–Zygmund operators.

In analogy to the usual Calderón–Zygmund operators we may consider the discrete counterpart of (1.12). Let $f \colon \mathbb{Z}^d \to \mathbb{C}$ be a finitely supported function. The *discrete singular Radon transform* of f is defined as

$$H^{\mathcal{P}} f(x) := \sum_{m \in \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(m)) K(m), \quad x \in \mathbb{Z}^d, \tag{1.14}$$

where $K \colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a Calderón–Zygmund kernel. Despite the obvious similarity to the discrete Calderón–Zygmund operators H_{CZ} the operators $H^{\mathcal{P}}$ are much more difficult objects to study. For example, the question of boundedness of $H^{\mathcal{P}}$ on $\ell^p(\mathbb{Z}^d)$ with $p \in (1, \infty)$ was a very challenging problem. First of all, we cannot repeat the argument from the continuous setting and use the boundedness of the discrete Calderón–Zygmund operators H_{CZ}. This is due the fact that in the discrete setting we do not have the substitution principle. Secondly, we cannot deduce the ℓ^p-boundedness of $H^{\mathcal{P}}$ by using the L^p-boundedness of $\mathcal{H}^{\mathcal{P}}$ – for more details see the discussion after the proof of Proposition 1.15. This is a completely different situation than in the case of standard Calderón–Zygmund operators.

The first partial answer about ℓ^p-boundedness of the operator $H^{\mathcal{P}}$ was given by Stein and Wainger in [58] where they managed to prove that $H^{\mathcal{P}}$ is bounded on $\ell^p(\mathbb{Z}^d)$ for p in a certain neighborhood of 2. The full range of $p \in (1, \infty)$ was obtained by Ionescu and Wainger [26] in 2005, see also [37] for a different approach.

In a similar fashion we may generalize the Hardy–Littlewood maximal function. Let $\mathcal{P}$ be a polynomial mapping (1.11). For any $f \in C_c^\infty(\mathbb{R}^d)$ we define the *maximal function of Radon averages* as

$$\mathcal{M}^{\mathcal{P}} f(x) := \sup_{t > 0} \frac{1}{|B(0, t)|} \int_{B(0,t)} f(x - \mathcal{P}(y)) \mathrm{d}y, \quad x \in \mathbb{R}^d.$$

It can be easily seen that the above definition is a natural extension of the Hardy–Littlewood maximal function. It is known that for any $p \in (1, \infty]$ there is a constant $C_{p,d,k,\deg \mathcal{P}} > 0$ such that

$$\|\mathcal{M}^{\mathcal{P}} f\|_{L^p(\mathbb{R}^d)} \le C_{p,d,k,\deg \mathcal{P}} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).$$

Again, although it is a non-trivial complicated task, the above inequality can be deduced from the boundedness of the Hardy–Littlewood maximal function $\mathcal{M}_{\mathrm{HL}}$. As in the case of the discrete singular Radon transform we define the discrete analogue of $\mathcal{M}^{\mathcal{P}}$. For any bounded function $f \colon \mathbb{Z}^d \to \mathbb{C}$ the *discrete maximal function of Radon averages* is given by

$$M^{\mathcal{P}} f(x) := \sup_{t>0} \frac{1}{|B(0,t) \cap \mathbb{Z}^k|} \sum_{m \in B(0,t) \cap \mathbb{Z}^k} f(x - \mathcal{P}(m)), \quad x \in \mathbb{Z}^d.$$

As before, we cannot deduce the ℓ^p-boundedness by using the discrete Hardy–Littlewood maximal function M_{HL} neither the continuous counterpart $\mathcal{M}^{\mathcal{P}}$.

The first proof of the ℓ^p-boundedness of $M^{\mathcal{P}}$ (in the case when $d = k = 1$) was given by Bourgain at the end of 80's in his groundbreaking series of works [4, 5, 6] about pointwise convergence of the ergodic averages along polynomial orbits – see more details in Section 1.2. In his work, Bourgain has introduced tools that capture the arithmetic nature of the operator $M^{\mathcal{P}}$. Bourgain's work has greatly influenced the field of discrete analogues and his ideas are still used today.

Transference of bounds between discrete and continuous setting

In previous section we stated that the boundedness of the discrete Calderón–Zygmund operators can be deduced from the estimates for their continuous counterparts. However, we noted that this is impossible for the general Radon operators. Below we try to illustrate this phenomenon. At first we show how to transfer bounds between the standard Calderón–Zygmund operators.

Proposition 1.15. *Then for $p \in (1, \infty)$ there is a constant $C_{p,k,\sigma} > 0$ such that*

$$\|H_{\mathrm{CZ}}\|_{\ell^p(\mathbb{Z}^k) \to \ell^p(\mathbb{Z}^k)} \le C_{p,k,\sigma} \|\mathcal{H}_{\mathrm{CZ}}\|_{L^p(\mathbb{R}^k) \to L^p(\mathbb{R}^k)}. \tag{1.16}$$

Proof. Let $p \in (1, \infty)$ and let p' be its dual. Let $\mathbf{Q} := [-1/2, 1/2)^k$. For any $f \in \ell^p(\mathbb{Z}^k)$ and $g \in \ell^{p'}(\mathbb{Z}^k)$ we define its extension to $\mathbb{R}^k$ by

$$F(x) := \sum_{n \in \mathbb{Z}^k} f(n) \mathbb{1}_{\mathbf{Q}}(x - n) \quad \text{and} \quad G(x) := \sum_{n \in \mathbb{Z}^k} g(n) \mathbb{1}_{\mathbf{Q}}(x - n), \quad x \in \mathbb{R}^k.$$

Clearly, we have $F(n) = f(n)$ and $G(n) = g(n)$ for $n \in \mathbb{Z}^k$. Moreover, we have $\|F\|_{L^p(\mathbb{R}^k)} = \|f\|_{\ell^p(\mathbb{Z}^k)}$. The same holds for functions G and g. Let us observe that one has

$$\begin{aligned}
\int_{\mathbb{R}^k} \mathcal{H}_{\mathrm{CZ}}(F)(x) G(x) \mathrm{d}x &= \sum_{m \in \mathbb{Z}^k} g(m) \int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}(F)(x) \mathrm{d}x \\
&= \sum_{m \in \mathbb{Z}^k} g(m) \sum_{n \neq m} f(n) \int_{\mathbf{Q}+m} \int_{\mathbf{Q}+n} K(x-y) \mathrm{d}y \mathrm{d}x \\
&\quad + \sum_{m \in \mathbb{Z}^k} g(m) \int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}\big(f(m) \mathbb{1}_{\mathbf{Q}+m}\big)(x) \mathrm{d}x \\
&= \sum_{m \in \mathbb{Z}^k} (H_{\mathrm{CZ}} f)(m) g(m) + \sum_{\substack{n,m \in \mathbb{Z}^k \\ m \neq n}} \widetilde{K}(m-n) f(n) g(m) \\
&\quad + \sum_{m \in \mathbb{Z}^k} g(m) \int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}\big(f(m) \mathbb{1}_{\mathbf{Q}+m}\big)(x) \mathrm{d}x,
\end{aligned} \tag{1.17}$$

where

$$\widetilde{K}(m-n) := \int_{m+\mathbf{Q}} \int_{n+\mathbf{Q}} K(x-y) - K(m-n)\mathrm{d}y\mathrm{d}x$$
$$= \int_{\mathbf{Q}} \int_{\mathbf{Q}} K(m-n+x-y) - K(m-n)\mathrm{d}y\mathrm{d}x.$$

By (1.17) we see that it is enough to estimate

$$\sum_{\substack{n,m\in\mathbb{Z}^k\\ m\neq n}} \widetilde{K}(m-n)f(n)g(m) + \sum_{m\in\mathbb{Z}^k} g(m)\int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}\big(f(m)\mathbb{1}_{\mathbf{Q}+m}\big)(x)\mathrm{d}x,$$

Observe that the double application of Hölder's inequality yields

$$\Big|\sum_{m\in\mathbb{Z}^k} g(m)\int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}\big(f(m)\mathbb{1}_{\mathbf{Q}+m}\big)(x)\mathrm{d}x\Big|$$
$$\leq \|g\|_{\ell^{p'}(\mathbb{Z}^k)}\Big(\sum_{m\in\mathbb{Z}^d}\Big|\int_{\mathbf{Q}+m} \mathcal{H}_{\mathrm{CZ}}\big(f(m)\mathbb{1}_{\mathbf{Q}+m}\big)(x)\mathrm{d}x\Big|^p\Big)^{1/p}$$
$$\leq \|g\|_{\ell^{p'}(\mathbb{Z}^k)}\Big(\sum_{m\in\mathbb{Z}^k}\int_{\mathbf{Q}+m} \big|\mathcal{H}_{\mathrm{CZ}}\big(f(m)\mathbb{1}_{\mathbf{Q}+m}\big)(x)\big|^p\mathrm{d}x\Big)^{1/p}$$
$$\leq \|g\|_{\ell^{p'}(\mathbb{Z}^k)}\|f\|_{\ell^p(\mathbb{Z}^k)}\|\mathcal{H}_{\mathrm{CZ}}\|_{L^p\to L^p}.$$

By the condition (1.6) we have

$$|K(m-n+x-y) - K(m-n)| \leq |x-y|^{\sigma}|m-n|^{-k-\sigma}$$

which implies that

$$|\widetilde{K}(n-m)| \leq |m-n|^{-k-\sigma}.$$

Therefore, again by Hölder's inequality, we get

$$\Big|\sum_{\substack{n,m\in\mathbb{Z}^k\\ m\neq n}} \widetilde{K}(m-n)f(n)g(m)\Big| \leq \Big(\sum_{\substack{n,m\in\mathbb{Z}^k\\ m\neq n}} \widetilde{K}(m-n)||f(n)|^p\Big)^{1/p}\Big(\sum_{\substack{n,m\in\mathbb{Z}^k\\ m\neq n}} |\widetilde{K}(m-n)||g(m)|^{p'}\Big)^{1/p'}$$
$$\lesssim_{k,\sigma} \|f\|_{\ell^p(\mathbb{Z}^k)}\|g\|_{\ell^{p'}(\mathbb{Z}^k)}$$

since $\sum_{k\in\mathbb{Z}^k\setminus\{0\}} |k|^{-k-\sigma} < \infty$. This gives (1.16). □

A similar result can be stated for the Hardy–Littlewood maximal function (although the proof is different). However, for the sake of clarity, we focus only on the singular integrals, noting that similar reasoning can be done for the maximal functions associated with averages.

Proposition 1.15 shows that in the case of standard Calderón–Zygmund operators the discrete and continuous cases are equivalent[2]. However, things get complicated when one studies Radon type operators associated with polynomials with degree greater than one. In order to illustrate this issue we use the following example. Let us consider the following Radon type operators

$$\mathcal{H}_{\mathrm{cont}}f(x) := \mathrm{p.v.}\int_{\mathbb{R}} f(x-y^2)K(y)\mathrm{d}y, \quad x\in\mathbb{Z},$$

[2]It can be shown that the reverse inequality (1.16) also holds.

and

$$H_{\mathrm{dis}}f(m) := \sum_{n\in\mathbb{Z}\setminus\{0\}} f(m-n^2)K(n), \quad m \in \mathbb{Z}.$$

It is clear that $\mathcal{H}_{\mathrm{cont}}$ is continuous counterpart of H_{dis} and vice versa. Let us see whether we can repeat the proof of Proposition 1.15 for $\mathcal{H}_{\mathrm{cont}}$ and H_{dis}. Clearly, we can write the decomposition (1.17). The problem arises when one needs to estimate the kernel

$$\widetilde{K}(m-n^2) := \int_{\mathbf{Q}}\int_{\mathbf{Q}} K(x+m-(n+y)^2) - K(m-n^2)\mathrm{d}y\mathrm{d}x.$$

By using condition (1.6) we write that

$$\big|K(x+m-(n+y)^2) - K(m-n^2)\big| \le \frac{|x-2yn-y^2|^{\sigma}}{|m-n^2|^{1+\sigma}}.$$

Unfortunately, the right hand side of the above inequality is unsummable in $n \in \mathbb{Z}$ hence the proof does not work. The main reason why this happens is because for any polynomial $\mathcal{P}:\mathbb{Z}\to\mathbb{Z}$ with degree greater than 1 we have

$$\mathcal{P}(n+t) - \mathcal{P}(n) = \mathcal{O}(n^{\deg\mathcal{P}-1}), \quad n\in\mathbb{Z}, \quad t\in[0,1]. \tag{1.18}$$

This problem does not occur in the case when $\mathcal{P}(n) = an$ since then $\mathcal{P}(n+t)-\mathcal{P}(n) = at$ which is bounded in $n \in \mathbb{Z}$. Similar issues occur when one tries other transference methods – the main obstacle is the fact that $\mathcal{P}:\mathbb{Z}\to\mathbb{Z}$ may have unbounded gaps (1.18). Therefore, in the case of Radon type operators, we cannot simply transfer the bounds from the continuous to the discrete setting. Consequently, one needs to develop completely new methods to deal with discrete operators associated with arbitrary polynomials. An appropriate set of tools which are capable of dealing with discrete problems was introduced by Bourgain in late 80's in his groundbreaking work about pointwise convergence of ergodic averages along squares – see the next section.

1.2 The problem of the pointwise convergence and the circle method of Hardy and Littlewood

Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space. Let $T_t\colon L^p(X)\to L^p(X)$ be linear operators indexed by $t\in\mathbb{R}_+$ or $t\in\mathbb{N}$. In many contexts, a natural question that one may ask about the whole family of operators is what happens with T_tf when $t\to\infty$ (if $t\in\mathbb{R}_+$ or $t\in\mathbb{N}$) or when $t\to 0$ (only if $t\in\mathbb{R}_+$). In other words, we are asking if the limit

$$\lim_{t\to\infty} T_tf \quad \text{or} \quad \lim_{t\to 0} T_tf$$

exists and in what sense (norm convergence, pointwise, etc.). In the thesis we are particularly interested in the pointwise convergence. Namely, we want to know whether the limit

$$\lim_{t\to\infty} T_tf(x) \quad \text{or} \quad \lim_{t\to 0} T_tf(x), \qquad x\in X,$$

exists μ-almost everywhere. The classical approach for verifying of pointwise convergence (sometimes called Banach's principle) consists of two steps:

(a) Establishing L^p-boundedness for the maximal function given by

$$T_*f(x) := \sup_t |T_tf(x)|,$$

where, depending on the set of indices, the supremum is taken over $t\in\mathbb{R}_+$ or $t\in\mathbb{N}$;

(b) Finding a dense class of functions in $L^p(X,\mu)$ for which we know that the pointwise convergence holds.

For the proof that these two conditions are indeed sufficient see [20, Theorem 2.1.14]. The classical application of the described procedure is the proof of Lebesgue's differentiation theorem.

Theorem 1.19 **(Lebesgue's differentiation theorem).** *Let $p > 1$. For any function $f \in L^p(\mathbb{R}^d)$ we have*[3]

$$\lim_{t\to 0} \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\mathrm{d}y = f(x) \tag{1.20}$$

for almost all $x \in \mathbb{R}^d$.

Indeed, let

$$T_t f(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Then, $T_* f(x) = \sup_{t>0} |T_t f(x)|$ is the Hardy–Littlewood maximal function and we know that it is L^p-bounded, that is

$$\|T_* f\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R}^d),$$

which shows that the step (a) is satisfied. On the other hand, it is easy to verify that (1.20) holds for functions $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. Since the set $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ is dense in every $L^p(\mathbb{R}^d)$ this establishes (b).

Another example of the Banach principle is the proof of Birkhoff's ergodic theorem.

Theorem 1.21 **(Birkhoff's ergodic theorem).** *Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space. Let $T\colon X \to X$ be an invertible measure preserving transformation which means that*

$$\mu(T^{-1}A) = \mu(A) \text{ for each } A \in \mathcal{B}.$$

Let $p \in (1,\infty)$. Then for any $f \in L^p(X,\mu)$ the averages

$$M_N^{\mathrm{Birk}} f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^n x)$$

converge, as $N \to \infty$, for μ-almost every $x \in X$.

In the case of Birkhoff's averages M_N^{Birk}, the Calderón transference principle (see Section 2.2) allows one to deduce the estimate

$$\|\sup_{N\in\mathbb{N}} |M_N f|\|_{L^p(X,\mu)} \lesssim_p \|f\|_{L^p(X,\mu)}$$

for $p \in (1,\infty]$ from the estimate for the discrete Hardy–Littlewood maximal function (1.9). This establishes the first step (a). For the second step, one can use the idea of F. Riesz decomposition [51] to analyze the space $\mathbb{I}_T \oplus \mathbb{J}_T \subseteq L^2(X,\mu)$, where

$$\mathbb{I}_T := \{f \in L^2(X,\mu) : f\circ T = f\} \qquad \text{and} \qquad \mathbb{J}_T := \{h\circ T - h : h \in L^2(X,\mu) \cap L^\infty(X,\mu)\}.$$

We see that $M_N^{\mathrm{Birk}} f = f$ for $f \in \mathbb{I}_T$ and, for $g = h\circ T - h \in \mathbb{J}_T$, we have

$$M_N^{\mathrm{Birk}} g(x) = \frac{1}{2N+1}\big(h(T^{N+1}x) - h(T^N x)\big)$$

[3] Actually, the theorem is true when $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ since the Hardy–Littlewood maximal function is of weak type (1,1). However, in the presentation we focus only on L^p spaces with $p > 1$ hence the formulation for $p > 1$.

by telescoping. Consequently, we see that $M_N g \to 0$ as $N \to \infty$. This establishes μ-almost everywhere pointwise convergence of M_N^{Birk} on $\mathbb{I}_T \oplus \mathbb{J}_T$, which turns out to be dense in $L^2(X,\mu)$. Since $L^2(X,\mu)$ is dense in $L^p(X,\mu)$ for every $p \in (1,\infty)$, this establishes (b).

We have just seen that the so-called Banach principle is proving to be a very effective tool when showing the pointwise convergence. However, not every problem can be handled easily by using this approach. The most known example is the pointwise convergence of the ergodic averages along monomials given by

$$T_N^b f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^{n^b} x), \quad b \in \mathbb{N}.$$

In the case of the operator T_N^b it is not easy to find an appropriate dense class for which the pointwise convergence is *a priori* known. The approach taken in the case of Birkhoff's averages is insufficient here since it is difficult to establish if the family $\mathbb{I}_T \oplus \mathbb{J}_T$ is a dense class of functions in $L^2(X,\mu)$ for which the averages along the squares converge pointwise. The problem is caused by the fact that $(n+1)^b - n^b$ is unbounded and we loose the telescoping nature of the averaging operators on $\mathbb{J}_T$.

At the end of the 1980's, Bourgain established the pointwise convergence of the averages T_N^b in a series of groundbreaking articles [4, 5, 6]. By using the Hardy–Littlewood circle method from analytic number theory, he showed L^p-bounds for the maximal function

$$\sup_{N\in\mathbb{N}} |T_N^b f(x)|,$$

which is the step (a). He then bypassed the problem of finding the requisite dense class of functions by using the oscillation seminorm (1.22).

Seminorm approach to the pointwise convergence

Let us recall the definition of the oscillation seminorm. Let $\mathbb{I} \subseteq \mathbb{R}_+$. For any increasing sequence $I = (I_j : j \in \mathbb{N}) \subseteq \mathbb{I}$ and any $N \in \mathbb{N} \cup \{\infty\}$, the oscillation seminorm of a function $f\colon \mathbb{I} \to \mathbb{C}$ is defined by

$$O^2_{I,N}(f(t) : t \in \mathbb{I}) := \Big(\sum_{j=1}^{N} \sup_{\substack{I_j \le t < I_{j+1} \\ t \in \mathbb{I}}} |f(t) - f(I_j)|^2 \Big)^{1/2}. \tag{1.22}$$

Although not apparent at the first glance the above object is very much related to pointwise convergence. This was first noted by Bourgain and was used to show that for any $f \in L^p(x,\mu)$, $p \in (1,\infty)$, the averages $T_N^b f$ converge μ-almost everywhere. He did it by proving that for a lacunary sequence $I = (I_j : j \in \mathbb{N})$ and for any $J \in \mathbb{N}$ one has

$$\big\| O^2_{I,J}(T_N^b f : N \in \mathbb{N}) \big\|_{L^2(X,\mu)} \lesssim_I J^c \|f\|_{L^2(X)} \tag{1.23}$$

for some $c < 1/2$. From the above inequality one may deduce that $T_N^b f$ converge pointwise for $f \in L^2(X,\mu)$ – see Proposition 2.3.

In order to establish (1.23) Bourgain used variety of tools: Calderón principle (see Section 2.2), The Hardy–Littlewood circle method (see the next section), the r-variation seminorms V^r and jump quasi-seminorm (for more details see Section 2.1).

Let $r \in [1,\infty)$. Let us recall that the r-variation seminorm V^r of a function $f\colon \mathbb{I} \to \mathbb{C}$ is defined by

$$V^r(f(t) : t \in \mathbb{I}) := \sup_{N\in\mathbb{N}} \sup_{\substack{t_1 \le \cdots \le t_{N+1} \\ t_j \in \mathbb{I}}} \Big(\sum_{j=1}^{N} |f(t_{j+1}) - f(t_j)|^2 \Big)^{1/2}. \tag{1.24}$$

Bourgain observed that the V^r seminorm can be used to obtain (1.23). This is because we have

$$O^2_{I,N}(f(t) : t \in \mathbb{I}) \le N^{1/2-1/r} V^r(f(t) : t \in \mathbb{I})$$

for $r \ge 2$ – see Section 2.1. In order to prove the r-variational inequality for the averages T^b_N, Bourgain used the jump quasi-seminorm. In order to define it we need the notion of the λ-jump counting function. For any $\lambda > 0$ and $\mathbb{I} \subseteq \mathbb{R}$, the λ-jump counting function of $f : \mathbb{I} \to \mathbb{C}$ is defined by

$$N_\lambda(f(t) : t \in \mathbb{I}) := \sup\{J \in \mathbb{N} \mid \exists_{\substack{t_0<\cdots<t_J \\ t_j \in \mathbb{I}}} : \min_{0<j\le J} |f(t_j) - f(t_{j-1})| \ge \lambda\}.$$

The jump quasi-seminorm of a function $f : X \times \mathbb{I} \to \mathbb{C}$ is the following quantity

$$\sup_{\lambda>0} \big\| \lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I})^{1/2} \big\|_{L^p(X)}. \tag{1.25}$$

It is not hard to obtain that one has

$$\sup_{\lambda>0} \big\| \lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I})^{1/2} \big\|_{L^p(X)} \le \|V^2(f(t) : t \in \mathbb{I})\|_{L^p(X)}. \tag{1.26}$$

The remarkable feature of the λ-jumps, observed by Bourgain [6], is that, in some sense, the inequality (1.26) can be reversed. Namely, *a priori* uniform λ-jump estimates

$$\sup_{\lambda>0} \big\| \lambda N_\lambda(f(\cdot, t) : t \in \mathbb{I})^{1/2} \big\|_{L^p(X)} \tag{1.27}$$

for some $p \in [1, \infty)$ imply weak r-variational estimates

$$\|V^r(f(t) : t \in \mathbb{I})\|_{L^{p,\infty}(X)} \le C_{p,r}$$

for the same value of p and for all $r \in (2, \infty]$. Those observations made by Bourgain were the starting point of comprehensive investigations in ergodic theory and harmonic analysis, which resulted in many papers. In particular, they have attracted the attention of researchers to the notion of the oscillation seminorm $O_{I,N}$, r-variation seminorms V^r and jump quasi-seminorm.

For more details and properties of the quantities (1.22), (1.24) and (1.25) we refer to Section 2.1 where we present detailed proofs of some selected facts and properties.

1.2.1 Waring problem and the circle method of Hardy and Littlewood

The exposition of this section is based on [17] and [38].

Let $\mathbb{N}_0$ denote the set of nonnegative natural numbers $\mathbb{N}_0 = \{0, 1, 2, \ldots, \}$. In 1770 Waring made the statement that for each $k \in \mathbb{N}$ there exist $d \in \mathbb{N}$ such that every natural number N can be expressed as

$$N = n_1^k + n_2^k + \cdots + n_d^k, \quad \text{for } n_i \in \mathbb{N}_0. \tag{1.28}$$

The first proof which concerns every $k \in \mathbb{N}$ was given by Hilbert [24] in 1909. In the 1920' Hardy and Littlewood [21] began the study of questions related to Waring's problem from a quantitative perspective. Namely, for any $N \in \mathbb{N}$ let $r_k(N)$ denote the number of d-tuples $(n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ which solve the equation (1.28). The circle method was pioneered by Hardy and Littlewood in order to prove that for $k \ge 2$, $d \ge 2^k + 1$ we have

$$r_k(N) = \mathfrak{S}(N) \frac{\Gamma\left(1 + \frac{1}{k}\right)^d}{\Gamma\left(\frac{d}{k}\right)} N^{d/k-1} + \mathcal{O}\big(N^{d/k-1-\delta}\big), \tag{1.29}$$

for some $\delta > 0$. Here Γ is the Gamma function, $\mathfrak{S}(N)$ is the *singular series* given by

$$\mathfrak{S}(N) := \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} G(a/q)^d \boldsymbol{e}(-Na/q) \tag{1.30}$$

with $\boldsymbol{e}(z) = \exp(2\pi \boldsymbol{i} z)$, and $G(a/q)$ is the Gaussian sum

$$G(a/q) := \frac{1}{q} \sum_{r=1}^{q} \boldsymbol{e}\big(\frac{a}{q} r^k\big).$$

Let us show how to derive the asymptotic formula (1.29) with the aid of the circle method. Let $k \geq 2$ be a fixed integer and denote

$$S_N = \{(n_1, n_2, \ldots, n_d) \in \mathbb{N}^d : \ n_1^k + n_2^k + \cdots + n_d^k = N\}.$$

Observe, that for $X_N := \lfloor N^{1/k} \rfloor$ one can write

$$\begin{aligned} r_k(N) &= \sum_{(n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d} \mathbb{1}_{S_N}(n_1, n_2, \ldots, n_d) \\ &= \sum_{n_1=1}^{X_N} \cdots \sum_{n_d=1}^{X_N} \int_0^1 \boldsymbol{e}\big(\xi(n_1^k + n_2^k + \cdots + n_d^k)\big) e^{-2\pi i \xi N} \mathrm{d}\xi \\ &= \int_0^1 \big(f_{X_N}(\xi)\big)^d \boldsymbol{e}(-\xi N) \mathrm{d}\xi, \end{aligned} \tag{1.31}$$

where the function f_{X_N} is given by

$$f_{X_N}(\xi) := \sum_{n=0}^{X_N} \boldsymbol{e}(\xi n^k). \tag{1.32}$$

Therefore, our task is to find the asymptotics for the integral (1.31). The main idea is to approximate f_{X_N} by its integral counterpart

$$\int_0^{X_N} \boldsymbol{e}(\xi x^k) \mathrm{d}x.$$

However, we cannot do it in a standard way, since the derivative of the phase function is equal to $kx^{k-1}\xi$ and may be large. In consequence, we are not able to control the quantity

$$\Big| \sum_{n=0}^{X_N} \boldsymbol{e}(\xi n^k) - \int_0^{X_N} \boldsymbol{e}(\xi x^k) \mathrm{d}x \Big|$$

in a satisfactory way. This obstacle was bypassed by Hardy and Littlewood. We follow their approach and decompose the unit interval $[0,1]$ into two disjoint sets, called the major arcs $\mathfrak{M}_{X_N}$ and the minor arcs $\mathfrak{m}_{X_N}$, and evaluate the integral over both sets separately. The major arcs consist of such real numbers $\xi \in [0,1]$ which can be "well approximated" by rational numbers a/q with $(a,q) = 1$. For $\xi \in \mathfrak{M}_{X_N}$ we are able to show that

$$f_{X_N}(\xi) \approx G(a/q) \int_0^{N^{1/k}} \boldsymbol{e}\big((\xi - a/q) x^k\big) \mathrm{d}x,$$

where a/q is a rational number which is a good approximation of ξ. On the other hand, on the minor arcs, which are the complement of the major arcs, the integral (1.31) is negligible.

Following this idea, for fixed $N \in \mathbb{N}$ and $\alpha \in (0, 1/4)$ we define the family of the *major arcs*

$$\mathfrak{M}_{X_N} := \bigcup_{1 \le q \le X_N^\alpha} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_{X_N}(a/q),$$

where

$$\mathfrak{M}_{X_N}(a/q) := \big\{\xi \in [0,1] : |\xi - a/q| \le X_N^{-k+\alpha}\big\} \quad \text{with} \quad q \le X_N^\alpha.$$

We see that if a/q varies over the rational fractions with small denominators ($1 \le q \le X_N^\alpha$ and $(a,q) = 1$) then $\mathfrak{M}_{X_N}(a/q)$ are disjoint. The *minor arcs* is the set

$$\mathfrak{m}_{X_N} = [0,1] \setminus \mathfrak{M}_{X_N}.$$

In view of this partition we obtain that

$$\begin{aligned} r_k(N) &= \int_{\mathfrak{M}_{X_N}} \big(f_{X_N}(\xi)\big)^d \boldsymbol{e}(-\xi N) \mathrm{d}\xi + \int_{\mathfrak{m}_{X_N}} \big(f_{X_N}(\xi)\big)^d \boldsymbol{e}(-\xi N) \mathrm{d}\xi \\ &:= M_k(N) + m_k(N). \end{aligned} \tag{1.33}$$

Now our task is to estimate $M_k(N)$ and $m_k(N)$ separately.

We start with showing that the contribution from the minor arcs is negligible that is

$$\big|m_k(N)\big| = \mathcal{O}\big(N^{d/k-1-\delta}\big),$$

for some $\delta > 0$. In order to do so we make use of **Weyl's inequality**.

Lemma 1.34 ([17, Lemma 3.1])**.** *Suppose that $\xi \in [0,1]$ has a rational approximation a/q satisfying*

$$(a,q) = 1, \quad q \in \mathbb{N}, \quad \Big|\xi - \frac{a}{q}\Big| \le \frac{1}{q^2}. \tag{1.35}$$

Then for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|f_{X_N}(\xi)| \le C_\varepsilon {X_N}^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{X_N} + \frac{q}{{X_N}^k}\right)^{\frac{1}{2^{k-1}}}. \tag{1.36}$$

The above inequality was established, in a less explicit form, in Weyl's groundbreaking work on the uniform distribution of sequences.

Let us observe that if $q \in \mathbb{N}$ from the condition (1.35) satisfies $X_N^\alpha < q \le X_N^{k-\alpha}$ then

$$|f_{X_N}(\xi)| \lesssim X_N^{1-\delta'}, \tag{1.37}$$

for some $\delta' > 0$. Now, if $\xi \in \mathfrak{m}_{X_N}$ then by Dirichlet's principle (Lemma 3.35) one can always find $1 \le q \le X_N^{k-\alpha}$ and $0 \le a \le q$ such that $(a,q) = 1$ and

$$\Big|\xi - \frac{a}{q}\Big| \le \frac{1}{qX_N^{k-\alpha}} \le \frac{1}{q^2}.$$

Hence the condition (1.35) is satisfied. Next, if we would have $q \le X_N^\alpha$ then $\xi \in \mathfrak{M}_{X_N}$ but it would contradict to that $\xi \in \mathfrak{m}_{X_N}$. Thus $q > X_N^\alpha$. Therefore, we see that for any $\xi \in \mathfrak{m}_{X_N}$ the inequality (1.37) holds. Now, let us write

$$\big|m_k(N)\big| = \Big|\int_{\mathfrak{m}_{X_N}} \big(f_{X_N}(\xi)\big)^d \boldsymbol{e}(-\xi N) \mathrm{d}\xi\Big| \le \sup_{\xi \in \mathfrak{m}_{X_N}} |f_{X_N}(\xi)|^{d-2^k} \int_0^1 |f_{X_N}(\xi)|^{2^k} \mathrm{d}\xi. \tag{1.38}$$

By Hua's lemma [17, Lemma 3.2] we know that for any $\varepsilon > 0$ one has

$$\int_0^1 |f_{X_N}(\xi)|^{2^k} \mathrm{d}\xi \lesssim_\varepsilon X_N^{2^k-k+\varepsilon}.$$

Combining the above estimate with (1.37) and applying in (1.38) yields

$$\big|m_k(N)\big| \lesssim X_N^{(1-\delta')(d-2^k)} X_N^{2^k-k+\varepsilon} = X_N^{d-k-\gamma} \lesssim N^{d/k-1-\gamma/k}$$

with $\gamma := \delta(d-2^k) - \varepsilon > 0$ for some small enough $\varepsilon > 0$. This shows the estimate for the minor arcs part.

Now we briefly sketch how to handle the major arcs part $M_k(N)$. Let $\xi \in \mathfrak{M}_{X_N}(a/q)$ with $q \le X_N^\alpha$. By splitting the set $\{0, \dots, X_N\}$ into congruence classes modulo q we may write

$$f_{X_N}(\xi) = \sum_{r=1}^{q} \boldsymbol{e}\big(\frac{a}{q} r^k\big) \sum_{-\frac{r}{q} < n \le \frac{X_N - r}{q}} \boldsymbol{e}\big((\xi - \frac{a}{q})(qn+r)^k\big). \tag{1.39}$$

Our aim is to replace the last sum by some integral. To do this we need to estimate the size of the error. The following approximation is a simple consequence of the mean value theorem. Let f be a differentiable function. Then for any $b > a$ we have

$$\left| \int_a^b f(z)dz - \sum_{a<n<b} f(k) \right| \lesssim (b-a) \max|f'(y)| + \max|f(y)|.$$

In our case $f(z) = \boldsymbol{e}\big((\xi - \frac{a}{q})(qn+r)^k\big)$ and since we are on the major arcs we have $|\xi - a/q| \le X_N^{k-\alpha}$. This implies that the approximation error between the sum and the integral is $\mathcal{O}(N^\delta)$ for some $\delta > 0$. Therefore, for any $\xi \in \mathfrak{M}_{X_N}(a/q)$ we have

$$f_{X_N}(\xi) = G(a/q) \int_0^{N^{1/k}} \boldsymbol{e}\big((\xi - a/q)x^k\big) \mathrm{d}x + \mathcal{O}(N^\delta).$$

If we use this estimate in $M_k(N)$ to replace f_{X_N} we get

$$M_k(N) = \sum_{q=1}^{X_N^\alpha} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} G(a/q)^d \int_{\mathfrak{M}(a/q)} \left(\int_0^{N^{1/k}} \boldsymbol{e}\big((\xi - a/q)x^k\big)\mathrm{d}x \right)^d \boldsymbol{e}(-\xi N)\mathrm{d}\xi + \mathcal{O}(X_N^{d-1-\delta}).$$

By the change of variables the integral is equal to

$$\boldsymbol{e}(-a/qN) \int_{|\xi| \le X_N^{k-\alpha}} \left(\int_0^{N^{1/k}} \boldsymbol{e}\big(\xi x^k\big)\mathrm{d}x \right)^d \boldsymbol{e}(-\xi N)\mathrm{d}\xi.$$

By slightly worsening the approximation error we may increase the range of integration in the above integral to $(-\infty, \infty)$ and the range of summation to $q \in [1, \infty)$ which gives

$$M_k(N) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} G(a/q)^d \boldsymbol{e}(-a/qN) \int_{-\infty}^{\infty} \left(\int_0^{N^{1/k}} \boldsymbol{e}\big(\xi x^k\big)\mathrm{d}x \right)^d \boldsymbol{e}(-\xi N)\mathrm{d}\xi + \mathcal{O}(X_N^{d-1-\delta'}),$$

for some $\delta' > 0$. It can be shown [17, Theorem 4.1] that one has

$$\int_{-\infty}^{\infty} \left(\int_0^{N^{1/k}} \boldsymbol{e}\big(\xi x^k\big)\mathrm{d}x \right)^d \boldsymbol{e}(-\xi N)\mathrm{d}\xi = \frac{\Gamma\left(1+\frac{1}{k}\right)^d}{\Gamma\left(\frac{d}{k}\right)} N^{d/k-1}.$$

This finishes the brief sketch of the Hardy–Littlewood circle method.

The Hardy–Littlewood circle method was used by Bourgain [4, 5, 6] to study the Fourier multipliers related to the averages

$$\frac{1}{2N+1}\sum_{n=-N}^{N} f(x-n^b), \quad x \in \mathbb{Z},$$

which are given by

$$\frac{1}{2N+1}\sum_{n=-N}^{N} \boldsymbol{e}(\xi n^b), \quad \xi \in \mathbb{T}.$$

It is easy to see the similarity to (1.32) which suggests that the approach described above is a suitable tool to study such multipliers.

1.3 Main results of the thesis

The thesis is based on the results from the following papers:

[D1] Mirek, M., Słomian, W., Szarek, T.Z. Some remarks on oscillation inequalities. *Ergodic Theory and Dynamical Systems*, 1–30 (2022). doi:10.1017/etds.2022.77

[D2] Słomian, W. Oscillation Estimates for Truncated Singular Radon Operators. *J. Fourier Anal. Appl.* **29**, 4 (2023).

[D3] Słomian, W. Bootstrap methods in bounding discrete Radon operators. *J. Funct. Anal.* **283**, 9 (2022).

Here we give a brief summary of each paper. In order to do so we introduce some notation. Let $d, k \in \mathbb{N}$ be fixed natural numbers. Let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_d)\colon \mathbb{Z}^k \to \mathbb{Z}^d \tag{1.40}$$

be a polynomial mapping, where each $\mathcal{P}_j\colon \mathbb{Z}^k \to \mathbb{Z}$ is a polynomial of k variables with integer coefficients such that $\mathcal{P}_j(0) = 0$. Let Ω be a non-empty bounded open convex subset of $\mathbb{R}^k$. Moreover, we assume that $B(0, c_\Omega) \subseteq \Omega \subseteq B(0,1) \subset \mathbb{R}^k$ for some $c_\Omega \in (0,1)$, where $B(x,t)$ denotes an open Euclidean ball in $\mathbb{R}^k$. For a given set Ω we define its dilates by setting

$$\Omega_t := \{x \in \mathbb{R}^k : t^{-1}x \in \Omega\}, \quad t > 0.$$

A typical choice of Ω_t is a ball of radius t for some norm on $\mathbb{R}^k$.

Now, for finitely supported functions $f\colon \mathbb{Z}^d \to \mathbb{C}$ and $t > 0$, we define the *discrete Radon average* by setting

$$M_t^{\mathcal{P}} f(x) := \frac{1}{|\Omega_t \cap \mathbb{Z}^k|}\sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(x - \mathcal{P}(y)), \quad x \in \mathbb{Z}^d, \tag{1.41}$$

where $|\Omega_t \cap \mathbb{Z}^k|$ denotes the number of lattice points from $\mathbb{Z}^k$ which are contained in Ω_t. In a similar fashion, we define the *discrete truncated Radon singular operator* by setting

$$H_t^{\mathcal{P}} f(x) := \sum_{y \in \Omega_t \cap \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y))K(y), \quad x \in \mathbb{Z}^d, \tag{1.42}$$

where $K\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a Calderón–Zygmund kernel which satisfies conditions (1.4), (1.5) and (1.6). In an analogous way, we define the continuous Radon operators. For a given smooth compactly supported function $f\colon \mathbb{R}^d \to \mathbb{C}$ the *continuous Radon average* of f is defined as

$$\mathcal{M}_t^{\mathcal{P}} f(x) := \frac{1}{|\Omega_t|}\int_{\Omega_t} f(x - \mathcal{P}(y))\mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{1.43}$$

and the *continuous singular Radon operator* of f is defined by setting

$$\mathcal{H}_t^{\mathcal{P}} f(x) := \mathrm{p.v.} \int_{\Omega_t} f(x - \mathcal{P}(y)) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^d, \tag{1.44}$$

where $K\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a Calderón–Zygmund kernel which satisfies conditions (1.4), (1.5) and (1.6).

The uniform oscillation inequalities for the Radon averages – the main result of [D1]

The article was written in cooperation of the author with M. Mirek and T.Z. Szarek. Main results of this paper are the uniform oscillation inequalities for the Radon averages. We state this result below. See (1.57) for the definition of the set $\mathfrak{S}_N(\mathbb{R}_+)$.

Theorem 1.45 ([D1,Theorem 1.4]). *Let $d, k \geq 1$ and let $\mathcal{P}$ be a polynomial mapping (1.40). For any $p \in (1, \infty)$ there is a constant $C_{p,d,k,\deg\mathcal{P}} > 0$ such that*

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O_{I,N}^2 (M_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,d,k,\deg\mathcal{P}} \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d), \tag{1.46}$$

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O_{I,N}^2 (\mathcal{M}_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{L^p(\mathbb{R}^d)} \leq C_{p,d,k,\deg\mathcal{P}} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d). \tag{1.47}$$

In particular, the implied constants in the inequalities above are independent of the coefficients of the polynomial mapping $\mathcal{P}$.

The proof of the above theorem is entirely up to the author and was his main contribution to the paper [D1]. It is worth noting that in [D1] this result is formulated for the ergodic averages. However by the Calderón transference principle, see Section 2.2, the above formulation is equivalent to [D1, Theorem 1.4]. Also, the formulation of [D1, Theorem 1.4] concerns only discrete averages, the inequality for the continuous averages (1.47) is proved along the way and is not explicitly formulated in [D1, Theorem 1.4]. Here we decided to state it as a separate result since it is more in line with the rest of the presentation.

The proof of the inequality (1.46) uses the methods developed by Mirek, Stein, Trojan and Zorin-Kranich [40, 43]. The main tools are the Hardy–Littlewood circle method applied with the Ionescu–Wainger multiplier theory (Theorem 2.71) and the Rademacher–Menshov inequality (2.36). In the proof of (1.47) we use the ideas of Jones, Seeger and Wright [32] to approximate the operator $\mathcal{M}_t^{\mathcal{P}}$ by Christ's dyadic martingales which are related to the group of dilations induced by the polynomial $\mathcal{P}$ – see Section 3.2.2. The detailed proof of Theorem 1.45 and the history of the problem are presented at the beginning of Chapter 3.

The uniform oscillation inequalities for the Radon singular integrals – the content of [D2]

The aim of the article [D2] was to establish a counterpart of Theorem 1.45 in the context of the singular integrals of Radon type $H_t^{\mathcal{P}}$ and $\mathcal{H}_t^{\mathcal{P}}$. The following theorem summarizes the main results of [D2].

Theorem 1.48 ([D2, Theorem 1.14]). *Let $d, k \geq 1$ and let $\mathcal{P}$ be a polynomial mapping (1.40). For any $p \in (1, \infty)$ there is a constant $C_{p,d,k,\deg\mathcal{P}} > 0$ such that*

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O_{I,N}^2 (H_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,d,k,\deg\mathcal{P}} \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d), \tag{1.49}$$

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O_{I,N}^2 (\mathcal{H}_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{L^p(\mathbb{R}^d)} \leq C_{p,d,k,\deg\mathcal{P}} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d). \tag{1.50}$$

In particular, the implied constants in the inequalities above are independent of the coefficients of the polynomial mapping $\mathcal{P}$.

Again, the proof of the inequality (1.49) uses the methods developed by Mirek, Stein, Trojan and Zorin-Kranich [40, 43]. In order to handle the oscillatory nature of the singular integral $H_t^{\mathcal{P}}$ we use the fact that the oscillation seminorm $O_{I,N}^2$ is translation invariant and we express this operator as an appropriate telescoping sum. This step, roughly speaking, reduces matters to study the difference operator

$$H_{n+1}^{\mathcal{P}} - H_n^{\mathcal{P}}, \quad n \in \mathbb{N}.$$

The Calderón–Zygmund conditions (1.4)–(1.6) which are satisfied by the kernel associated with $H_t^{\mathcal{P}}$ ensure that this operator have nice behavior. In particular, one obtains good decay estimates for the related Fourier multipliers. This fact combined with a careful analysis of the approximation errors allows us to handle the problem in the discrete setting.

In the proof of the inequality (1.50) we use the ideas of Jones, Seeger and Wright [32] which originates in the groundbreaking work of Duoandikoetxea and Rubio de Francia [18] about square function estimates for singular integral operators. As in the discrete setting we express $\mathcal{H}_t^{\mathcal{P}}$ as a telescoping sum

$$\mathcal{H}_n^{\mathcal{P}} = \sum_{k \geq n} T_k, \quad n \in \mathbb{N},$$

where each T_k is, roughly speaking, equal to $\mathcal{H}_{k+1}^{\mathcal{P}} - \mathcal{H}_k^{\mathcal{P}}$. Then we may employ a decomposition of the type

$$\mathcal{H}_n^{\mathcal{P}} f = \varphi_n * \Big(\sum_{k \in \mathbb{Z}} T_k f \Big) - \varphi_n * \Big(\sum_{k < n} T_k f \Big) + \sum_{j \geq n} (\delta_0 - \varphi_n) * T_k f,$$

where δ_0 is the Dirac measure at 0 and φ_n is an appropriate smooth function. As it turns out, each term of the above decomposition has behavior good enough to obtain the desired estimates. The detailed proof of Theorem 1.48 is given in Section 3.3.

Bootstrapping approach to seminorm estimates for discrete Radon averages – the content of [D3]

The aim of this paper was to give a new proof of known results about discrete Radon averages by using the so-called bootstrap approach – see Chapter 4 for more details. This paper is motivated by the work of Mirek, Stein and Zorin-Kranich [42] in which they proved, among others, that the bootstrap approach can be used to prove jump inequalities for continuous Radon operators. In [D1] we develop a new method of handling the seminorm inequalities, by using bootstrap approach. The main result of this paper is the proof of the following.

Theorem 1.51 ([D3,Theorem 1.6])**.** *Let $d, k \geq 1$ and let $\mathcal{P}$ be a polynomial mapping (1.40). Then for any $p \in (1, \infty)$ there is a constant $C_{p,d,k} > 0$ such that for any $f \in \ell^p(\mathbb{Z}^d)$ we have*

$$\sup_{\lambda > 0} \big\| \lambda N_\lambda (M_t^{\mathcal{P}} f : t \in \mathbb{R}_+)^{1/2} \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,d,k} \|f\|_{\ell^p(\mathbb{Z}^d)}, \tag{1.52}$$

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O_{I,N}^2 (M_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,d,k} \|f\|_{\ell^p(\mathbb{Z}^d)}. \tag{1.53}$$

See (1.57) for the definition of the set $\mathfrak{S}_N(\mathbb{R}_+)$. Moreover, for any $r \in (2, \infty)$ there is a constant $C_{p,d,k,r} > 0$ such that

$$\big\| V^r (M_t^{\mathcal{P}} f : t \in \mathbb{R}_+) \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_{p,d,k,r} \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d). \tag{1.54}$$

In addition, the constants mentioned above can be chosen to depend only on the degree of $\mathcal{P}$ and not on the coefficients of the polynomials $\mathcal{P}_j$.

Again, we want to emphasize that the novelty lies in the proof of the Theorem 1.51 and not in the theorem itself. The jump inequality (1.52) was proven by Mirek, Stein and Zorin-Kranich [43, Theorem 1.9] in 2020. The oscillation inequality (1.53) was first proven in [D1]. Therefore we give the a new proof of Theorem 1.45 in the case of the discrete Radon averages $M_t^{\mathcal{P}}$. The first proof of the r-variation estimate (1.54) in the full range $r > 2$ was given by Mirek, Stein and Trojan [40] (see also [65] for previous results). It is worth noting that the inequality (1.52) implies the r-variation estimates (1.54) for $r \in (2, \infty)$. Only the oscillation inequality (1.53) is not implied by the former ones.

The novelty of the presented approach lies in the fact that it is more "standalone" than the previous methods. Namely, if one wants to prove (1.52) and (1.53) by using the approach presented in [43] and [D1] (see Chapter 3 where the proof of (1.53) is based on the exposition from [D1]) one needs to show that for the continuous Radon averages $\mathcal{M}_t^{\mathcal{P}}$ the estimate

$$\Big\| \Big(\sum_{k \in \mathbb{N}} \big| (\mathcal{M}_{t_{k+1}}^{\mathcal{P}} - \mathcal{M}_{t_k}^{\mathcal{P}}) f \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}, \tag{1.55}$$

holds for every increasing sequence $0 < t_1 \le t_2 \le \cdots$ with $C_p > 0$ independent of the choice of that sequence. The inequality (1.55) can be proven by using the results from [42] and the detailed proof is quite long and relies heavily on the Littlewood–Paley theory. On the other hand, if one decides to prove the inequality (1.54) by following the approach presented in [40], then one needs to establish a vector-valued estimate of the form

$$\Big\| \Big(\sum_{n \in \mathbb{Z}} \sup_{t>0} \big| M_t^{\mathcal{P}} f_n \big|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^d)} \le C_p \Big\| \Big(\sum_{n \in \mathbb{Z}} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^d)}. \tag{1.56}$$

A whole separate paper [39] is devoted to proving the vector-valued inequality (1.56).

In the proof presented in Chapter 4 we do not use neither (1.55) nor (1.56) which makes the proof more elementary and self-contained, since it does not refer to vector-valued inequalities which are difficult to prove.

In order to prove Theorem 1.51 we exploit some methods introduced in [40] in the context of r-variations. The key ingredient is the discrete Littlewood–Paley theory which was formulated by Mirek [37]. We connect those tools with a variant of the bootstrapping lemma (Lemma 4.43) of Duoandikoetxea and Rubio de Francia [18].

1.4 Notation

Throughout the thesis we consistently use the notation introduced here.

Basic notation

We denote $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ := (0, \infty)$. For $d \in \mathbb{N}$ the sets $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{T}^d \equiv [-1/2, 1/2)^d$ have the usual meaning. For every $N \in \mathbb{N}$ we define

$$\mathbb{N}_N := \{1, \ldots, N\}.$$

For any $x \in \mathbb{R}$ the floor function is defined by

$$\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \le x\}.$$

For $u \in \mathbb{N}$ we define set

$$2^{u\mathbb{N}} := \{2^{un} : n \in \mathbb{N}\}.$$

We write $A \lesssim B$ to indicate that $A \le CB$ with a constant $C > 0$. The constant C may change from line to line. We write $\lesssim_\delta$ if the implicit constant depends on δ. Sometimes we will omit the subscript when a possible dependence on the related parameter is clearly allowed. For two functions $f\colon X \to \mathbb{C}$ and $g\colon X \to [0,\infty)$, we write $f = \mathcal{O}(g)$ if there is a constant $C > 0$ such that $|f(x)| \le Cg(x)$ for all $x \in X$.

Let $\mathbb{I} \subseteq \mathbb{R}$. For $N \in \mathbb{N} \cup \{\infty\}$ we denote by $\mathfrak{S}_N(\mathbb{I})$ the family of all strictly increasing sequences of length $N+1$ contained in $\mathbb{I}$. In other words

$$\mathfrak{S}_N(\mathbb{I}) := \big\{(I_1, I_2, \ldots, I_{N+1}) \in \mathbb{I}^{N+1} : I_1 < I_2 < \cdots < I_{N+1}\big\} \tag{1.57}$$

with the appropriate modification when $N = \infty$.

Throughout the thesis the symbol Ω will always denote a non-empty convex body (not necessarily symmetric) in $\mathbb{R}^k$, which simply means that Ω is a bounded convex open subset of $\mathbb{R}^k$. We will additionally assume that $B(0, c_\Omega) \subseteq \Omega \subseteq B(0,1) \subset \mathbb{R}^k$ for some $c_\Omega \in (0,1)$, where $B(x,t)$ denotes the open Euclidean ball in $\mathbb{R}^k$ centered at $x \in \mathbb{R}^k$ with radius $t > 0$. For $t > 0$, we define the dilate of Ω by

$$\Omega_t := \{x \in \mathbb{R}^k : t^{-1}x \in \Omega\}.$$

Later on, the symbol Ω_t will always refer to the dilate of the convex body Ω which satisfies the above conditions.

Euclidean and function spaces

The standard inner product, the corresponding Euclidean norm, and the maximum norm on $\mathbb{R}^d$ are denoted respectively, for any $x = (x_1, \ldots, x_d)$, $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, by

$$x \cdot \xi := \sum_{k=1}^{d} x_k \xi_k, \qquad \text{and} \qquad |x| := |x|_2 := \sqrt{x \cdot x}, \qquad \text{and} \qquad |x|_\infty := \max_{1 \le k \le d} |x_k|.$$

For any multi-index $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{N}^k$, by abuse of notation we will write $|\gamma| := \gamma_1 + \cdots + \gamma_k$. This will never cause confusions since the multi-indices will be always denoted by Greek letters.

Throughout the paper the d-dimensional torus $\mathbb{T}^d$ is a priori endowed with the periodic norm

$$\|\xi\| := \Big(\sum_{k=1}^{d} \|\xi_k\|^2\Big)^{1/2} \qquad \text{for} \qquad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{T}^d, \tag{1.58}$$

where $\|\xi_k\| = \operatorname{dist}(\xi_k, \mathbb{Z})$ for all $\xi_k \in \mathbb{T}$ and $k \in \{1, \ldots, d\}$. Identifying $\mathbb{T}^d$ with $[-1/2, 1/2)^d$ we see that the norm $\|\cdot\|$ coincides with the Euclidean norm $|\cdot|$ restricted to $[-1/2, 1/2)^d$.

In this paper all function spaces will be defined over $\mathbb{C}$. The triple $(X, \mathcal{B}(X), \mu)$ denotes a measure space X with a σ-algebra $\mathcal{B}(X)$ and a σ-finite measure μ. The space of all μ-measurable functions $f\colon X \to \mathbb{C}$ will be denoted by $L^0(X)$. The space of all functions in $L^0(X)$ whose modulus is integrable with p-th power is denoted by $L^p(X)$ for $p \in (0,\infty)$, whereas $L^\infty(X)$ denotes the space of all essentially bounded functions in $L^0(X)$. These notions can be extended to functions taking values in a normed vector space $(B, \|\cdot\|_B)$, for instance

$$L^p(X; B) := \big\{F \in L^0(X; B) : \|F\|_{L^p(X;B)} := \big\|\|F\|_B\big\|_{L^p(X)} < \infty\big\},$$

where $L^0(X;B)$ denotes the space of measurable functions from X to B (up to almost everywhere equivalence).

For any $p \in [1,\infty]$ we define the weak-L^p space of measurable functions on X by setting

$$L^{p,\infty}(X) := \{f : X \to \mathbb{C}\colon \|f\|_{L^{p,\infty}(X)} < \infty\},$$

where for any $p \in [1, \infty)$ we have

$$\|f\|_{L^{p,\infty}(X)} := \sup_{\lambda>0} \lambda\mu(\{x \in X : |f(x)| > \lambda\})^{1/p} \qquad \text{and} \qquad \|f\|_{L^{\infty,\infty}(X)} := \|f\|_{L^\infty(X)}.$$

In our case we will mainly have $X = \mathbb{R}^d$ or $X = \mathbb{T}^d$ equipped with the Lebesgue measure, and $X = \mathbb{Z}^d$ endowed with counting measure. If X is endowed with a counting measure we will abbreviate $L^p(X)$ to $\ell^p(X)$ and $L^p(X; B)$ to $\ell^p(X; B)$ and $L^{p,\infty}(X)$ to $\ell^{p,\infty}(X)$.

If $T\colon B_1 \to B_2$ is a continuous linear map between two normed vector spaces B_1 and B_2, we use $\|T\|_{B_1 \to B_2}$ to denote its operator norm.

Fourier transform and convolutions

We will use the convention that $\boldsymbol{e}(z) = e^{2\pi \boldsymbol{i} z}$ for every $z \in \mathbb{C}$, where $\boldsymbol{i}^2 = -1$. Let $\mathcal{F}_{\mathbb{R}^d}$ denote the Fourier transform on $\mathbb{R}^d$ defined for any $f \in L^1(\mathbb{R}^d)$ and for any $\xi \in \mathbb{R}^d$ as

$$\mathcal{F}_{\mathbb{R}^d} f(\xi) := \int_{\mathbb{R}^d} f(x)\boldsymbol{e}(x \cdot \xi)\mathrm{d}x.$$

Sometimes we will write $\hat{f}(\xi)$ instead of $\mathcal{F}_{\mathbb{R}^d}$. For $f \in L^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ the inverse Fourier transform on $\mathbb{R}^d$ is given by

$$\mathcal{F}^{-1}_{\mathbb{R}^d} f(x) := \int_{\mathbb{R}^d} f(\xi)\boldsymbol{e}(-\xi \cdot x)\mathrm{d}\xi.$$

If $f \in \ell^1(\mathbb{Z}^d)$ we define the discrete Fourier transform (Fourier series) $\mathcal{F}_{\mathbb{Z}^d}$, for any $\xi \in \mathbb{T}^d$, by setting

$$\mathcal{F}_{\mathbb{Z}^d} f(\xi) := \sum_{x \in \mathbb{Z}^d} f(x)\boldsymbol{e}(x \cdot \xi).$$

For $f \in L^1(\mathbb{T}^d)$ the inverse discrete Fourier transform (Fourier coefficients) is given by

$$\mathcal{F}^{-1}_{\mathbb{Z}^d} f(x) := \int_{\mathbb{T}^d} f(\xi)\boldsymbol{e}(-\xi \cdot x)\mathrm{d}\xi, \quad x \in \mathbb{Z}^d.$$

The continuous convolution of two functions $f\colon \mathbb{R}^d \to \mathbb{C}$ and $f\colon \mathbb{R}^d \to \mathbb{C}$ is given by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y), \quad x \in \mathbb{R}^d.$$

It is known that if $f, g \in L^1(\mathbb{R}^d)$ then the Fourier transform convolution intertwines with the convolution and one has

$$\mathcal{F}_{\mathbb{R}^d}(f * g) = \mathcal{F}_{\mathbb{R}^d}(f)\mathcal{F}_{\mathbb{R}^d}(g).$$

Similarly, we define the discrete convolution of two functions (sequences) $f, g \in \ell^1(\mathbb{Z}^d)$ by setting

$$(f * g)(x) := \sum_{y \in \mathbb{Z}^d} f(x - y)g(y), \quad y \in \mathbb{Z}^d.$$

We do not use different symbols for the convolutions since the meaning of the symbol $*$ will be always clear from the context. As in the case of the continuous convolution and the Fourier transform on $\mathbb{R}^d$, the discrete convolution intertwines with the Fourier transform on $\mathbb{Z}^d$. Namely, for $f, g \in \ell^1(\mathbb{Z}^d)$ one has

$$\mathcal{F}_{\mathbb{Z}^d}(f * g) = \mathcal{F}_{\mathbb{Z}^d}(f)\mathcal{F}_{\mathbb{Z}^d}(g).$$

Chapter 2

Preliminaries

In this chapter we present and discuss some general results concerning seminorms of the oscillation type like the oscillation seminorm $O_{I,N}$, r-variations V^r and the jump seminorm J_p^2. We pay special attention to some basic properties of those seminorms which are widely used in the following chapters. We also formulate and prove the Calderón transference principle [7] which allows us to deduce seminorm inequalities formulated in the language of ergodic theory from seminorm inequalities formulated in the language of discrete harmonic analysis. It turns out that this procedure leads to the discrete operators of Radon type given by (1.41) and (1.42). Radon operators associated with an arbitrary polynomial mapping may by problematic the work with. However, it turns out that by using so-called lifting procedure the study can be narrowed to the special class of polynomials, called canonical polynomials. We use this opportunity to discuss some properties of Radon operators related to canonical polynomials. We will be particularly interested in their Fourier multipliers and estimates for them. At the end of the chapter we state two sampling principles: one due to Magyar, Stein and Wainger, and the second one due to Ionescu and Wainger. Those two sampling principles are irreplaceable tools in proving Theorems 1.45, 1.48 and 1.51.

The organization of this chapter is as follows. In Section 2.1 we gather basic information about seminorms of oscillation type. We also introduce some notation which allows us to write the results in the sequel in a more concise way. Finally, we state the Calderón transference principle for dynamical systems. In Section 2.3 we state and prove the lifting lemma which In Section 2.4 we collect some basic information about Fourier multipliers related to Radon operators. Finally, in the last section we formulate, without proofs, the Magyar–Stein–Wainger sampling principle and Ionescu–Wainger sampling principle which are widely used in the following chapters.

2.1 Seminorms and the pointwise convergence

We begin with recalling the notion of a seminorm on a vector space. Let X be a vector space over $\mathbb{C}$. A real-valued function $p\colon X \to \mathbb{R}$ is called a seminorm if the following two conditions are satisfied:

1. For any $x, y \in X$ the function p satisfies the *triangle inequality*, that is

$$\rho(x+y) \leq \rho(x) + \rho(y).$$

2. The function p is *absolutely homogeneous*, that is for any $a \in \mathbb{C}$ and any $x \in X$ we have

$$\rho(ax) = |a|\rho(x).$$

Those conditions imply that $p(0) = 0$ and $p(x) \geq 0$ for any $x \in X$. The key difference between the norm and the seminorm is that the latter does not have the property of the point-separating. An important example of a seminorm which is not a norm is the following. For any $f \in \ell^\infty(\mathbb{N})$ we define

$$p(f) := \sup_{n\in\mathbb{N}} |f(n) - f(1)|.$$

Then p is a seminorm on $\ell^\infty(\mathbb{N})$ which does not separate the sequences differing by a constant.

Another example is the oscillation seminorm which is the most fundamental of the seminorms we consider in the sequel.

Definition 2.1. Let $\mathbb{I} \subseteq \mathbb{R}$. For an increasing sequence $I = (I_j : j \in \mathbb{N}) \subseteq \mathbb{I}$ and $N \in \mathbb{N} \cup \{\infty\}$, the *truncated oscillation seminorm* of a function $f : \mathbb{I} \to \mathbb{C}$ is defined by

$$O^2_{I,N}(f(t) : t \in \mathbb{I}) := \Big(\sum_{j=1}^{N} \sup_{\substack{I_j \le t < I_{j+1} \\ t \in \mathbb{I}}} |f(t) - f(I_j)|^2 \Big)^{1/2}. \tag{2.2}$$

One can easily check that for any $N \in \mathbb{N}$ and any increasing sequence $I \subseteq \mathbb{I}$ the function $O^2_{I,N}$ is a seminorm. It the late 80's Bourgain [4] observed that the oscillation seminorm $O_{I,N}$ can be effectively used to study the problem of convergence of a given sequence. Let $f \in \ell^\infty(\mathbb{N})$. Then it is easy to see that one has

$$O^2_{I,N}(f(t) : t \in \mathbb{N}) \le 2\|f\|_{\ell^\infty(\mathbb{N})} N^{1/2},$$

for any $N \in \mathbb{N}$ and any sequence $I \subseteq \mathbb{N}$. On the other hand, if we assume that for some $c \in [0, 1/2)$ we have that

$$O^2_{I,N}(f(t) : t \in \mathbb{N}) \lesssim \|f\|_{\ell^\infty(\mathbb{N})} N^c$$

for all $N \in \mathbb{N}$ and $I \subseteq \mathbb{N}$. Since $c < 1/2$, this suggests that

$$\sup_{\substack{I_j \le t < I_{j+1} \\ t \in \mathbb{N}}} |f(t) - f(I_j)| \to 0 \quad \text{as} \quad j \to \infty,$$

which we can indeed prove, referring to the fact that the assumed estimate should hold for all $I \subseteq \mathbb{N}$. Thus, f satisfies the Cauchy condition and as a consequence it is a convergent sequence.

Proposition 2.3. *Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space and let $(\mathfrak{a}_t(x) : t \in \mathbb{R}) \subseteq \mathbb{C}$ be a family of measurable functions on X. Suppose that there are $p \in [1, \infty)$ and constants $c \in [0, 1/2)$ and $C_p > 0$ such that*

$$\sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \|O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{R}_+)\|_{L^p(X)} \le N^c C_p.$$

Then the limit

$$\lim_{t\to\infty} \mathfrak{a}_t(x) \tag{2.4}$$

exists for μ-almost every $x \in X$.

Proof. Suppose for a contradiction that the limit in (2.4) does not exists. Since μ is a σ-finite measure then there exists $X_0 \subseteq X$ with $\mu(X_0) < \infty$ and small $\delta > 0$ such that

$$\mu\big(\{x \in X_0 : \lim_{n\to\infty} \sup_{n \le s,t} |\mathfrak{a}_s(x) - \mathfrak{a}_t(x)| > 2\delta\}\big) > 2\delta.$$

For $n \in \mathbb{N}$ we denote

$$A_n = \{x \in X_0 : \sup_{n \le s,t} |\mathfrak{a}_s(x) - \mathfrak{a}_t(x)| > 2\delta\}.$$

Note that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$, and by the continuity of measure one has

$$\lim_{n\to\infty} \mu\big(\{x \in X_0 \colon \sup_{n\le s,t} |\mathfrak{a}_s(x) - \mathfrak{a}_t(x)| > 2\delta\}\big) > 2\delta.$$

Observe that for any $n \in \mathbb{N}$ we have the following inclusion

$$\{x \in X_0 \colon \sup_{n\le s,t} |\mathfrak{a}_s(x) - \mathfrak{a}_t(x)| > 2\delta\} \subseteq \{x \in X_0 \colon \sup_{n\le t} |\mathfrak{a}_t(x) - \mathfrak{a}_n(x)| > \delta\}$$

and hence there is a $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ we have

$$\mu\big(\{x \in X_0 \colon \sup_{n\le t} |\mathfrak{a}_t(x) - \mathfrak{a}_n(x)| > \delta\}\big) > \delta.$$

Next, for $m, n \in \mathbb{N}$ we define

$$B_m^n = \{x \in X \colon \sup_{n\le t<m} |\mathfrak{a}_t(x) - \mathfrak{a}_n(x)| > \delta\}.$$

We observe that $B_m^n \subseteq B_{m+1}^n$ for every $m, n \in \mathbb{N}$ and once again using continuity of measure we get for every $n \ge n_0$ that

$$\lim_{m\to\infty} \mu(B_m^n) = \mu\big(\{x \in X \colon \sup_{n\le t} |\mathfrak{a}_t(x) - \mathfrak{a}_n(x)| > \delta\}\big) > \delta. \tag{2.5}$$

Consequently, there is $m_1 > n_0$ such that

$$\mu\big(\{x \in X \colon \sup_{n_0\le t<m_1} |\mathfrak{a}_t(x) - \mathfrak{a}_{n_0}(x)| > \delta\}\big) > \delta.$$

Using (2.5) recursively (in the next step we use $n = m_1$) one can construct a strictly increasing sequence $(I_j : j \in \mathbb{N}) \subset \mathbb{R}_+$ with $I_1 = n_0$ such that for every $j \in \mathbb{N}$ we have

$$\mu\big(\{x \in X \colon \sup_{I_j\le t<I_{j+1}} |\mathfrak{a}_t(x) - \mathfrak{a}_{I_j}(x)| > \delta\}\big) > \delta. \tag{2.6}$$

Then by (2.6) we obtain for every $N \in \mathbb{N}$ and $q = \min\{p, 2\}$ that

$$\begin{aligned} N\delta^{p+1} = \sum_{j=1}^{N} \delta^{p+1} &\le \int_X \sum_{j=1}^{N} \sup_{I_j\le t<I_{j+1}} |\mathfrak{a}_t(x) - \mathfrak{a}_{I_i}(x)|^p \mathrm{d}\mu(x) \\ &\le N^{1-q/2} \sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)} \big\|O_{I,N}^2(\mathfrak{a}_t : t \in \mathbb{R}_+)\big\|_{L^p(X)}^p. \end{aligned}$$

Thus

$$N\delta^{p+1} \le N^{1-q/2} \sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)} \big\|O_{I,N}^2(\mathfrak{a}_t : t \in \mathbb{R}_+)\big\|_{L^p(X)}^p \le N^{1-q/2} N^{c/p} C_p^p.$$

Since $c \in [0, 1/2)$, by letting $N \to \infty$ we get that $\delta = 0$ which gives us a contradiction. This completes the proof of Proposition 2.3. □

Let us note that the above proof works also for the oscillation seminorm taken over all sequences $I = (I_j : j \in \mathbb{N})$ such that $N_{j+1} > 2N_j$. It was this form of the oscillation seminorm that was first used by Bourgain [4]. Moreover, it can be easily seen that we are not restricted to the condition $t \in \mathbb{R}_+$. The proof works also when $t \in \mathbb{N}$ or $t \in 2^{\mathbb{N}}$.

Another remarkable feature of the oscillation seminorm is the fact that it dominates the maximal function.

Proposition 2.7. *Let $(X, \mathcal{B}(X), \mu)$ be a σ-finite measure space and let $(\mathfrak{a}_t(x) : t \in \mathbb{R}) \subseteq L^p(X)$ be a family of measurable functions on X. Let $\mathbb{I} \subseteq \mathbb{R}$ and $|\mathbb{I}| \geq 2$, then for every $p \in [1, \infty)$ and any $N \in \mathbb{N}$ such that $|\mathbb{I}| \geq N + 1$ we have*

$$\| \sup_{t \in \mathbb{I} \setminus \sup \mathbb{I}} |\mathfrak{a}_t| \|_{L^p(X)} \leq \sup_{t \in \mathbb{I}} \|\mathfrak{a}_t\|_{L^p(X)} + \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \big\| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \big\|_{L^p(X)}. \tag{2.8}$$

Proof. Let $a = \inf \mathbb{I}$ and $b = \sup \mathbb{I}$. Since $|\mathbb{I}| \geq 2$ we see that $a < b$. Next, one can choose a non-increasing sequence $(a_n : n \in \mathbb{N}) \subseteq \mathbb{I}$ and an increasing sequence $(b_n : n \in \mathbb{N}) \subseteq \mathbb{I}$ such that $a \leq a_n \leq b_n \leq b$ for every $n \in \mathbb{N}$ and satisfying

$$\lim_{n \to \infty} a_n = a \text{ and } \lim_{n \to \infty} b_n = b.$$

Moreover, if $a \in I$, then we assume that $a_n = a$ for all n. By the monotone convergence theorem we get

$$\begin{aligned}
\big\| \sup_{t \in \mathbb{I} \setminus \sup \mathbb{I}} |a_t| \big\|_{L^p(X)} &= \lim_{n \to \infty} \| \sup_{t \in [a_n, b_n) \cap \mathbb{I}} |\mathfrak{a}_t| \|_{L^p(X)} \\
&\leq \lim_{n \to \infty} \|\mathfrak{a}_{a_n}\|_{L^p(X)} + \lim_{n \to \infty} \big\| \sup_{t \in [a_n, b_n) \cap \mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{a_n}| \big\|_{L^p(X)} \\
&\leq \sup_{n \to \infty} \|\mathfrak{a}_{a_n}\|_{L^p(X)} + \lim_{n \to \infty} \big\| \sup_{t \in [a_n, b_n) \cap \mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{a_n}| \big\|_{L^p(X)}.
\end{aligned}$$

Now, let $n \in \mathbb{N}$ be fixed natural number. Let $I \in \mathfrak{S}_1(\mathbb{I})$ be defined as

$$I_1 = a_n \quad \text{and} \quad I_2 = b_n.$$

Then we can write

$$\big\| \sup_{t \in [a_n, b_n) \cap \mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{a_n}| \big\|_{L^p(X)} = \big\| O^2_{I,1}(\mathfrak{a}_t : t \in \mathbb{I}) \big\|_{L^p(X)} \leq \sup_{I \in \mathfrak{S}_1(\mathbb{I})} \big\| O^2_{I,1}(\mathfrak{a}_t : t \in \mathbb{I}) \big\|_{L^p(X)}$$

which ends the proof of (2.8) in the case when $N = 1$. For $N > 1$ it follows by the fact that for any $M, N \in \mathbb{N}$ with $M \leq N$ we have

$$\sup_{I \in \mathfrak{S}_M(\mathbb{I})} \big\| O^2_{I,M}(\mathfrak{a}_t : t \in \mathbb{I}) \big\|_{L^p(X)} \leq \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \big\| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \big\|_{L^p(X)},$$

provided that $|\mathbb{I}| \leq N + 1$. This completes the proof. □

A closely related concept to the oscillation seminorm is the r-variation seminorm. Let us recall its definition.

Definition 2.9. Let $\mathbb{I} \subseteq \mathbb{R}$. For any $r \in [1, \infty]$ the *r-variational seminorm* V^r of a function $f : \mathbb{I} \to \mathbb{C}$ is defined by

$$V^r(f(t) : t \in \mathbb{I}) := \sup_{N \in \mathbb{N}} \sup_{\substack{t_1 \leq \cdots \leq t_{N+1} \\ t_j \in \mathbb{I}}} \Big(\sum_{j=1}^{N} |f(t_{j+1}) - f(t_j)|^2 \Big)^{1/2}.$$

In the case of $r = \infty$ we consider an appropriate modification related to the ℓ^∞ norm.

Clearly, for any $r \in [1, \infty]$ the r-variation is a seminorm. Moreover, we have the following pointwise estimate for the maximal function

$$\sup_{t \in \mathbb{I}} |f(t)| \leq V^r(f(t) : t \in \mathbb{I}) + |f(t_0)|, \quad \text{for any} \quad t_0 \in \mathbb{I}. \tag{2.10}$$

The variation seminorm is closely related to the oscillation seminorm. This can be seen by from the following observation. Let $\mathbb{I} \subseteq \mathbb{R}$. Let $\mathbb{I}_s \subseteq \mathbb{I}$ be finite. Then for any $r \geq 2$ by Hölder's inequality, we have

$$O^2_{I,N}(f(t) : t \in \mathbb{I}_s) \leq N^{1/2 - 1/r} V^r(f(t) : t \in \mathbb{I}_s).$$

Taking an ascending net of sets $\mathbb{I}_s \subseteq \mathbb{I}$ such that $\lim_{s\to\infty} \mathbb{I}_s = \mathbb{I}$ we get that

$$O^2_{I,N}(f(t) : t \in \mathbb{I}) \le N^{1/2-1/r} V^r(f(t) : t \in \mathbb{I}). \tag{2.11}$$

The inequality (2.11) was first observed by Bourgain [4]. This inequality implies that if for the family of measurable functions $(\mathfrak{a}_t(x) : t \in \mathbb{I})$ we are able to show the following r-variation inequality

$$\Big\| V^r(\mathfrak{a}_t : t \in \mathbb{I}) \Big\|_{L^p(X)} \le C_p \tag{2.12}$$

for some $r \in [2, \infty)$ and $p \in [1, \infty]$. Then we have the following oscillation inequality

$$\sup_{I \in \mathfrak{S}_N(\mathbb{I})} \| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \|_{L^p(X)} \le N^{1/2-1/r} C_p$$

which by Proposition 2.3 implies the pointwise convergence of the family $(\mathfrak{a}_t)_{t\in\mathbb{I}}$. Consequently, the problem of establishing the pointwise convergence can reduced to proving the r-variational estimates.

The r-variation seminorm was known before Bourgain's work. The seminorm V^r is a well known object from the martingale theory and according to Qian [50] its origin can be traced back to Wiener [62]. The r-variations for a family of bounded martingales $(\mathfrak{f}_n : X \to \mathbb{C} : n \in \mathbb{N})$ were studied in mid 70's by Lépingle [34] who showed that for all $r \in (2, \infty)$ and $p \in (1, \infty)$ there is a constant $C_{p,r} > 0$ such that the following inequality holds

$$\| V^r(\mathfrak{f}_n : n \in \mathbb{N}) \|_{L^p(X)} \le C_{p,r} \sup_{n\in\mathbb{N}} \| \mathfrak{f}_n \|_{L^p(X)}. \tag{2.13}$$

It is worth noting that the range $r \in (2, \infty)$ in Lépingle inequality is sharp, that is we cannot take $r = 2$. A counterexample can be found in the work of Qian [50]. A similar thing happens for many families of operators in harmonic analysis. For instance, Jones and Wang [27] studied the r-variational estimates for the Fejér and Poisson kernels and they proved that the seminorm V^2 is an unbounded operator on $L^p(\mathbb{T})$. For this reason, we usually do not expect the estimates for V^2 to be finite. As we have already mentioned, for the problem of the pointwise convergence, it is not very relevant since it is enough to show the r-variation estimates for **some** $r \in [2, \infty)$. However, if we leave aside the problem of the pointwise convergence and look at the inequality (2.11) with $r = 2$ we get

$$O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \le V^2(\mathfrak{a}_t : t \in \mathbb{I}) \tag{2.14}$$

for any $N \in \mathbb{N}$ and any sequence $I \in \mathfrak{S}_N(\mathbb{I})$. Hence an L^p-estimate for the 2-variation V^2 would imply the following uniform oscillation inequality

$$\sup_{N\in\mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \Big\| O^2_{I,N}(\mathfrak{a}_t : t \in \mathbb{I}) \Big\|_{L^p(X)} \le C_p. \tag{2.15}$$

Although, in most cases we cannot expect the L^p-boundedness of the 2-variations V^2 it turns out that we may expect the uniform oscillation inequality. In the case of bounded martingales $(\mathfrak{f}_n : X \to \mathbb{C} : n \in \mathbb{N})$ it was shown by Jones, Kaufman, Rosenblatt and Wierdl [28] (see also [D1]) that for every $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\sup_{N\in\mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{N})} \| O^2_{I,N}(\mathfrak{f}_n : n \in \mathbb{N}) \|_{L^p(X)} \le C_p \sup_{n\in\mathbb{N}} \| \mathfrak{f}_n \|_{L^p(X)}. \tag{2.16}$$

This result motivated the investigation of the uniform oscillation inequalities for various operators, see [28, 29, 30, 31] and the references given there. In particular, Campbell, Jones, Reinhold and Wierdl [9] investigated oscillation inequalities for the truncated Hilbert transform $\mathcal{H}_t$ given by

$$\mathcal{H}_t f(x) := \text{p.v.} \frac{1}{\pi} \int_{|y|<t} \frac{f(x-y)}{y} \mathrm{d}y, \quad x \in \mathbb{R}, \quad t > 0. \tag{2.17}$$

They proved that for any $p \in (1, \infty)$ there is a constant $C_p > 0$ such that

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \big\| O^2_{I,N}(\mathcal{H}_t f : t \in \mathbb{R}_+) \big\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}, \quad f \in L^p(\mathbb{R}). \tag{2.18}$$

Those results suggest that the oscillation seminorm can be some kind of endpoint at $r = 2$ for r-variations V^r. However, what is the exact relation between those seminorms, except the inequality (2.11), is currently an open problem – see [D1] and [44] for more details.

As we have seen, in order to handle the non-uniform oscillation inequalities one can use the r-variation seminorms which are easier to study due to their closer relationship to the ℓ^r-norms. Bourgain in his groundbreaking series of works [4, 5, 6] observed that the r-variation seminorm is related to another object "of the seminorm type" called the jump quasi-seminorm. In order to define it we need the notion of the jump counting function. Let $\lambda > 0$ and $\mathbb{I} \subseteq \mathbb{R}$ be given. The λ-*jump counting function* of a function $f \colon \mathbb{I} \to \mathbb{C}$ is defined by

$$N_\lambda(f(t) : t \in \mathbb{I}) := \sup\{J \in \mathbb{N} \,|\, \exists_{\substack{t_0 < \cdots < t_J \\ t_j \in \mathbb{I}}} : \min_{0 < j \le J} |f(t_j) - f(t_{j-1})| \ge \lambda\}. \tag{2.19}$$

The function N_λ counts the maximal number of jumps, the size of each of them being at least λ.

Definition 2.20. Let $\mathbb{I} \subseteq \mathbb{R}$ and let $(\mathfrak{a}_t(x) : t \in \mathbb{I}) \subseteq \mathbb{C}$ be a family of measurable functions on X. The *jump quasi-seminorm* of the family $(\mathfrak{a}_t(x) : t \in \mathbb{I})$ is defined by

$$J^2_{L^p(X)}(\mathfrak{a}_t : t \in \mathbb{I}) := \sup_{\lambda > 0} \big\| \lambda \big(N_\lambda(\mathfrak{a}_t(x) : t \in \mathbb{I})\big)^{1/2} \big\|_{L^p(X)}. \tag{2.21}$$

It is easy to see that $J^2_{L^p(X)}$ is absolutely homogeneous. Unfortunately, it does not satisfy the triangle inequality. However, it was proven by Mirek, Stein and Zorin-Kranich [41, Corollary 2.2] that there is a constant $C > 0$ such for any $N \in \mathbb{N} \cup \{\infty\}$ and any sequence of families $(\mathfrak{a}^n_t(x) : t \in \mathbb{I})_{n \in \mathbb{N}}$ we have

$$J^2_{L^p(X)}\Big(\sum_{n=1}^N \mathfrak{a}^n_t : t \in \mathbb{I} \Big) \le C \sum_{n=1}^N J^2_{L^p(X)}(\mathfrak{a}^n_t : t \in \mathbb{I}).$$

In particular, this justifies the name quasi-seminorm[1]. Since C does not depend on N, we say that this constant is absolute with respect to taking sums of more than two elements.

Although this is not apparent at first glance the jump quasi-seminorm is closely related to the 2-variation seminorm V^2. Namely, one has

$$J^2_{L^p(X)}(\mathfrak{a}_t : t \in \mathbb{I}) \le \|V^2(\mathfrak{a}_t : t \in \mathbb{I})\|_{L^p(X)} \tag{2.22}$$

The remarkable feature of the jump quasi-seminorm, observed by Bourgain [6], is that, in some sense, the inequality (2.22) can be reversed. Namely, for any $r \in (2, \infty]$ and any $p \in [1, \infty)$ we have the following inequality

$$\|V^r(\mathfrak{a}_t : t \in \mathbb{I})\|_{L^{p,\infty}(X)} \lesssim_{p,r} \sup_{\lambda > 0} \|\lambda N_\lambda(\mathfrak{a}_t : t \in \mathbb{I})^{1/2}\|_{L^{p,\infty}(X)} \tag{2.23}$$

where the implicit constant depends only on r and p. It can be shown that we can not replace the weak $L^{p,\infty}$ spaces by L^p. The proof of the inequality (2.23) involves the notion of the interpolation spaces and can be found in [41, Lemma 2.12]. Now if $(T_t)_{t \in \mathbb{I}}$ is a family of linear operators acting on $L^1(X) + L^\infty(X)$ then by the inequality (2.23) *a priori* jump estimates

$$J^2_{L^p(X)}(T_t f : t \in \mathbb{I}) \lesssim_p \|f\|_{L^p(X)}, \tag{2.24}$$

[1]The name quasinorm itself refers to the function $p : X \to \mathbb{R}$ which is homogeneous and satisfies $p(x+y) \le K(p(x)+p(y))$ for some $K > 0$. The latter condition is weaker than $p\big(\sum_{n=1}^N x_n\big) \le K \sum_{n=1}^N p(x_n)$ uniformly in $N \in \mathbb{N}$.

in some range $p \in (p_0, p_1)$ with $p_0 < p_1$, imply the weak r-variational estimates

$$\|V^r(T_t f : t \in \mathbb{I})\|_{L^{p,\infty}(X)} \lesssim_{p,r} \|f\|_{L^p(X)}, \tag{2.25}$$

for the same range $p \in (p_0, p_1)$ and for all $r \in (2, \infty]$. By Marcinkiewicz's interpolation theorem the estimate (2.25) implies that in the same range of p's and for all $r \in (2, \infty]$ one has

$$\|V^r(T_t f : t \in \mathbb{I})\|_{L^p(X)} \lesssim_{p,r} \|f\|_{L^p(X)}.$$

This argument shows that the jump estimates (2.24) can be interpreted as an endpoint for the r-variations when $r \to 2$ from above. As in the case of the oscillation seminorm many family of operators satisfy jump inequalities even though the 2-variations may be unbounded. For example, the martingale case $(\mathfrak{f}_n : X \to \mathbb{C} : n \in \mathbb{N})$ was studied by Pisier and Xu [49] on $L^2(X)$ and by Bourgain [6, Inequality (3.5)] on $L^p(X)$ with $p \in (1, \infty)$. More precisely, for every $p \in (1, \infty)$ there exists a constant $C_p > 0$ such that

$$J^2_{L^p(X)}(\mathfrak{f}_n : n \in \mathbb{N}) \le C_p \sup_{n \in \mathbb{N}} \|\mathfrak{f}_n\|_{L^p(X)}. \tag{2.26}$$

Those results motivated the study of the jump inequalities in harmonic theory. At this point is it is worth mentioning the work of Jones, Seeger and Wright [32] in which they established the jump inequalities for a wide range of operators in harmonic analysis including continuous operators of Radon type.

Now, let us state some properties of the mentioned seminorms. Most of them can be expressed in an unified way and in order to do so we introduce some common notation. Let $\mathbb{I} \subseteq \mathbb{R}$. For a given family of measurable functions $(\mathfrak{a}_t : t \in \mathbb{I}) \subset L^p(X)$ (in the thesis we use only $X = \mathbb{Z}^d$ or $X = \mathbb{R}^d$) we write

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{I})$$

to represent one of the following quantities:

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \big\|O^2_{I,N}(\mathfrak{a}_t(x) : t \in \mathbb{I})\big\|_{L^p(X)}, \quad J^2_{L^p(X)}(\mathfrak{a}_t : t \in \mathbb{I}) \text{ or } \|V^r(\mathfrak{a}_t(x) : t \in \mathbb{I})\|_{L^p(X)}$$

where $r \in (2, \infty]$ is fixed. In the sequel we will keep this notation in order to say that some properties and facts holds for all kind of introduced seminorms or quasi-seminorms.

It is clear that one has

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{I}) \le \|V^2(\mathfrak{a}_t(x) : t \in \mathbb{I})\|_{L^p(X)} \tag{2.27}$$

and if $\mathbb{J} \subset \mathbb{R}$ is countable then

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{J}) \le \|V^2(\mathfrak{a}_t(x) : t \in \mathbb{J})\|_{L^p(X)} \le 2\Big\|\Big(\sum_{t \in \mathbb{J}} |\mathfrak{a}_t|^2\Big)^{1/2}\Big\|_{L^p(X)}. \tag{2.28}$$

Obviously, $\mathcal{S}^p_X$ is monotonous with respect to$\mathbb{I}$. Namely, if $\mathbb{I}_1 \subseteq \mathbb{I}_2$, then

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{I}_1) \le \mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{I}_2).$$

The next important propriety is subadditivity.

Fact 2.29. *Let $p \in [1, \infty)$ and $\mathbb{I} \subseteq \mathbb{R}$. Let $N \in \mathbb{N} \cup \{\infty\}$. For any sequence of families $(\mathfrak{a}^n_t(x) : t \in \mathbb{I})_{n \in \mathbb{N}}$ we have*

$$\mathcal{S}^p_X\Big(\sum_{n=1}^N \mathfrak{a}^n_t : t \in \mathbb{I}\Big) \lesssim \sum_{n=1}^N \mathcal{S}^p_X(\mathfrak{a}^n_t : t \in \mathbb{I})$$

where the implied constant is independent of $N \in \mathbb{N} \cup \{\infty\}$, the set $\mathbb{I}$ and the families $(\mathfrak{a}^n_t(x) : t \in \mathbb{I})_{n \in \mathbb{N}}$.

Proof. In the case of the oscillation seminorm and r-variations the result follows by the fact that these object are seminorms and in these cases one may take the implied constant to be equal to 1. The jump quasi-seminorm $J^2_{L^p(X)}$ is a bit more problematic. However, by [41, Corollary 2.2] we know that it admits an equivalent subadditive seminorm which yields the desired result. □

The next proposition describes the splitting property $\mathcal{S}^p_X$.

Proposition 2.30. *Let $p \in [1, \infty)$ and let us consider $\mathcal{S}^p_X$. Then for $-\infty \le u < w < v \le \infty$ we have*

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in [u, v]) \lesssim \mathcal{S}^p_X(\mathfrak{a}_t \in [u, w+1]) + \mathcal{S}^p_X(\mathfrak{a}_t : t \in [w, v]), \tag{2.31}$$

where the implied constant depends only the choice of $\mathcal{S}^p_X$.

Proof. In the case of the oscillation seminorm we fix $N \in \mathbb{N}$ and $I \in \mathfrak{S}_N([u, v])$. We see that it is enough to consider sequences such that $I_k \le w < I_{k+1}$ for some $k = 0, 1, \ldots, N$. Then one has

$$\sup_{I_k \le t < I_{k+1}} |\mathfrak{a}_t - \mathfrak{a}_{I_k}| \le 2 \sup_{I_k \le t < w+1} |\mathfrak{a}_t - \mathfrak{a}_{I_k}| + \sup_{w \le t < I_{k+1}} |\mathfrak{a}_t - \mathfrak{a}_w|$$

and then we use the well-known inequality $(a + b)^2 \le 2(a^2 + b^2)$. In a similar way one can handle the r-variation seminorm. In the case of the jump quasi-seminorm one uses the inequality

$$N_\lambda(\mathfrak{a}_t : t \in [u, v]) \le N_{\lambda/2}(\mathfrak{a}_t : t \in [u, w+1]) + N_{\lambda/2}(\mathfrak{a}_t : t \in [w, v]).$$

Replacing λ by $\lambda/2$ only produces a numerical constant in the studied inequality. □

The following result describes the cut-off feature of $\mathcal{S}^p_X$.

Proposition 2.32. *Let $p \in (1, \infty)$ and let us consider $\mathcal{S}^p_X$. Then for $-\infty \le w < u \le \infty$ one has*

$$\mathcal{S}^p_X(\mathfrak{a}_t \mathbb{1}_{(w,\infty)}(t) : t \in [0, u]) \lesssim \mathcal{S}^p_X(\mathfrak{a}_t : t \in [w, u]) + \|\mathfrak{a}_w\|_{L^p(X)},$$

where the implied constant depends only the choice of $\mathcal{S}^p_X$.

Proof. In the case of the oscillation seminorm we fix $N \in \mathbb{N}$ and $I \in \mathfrak{S}_N([0, u])$. We see that it is enough to consider sequences such that $I_k < w < I_{k+1}$ for some $k = 0, 1, \ldots, N$. Then the desired inequality follows from the fact that

$$\sup_{I_k \le t < I_{k+1}} |\mathfrak{a}_t \mathbb{1}_{(w,\infty)}(t) - \mathfrak{a}_{I_k} \mathbb{1}_{(w,\infty)}(I_k)| \le \sup_{w \le t < I_{k+1}} |\mathfrak{a}_t - \mathfrak{a}_w| + |\mathfrak{a}_w|.$$

The case of the r-variation can be handled in a similar way. For the jump quasi-seminorm we observe that for any $\lambda > 0$ one has

$$\lambda N_\lambda(\mathfrak{a}_t \mathbb{1}_{(w,\infty)}(t) : t \in [0, u]) \le \lambda N_{\lambda/2}(\mathfrak{a}_t : t \in [w, u]) + 2|\mathfrak{a}_w|.$$

As before, replacing λ by $\lambda/2$ only produces a numerical constant in the studied inequality. □

The next result is a well-known decomposition into the dyadic scales and short variations from [32] (see also [40]).

Proposition 2.33 ([32, Lemma 1.3])**.** *Let $(\mathfrak{a}_t : t \in \mathbb{R})$ be a family of measurable functions on X. Let $\mathbb{I} \subset \mathbb{R}$. Then for any $\tau > 0$ we have*

$$\mathcal{S}^p_X(\mathfrak{a}_t : t \in \mathbb{I}) \lesssim \mathcal{S}^p_X(\mathfrak{a}_{2^{n^\tau}} : n \in \mathbb{Z}) + \Big\| \Big(\sum_{n \in \mathbb{Z}} V^2(\mathfrak{a}_t : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})^2 \Big)^{1/2} \Big\|_{L^p(X)}.$$

Proof. In the proof we focus only on the case of the oscillation seminorm. The case of r-variations is similar and the case of jump quasi-seminorm was handled in [32, Lemma 1.3]. We will exploit some ideas presented during the proof of [32, Lemma 1.3]. Fix $N \in \mathbb{N}$ and let $I_1 < I_2 < \cdots < I_N$ be any sequence contained in $\mathbb{I}$. We consider two disjoint sets:

$$J_{\mathrm{S}} := \{j \colon [I_j, I_{j+1}) \subseteq [2^{n^\tau}, 2^{(n+1)^\tau}) \text{ for some } n \in \mathbb{Z}\},$$
$$J_{\mathrm{L}} := \{j \colon I_j \le 2^{n^\tau} < I_{j+1} \text{ for some } n \in \mathbb{Z}\}.$$

The sets J_{S} and J_{L} correspond to the so-called short and long jumps, respectively. Now, for the short jumps it is easy to see that

$$\Big(\sum_{j\in J_{\mathrm{S}}} \sup_{\substack{I_j\le t<I_{j+1}\\ t\in\mathbb{I}}} |\mathfrak{a}_t - \mathfrak{a}_{I_j}|^2\Big)^{1/2} \le \Big(\sum_{n\in\mathbb{Z}} V^2(\mathfrak{a}_t : t\in[2^{n^\tau}, 2^{(n+1)^\tau})\cap\mathbb{I})^2\Big)^{1/2}. \tag{2.34}$$

Next we handle the long jumps. Let $j \in J_L$. We denote by $k_j \in \mathbb{Z}$ the largest number, and by $m_j \in \mathbb{Z}$ the smallest number, such that

$$2^{k_j^\tau} \le I_j < I_{j+1} < 2^{m_j^\tau}.$$

Now, for $t \in [I_j, I_{j+1}) \cap \mathbb{I}$ we have the following simple bound

$$|\mathfrak{a}_t - \mathfrak{a}_{I_j}| \le 2 \sup_{t\in[2^{k_j^\tau}, 2^{m_j^\tau})\cap\mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{2^{k_j^\tau}}|.$$

Moreover, one has

$$\begin{aligned}\sup_{t\in[2^{k_j^\tau}, 2^{m_j^\tau})\cap\mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{2^{k_j^\tau}}| &\le \sup_{n\in[k_j,m_j)} |\mathfrak{a}_{2^{n^\tau}} - \mathfrak{a}_{2^{k_j^\tau}}| + \Big(\sum_{n=k_j}^{m_j-1} \sup_{t\in[2^{n^\tau}, 2^{(n+1)^\tau})\cap\mathbb{I}} |\mathfrak{a}_t - \mathfrak{a}_{2^{n^\tau}}|^2\Big)^{1/2}\\ &\le \sup_{n\in[k_j,m_j)} |\mathfrak{a}_{2^{n^\tau}} - \mathfrak{a}_{2^{k_j^\tau}}| + \Big(\sum_{n=k_j}^{m_j-1} V^2(\mathfrak{a}_{2^t} : t\in[n^\tau, (n+1)^\tau)\cap\mathbb{I})^2\Big)^{1/2}.\end{aligned}$$

Therefore, we can estimate

$$\begin{aligned}&\Big(\sum_{j\in J_{\mathrm{L}}} \sup_{I_j\le t<I_{j+1}} |\mathfrak{a}_t - \mathfrak{a}_{I_j}|^2\Big)^{1/2}\\ &\qquad\le 2\Big(\sum_{j\in J_{\mathrm{L}}} \sup_{n\in[k_j,m_j)} |\mathfrak{a}_{2^{n^\tau}} - \mathfrak{a}_{2^{k_j^\tau}}|^2\Big)^{1/2} + 2\Big(\sum_{j\in J_{\mathrm{L}}} \sum_{n=k_j}^{m_j-1} V^2(\mathfrak{a}_t : t\in[2^{n^\tau}, 2^{(n+1)^\tau})\cap\mathbb{I})^2\Big)^{1/2}\\ &\qquad\le 2\Big(\sum_{j\in J_{\mathrm{L}}} \sup_{n\in[k_j,m_j)} |\mathfrak{a}_{2^{n^\tau}} - \mathfrak{a}_{2^{k_j^\tau}}|^2\Big)^{1/2} + 4\Big(\sum_{n=0}^{\infty} V^2(\mathfrak{a}_t : t\in[2^{n^\tau}, 2^{(n+1)^\tau})\cap\mathbb{I})^2\Big)^{1/2},\end{aligned}$$

where the last inequality follows from the fact that in the second term, for each $n \in \mathbb{Z}$, we can count $V^2(\mathfrak{a}_t : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})$ at most twice. Combining the above estimate with (2.34), taking norms and the appropriate suprema yield the desired result. □

The next result is crucial in our investigations and says that the 2-variations can be bounded by the sum of square functions of differences at dyadic points.

Lemma 2.35 (Rademacher–Menshov inequality). *Let b and s be fixed positive integers. Then for any complex-valued sequence $(a_j : b \le j \le 2^s)$ we have*

$$V^2(a_j \colon b \le j \le 2^s) \le \sqrt{2} \sum_{i=0}^{s} \Big(\sum_j |a_{u^i_{j+1}} - a_{u^i_j}|^2\Big)^{1/2}, \tag{2.36}$$

where $[u^i_j, u^i_{j+1})$ *are dyadic intervals of the form* $[j2^i, (j+1)2^i)$ *for some* $0 \le i \le s$, $0 \le j \le 2^{s-i}-1$, *contained in* $[b, 2^s]$ *(in particular, the number of intervals occurring in the inner sum is finite).*

Proof. The proof comes from [45]. At the beginning we observe that any interval $[m,n)$ for $m,n \in \mathbb{N}$ such that $0 \le m < n \le 2^s$ is a finite disjoint union of dyadic subintervals, i.e. intervals belonging to some $\mathcal{I}_i$ for $0 \le i \le s$, where

$$\mathcal{I}_i := \left\{[j2^i, (j+1)2^i) : 0 \le j \le 2^{s-i}-1\right\}$$

and intervals of each length appears at most twice.

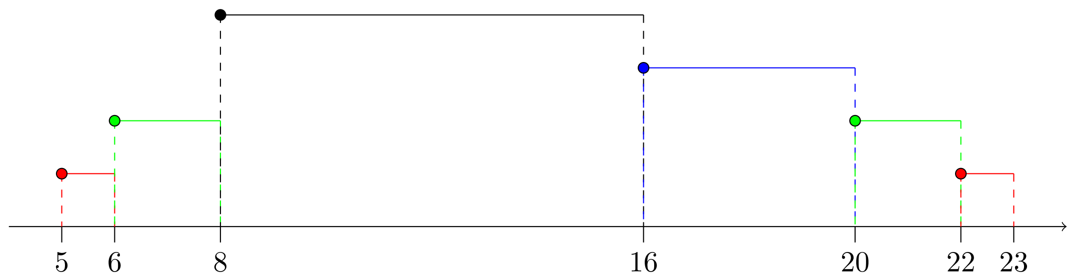

Figure 2.1: The dyadic decomposition of the interval $[5, 23)$ into dyadic intervals from $\mathcal{I}_i$.

For the proof of this fact, let us set $m_0 = m$. If we have chosen m_l then we select m_{l+1} in such a way that $[m_l, m_{l+1})$ is the longest dyadic interval starting at m_l and contained inside $[m_l, n)$. If the lengths of the selected dyadic intervals increase then we continue by repeating this procedure. Otherwise, there is l such that $m_{l+1} - m_l \ge m_{l+2} - m_{l+1}$.

We show that this implies that $m_{l+2} - m_{l+1} > m_{l+3} - m_{l+2}$. Suppose for a contradiction that $m_{l+2} - m_{l+1} \le m_{l+3} - m_{l+2}$. In that case we have following inclusions

$$[m_{l+1}, m_{l+2}) \subset [m_{l+1}, 2m_{l+2} - m_{l+1}) \subseteq [m_{l+1}, m_{l+3}) \subseteq [m_{l+1}, n).$$

Therefore, if we show that $2(m_{l+2} - m_{l+1})$ divides m_{l+1} then $[m_{l+1}, 2m_{l+2} - m_{l+1})$ is a dyadic interval contained in $[m_{l+1}, n)$ which starts at m_{l+1} and ends at $2m_{l+2} - m_{l+1} > m_{l+2}$ which contradicts with the choice of m_{l+2}.The task is easy if $m_{l+1} - m_l > m_{l+2} - m_{l+1}$ since then we have $m_{l+1} = k2^i$ and $m_{l+2} - m_{l+1} = 2^j$ for some $i, j, k \in \mathbb{N}$ with $i > j$. When one has $m_{l+1} - m_l = m_{l+2} - m_{l+1}$ then, by maximality of $[m_l, m_{l+1})$, we have that $2(m_{l+2} - m_{l+1})$ cannot divide m_l, thus divides m_{l+1}.

Now we can prove the inequality (2.36). Let $N \in \mathbb{N}$ be fixed and let $b < t_1 < \ldots < t_{N+1} \le 2^s$ be any increasing sequence. By the first part of the proof, for each $j \in \{1, \cdots, N\}$ we write

$$[t_j, t_{j+1}) = \bigcup_{l=0}^{L_j} [u^j_l, u^j_{l+1})$$

for some $L_j \ge 1$ where each interval $[u^j_l, u^j_{l+1}) \subseteq [b, 2^s)$ is dyadic. Then

$$|a_{t_{j+1}} - a_{t_j}| \le \sum_{l=0}^{L_j} |a_{u^j_{l+1}} - a_{u^j_l}| = \sum_{i=0}^{s} \sum_{l:\,[u^j_l, u^j_{l+1}) \in \mathcal{I}_i} |a_{u^j_{l+1}} - a_{u^j_l}|.$$

Hence, by Minkowski's inequality

$$\begin{aligned}
\Big(\sum_{j=1}^{N} |a_{t_{j+1}} - a_{t_j}|^2\Big)^{1/2} &\le \Big(\sum_{j=1}^{N}\Big(\sum_{i=0}^{s} \sum_{l:\,[u^j_l, u^j_{l+1}) \in \mathcal{I}_i} |a_{u^j_l} - a_{u^j_{l+1}}|\Big)^2\Big)^{1/2} \\
&\le \sum_{i=0}^{s}\Big(\sum_{j=1}^{N}\Big(\sum_{l:\,[u^j_l, u^j_{l+1}) \in \mathcal{I}_i} |a_{u^j_l} - a_{u^j_{l+1}}|\Big)^2\Big)^{1/2}.
\end{aligned}$$

Since for a given $i \in \{0, 1, \dots, 2^s\}$ and $j \in \{1, 1, \dots, N\}$ the inner sums contain at most two elements we can use the inequality $(a+b)^2 \le 2(a^2+b^2)$ to get that

$$\Big(\sum_{j=1}^{N} |a_{t_{j+1}} - a_{t_j}|^2\Big)^{1/2} \le \sqrt{2} \sum_{i=0}^{s} \Big(\sum_{j=1}^{N} \sum_{l:\, [u_l^j, u_{l+1}^j) \in \mathcal{I}_i} |a_{u_l^j} - a_{u_{l+1}^j}|^2\Big)^{1/2}$$

which is bounded by the right-hand side of (2.36). Taking appropriate suprema completes the proof. □

The inequality (2.36) originates in the paper of Lewko and Lewko [35] where it was used to obtain a variational version of the Rademacher–Menshov theorem. A few years later the inequality (2.36) was independently discovered by Mirek and Trojan [45] in the context of the maximal estimates for the ergodic averages. We note that by the inequality (2.27) we get that $\mathcal{S}_X^p$ is bounded by the 2-variations hence the Rademacher–Menshov inequality holds for $\mathcal{S}_X^p$. More precisely, we have the following observation.

Remark 2.37. By inequality (2.27) we deduce that the Rademacher–Menshov inequality holds for $\mathcal{S}_X^p$, namely for any sequence $(f_j(x) : b \le j \le 2^s)$ of functions from $L^p(X)$ one has

$$\mathcal{S}_X^p\big(f_n : b \le n \le 2^s\big) \le \sqrt{2} \Big\| \sum_{i=1}^{s} \Big(\sum_{j} |f_{u_{j+1}^i} - f_{u_j^i}|^2 \Big)^{1/2} \Big\|_{L^p(X)} \tag{2.38}$$

where $[u_j^i, u_{j+1}^i)$ are dyadic intervals of the form $[j2^i, (j+1)2^i)$ for some $0 \le i \le s$, $0 \le j \le 2^{s-i} - 1$, contained in $[b, 2^s]$.

2.2 Calderón transference principle

Bourgain in his groundbreaking series of papers [4, 5, 6] was interested in the pointwise convergence of the ergodic averages along the squares given by

$$T_N f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^{n^2} x) \quad , x \in X, \quad f \in L^p(X),$$

where $T \colon X \to X$ is a measure preserving transformation. If we consider $X = \mathbb{Z}$ and $T(x) = x - 1$ we obtain a special case of such averages, namely

$$M_N f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(x - n^2), \quad x \in \mathbb{Z}, \quad f \in \ell^p(\mathbb{Z}),$$

which is an example of discrete Radon averages. Consequently, we see that Radon type operators are special cases of more general ergodic averages. It turns out that, in the case of the convergence problems, this is the only relevant case of the ergodic averages.

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space with a family of invertible, commuting and measure preserving transformations $T_1, T_2, \dots, T_d$ which means that

$$\mu(T_i^{-1} A) = \mu(A) \text{ for each } A \in \mathcal{B} \text{ and each } i = 1, \dots, d.$$

For a given polynomial mapping $\mathcal{P}$ of the form (1.40) and a non-empty convex body Ω we define

$$M_t^{\mathcal{P},\mathrm{erg}} f(x) := \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(T_1^{\mathcal{P}_1(y)} \cdots T_d^{\mathcal{P}_d(y)} x) K_t(y), \quad x \in X, \tag{2.39}$$

where $K_t : (0,\infty) \times \mathbb{R}^k \to \mathbb{C}$ is a fixed function. For example, when

$$K_t(y) = \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \mathbb{1}_{\Omega_t}(y), \quad y \in \mathbb{R}^k, \tag{2.40}$$

then $M_t^{\mathcal{P},\mathrm{erg}}$ became the "standard" ergodic averages. In the case when

$$K_t(y) := K(y)\mathbb{1}_{\Omega_t \setminus \{0\}}(y), \quad y \in \mathbb{R}^k, \tag{2.41}$$

with $K\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ being a Calderón–Zygmund kernel we get that $M_t^{\mathcal{P},\mathrm{erg}}$ are the Cotlar type ergodic average. Again, we are particularly interested in the integer shift setting. Let us consider the dynamical system of $\mathbb{Z}^d$ equipped with counting measure and the shift operators $S_j\colon \mathbb{Z}^d \to \mathbb{Z}^d$ given by $S_j(x_1,\ldots,x_d) := (x_1,\ldots,x_j-1,\ldots,x_d)$. Then the average $M_t^{\mathcal{P},\mathrm{erg}}$ can be written as

$$M_t^{\mathcal{P},\mathrm{shift}} f(n) := \sum_{y\in\Omega_t\cap\mathbb{Z}^k} f(n-\mathcal{P}(y))K_t(y), \quad n \in \mathbb{Z}^d. \tag{2.42}$$

It can be easily seen that the discrete Radon type operators (1.41) and (1.42) are special cases of $M_t^{\mathcal{P},\mathrm{shift}}$.

In 1968, Calderón [7] made an important observation that some results in ergodic theory can be easily deduced from known results in harmonic analysis. Namely, the boundedness of the maximal function of Birkhoff's averages given by

$$\sup_{N\in\mathbb{N}} \frac{1}{N}\sum_{n=0}^{N} |f(T^n x)|, \quad f \in L^p(X), \quad x \in X,$$

can be deduced from the boundedness of the Hardy–Littlewood maximal function

$$\sup_{N\in\mathbb{N}} \frac{1}{N}\sum_{n=0}^{N} |f(x-n)|, \quad f \in \ell^p(\mathbb{Z}), \quad x \in \mathbb{Z}.$$

It turns out that Calderón's observation can be extended to the setting of seminorms and averages $M_t^{\mathcal{P},\mathrm{erg}}$.

Theorem 2.43. *Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space with a family of invertible, commuting and measure preserving transformations $T_1, T_2, \ldots, T_d$. Let $M_t^{\mathcal{P},\mathrm{erg}}$ and $M_t^{\mathcal{P},\mathrm{shift}}$ be defined as in (2.39) and (2.42). If for some $p \in [1,\infty)$ there is a constant $C_p > 0$ such that*

$$\mathcal{S}^p_{\mathbb{Z}^d}(M_t^{\mathcal{P},\mathrm{shift}} f : t > 0) \le C_p \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d). \tag{2.44}$$

Then we have

$$\mathcal{S}^p_X(M_t^{\mathcal{P},\mathrm{erg}} f : t > 0) \le C_p \|f\|_{L^p(X)}, \quad f \in L^p(X).$$

Proof. Let $p \in [1,\infty)$ be fixed. For a family of functions $(\mathfrak{a}_t : t \in \mathbb{I}) \subset L^p(X)$ indexed by the set $\mathbb{I} \subseteq \mathbb{R}$ we write

$$\mathcal{R}(\mathfrak{a}_t : t\in\mathbb{I}) := \begin{cases} O^2_{I,N}(\mathfrak{a}_t : t\in\mathbb{I}) \\ V^r(\mathfrak{a}_t : t\in\mathbb{I}) \\ \lambda(N_\lambda(\mathfrak{a}_t : t\in\mathbb{I}))^{1/2} \end{cases} \tag{2.45}$$

to represent one of the quantities on the right hand side of (2.45). Clearly, $\mathcal{R}$ depends on some parameters but we will not be using their exact form so, for the sake of simplicity, we omit them. It is easy to see that after taking the appropriate suprema, we get that

$$\sup \big\|\mathcal{R}(\mathfrak{a}_t : t\in\mathbb{I})\big\|_{L^p(X)} = \mathcal{S}^p_X(\mathfrak{a}_t : t\in\mathbb{I}).$$

In the case of the r-variation norm V^r we do not need to take any supremum.

Now, let $N \in \mathbb{N}$ be a fixed large natural number and let

$$\overline{N} := \max_{y \in \Omega_N \cap \mathbb{Z}^k} |\mathcal{P}(y)|.$$

Clearly $\overline{N} < \infty$ since Ω_N is bounded and $\mathcal{P}$ is a polynomial mapping.

Now let $J > \overline{N}$ be a natural number. For $f \in L^p(X)$ and $x \in X$ we define a sequence on $\mathbb{Z}^d$ by setting

$$\varphi(n) := \begin{cases} f(T_1^{-n_1} \cdots T_d^{-n_d} x), & \text{if } 0 \le |n| \le J, \\ 0, & \text{otherwise}. \end{cases} \tag{2.46}$$

Observe that for $t \in (0, N]$ and for $m \in \mathbb{Z}^d$ such that $0 \le |m| \le J - \overline{N}$ we have

$$\begin{aligned} M_t^{\mathcal{P},\text{shift}} \varphi(m) = \sum_{y \in \Omega_t \cap \mathbb{Z}^k} \varphi(m - \mathcal{P}(y)) K_t(y) &= \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f\big(T_1^{\mathcal{P}_1(y)} \cdots T_d^{\mathcal{P}_d(y)} (T_1^{-m_1} \cdots T_d^{-m_d} x)\big) K_t(y) \\ &= M_t^{\mathcal{P},\text{erg}} f(T_1^{-m_1} \cdots T_d^{-m_d} x) \end{aligned}$$

since $T_1, \dots, T_d$ are commuting. Therefore, we have

$$\mathcal{R}\big(M_t^{\mathcal{P},\text{shift}} \varphi(m) : t \in (0, N]\big) = \mathcal{R}\big(M_t^{\mathcal{P},\text{erg}} f(T_1^{-m_1} \cdots T_d^{-m_d} x) : t \in (0, N]\big).$$

Hence, from the seminorm estimate for the shift (2.44) we get

$$\sum_{0 \le |m| \le J - \overline{N}} \Big| \mathcal{R}\big(M_t^{\mathcal{P},\text{erg}} f(T_1^{-m_1} \cdots T_d^{-m_d} x) : t \in (0, N]\big) \Big|^p \le C_p^p \sum_{0 \le |n| \le J} \big| f(T_1^{-n_1} \cdots T_d^{-n_d} x) \big|^p, \quad x \in X.$$

Now, if we average the above inequality in X and use the fact that each T_i is measure preserving we obtain that

$$\sum_{0 \le |m| \le J - \overline{N}} \big\| \mathcal{R}\big(M_t^{\mathcal{P},\text{erg}} f : t \in (0, N]\big) \big\|_{L^p(X)}^p \le C_p^p \sum_{0 \le |n| \le J} \|f\|_{L^p(X)}^p.$$

This implies that

$$\big\| \mathcal{R}\big(M_t^{\mathcal{P},\text{erg}} f : t \in (0, N]\big) \big\|_{L^p(X)} \le \Big(\frac{2J + 1}{2J - 2\overline{N} + 1} \Big)^{d/p} C_p \|f\|_{L^p(X)}$$

for any $J > \overline{N}$. Letting $J \to \infty$ gives us

$$\big\| \mathcal{R}\big(M_t^{\mathcal{P},\text{erg}} f : t \in (0, N]\big) \big\|_{L^p(X)} \le C_p \|f\|_{L^p(X)}.$$

Now we may take the appropriate suprema to get that

$$\mathcal{S}_X^p\big(M_t^{\mathcal{P},\text{erg}} f : t \in (0, N]\big) \le C_p \|f\|_{L^p(X)}.$$

Due to the monotonicity of $\mathcal{S}_X^p$, taking $N \to \infty$ yields the desired result. □

The above result shows that the discrete operators of Radon type defined in Sections 1.1 and 1.3 are closely related to the ergodic averages $M_t^{\mathcal{P},\text{erg}}$ associated with the kernels (2.40) and (2.41). In those cases, by the Caderón transference principle, the seminorm estimates

$$\mathcal{S}_X^p(M_t^{\mathcal{P},\text{erg}} f : t > 0) \le C_p \|f\|_{L^p(X)}, \quad f \in L^p(X),$$

follows by Theorem 1.45 and Theorem 1.48.

2.3 Lifting procedure and canonical mappings

Let d, k be fixed natural numbers. As we know, the polynomial mapping $\mathcal{P}$ from $\mathbb{Z}^k$ to $\mathbb{Z}^d$ is defined as a transformation of the form

$$\mathcal{P} := (\mathcal{P}_1, \ldots, \mathcal{P}_d) \colon \mathbb{Z}^k \to \mathbb{Z}^d \tag{2.47}$$

where each $\mathcal{P}_j \colon \mathbb{Z}^k \to \mathbb{Z}$ is a polynomial of k variables with integer coefficients such that $\mathcal{P}_j(0) = 0$. For example the mapping

$$\mathbb{Z}^3 \ni (x, y, z) \mapsto (x^3 y + 2yz, x^2 z + 5y^4) \in \mathbb{Z}^2$$

is a polynomial mapping between $\mathbb{Z}^3$ and $\mathbb{Z}^2$.

Let $\mathcal{P}$ be a polynomial mapping (2.47). We define its degree by setting

$$\deg \mathcal{P} := \max\{\deg \mathcal{P}_j \colon 1 \leq j \leq d\}.$$

Let us consider the set of multi-indices

$$\Gamma := \left\{\gamma \in \mathbb{N}_0^k \setminus \{0\} \colon 0 < |\gamma| \leq \deg \mathcal{P}\right\} \tag{2.48}$$

equipped with the lexicographic order. It is easy to see that for each $j \in \{1, \ldots, d\}$ there is a sequence $(c_j^\gamma : \gamma \in \Gamma) \subset \mathbb{Z}$ such that

$$\mathcal{P}_j(x) = \sum_{\gamma \in \Gamma} c_j^\gamma x^\gamma,$$

where

$$x^\gamma := x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_k^{\gamma_k}.$$

Further, we denote by $\mathbb{Z}^\Gamma$ the space of tuples of integer numbers labeled by multi-indices $\gamma = (\gamma_1, \ldots, \gamma_k)$, so that $\mathbb{Z}^\Gamma \cong \mathbb{Z}^{|\Gamma|}$. In a similar fashion, we denote $\mathbb{R}^\Gamma \cong \mathbb{R}^{|\Gamma|}$. Finally, we define the *canonical polynomial mapping*

$$\mathbb{Z}^k \ni x = (x_1, \ldots, x_k) \mapsto (x)^\Gamma := (x^\gamma : \gamma \in \Gamma) \in \mathbb{Z}^\Gamma. \tag{2.49}$$

It is easy to see that the coefficients $\big(c_j^\gamma : \gamma \in \Gamma, j \in \{1, \ldots, d\}\big)$ determine a linear transformation $L \colon \mathbb{R}^\Gamma \to \mathbb{R}^d$ such that $L\big((y)^\Gamma\big) = \mathcal{P}(y)$ for $y \in \mathbb{Z}^k$. Indeed, let L be given by

$$L(x) := \big(L_1(x), \ldots, L_d(x)\big), \quad x \in \mathbb{R}^\Gamma, \tag{2.50}$$

where for each $j \in \{1, \ldots, d\}$ we set

$$L_j(x) := \sum_{\gamma \in \Gamma} c_j^\gamma x_\gamma. \tag{2.51}$$

Clearly, for any $y \in \mathbb{Z}^k$ we have $L_j\big((y)^\Gamma\big) = \mathcal{P}_j(y)$.

It turns out that the study of the seminorm inequalities related to the operators of the form

$$\sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(n - \mathcal{P}(y)) K_t(y)$$

can be reduced to the setting of the canonical polynomials. For any set $\Gamma \subset \mathbb{N}^k \setminus \{0\}$ we define

$$M_t^{\Gamma,\text{shift}} f(x) := \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(x - (y)^\Gamma) K_t(y), \quad x \in \mathbb{Z}^\Gamma,$$

where $K_t \colon (0, \infty) \times \mathbb{R}^k \to \mathbb{C}$ is some fixed function. Let $M_t^{\mathcal{P},\text{shift}}$ be the average defined in (2.42) associated with the kernel K_t. Then the following result holds.

Lemma 2.52 ([39, Lemma 2.2])**.** *Let $d, k \in \mathbb{N}$ be fixed. Let $\mathcal{P}$ be a fixed polynomial mapping (2.47) and let Γ be defined as in (2.48). Suppose that for some $p \in [1, \infty)$ there is a constant $C_p > 0$ such that*

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(M_t^{\Gamma,\mathrm{shift}} f : t > 0\big) \leq C_p \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \quad f \in \ell^p(\mathbb{Z}^\Gamma). \tag{2.53}$$

Then

$$\mathcal{S}_{\mathbb{Z}^d}\big(M_t^{\mathcal{P},\mathrm{shift}} f : t > 0\big) \leq C_p \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d)$$

with the same constant as in (2.53).

Proof. Let $p \in [1, \infty)$ and $f \in \ell^p(\mathbb{Z}^d)$ be fixed. Recall the notion of $\mathcal{R}(\mathfrak{a}_t : t \in \mathbb{I})$ defined in (2.45). Let $R > 0$ and $N > 0$ be fixed. For any $x \in \mathbb{Z}^d$ we define the function $F_x \colon \mathbb{Z}^\Gamma \to \mathbb{C}$ by setting

$$F_x(z) := \begin{cases} f(x + L(z)) & \text{if } |z| \leq R + N^{k \deg \mathcal{P}}, \\ 0 & \text{otherwise,} \end{cases} \qquad z \in \mathbb{Z}^\Gamma,$$

where $L \colon \mathbb{R}^\Gamma \to \mathbb{R}^d$ is the linear transformation (2.50) associated with the mapping $\mathcal{P}$. Let $t \leq N$. For any $y \in \mathbb{Z}^k$ with $|y|_\infty \leq t$ and any $u \in \mathbb{Z}^\Gamma$ with $|u|_\infty \leq R$ we have

$$\big|u - (y)^\Gamma\big|_\infty \leq R + \max_{\gamma \in \Gamma} |t^{k|\gamma|}| \leq R + N^{k \deg \mathcal{P}}.$$

Consequently, for each $x \in \mathbb{Z}^d$ and any $u \in \mathbb{Z}^\Gamma$ with $|u|_\infty \leq R$ we have

$$M_t^{\mathcal{P},\mathrm{shift}} f(x + L(u)) = \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f\big(x + L(u - (y)^\Gamma)\big) K_t(y) = M_t^{\Gamma,\mathrm{shift}} F_x(u),$$

provided that $t \leq N$. Therefore, we may write

$$\mathcal{R}\big(M_t^{\mathcal{P},\mathrm{shift}} f(x + L(u)) : t \in (0, N]\big) = \mathcal{R}\big(M_t^{\Gamma,\mathrm{shift}} F_x(u) : t \in (0, N]\big).$$

Now, since the ℓ^p-norm is translation invariant, we have

$$\begin{aligned} \Big\|\mathcal{R}\big(M_t^{\mathcal{P},\mathrm{shift}} f : t \in (0, N]\big)\Big\|^p_{\ell^p(\mathbb{Z}^\Gamma)} &= \frac{1}{(2R+1)^{|\Gamma|}} \sum_{x \in \mathbb{Z}^d} \sum_{\substack{u \in \mathbb{Z}^\Gamma \\ |u|_\infty \leq R}} \big|\mathcal{R}\big(M_t^{\mathcal{P},\mathrm{shift}} f(x + L(u)) : t \in (0, N]\big)\big|^p \\ &= \frac{1}{(2R+1)^{|\Gamma|}} \sum_{x \in \mathbb{Z}^d} \sum_{\substack{u \in \mathbb{Z}^\Gamma \\ |u|_\infty \leq R}} \big|\mathcal{R}\big(M_t^{\Gamma,\mathrm{shift}} F_x(u) : t \in (0, N]\big)\big|^p. \end{aligned}$$

By the inequality (2.53) one has

$$\frac{1}{(2R+1)^{|\Gamma|}} \sum_{x \in \mathbb{Z}^d} \sum_{\substack{u \in \mathbb{Z}^\Gamma \\ |u|_\infty \leq R}} \big|\mathcal{R}\big(M_t^{\Gamma,\mathrm{shift}} F_x(u) : t \in (0, N]\big)\big|^p \leq \frac{C_p^p}{(2R+1)^{|\Gamma|}} \sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{Z}^\Gamma} |F_x(u)|^p.$$

Let us observe that by the definition of the function F_x and by the fact that the ℓ^p-norm is translation invariant one has

$$\sum_{x \in \mathbb{Z}^d} \sum_{u \in \mathbb{Z}^\Gamma} |F_x(u)|^p = \sum_{x \in \mathbb{Z}^d} \sum_{\substack{u \in \mathbb{Z}^\Gamma \\ |u|_\infty \leq R + N^{k \deg \mathcal{P}}}} |f(x + L(u))|^p = (2R + 2N^{k \deg \mathcal{P}} + 1)^{|\Gamma|} \|f\|^p_{\ell^p(\mathbb{Z}^\Gamma)}.$$

As a consequence we obtain

$$\Big\|\mathcal{R}\big(M_t^{\mathcal{P},\mathrm{shift}} f : t \in (0, N]\big)\Big\|^p_{\ell^p(\mathbb{Z}^\Gamma)} \leq C_p^p \frac{(2R + 2N^{k \deg \mathcal{P}} + 1)^{|\Gamma|}}{(2R+1)^{|\Gamma|}} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which, by taking $R \to \infty$, implies

$$\Big\| \mathcal{R}\big(M_t^{\mathcal{P},\text{shift}} f : t \in (0,N]\big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_p \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

Now we may take the appropriate suprema to get that

$$\mathcal{S}^p_{\mathbb{Z}^d}\big(M_t^{\mathcal{P},\text{shift}} f : t \in (0,N]\big) \le C_p \|f\|_{L^p(X)}.$$

Due to the monotonicity of $\mathcal{S}^p_{\mathbb{Z}^d}$, taking $N \to \infty$ yields the desired result. □

The procedure presented in the proof of the above lemma is called *lifting* or *method of descent*. As one may expect a similar result holds in the continuous setting. For any $C_c^\infty(\mathbb{R}^d)$ we define

$$\mathcal{M}_t^{\Gamma,\text{shift}} f(x) := \int_{\Omega_t} f(x - \mathcal{P}(y)) K_t(y) \mathrm{d}y, \quad x \in \mathbb{R}^d,$$

where $K_t \colon (0,\infty) \times \mathbb{R}^k \to \mathbb{C}$ is some fixed function and the integral may be understood in the principal value sense. For any set $\Gamma \subset \mathbb{N}^k \setminus \{0\}$ and any $C_c^\infty(\mathbb{R}^\Gamma)$ we define

$$\mathcal{M}_t^{\Gamma,\text{shift}} f(x) := \int_{\Omega_t} f(x - (y)^\Gamma) K_t(y) \mathrm{d}y, \quad x \in \mathbb{Z}^\Gamma.$$

Then the following holds.

Lemma 2.54 ([56, Section 2.4, p. 483])**.** *Let $d, k \in \mathbb{N}$ be fixed. Let $\mathcal{P}$ be a fixed polynomial mapping* (2.47) *and let Γ be defined as in* (2.48)*. Suppose that for some $p \in [1,\infty)$ there is a constant $C_p > 0$ such that*

$$\mathcal{S}^p_{\mathbb{R}^\Gamma}\big(\mathcal{M}_t^{\Gamma,\text{shift}} f : t > 0\big) \le C_p \|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f \in L^p(\mathbb{R}^\Gamma).$$

Then

$$\mathcal{S}_{\mathbb{R}^d}\big(\mathcal{M}_t^{\mathcal{P},\text{shift}} f : t > 0\big) \le C_p \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d)$$

with the same constant as in (2.53)*.*

2.4 Radon type operators

As we seen in Section 2.2 the discrete Radon averages $M_t^{\mathcal{P}}$ and $H_t^{\mathcal{P}}$ arise naturally upon applying Calderón's transference principle to the ergodic averages (2.39). Nonetheless, the above observation is not the only reason to consider Radon averages. Namely, the operators $M_t^{\mathcal{P}}$ can be seen as discrete counterparts of the continuous Radon operators defined in (1.43) and (1.44). In turn, those operators are natural generalisations of the Hardy–Littlewood operators

$$\frac{1}{|\Omega_t|} \int_{\Omega_t} f(x-y) \mathrm{d}y$$

and the Calderón–Zygmund singular integrals

$$\text{p.v.} \int_{\Omega_t} f(x-y) K(y) \mathrm{d}y,$$

where $K \colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is a Calderón–Zygmund kernel which satisfy conditions (1.4), (1.5) and (1.6). The idea of considering such operators related to the polynomial trajectories originates in the work of Stein and collaborators, related to curvatures and parabolic differential equations, see [15, 25, 59, 58]. Since then those operators became widely known in harmonic analysis. Of particular interest were the seminorm estimates for those operators – for more details see Chapters 3 and 4.

We will now proceed to present some basic properties of Radon type operators which will we be used later. By the results of Section 2.3 we may consider the Radon operators related to the canonical mappings (2.49) only. Let $\Omega \subset \mathbb{R}^k$ be a convex body and let Ω_t with $t > 0$ denote its dilation. For finitely supported functions $f\colon \mathbb{Z}^\Gamma \to \mathbb{C}$ we denote

$$M_t f(x) := \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(x - (y)^\Gamma), \quad x \in \mathbb{Z}^\Gamma, \tag{2.55}$$

and

$$H_t f(x) := \sum_{y \in \Omega_t \cap \mathbb{Z}^k \setminus \{0\}} f(x - \mathcal{P}(y)) K(y), \quad x \in \mathbb{Z}^\Gamma, \tag{2.56}$$

where $K\colon \mathbb{R}^k \setminus \{0\} \to \mathbb{C}$ is the Calderón–Zygmund kernel which satisfy conditions (1.4), (1.5) and (1.6). From now on, M_t and H_t will always refer to the operators defined above. In a similar fashion, we denote the continuous Radon operators. For smooth compactly supported function $f\colon \mathbb{R}^\Gamma \to \mathbb{C}$ we denote

$$\mathcal{M}_t f(x) := \frac{1}{|\Omega_t|} \int_{\Omega_t} f(x - (y)^\Gamma) \mathrm{d}y, \quad x \in \mathbb{R}^\Gamma, \tag{2.57}$$

and

$$\mathcal{H}_t f(x) = \text{p.v.} \int_{\Omega_t} f(x - (y)^\Gamma) K(y) \mathrm{d}y, \quad x \in \mathbb{R}^\Gamma, \tag{2.58}$$

where again K is a Calderón–Zygmund kernel which satisfy conditions (1.4), (1.5) and (1.6).

It is easy to see that M_t and H_t are multiplier operators related to the Fourier transform on $\mathbb{Z}^\Gamma$, that is, for any $x \in \mathbb{Z}^\Gamma$ we have

$$M_t f(x) = \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_t \mathcal{F}_{\mathbb{Z}^\Gamma} f)(x) \text{ and } H_t f(x) = \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(n_t \mathcal{F}_{\mathbb{Z}^\Gamma} f)(x)$$

where

$$m_t(\xi) := \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{y \in \Omega_t \cap \mathbb{Z}^k} \boldsymbol{e}\big(\xi \cdot (y)^\Gamma\big), \quad \xi \in \mathbb{T}^\Gamma, \tag{2.59}$$

and

$$n_t(\xi) := \sum_{y \in \Omega_t \cap \mathbb{Z}^k \setminus \{0\}} \boldsymbol{e}\big(\xi \cdot (y)^\Gamma\big) K(y), \quad \xi \in \mathbb{T}^\Gamma. \tag{2.60}$$

Similarly, the operators $\mathcal{M}_t$ and $\mathcal{H}_t$ are multiplier operators for the Fourier transform on $\mathbb{R}^\Gamma$, namely

$$\mathcal{M}_t f(x) = \mathcal{F}^{-1}_{\mathbb{R}^\Gamma}(\Phi_t \mathcal{F}_{\mathbb{R}^\Gamma} f)(x) \text{ and } \mathcal{H}_t f(x) = \mathcal{F}^{-1}_{\mathbb{R}^\Gamma}(\Psi_t \mathcal{F}_{\mathbb{R}^\Gamma} f)(x)$$

where

$$\Phi_t(\xi) := \frac{1}{|\Omega_t|} \int_{\Omega_t} \boldsymbol{e}(\xi \cdot (t)^\Gamma) \mathrm{d}y, \quad \xi \in \mathbb{R}^\Gamma, \tag{2.61}$$

and

$$\Psi_t(\xi) := \text{p.v.} \int_{\Omega_t} \boldsymbol{e}(\xi \cdot (t)^\Gamma) K(y) \mathrm{d}y, \quad \xi \in \mathbb{R}^\Gamma. \tag{2.62}$$

Let A be the diagonal $|\Gamma| \times |\Gamma|$ matrix satisfying

$$(Av)_\gamma = |\gamma| v_\gamma. \tag{2.63}$$

For $t > 0$ we set

$$t^A := \exp(A \log t),$$

which means that $t^A x = \big(t^{|\gamma|}x_\gamma : \gamma \in \Gamma\big)$ for every $x \in \mathbb{R}^\Gamma$. In the sequel we frequently exploit the following decay estimates for the multiplier Φ_t,

$$|\Phi_t(\xi) - 1| \lesssim |t^A\xi|_\infty \quad \text{and} \quad |\Phi_t(\xi)| \lesssim |t^A\xi|_\infty^{-1/|\Gamma|} \tag{2.64}$$

where A is the matrix of the form (2.63). The first inequality is a straightforward consequence of the mean value theorem. The second estimate is a consequence of the refined van der Corput's oscillatory integral lemma with a rough amplitude function proven by Zorin-Kranich [64, Lemma A.1].

Proposition 2.65 (Van der Corput Lemma)**.** *Let $d, k \in \mathbb{N}$ be given and let $P(x) = \sum_{1\le|\alpha|\le d}\lambda_\alpha x^\alpha$ be a polynomial in k variables of degree at most d with real coefficients. Let $R > 0$ and let $\psi\colon \mathbb{R}^k \to \mathbb{C}$ be an integrable function supported in $B(0, R/2)$. Then*

$$\left|\int_{\mathbb{R}^k} e^{iP(x)}\psi(x)\mathrm{d}x\right| \lesssim_{d,k} \sup_{v\in\mathbb{R}^k : |v|\le R\Lambda^{-1/d}} \int_{\mathbb{R}^k} |\psi(x) - \psi(x - v)|\mathrm{d}x,$$

where $\Lambda := \sum_{1\le|\alpha|\le d} R^{|\alpha|}|\lambda_\alpha|$.

Thanks to the above proposition we may write

$$|\Phi_t(\xi)| \lesssim_{|\Gamma|,k} |\Omega_t|^{-1} \sup_{v\in\mathbb{R}^k:|v|\le t\Lambda^{-1/|\Gamma|}} \int_{\mathbb{R}^k} \big|\mathbb{1}_{\Omega_t}(x) - \mathbb{1}_{\Omega_t}(x - v)\big|\mathrm{d}x = \sup_{v\in\mathbb{R}^k:|v|\le t\Lambda^{-1/|\Gamma|}} \frac{|(\Omega_t + v)\setminus\Omega_t|}{|\Omega_t|}$$

where $\Lambda \simeq_{|\Gamma|} |t^A\xi|_\infty$. In order to estimate the last quantity we make use of the following lemma which allows us to control the measure of neighborhoods of the boundaries of convex sets – see also Proposition 3.15 in Section 3.2.

Lemma 2.66 ([42, Lemma A.1])**.** *Let $G \subset \mathbb{R}^k$ be a bounded and convex set and let $0 < s \lesssim \operatorname{diam}(G)$. Then*

$$\big|\{x \in \mathbb{R}^k : \operatorname{dist}(x, \partial G) < s\}\big| \lesssim_k s\operatorname{diam}(G)^{k-1}.$$

The implicit constant depends only on the dimension k, but not on the convex set G.

If we apply the above lemma to our setting we see that

$$|(\Omega_t + v)\setminus\Omega_t| \lesssim |v|t^{k-1}$$

and consequently

$$|\Phi_t(\xi)| \lesssim_{|\Gamma|,k} \sup_{v\in\mathbb{R}^k:|v|\le t\Lambda^{-1/|\Gamma|}} \frac{|v|t^{k-1}}{|\Omega_t|} \lesssim_{|\Gamma|,k,\Omega} |t^A\xi|_\infty^{-1/|\Gamma|}$$

which ends the proof of the second estimate in (2.64).

We have analogous estimates for the multiplier Ψ_t. For a fixed $c \in (0, 1)$ and any real number $t > 0$ we have

$$|\Psi_t(\xi) - \Psi_{ct}(\xi)| \lesssim |t^A\xi|_\infty \quad \text{and} \quad |\Psi_t(\xi) - \Psi_{ct}(\xi)| \lesssim |t^A\xi|_\infty^{-\sigma/|\Gamma|}, \tag{2.67}$$

where $\sigma > 0$ is from the continuity condition (1.6). The first estimate follows from the cancellation condition (1.5) and the second one is a consequence of Proposition 2.65 and condition (1.6) – see [42, Section 3.3] for more details.

2.5 Sampling principles of Magyar–Stein–Wainger and Ionescu–Wainger

In order to handle the discrete Radon type operators we follow Bourgain [6] approach and we use the Hardy–Littlewood circle method (Section 1.2.1) to localize multipliers m_t and n_t around appropriate rational frequencies, and replace them by their continuous counterparts. Then we want to utilize some known results obtained in the continuous setting but in order to do so we need some sampling (or transference) principles which will show us how to do it.

The sampling principles (or transference) are invaluable tools in harmonic analysis thanks to which some results obtained in one setting can also be used in another one (usually results from $\mathbb{R}^d$ are used in $\mathbb{Z}^d$ or vice versa). There are many transference results in harmonic analysis and in our case a particularly important will be a sampling principle of Magyar, Stein and Wainger [36]. In 2002 Magyar, the authors have proved an ℓ^p-estimates for the maximal discrete spherical averages on $\mathbb{Z}^d$. Recall that the discrete spherical average for $f \in \ell^p(\mathbb{Z}^d)$ is defined as

$$A_\lambda f(n) := \frac{1}{N(\lambda)} \sum_{|m|=\lambda} f(n-m), \quad n \in \mathbb{Z}^d,$$

where $N(\lambda)$ denotes the number of $m \in \mathbb{Z}$ such that $|m| = \lambda$.

Theorem 2.68 ([36, Theorem 1])**.** *Let $d \geq 5$. Then the inequality*

$$\big\| \sup_{\lambda>0} |A_\lambda f| \big\|_{\ell^p(\mathbb{Z}^d)} \leq C_d \|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f \in \ell^p(\mathbb{Z}^d), \tag{2.69}$$

holds for $p > \frac{d}{d-2}$.

It is impossible to deduce the above inequality directly from its continuous counterpart. In order to prove (2.69) Magyar, Stein and Wainger have to use entirely different methods than in the continuous case. The key tool that allowed them to handle the above problem was the modified version of the Hardy–Littlewood circle method. Moreover, the authors have developed the following transference principle [36, Corollary 2.1] that allowed them, nevertheless, to use the results from the continuous setting.

Proposition 2.70. *Let $d \in \mathbb{N}$ be fixed. There exists an absolute constant $C_d > 0$ such that the following holds. Let $p \in [1, \infty]$ and $q \in \mathbb{N}$, and let B_1, B_2 be finite-dimensional Banach spaces. Let $\mathfrak{m}\colon \mathbb{R}^d \to L(B_1, B_2)$ be a bounded operator-valued function supported on $q^{-1}[-1/2, 1/2]^d$ and denote the associated Fourier multiplier operator over $\mathbb{R}^d$ by $T_{\mathbb{R}^d}[\mathfrak{m}]$. Let $\mathfrak{m}^q_{\mathrm{per}}$ be the periodic multiplier*

$$\mathfrak{m}^q_{\mathrm{per}}(\xi) := \sum_{n \in \mathbb{Z}^d} \mathfrak{m}(\xi - n/q), \qquad \xi \in \mathbb{T}^d,$$

and denote by $T_{\mathbb{Z}^d}[\mathfrak{m}^q_{\mathrm{per}}]$ the associated Fourier multiplier operator over $\mathbb{Z}^d$. Then

$$\|T_{\mathbb{Z}^d}[\mathfrak{m}^q_{\mathrm{per}}]\|_{\ell^p(\mathbb{Z}^d;B_1) \to \ell^p(\mathbb{Z}^d;B_2)} \leq C_d \|T_{\mathbb{R}^d}[\mathfrak{m}]\|_{L^p(\mathbb{R}^d;B_1) \to L^p(\mathbb{R}^d;B_2)}.$$

The proof can be found in [36, Corollary 2.1, p. 196]. Roughly speaking, the above proposition allows us to control the periodic multiplier $\mathfrak{m}^q_{\mathrm{per}}$ on $\mathbb{Z}^d$ by its single peak $\mathfrak{m}$ on $\mathbb{R}^d$. It is important that Proposition 2.70 covers the case of the finite-dimensional Banach spaces which allows us to work with the supremum norm. We also refer to [41] for a generalization of Proposition 2.70 to real interpolation spaces, which in particular covers the case of jump inequalities.

The second important sampling result is the Ionescu–Wainger theorem which is a sort of generalization of the Magyar–Stein–Wainger sampling principle which covers the case of multipliers localized around fractions with different denominators.

The origin of the Ionescu–Wainger theory takes place in 2004 with their groundbreaking paper [26] in which they proved that the discrete singular Radon transform

$$H^{\mathcal{P}} f(x) := \sum_{y\in\mathbb{Z}^k\setminus\{0\}} f(x-\mathcal{P}(y))K(y),$$

is bounded on ℓ^p with $p>1$. It was a challenging problem to establish the boundedness of $H^{\mathcal{P}}$ on $\ell^p(\mathbb{Z}^d)$ with $p\in(1,\infty)$. The first partial answer was given by Stein and Wainger in [58] where they managed to prove that $H^{\mathcal{P}}$ is bounded on $\ell^p(\mathbb{Z}^d)$ for p in a certain neighborhood of 2. The full range of $p\in(1,\infty)$ was obtained by Ionescu and Wainger [26] by constructing a special set of fractions which allowed them to exhibit some orthogonality properties on ℓ^p. Below we present the vector-valued version of their sampling theorem whose proof can be found in [43, Theorem 2.1].

Theorem 2.71. *For every $\varrho>0$, there exists a family $(P_{\le N})_{N\in\mathbb{N}}$ of subsets of $\mathbb{N}$ such that:*

(i) $\mathbb{N}_N\subseteq P_{\le N}\subseteq \mathbb{N}_{\max\{N,e^{N^\varrho}\}}$.

(ii) *If $N_1\le N_2$, then $P_{\le N_1}\subseteq P_{\le N_2}$.*

(iii) *If $q\in P_{\le N}$, then all factors of q also lie in $P_{\le N}$.*

(iv) $\operatorname{lcm}(P_N)\le 3^N$.

Furthermore, for every $p\in(1,\infty)$, there exists $0<C_{p,\varrho,d}<\infty$ such that, for every $N\in\mathbb{N}$, the following holds:

Let $0<\varepsilon_N\le e^{-N^{2\varrho}}$ and let $\mathbf{Q}:=[-1/2,1/2)^d$ be a unit cube. Let $\mathfrak{m}\colon\mathbb{R}^d\to L(H_0,H_1)$ be a measurable function supported on $\varepsilon_N\mathbf{Q}$ taking values in $L(H_0,H_1)$, the space of bounded linear operators between separable Hilbert spaces H_0 and H_1. Let $0<\mathbf{A}_p\le\infty$ denote the smallest constant such that

$$\big\|\mathcal{F}^{-1}_{\mathbb{R}^d}\big(\mathfrak{m}\,\mathcal{F}_{\mathbb{R}^d}f\big)\big\|_{L^p(\mathbb{R}^d;H_1)}\le \mathbf{A}_p\|f\|_{L^p(\mathbb{R}^d;H_0)}$$

for every function $f\in L^2(\mathbb{R}^d;H_0)\cap L^p(\mathbb{R}^d;H_0)$. Then, the multiplier

$$\Delta_N(\xi):=\sum_{b\in\Sigma_{\le N}}\mathfrak{m}(\xi-b),$$

where $\Sigma_{\le N}$ is defined by

$$\Sigma_{\le N}:=\Big\{\frac{a}{q}\in\mathbb{Q}^d\cap\mathbb{T}^d : q\in P_{\le N}\ \text{and}\ \gcd(a,q)=1\Big\},$$

satisfies

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^d}\big(\Delta_N\,\mathcal{F}_{\mathbb{Z}^d}f\big)\big\|_{\ell^p(\mathbb{Z}^d;H_1)}\le C_{p,\varrho,d}(\log N)\mathbf{A}_p\|f\|_{\ell^p(\mathbb{Z}^d;H_0)} \tag{2.72}$$

for every $f\in\ell^p(\mathbb{Z}^d;H_0)$.

At first, it is easy to see that in Theorem 2.71 the multiplier Δ_N is localized around fractions with different denominators and by property (i) we know that among these denominators are numbers $\{1,\ldots,N\}$ which clearly generalizes Proposition 2.70 where we have only one denominator q. Unfortunately, the set of fractions $\Sigma_{\le N}$ does not consist solely of fractions with denominators from the set $\{1,\ldots,N\}$. There are some bigger denominators but fortunately they are bounded by e^{N^ρ}. It is not easy to describe in a few words why one cannot just take the fractions with denominators from the set $\{1,\ldots,N\}$. We refer to [26, 37, 61] for more details. The second thing is the limitation the underlying setting. In the Magyar–Stein–Wainger sampling principle we are able to consider Banach spaces but in the Ionescu–Wainger theorem we are limited to the Hilbert spaces only. Fortunately, it is enough in our case since this covers the case of the square functions.

In the first version of Theorem 2.71 in the inequality (2.72) there was a factor $(\log N)^D$ with some $D > 0$ related to ρ. In mid 2010's Mirek [37] managed to improve the loss in the Ionescu–Wainger theorem to $\log N$ and he has used it to provide an analogue of the Littlewood–Paley theory adapted to major arcs. In late 2010's Mirek, Stein and Zorin-Kranich in their work [43] about the jump inequalities for the Radon operators have developed the vector-valued version with the $\log N$ loss. In 2020 Tao [61] has used the last progression on the so-called sunflower conjecture to remove the factor $\log N$[2].

[2] In this thesis we present the Ionescu–Wainger with the $\log N$ loss because the results presented here were obtained before the work of Tao and have used the version due to Mirek, Stein and Zorin-Kranich.

Chapter 3

Uniform oscillation estimates for Radon operators

The results of this chapters are based on [D1] and [D2]. Our goal is to prove the uniform oscillation inequalities for the Radon operators (Theorems 1.45 and 1.48). The chapter is organized as follows. In Section 3.1 we present a brief history of the problem and place our results among other known theorems. In Section 3.2 we present the proof of the uniform oscillation inequality for the Radon averages M_t and $\mathcal{M}_t$ – see Section 2.4 for definitions. In Section 3.3 we present the proof of the uniform oscillation inequality for the singular integrals of Radon type H_t and $\mathcal{H}_t$.

3.1 Brief history of the problem

At the beginning of the 1980's, Bellow [3] and independently Furstenberg [19] posed the problem about pointwise convergence of the ergodic averages along monomials given by

$$T_N^b f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(T^{n^b} x),$$

where T is some measure preserving transformation. At the end of the 1980's, Bourgain established the pointwise convergence of the averages T_N^b in a series of groundbreaking articles [4, 5, 6]. By using the Hardy–Littlewood circle method Bourgain [6] proved that, for any $\lambda > 1$ and any sequence of integers $I = (I_j : j \in \mathbb{N})$ with $I_{j+1} > 2I_j$ for all $j \in \mathbb{N}$, we have

$$\big\| O_{I,N}^2 (T_{\lambda^n}^b f : n \in \mathbb{N}) \big\|_{L^2(X,\mu)} \leq C_{I,\lambda}(N) \|f\|_{L^2(X,\mu)}, \qquad N \in \mathbb{N}, \tag{3.1}$$

for all $f \in L^2(X,\mu)$ with $\lim_{N\to\infty} N^{-1/2} C_{I,\lambda}(N) = 0$. By Proposition 2.3 this non-uniform inequality (3.1) suffices to establish the pointwise convergence of the averaging operators $T_N^b f$ for all $f \in L^2(X,\mu)$. In order to prove (3.1) one may follow Bourgain's approach and use the Calderón transference principle (Theorem 2.43) in order to reduce the matter to the case of the shift-related averages given by

$$M_N^b f(x) := \frac{1}{2N+1} \sum_{n=-N}^{N} f(x - n^b), \quad x \in \mathbb{Z}, \quad f \in \ell^p(\mathbb{Z}), \quad x \in \mathbb{Z}, \quad N \in \mathbb{N}.$$

Then (3.1) is just a consequence of the oscillation inequality for M_N^b, namely

$$\big\| O_{I,N}^2 (M_{\lambda^n}^b f : n \in \mathbb{N}) \big\|_{\ell^2(\mathbb{Z})} \leq C_{I,\lambda}(N) \|f\|_{\ell^2(\mathbb{Z})}, \qquad N \in \mathbb{N}. \tag{3.2}$$

Shortly after the groundbreaking work of Bourgain, Lacey [55, Theorem 4.23, p. 95] improved (3.2) by showing that, for every $\lambda > 1$, there is a constant $C_\lambda > 0$ such that

$$\sup_{J\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{L}_\tau)} \big\|O^2_{I,N}(M^b_{\lambda^n} f : n\in\mathbb{N})\big\|_{\ell^2(\mathbb{Z})} \le C_\lambda(N)\|f\|_{\ell^2(\mathbb{Z})}, \qquad N\in\mathbb{N}. \tag{3.3}$$

where $\mathbb{L}_\tau := \{\tau^n : n\in\mathbb{N}\}$. This result motivated the question about uniform estimates independent of $\lambda > 1$ in (3.3). In the case of the averages M^1_N, corresponding to the standard Birkhoff's averages, this question was explicitly formulated in [55, Problem 4.12, p. 80]. In 1998, Jones, Kaufman, Rosenblatt, and Wierdl [28] established the uniform oscillation inequality on $\ell^p(\mathbb{Z})$ for Birkhoff's averages M^1_N. Namely, there is a constant $C_p > 0$ such that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \big\|O^2_{I,N}(M^1_N f : N\in\mathbb{N})\big\|_{\ell^p(\mathbb{Z})} \le C_p\|f\|_{\ell^p(\mathbb{Z})}, \quad f\in\ell^p(\mathbb{Z}),$$

which gives an affirmative answer to [55, Problem 4.12, p. 80]. In 2003, Jones, Rosenblatt, and Wierdl [31] proved uniform oscillation inequalities on $\ell^p(\mathbb{Z}^d)$ with $p\in(1,2]$ for the Birkhoff averages over cubes given by

$$\sum_{n\in[-N,N]^d} f(x-n), \quad x\in\mathbb{Z}^d, \quad f\in\ell^p(\mathbb{Z}^d).$$

However in the case of polynomial averages, even one-dimensional, the problem of uniformity was open until the recent work [D1]. At this moment, it is worth mentioning that **non-uniform** variants of the oscillation inequality for the Radon averages are known. Let $M^{\mathcal{P}}_t$ be the Radon average given by (1.41). By using the r-variational estimates for $r > 2$ established by Mirek, Stein and Trojan [40], the inequality (2.11) implies that for any $p\in(1,\infty)$ and any $r\in(2,\infty)$ we have

$$\sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \big\|O^2_{I,J}(M^{\mathcal{P}}_t f : t>0)\big\|_{\ell^p(\mathbb{Z}^d)} \le C_p\frac{r}{r-2}N^{\frac12-\frac1r}\|f\|_{\ell^p(\mathbb{Z}^d)}, \qquad N\in\mathbb{N}.$$

Unfortunately, we are note able to take $r=2$ in the above estimate which forces us to take a different approach to obtain uniform oscillation estimates.

In the case of the continuous averages $\mathcal{M}^{\mathcal{P}}_t$ given by (1.43) it was only known that there is a **non-uniform** variant of the oscillation inequality,

$$\sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \big\|O^2_{I,J}(\mathcal{M}^{\mathcal{P}}_t f : t>0)\big\|_{L^p(\mathbb{R}^d)} \le C_p\frac{r}{r-2}N^{\frac12-\frac1r}\|f\|_{L^p(\mathbb{R}^d)}, \qquad N\in\mathbb{N},$$

which follows by (2.11) and the r-variational estimates obtained by Jones, Seeger and Wright [32].

In this place we can put the main result of [D1] which we formulate as a separate theorem.

Theorem 3.4. *Let $d,k\ge 1$ and let $\mathcal{P}$ be a polynomial mapping (1.40). For any $p\in(1,\infty)$ there is a constant $C_{p,d,k,\deg\mathcal{P}} > 0$ such that*

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)} \big\|O^2_{I,N}(M_t f : t\in\mathbb{R}_+)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,d,k,\deg\mathcal{P}}\|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f\in\ell^p(\mathbb{Z}^d), \tag{3.5}$$

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)} \big\|O^2_{I,N}(\mathcal{M}_t f : t\in\mathbb{R}_+)\big\|_{L^p(\mathbb{R}^\Gamma)} \le C_{p,d,k,\deg\mathcal{P}}\|f\|_{L^p(\mathbb{R}^d)}, \quad f\in L^p(\mathbb{R}^\Gamma). \tag{3.6}$$

In particular, the implied constants in the inequalities above are independent of the coefficients of the polynomial mapping $\mathcal{P}$.

The above theorem answers in the affirmative to the question about uniform oscillation inequalities for the both Radon averages related to any polynomial mapping (1.40) and any convex body Ω.

In 2000, Campbell, Jones, Reinhold and Wierdl [9] investigated oscillation inequalities for the truncated Hilbert transform $\mathcal{H}_t$ given by

$$\mathcal{H}_t f(x) := \text{p.v.}\frac{1}{\pi}\int_{|y|<t}\frac{f(x-y)}{y}\mathrm{d}y, \quad x\in\mathbb{R}, \quad t>0.$$

They proved that for any $p\in(1,\infty)$ there is a constant $C_p>0$ such that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\big\|O^2_{I,N}(\mathcal{H}_t f : t>0)\big\|_{L^p(\mathbb{R})}\le C_p\|f\|_{L^p(\mathbb{R})}, \quad f\in L^p(\mathbb{R}).$$

Three years later, the same authors [10] managed to extend the above result to the case of multidimensional singular integrals of the Calderón–Zygmund type defined by

$$T_t^{\text{ball}} f(x) := \text{p.v.}\int_{|y|<t} f(x-y)K(y)\mathrm{d}y, \quad x\in\mathbb{R}^d.$$

They proved [10, Theorem A, p. 2116] that for every $p\in(1,\infty)$, there is a constant $C_{p,d}>0$ such that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\big\|O^2_{I,N}(T_t^{\text{ball}} f : t>0)\big\|_{L^p(\mathbb{R}^d)}\le C_{p,d}\|f\|_{L^p(\mathbb{R}^d)}, \quad f\in L^p(\mathbb{R}^d).$$

Again, in the general case of the continuous Radon type singular integrals $\mathcal{H}_t^{\mathcal{P}}$ given by (1.44) it is known that

$$\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,J}(\mathcal{H}_t^{\mathcal{P}} f : t>0)\big\|_{L^p(\mathbb{R}^d)}\le C_p\frac{r}{r-2}N^{\frac{1}{2}-\frac{1}{r}}\|f\|_{L^p(\mathbb{R}^d)}, \qquad N\in\mathbb{N}.$$

The above estimate follows by the r-variational estimates obtained by Jones, Seeger and Wright [32].

In the case of the discrete Radon type singular integrals $H_t^{\mathcal{P}}$ it is only known that

$$\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,J}(H_t^{\mathcal{P}} f : t>0)\big\|_{\ell^p(\mathbb{Z}^d)}\le C_p\frac{r}{r-2}N^{\frac{1}{2}-\frac{1}{r}}\|f\|_{\ell^p(\mathbb{Z}^d)}, \qquad N\in\mathbb{N},$$

which follows by the r-variational estimates obtained by Mirek, Stein and Zorin-Kranich [42]. It appears that until the work [D2] there were no known uniform oscillation inequalities for the discrete singular integrals $H_t^{\mathcal{P}}$, even in the case $d=k=1$ and $\mathcal{P}(y)=y$. The next theorem which comes from [D2] completely solves the problem of the uniform oscillation inequalities for the singular operators of Radon type in continuous and discrete settings.

Theorem 3.7. *Let $d,k\ge 1$ and let $\mathcal{P}$ be a polynomial mapping (1.40). For any $p\in(1,\infty)$ there is a constant $C_{p,d,k,\deg\mathcal{P}}>0$ such that*

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\big\|O^2_{I,N}(H_t^{\mathcal{P}} f : t\in\mathbb{R}_+)\big\|_{\ell^p(\mathbb{Z}^d)}\le C_{p,d,k,\deg\mathcal{P}}\|f\|_{\ell^p(\mathbb{Z}^d)}, \quad f\in\ell^p(\mathbb{Z}^d), \tag{3.8}$$

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\big\|O^2_{I,N}(\mathcal{H}_t^{\mathcal{P}} f : t\in\mathbb{R}_+)\big\|_{L^p(\mathbb{R}^d)}\le C_{p,d,k,\deg\mathcal{P}}\|f\|_{L^p(\mathbb{R}^d)}, \quad f\in L^p(\mathbb{R}^d). \tag{3.9}$$

In particular, the implied constants in the inequalities above are independent of the coefficients of the polynomial mapping $\mathcal{P}$.

In the next sections we focus on proving Theorems 3.4 and 3.7.

3.2 Oscillation inequality for averages of Radon type – proof of Theorem 3.4

In this section we give the proof of the uniform oscillation inequality for the Radon averages. The results in this section are based on results from [D1]. At first we focus on the discrete averages

$$M_t f(x) = \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{y \in \Omega_t \cap \mathbb{Z}^k} f(x - (y)^\Gamma), \quad x \in \mathbb{Z}^\Gamma.$$

Next we establish the uniform oscillation inequality for the continuous averages

$$\mathcal{M}_t f(x) = \frac{1}{|\Omega_t|} \int_{\Omega_t} f(x - (y)^\Gamma) \mathrm{d}y, \quad x \in \mathbb{R}^\Gamma.$$

By invoking the lifting procedure for the Radon averages described in Section 2.3 it is enough to prove Theorem 3.4 only for the canonical mappings.

3.2.1 Discrete Radon averages

Assume that $p \in (1, \infty)$ and let $f \in \ell^p(\mathbb{Z}^\Gamma)$ be a function with a compact support. Our aim is to prove that there is a constant $C_{p,k,|\Gamma|}$ such that

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{R}_+)} \|O^2_{I,N}(M_t f : t > 0)\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,k,|\Gamma|} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.10}$$

By using the monotone convergence theorem and standard density arguments to prove (3.10) it is enough to establish

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \|O^2_{I,N}(M_t f : t \in \mathbb{I})\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,k,|\Gamma|} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.11}$$

for every finite subset $\mathbb{I} \subset \mathbb{R}_+$ with a constant $C_{p,k,|\Gamma|} > 0$ that is independent of the set $\mathbb{I}$. Let us choose $p_0 > 1$, close to 1 such that $p \in (p_0, p_0')$. Then we take $\tau \in (0,1)$ such that

$$\tau < \frac{1}{2} \min\{p_0 - 1, 1\}. \tag{3.12}$$

By Proposition 2.33 we split (3.11) into long oscillations and short variations,

$$\begin{aligned} \sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{I})} \|O^2_{I,N}(M_t f : t \in \mathbb{I})\|_{\ell^p(\mathbb{Z}^\Gamma)} &\lesssim \sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(M_{2^{n^\tau}} f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &\quad + \Big\| \Big(\sum_{n=0}^{\infty} V^2\big(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \end{aligned} \tag{3.13}$$

since $M_t f \equiv f$ for $t \in (0,1)$. Now, we separately estimate the each term on right hand side of (3.13).

Estimates for short variations

Estimates for short variations for the discrete Radon averages were obtained by Mirek, Stein and Zorin-Kranich in [43] by using the techniques developed in the work of Zorin-Karnich [65]. For the sake of completeness we present that argument. The key observation is that short variations can be controlled by the ℓ^p norm of the 1-variations V^1. In our case it will be sufficient to prove that

$$\big\|V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim n^{\tau - 1} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \quad n \in \mathbb{N}. \tag{3.14}$$

Let $t_1 < t_2 < \cdots < t_{J(n)}$ be a sequence of elements of $[2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}$. Since the number of elements in $[2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}$ is finite, it is easy to see that

$$\big\|V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \le \Big\| \sum_{j=1}^{J(n)} |M_{t_j} f - M_{t_{j-1}} f| \Big\|_{\ell^1(\mathbb{Z}^\Gamma)}$$

for any $n \in \mathbb{N}_0$. Hence, by the monotonicity of the sets Ω_t and by the fact that $|\Omega \cap \mathbb{Z}^k| \simeq 2^{kn^\tau}$ we have

$$\big\|V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-kn^\tau} \big|(\Omega_{2^{(n+1)^\tau}} \setminus \Omega_{2^{n^\tau}}) \cap \mathbb{Z}^k\big| \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

In order to estimate the number of lattice points in the set $\Omega_{2^{(n+1)^\tau}} \setminus \Omega_{2^{n^\tau}}$ we make use of the following discrete counterpart of Lemma 2.66.

Proposition 3.15 (cf. [43, Proposition 4.16])**.** *Let $G \subset \mathbb{R}^k$ be a bounded and convex set and let $1 \le s \le \operatorname{diam}(G)$. Then*

$$\#\{x \in \mathbb{Z}^k : \operatorname{dist}(x, \partial G) < s\} \lesssim_k s \operatorname{diam}(G)^{k-1}. \tag{3.16}$$

The implicit constant depends only on the dimension k, but not on the convex set G.

Consequently, for large $n \in \mathbb{N}$ we have

$$2^{-kn^\tau} |\mathbb{Z}^k \cap (\Omega_{2^{(n+1)^\tau}} \setminus \Omega_{2^{n^\tau}})| \lesssim_{k,\Omega} n^{\tau-1}$$

and we see that (3.14) holds. Hence, one can estimate

$$\begin{aligned}
&\Big\|\Big(\sum_{n=0}^\infty V^2\big(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}\big)^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} && (3.17)\\
&\quad\le \Big\|\Big(\sum_{n=0}^\infty V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\
&\quad\le \Big\|\Big(\sum_{n=0}^\infty V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})^q\Big)^{1/q}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} && \text{with } q = \min\{p, 2\}\\
&\quad\le \Big(\sum_{n=0}^\infty \big\|V^1(M_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)}^q\Big)^{1/q} && \text{by Minkowski's inequality}\\
&\quad\lesssim \Big(\sum_{n=0}^\infty n^{-q(1-\tau)}\Big)^{1/q} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} && \text{by (3.14)}\\
&\quad\lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)},
\end{aligned}$$

since $q(1-\tau) > 1$ by (3.12). This proves the estimate for the short variations in (3.13).

Estimates for long oscillations

The aim of this subsection is to give a proof of the estimate for the long oscillations,

$$\sup_{N \in \mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(M_{2^{n^\tau}} f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,k,|\Gamma|} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.18}$$

First of all, let us note that it is enough to consider the operator

$$\widetilde{M}_{2^{n^\tau}} f(x) := \frac{1}{|\Omega_{2^{n^\tau}}|} \sum_{y \in \Omega_{2^{n^\tau}} \cap \mathbb{Z}^k} f(x - (y)^\Gamma), \quad x \in \mathbb{Z}^\Gamma, \tag{3.19}$$

in the place of $M_{2^{n^\tau}}$. This follows since by Davenport's result [17] (see also Proposition 3.41) we know that

$$|\Omega_{2^{n^\tau}} \cap \mathbb{Z}^k| = |\Omega_{2^{n^\tau}}| + \mathcal{O}(2^{n^\tau(k-1)}).$$

Consequently, one has the following estimate

$$\big\|(M_{2^{n^\tau}} - \widetilde{M}_{2^{n^\tau}})f\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-n^\tau}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Hence, by (2.28) we can control the error term by

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\big\|O^2_{I,N}((M_{2^{n^\tau}} - \widetilde{M}_{2^{n^\tau}})f : n\in\mathbb{N}_0)\big\|_{L\ell^p(\mathbb{Z}^\Gamma)} \lesssim \Big\|\Big(\sum_{n=0}^\infty \big|(M_{2^{n^\tau}} - \widetilde{M}_{2^{n^\tau}})f\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Therefore we may prove

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\big\|O^2_{I,N}(\widetilde{M}_{2^{n^\tau}} f : n\in\mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.20}$$

instead of (3.18). It can be easily noted that

$$\widetilde{M}_{2^{n^\tau}} f(x) = \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}\mathcal{F}_{\mathbb{Z}^\Gamma} f)(x), \quad x\in\mathbb{Z}^\Gamma,$$

where

$$\widetilde{m}_{2^{n^\tau}}(\xi) := \frac{1}{|\Omega_{2^{n^\tau}}|}\sum_{y\in\Omega_{2^{n^\tau}}\cap\mathbb{Z}^k} e(\xi\cdot(y)^\Gamma), \quad \xi\in\mathbb{T}^\Gamma.$$

Now we use the Hardy–Littewood circle method to establish (3.20). Let $\chi\in(0,1/10)$ be a fixed number. The proof of the inequality (3.20) will require a several appropriately chosen parameters. Let us choose $\alpha>0$ such that

$$\alpha > \Big(\frac{1}{p_0}-\frac{1}{2}\Big)\Big(\frac{1}{p_0}-\frac{1}{\min\{p,p'\}}\Big)^{-1},$$

where p_0 is the parameter set at the beginning of the proof. Let $u\in\mathbb{N}$ be a large natural number which will be specified later. We set

$$\varrho := \min\Big\{\frac{1}{10u}, \frac{\delta}{8\alpha}\Big\}, \tag{3.21}$$

where $\delta>0$ is from the estimate for the Gauss sum (3.40). Now, let us consider $\tilde{S} = \max(2^{u\mathbb{N}}\cap[1,n^{\tau u}])$. We recall the family of rational fractions $\Sigma_{\le\tilde{S}}$ related to parameter ϱ from the Ionescu–Wainger (Theorem 2.71). For simplicity we will write

$$\Sigma_{\le n^{\tau u}} := \Sigma_{\le\tilde{S}}.$$

Next, for dyadic integers $S\in 2^{u\mathbb{N}}$ we define "annulus" sets of fractions by

$$\Sigma_S = \begin{cases}\Sigma_{\le S}, & \text{if} \quad S=2^u,\\ \Sigma_{\le S}\setminus\Sigma_{\le S/2^u}, & \text{if} \quad S>2^u.\end{cases}$$

We note that by property (ii) from Theorem 2.71 the above definition makes sense. It is easy to see that

$$\Sigma_{\le n^{\tau u}} = \bigcup_{\substack{S\le n^{\tau u},\\ S\in 2^{u\mathbb{N}}}} \Sigma_S. \tag{3.22}$$

Now, we are able to define the Ionescu–Wainger projection multipliers. Let $\eta\colon \mathbb{R}^\Gamma \to [0,1]$ be a smooth function such that

$$\eta(x) = \begin{cases} 1, & |x| \le 1/(32|\Gamma|), \\ 0, & |x| \ge 1/(16|\Gamma|). \end{cases}$$

For any $n \in \mathbb{N}$ we define

$$\Pi_{\le n^\tau, n^\tau(A-\chi I)}(\xi) := \sum_{a/q \in \Sigma_{\le n^\tau u}} \eta\big(2^{n^\tau(A-\chi I)}(\xi - a/q)\big), \quad \xi \in \mathbb{T}^\Gamma, \tag{3.23}$$

where A is the matrix introduced in (2.63). We also define annulus projections by setting

$$\Pi_{S, n^\tau(A-\chi I)}(\xi) := \sum_{a/q \in \Sigma_S} \eta\big(2^{n^\tau(A-\chi I)}(\xi - a/q)\big), \quad \xi \in \mathbb{T}^\Gamma. \tag{3.24}$$

Note that by Theorem 2.71 we have that

$$\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\Pi_{\le n^\tau, n^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim_{\tau,u} \log(n)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.25}$$

and

$$\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\Pi_{S, n^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.26}$$

since $2^{n^\tau(|\gamma|-\chi)} \le e^{-n^{\tau/10}} \le e^{-\tilde{S}^\varrho}$ for sufficiently large $n \in \mathbb{N}$.

Projections defined in (3.23) allows us to partition the multiplier $\widetilde{m}_{2^{n^\tau}}$ and estimate (3.20) by

$$\lesssim \sup_{N\in\mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}\Pi_{\le n^\tau, n^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.27}$$

$$+ \sup_{N\in\mathbb{N}} \sup_{I \in \mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.28}$$

The first and second terms in the above inequality corresponds to major and minor arcs, respectively.

Minor arcs estimates – Weyl's inequality

As in the Waring problem we handle minor arcs estimate (3.28) by using some version of Weyl's inequality. First, we note that the oscillation seminorm is controlled by the 2-variations V^2 and moreover

$$\big\|V^2(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \sum_{n=0}^{\infty}\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

As a consequence it is enough to show

$$\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)}))\mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.29}$$

for any $n \in \mathbb{N}$. Let us note that for any $p \in (1,\infty)$ by the inequality (3.25) we obtain

$$\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)}))\mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.30}$$

since operators $\widetilde{M}_t$ have uniformly bounded ℓ^p-norm. It turns out that for $p = 2$ we have a much better estimate. Let us recall a result from [39], based on Weyl's inequality, that allows us to bound exponential sums over convex sets.

Theorem 3.31 ([39, Theorem 3.1]). *For every $d, k \in \mathbb{N}$ and $\overline{\alpha} > 0$ there are $\beta_{\overline{\alpha}} = \beta_{\overline{\alpha}}(d, k)$ and $C > 0$ such that for every $\beta > \beta_{\overline{\alpha}}$, every $N > 1$, every polynomial*

$$P(x) = \sum_{\substack{\gamma \in \mathbb{N}_0^k, \\ 0 < |\gamma| \leq d}} \xi_\gamma x^\gamma, \quad \text{with} \quad P(0) = 0, \quad \xi_\gamma \in \mathbb{R},$$

and every convex set $\Omega \subseteq B(0, N)$ the following holds. Suppose that for some multi-index $\gamma_0 \in \mathbb{N}_0^k$ such that $0 < |\gamma| \leq d$, there are integers $0 \leq a \leq q$ with $\gcd(a, q) = 1$, and

$$\left| \xi_{\gamma_0} - \frac{a}{q} \right| \leq \frac{1}{q^2}, \tag{3.32}$$

where q satisfies

$$(\log N)^\beta \leq q \leq N^{|\gamma_0|} (\log N)^{-\beta}. \tag{3.33}$$

Then

$$\Big| \sum_{y \in \Omega \cap \mathbb{Z}^k} e(P(y)) \Big| \leq C N^k (\log N)^{-\overline{\alpha}}. \tag{3.34}$$

The implied constant C may depend on $d, k, \overline{\alpha}$, but is independent of a, q, N, the set Ω and the coefficients of P.

Clearly, both the set $\Omega_{2^{n^\tau}}$ and the polynomial $\xi \cdot (x)^\Gamma$ satisfy the assumptions of Theorem 3.31. Thus, if we show that there are ξ_{γ_0}, a, q for which the conditions (3.32) and (3.33) hold, then

$$|\widetilde{m}_{2^{n^\tau}}(\xi)| \lesssim_k \frac{2^{n^\tau k}}{|\Omega_{2^{n^\tau}}|} (n+1)^{-\overline{\alpha}\tau} \lesssim_\Omega (n+1)^{-\overline{\alpha}\tau},$$

since $|\Omega_{2^{n^\tau}}| \gtrsim_\Omega 2^{n^\tau k}$. Consequently, by Parseval's theorem we have

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\widetilde{m}_{2^{n^\tau}}(1 - \Pi_{\leq n^\tau, n^\tau(A-\chi I)}))\mathcal{F}_{\mathbb{Z}^\Gamma} f\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-\overline{\alpha}\tau} \|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

Now, if we take $p = p_0$ in (3.30) and interpolate with the above inequality, then we obtain

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\widetilde{m}_{2^{n^\tau}}(1 - \Pi_{\leq n^\tau, n^\tau(A-\chi I)}))\mathcal{F}_{\mathbb{Z}^\Gamma} f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-\overline{\alpha}\tau/\alpha} \log(n+1) \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

For appropriately large $\overline{\alpha} > 0$ we get (3.29). It remains to show that conditions (3.32) and (3.33) hold whenever $1 - \Pi_{\leq n^\tau, n^\tau(A-\chi I)}(\xi) \neq 0$.

In order to do so we use the so-called Dirichlet's principle which proof can be found in [48, Theorem 4.1].

Lemma 3.35 (Dirichlet's principle). *Let ξ and Q be real numbers, $Q \geq 1$. There exists integers a and q such that*

$$1 \leq q \leq Q, \quad \gcd(a, q) = 1$$

and

$$\left| \xi - \frac{a}{q} \right| < \frac{1}{qQ}.$$

For each ξ_γ by Dirichlet's principle we have

$$\left| \xi_\gamma - \frac{a_\gamma}{q_\gamma} \right| \leq \frac{1}{q_\gamma 2^{n^\tau |\gamma|} n^{-\beta\tau}} \leq \frac{1}{q_\gamma^2}$$

with $q_\gamma \le 2^{n^\tau|\gamma|}n^{-\beta\tau}$. We claim that if $1 - \Pi_{\le n^\tau, n^\tau(A-\chi I)}(\xi) \ne 0$, then $q_{\gamma_0} \ge n^{\beta\tau}$ holds for some $\gamma_0 \in \Gamma$. Suppose for a contradiction that for any $\gamma \in \Gamma$ we have $1 \le q_\gamma < n^{\beta\tau}$. Then for $q' = \operatorname{lcm}(q_\gamma \colon \gamma \in \Gamma) \le n^{\beta\tau|\Gamma|}$ we have

$$\left|\xi_\gamma - \frac{a'_\gamma}{q'}\right| \le \frac{1}{2^{n^\tau|\gamma|}n^{-\beta\tau}},$$

where $a'_\gamma = a_\gamma q_\gamma^{-1} q'$. We see that $\gcd(q', (a'_\gamma)_{\gamma\in\Gamma}) = 1$. Hence, taking $a' = (a'_\gamma \colon \gamma \in \Gamma)$ and u so large that $n^{\beta\tau|\Gamma|} \le n^{\tau u}$ we get that $a'/q' \in \Sigma_{\le n^{\tau u}}$. On the other hand, if $1 - \Pi_{\le n^\tau, n^\tau(A-\chi I)}(\xi) \ne 0$ then for any $a/q \in \Sigma_{\le n^{\tau u}}$, (thus in particular for a'/q') there exists $\gamma \in \Gamma$ for which

$$\left|\xi_\gamma - \frac{a_\gamma}{q}\right| > \frac{1}{(32|\Gamma|)2^{n^\tau(|\gamma|-\chi)}}.$$

This leads to the inequality

$$(32|\Gamma|)n^{\beta\tau} > 2^{n^\tau\chi},$$

which is false for large n. Therefore, we see that there is $\gamma_0 \in \Gamma$ such that the conditions (3.32) and (3.33) are satisfied and consequently (3.29) follows. This shows that if $u > \beta|\Gamma|$, where β is from Theorem 3.31, then we have

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\widetilde{m}_{2^{n^\tau}}(1-\Pi_{\le n^\tau, n^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which ends the proof of estimates for minor arcs.

Major arcs and kernel approximation

Now we can focus on major arcs. Our aim is to show that the inequality

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\widetilde{m}_{2^{n^\tau}}\Pi_{\le n^\tau, n^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.36}$$

holds. For simplicity, we denote

$$T^\chi_{n^\tau} f(x) := \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\widetilde{m}_{2^{n^\tau}}\Pi_{\le n^\tau, n^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)(x), \quad x \in \mathbb{Z}^\Gamma,$$

and we note that the operator $T^\chi_{n^\tau}$ have the Fourier symbol given by

$$\sum_{a/q\in\Sigma_{\le n^{\tau u}}} \widetilde{m}_{2^{n^\tau}}(\xi)\eta(2^{n^\tau(A-\chi I)}(\xi - a/q)), \quad \xi \in \mathbb{T}^\Gamma. \tag{3.37}$$

Now, our aim is to show that (3.37) is, up to an acceptable error term, equal to

$$\boldsymbol{m}_n(\xi) := \sum_{a/q\in\Sigma_{\le n^{\tau u}}} G(a/q)\Phi_{2^{n^\tau}}(\xi - a/q)\eta(2^{n^\tau(A-\chi I)}(\xi - a/q)) \tag{3.38}$$

where Φ_t is continuous version of multiplier $m_{2^{n^\tau}}$ given by (2.61) and $G(a/q)$ is the Gauss sum defined by

$$G(a/q) := \frac{1}{q^k}\sum_{r\in\mathbb{N}_q^k} \boldsymbol{e}((a/q)\cdot(r)^\Gamma). \tag{3.39}$$

We note that by the multidimensional version of Weyl's inequality [60, Proposition 3] we have

$$|G(a/q)| \lesssim_k q^{-\delta} \tag{3.40}$$

for some $\delta > 0$.

Proposition 3.41 ([43, Proposition 4.18]). *Let $\Omega \subseteq B(0,N) \subset \mathbb{R}^k$ be a convex set and let $\mathcal{K}\colon \Omega \to \mathbb{C}$ be a continuous function. Then for every $q \in \mathbb{N}$, $a \in \{\tilde{a} \in \mathbb{N}_q^d \colon (q, (\tilde{a}_\gamma \colon \gamma \in \Gamma)) = 1\}$ and for every $\xi = a/q + \theta \in \mathbb{R}^d$ we have*

$$\Big| \sum_{y\in\Omega\cap\mathbb{Z}^k} \boldsymbol{e}(\xi\cdot(y)^\Gamma)\mathcal{K}(y) - G(a/q)\int_\Omega \boldsymbol{e}(\theta\cdot(t)^\Gamma)\mathcal{K}(t)\mathrm{d}t\Big|$$
$$\lesssim_k \frac{q}{N}N^k\|\mathcal{K}\|_{L^\infty(\Omega)} + N^k\|\mathcal{K}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma}(q|\theta_\gamma|N^{|\gamma|-1})^{\varepsilon_\gamma} + N^k \sup_{x,y\in\Omega\colon |x-y|\le q}|\mathcal{K}(x)-\mathcal{K}(y)|,$$

for any sequence $(\varepsilon_\gamma \colon \gamma\in\Gamma) \subseteq [0,1]$. The implicit constant is independent of a, q, N, θ and the kernel $\mathcal{K}$.

Proof. We split the sum into congruence classes modulo q as follows:

$$\sum_{y\in\Omega\cap\mathbb{Z}^k}\boldsymbol{e}(\xi\cdot(y)^\Gamma)\mathcal{K}(y) = q^{-k}\sum_{r\in\mathbb{N}_q^k}\boldsymbol{e}((r)^\Gamma\cdot a/q)\cdot\Big(q^k\sum_{\substack{y\in\mathbb{Z}^k\\ qy+r\in\Omega}}\boldsymbol{e}(\theta\cdot(qy+r)^\Gamma)\mathcal{K}(qy+r)\Big).$$

In order to approximate the expression in the parentheses on the right hand side by an integral, we write

$$\Big|q^k\sum_{\substack{y\in\mathbb{Z}^k\\ qy+r\in\Omega}}\boldsymbol{e}(\theta\cdot(qy+r)^\Gamma)\mathcal{K}(qy+r) - \int_\Omega e(\theta\cdot(t)^\Gamma)\mathcal{K}(t)\mathrm{d}t\Big|$$
$$= \Big|q^k\sum_{y\in\mathbb{Z}^k}\boldsymbol{e}(\theta\cdot(qy+r)^\Gamma)\mathcal{K}(qy+r)\mathbb{1}_\Omega(qy+r) - \sum_{y\in\mathbb{Z}^k}\int_{qy+[0,q)^k}\boldsymbol{e}(\theta\cdot(t)^\Gamma)\mathcal{K}(t)\mathbb{1}_\Omega(t)\mathrm{d}t\Big|$$
$$\le \sum_{y\in\mathbb{Z}^k}\int_{[0,q)^k}\big|\boldsymbol{e}(\theta\cdot(qy+r)^\Gamma)\mathcal{K}(qy+r)\mathbb{1}_\Omega(qy+r) - \boldsymbol{e}(\theta\cdot(qy+t)^\Gamma)\mathcal{K}(qy+t)\mathbb{1}_\Omega(qy+t)\big|\mathrm{d}t.$$

Notice that

$$\big|\theta\cdot(qy+r)^\Gamma - \theta\cdot(qy+t)^\Gamma\big| \lesssim \sum_{\gamma\in\Gamma}\big(q|\theta_\gamma|N^{|\gamma|-1}\big)^{\varepsilon_\gamma},$$

and

$$\big|\mathcal{K}(qy+r)-\mathcal{K}(qy+t)\big| \lesssim \sup_{x,y\in\Omega:|x-y|\le q}|\mathcal{K}(x)-\mathcal{K}(y)|,$$

and

$$\sum_{y\in\mathbb{Z}^k}|\mathbb{1}_\Omega(qy+r)-\mathbb{1}_\Omega(qy+t)| \le |(q^{-1}(\Omega-r))\triangle(q^{-1}(\Omega-t))| \lesssim (N/q)^{k-1},$$

where the last inequality is a consequence of Proposition 3.15. Hence, we obtain the estimate

$$\Big|q^k\sum_{\substack{y\in\mathbb{Z}^k\\ qy+r\in\Omega}}\boldsymbol{e}(\theta\cdot(qy+r)^\Gamma)\mathcal{K}(qy+r) - \int_\Omega \boldsymbol{e}(\theta\cdot(t)^\Gamma)\mathcal{K}(t)\mathrm{d}t\Big|$$
$$\lesssim q^k\|\mathcal{K}\|_{L^\infty(\Omega)}(N/q)^{k-1} + N^k\|\mathcal{K}\|_{L^\infty(\Omega)}\sum_{\gamma\in\Gamma}\big(q|\theta_\gamma|N^{|\gamma|-1}\big)^{\varepsilon_\gamma} + N^k\sup_{x,y\in\Omega:|x-y|\le q}|\mathcal{K}(x)-\mathcal{K}(y)|.$$

Averaging in r, we obtain the claim. □

We use Proposition 3.41 with $\Omega_{2^{n^\tau}} \subseteq B(0,2^{n^\tau})$, $\mathcal{K} := |\Omega_{2^{n^\tau}}|^{-1}\mathbb{1}_{\Omega_{2^{n^\tau}}}$ and $\varepsilon_\gamma = 1$. Note that $\|\mathcal{K}\|_{L^\infty(\Omega)} \lesssim 2^{-n^\tau k}$ and $\sup_{x,y\in\Omega\colon |x-y|\le q}|\mathcal{K}(x)-\mathcal{K}(y)| = 0$. Therefore, on the support of $\Pi_{\le n^\tau, n^\tau(A-\chi I)}$,

$$\big|\widetilde{m}_{2^{n^\tau}}(\xi) - G(a,q)\Phi_{2^{n^\tau}}(\xi-a/q)\big| \lesssim q2^{-n^\tau} + \sum_{\gamma\in\Gamma}q|\xi_\gamma - a_\gamma/q|2^{n^\tau(|\gamma|-1)} \lesssim 2^{-n^\tau/2} \tag{3.42}$$

for $\chi \in (0, 1/10)$, since $q \lesssim e^{n^{\tau/10}}$ and for any $\gamma \in \Gamma$ we have $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n^\tau(|\gamma|-\chi)}$. By the disjointness of the supports of $\eta(2^{n^\tau(A-\chi I)}(\xi - a/q))$ we have

$$\sum_{a/q\in\Sigma_{\le n^{\tau u}}} \widetilde{m}_{2^{n^\tau}}(\xi)\eta(2^{n^\tau(A-\chi I)}(\xi - a/q)) = \boldsymbol{m}_n(\xi) + \mathcal{O}(2^{-n^\tau/2}). \tag{3.43}$$

For simplicity we denote

$$\mathcal{T}_{n^\tau} f(x) := \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}_n \mathcal{F}_{\mathbb{Z}^\Gamma} f)(x).$$

Let $p > 1$. Then we have the simple estimate

$$\big\|(T^\chi_{n^\tau} - \mathcal{T}_{n^\tau})f\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \big|\Sigma_{\le n^{\tau u}}\big|\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)n^{\tau/10}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.44}$$

since by property (i) from Theorem 2.71 the number of fractions in $\Sigma_{\le n^{\tau u}}$ is bounded by $e^{(|\Gamma|+1)n^{\tau/10}}$ and by [36, Proposition 2.1] each term in (3.37) and (3.38) defines a bounded multiplier on ℓ^p. Next, for $p = 2$ by using (3.43) and by Parseval's equality we obtain the following estimate

$$\big\|(T^\chi_{n^\tau} - \mathcal{T}_{n^\tau})f\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim 2^{-n^\tau/2}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{3.45}$$

Now, if we take $p = p_0$ in (3.44) and interpolate it with (3.45) we get

$$\big\|(T^\chi_{n^\tau} - \mathcal{T}_{n^\tau})f\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-n^\tau/4}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.46}$$

Therefore we can replace in (3.36) the multiplier $\widetilde{m}_{2^{n^\tau}}\Pi_{\le n^\tau, \le -n^\tau(A-\chi I)}$ by its continuous counterpart $\boldsymbol{m}_n$ since the error term can be handled by the estimate

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}((T^\chi_{n^\tau} - \mathcal{T}_{n^\tau})f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \Big\|\Big(\sum_{n=0}^\infty \big|(T^\chi_{n^\tau} - \mathcal{T}_{n^\tau})f\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

where the last inequality follows from (3.46). Consequently, to show (3.36) it is enough to prove that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}_n\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.47}$$

Now, we split our projection multiplier $\Pi_{\le n^\tau, n^\tau(A-\chi I)}$ into the sum of annulus projections (3.24). By (3.22) we see that

$$\Pi_{\le n^\tau, n^\tau(A-\chi I)}(\xi) = \sum_{\substack{S\le n^{\tau u},\\ S\in 2^{u\mathbb{N}}}} \Pi_{S, n^\tau(A-\chi I)}(\xi).$$

Hence, one has the following decomposition

$$\boldsymbol{m}_n(\xi) = \sum_{\substack{S\le n^{\tau u},\\ S\in 2^{u\mathbb{N}}}} \boldsymbol{m}^n_S(\xi), \tag{3.48}$$

where $\boldsymbol{m}^n_S$ is defined as

$$\boldsymbol{m}^n_S(\xi) := \sum_{a/q\in\Sigma_S} G(a/q)\Phi_{2^{n^\tau}}(\xi - a/q)\eta\big(2^{n^\tau(A-\chi I)}(\xi - a/q)\big). \tag{3.49}$$

By using the decomposition (3.48) combined with triangle's inequality from Fact 2.29 and with the cut-off Proposition 2.32 we obtain

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}_n\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\ &\quad\le \sum_{S\in 2^{u\mathbb{N}}}\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^S_\tau)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}^n_S\mathcal{F}_{\mathbb{Z}^\Gamma}) : n^\tau \ge S^{1/u})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} + \big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}^{S^{1/(\tau u)}}_S\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where $\mathbb{D}_\tau^S = \{n \in \mathbb{N} : n^\tau \geq S^{1/u}\}$. Hence it suffices to show that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau^S)} \big\|O_{I,N}^2(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\boldsymbol{m}_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f) : n^\tau \geq S^{1/u})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-4\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.50}$$

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\boldsymbol{m}_S^{S^{1/(\tau u)}} \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-6\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.51}$$

since both $S^{-4\varrho}$ and $S^{-6\varrho}$ are summable in $S \in 2^{u\mathbb{N}}$.

Gaussian multiplier and scale distinction

In order to apply the Ionescu–Wainger theory we need to manage the Gaussian part $G(a/q)$ in the multiplier (3.49). Let $\widetilde{\eta} := \eta(x/2)$. Then we have $\eta\widetilde{\eta} = \eta$ and since $n^\tau \geq S^{1/u}$ we also have

$$\eta(2^{n^\tau(A-\chi I)}\xi)\widetilde{\eta}(2^{S^{1/u}(A-\chi I)}\xi) = \eta(2^{n^\tau(A-\chi I)}\xi).$$

Next, we introduce new multipliers

$$v_S^n(\xi) = \sum_{a/q\in\Sigma_S} \Phi_{2^{n^\tau}}(\xi - a/q)\eta\big(2^{n^\tau(A-\chi I)}(\xi - a/q)\big),$$

$$\mu_S(\xi) = \sum_{a/q\in\Sigma_S} G(a/q)\widetilde{\eta}\big(2^{S^{1/u}(A-\chi I)}(\xi - a/q)\big).$$

Obviously, we have $\boldsymbol{m}_S^n = v_S^n\mu_S$ and we see that estimates (3.50) and (3.51) will follow if we show that for every $p \in (1,\infty)$ we have

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\mu_S \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-7\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.52}$$

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^{S^{1/(\tau u)}} \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.53}$$

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau^S)} \big\|O_{I,N}^2(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f) : n^\tau \geq S^{1/u})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{3\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.54}$$

It is easy to see that estimate (3.53) is a consequence of Theorem 2.71. Now we focus on proving (3.52). We may assume that S is so large that the functions $\widetilde{\eta}\big(2^{S^{1/u}(A-\chi I)}(\cdot - a/q)\big)$ have disjoint supports for $a/q \in \Sigma_S$. By Plancherel's theorem and by the estimate (3.40) we conclude

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\mu_S \mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim S^{-\delta}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{3.55}$$

Moreover, we will show that for any $p \in (1,\infty)$ the following holds

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\mu_S \mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.56}$$

If we interpolate (3.55) with the above bound for $p = p_0$ we obtain (3.52). We handle (3.56) by introducing certain approximation multipliers. Let $J = \lfloor 2^{S^{\frac{1}{2u}}} \rfloor$. We set

$$\widetilde{\mu}_{J,S}(\xi) := m_J(\xi) \sum_{a/q\in\Sigma_S} \widetilde{\eta}\big(2^{S^{1/u}(A-\chi I)}(\xi - a/q)\big),$$

$$h_{J,S}(\xi) := \sum_{a/q\in\Sigma_S} G(a/q)\Phi_J(\xi - a/q)\widetilde{\eta}\big(2^{S^{1/u}(A-\chi I)}(\xi - a/q)\big).$$

By Theorem 2.71 we have

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\tilde{\mu}_{J,S} \mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.57}$$

since ℓ^p-norm of M_J is uniformly bounded by 1. By Proposition 3.41 and by exploiting the same ideas presented during the proof of (3.46) one can prove that

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((\widetilde{\mu}_{J,S}-h_{J,S})\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-\frac{1}{2}S^{\frac{1}{2u}}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.58}$$

holds for $p\in(1,\infty)$. For $|\xi_\gamma-a_\gamma/q|\lesssim 2^{-S^{1/u}(|\gamma|-\chi)}$ the first estimate in (2.64) provides us with the bound

$$|1-\Phi_J(\xi-a/q)|\lesssim |J^A(\xi-a/q)|_\infty \lesssim 2^{-\frac{1}{2}S^{\frac{1}{2u}}}$$

and by Plancherel's theorem we get

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((\mu_S-h_{J,S})\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim 2^{-\frac{1}{2}S^{\frac{1}{2u}}}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{3.59}$$

Since $|\Sigma_S|\lesssim e^{(|\Gamma|+1)S^\varrho}$, for every $p\in(1,\infty)$ we have

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((\mu_S-h_{J,S})\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)S^\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.60}$$

Interpolating (3.59) with (3.60) leads to

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((\mu_S-h_{J,S})\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

The above estimate, (3.57) and (3.58) together ensure that (3.56) holds.

Now we may focus on proving the estimate (3.54). Let $\kappa_S=\lceil S^{2\varrho}\rceil$. By Proposition 2.30 we may split the left hand side of (3.54) at point 2^{κ_S} and write

$$\begin{aligned}\mathrm{LHS}(3.54) \lesssim &\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\leq S})}\big\|O^2_{I,N}(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^n\mathcal{F}_{\mathbb{Z}^\Gamma}f):n^\tau\in[S^{1/u},2^{\kappa_S+1}])\big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\ &+\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})}\big\|O^2_{I,N}(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^n\mathcal{F}_{\mathbb{Z}^\Gamma}f):n^\tau\geq 2^{\kappa_S})\big\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where $\mathbb{D}^\tau_{\leq S}:=\{n\in\mathbb{N}:n^\tau\in[S^{1/u},2^{\kappa_S+1}]\}$ and $\mathbb{D}^\tau_{\geq S}:=\{n\in\mathbb{N}:n^\tau\geq 2^{\kappa_S}\}$. The first term of the right hand side of the above inequality corresponds to small scales and the second one to large scales. We will deal with small scales by using the Rademacher–Menschov inequality (2.38) and Theorem 2.71. In the case of large scales we make use of the sampling principle of Magyar, Stein and Wainger (Proposition 2.70) to transfer estimates from the continuous case.

Estimates for small scales

We will rather closely follow arguments from [43] to prove the following estimate

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^S_\tau)}\big\|O^2_{I,N}(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^n\mathcal{F}_{\mathbb{Z}^\Gamma}f):n^\tau\in[S^{1/u},2^{\kappa_S+1}])\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \kappa_S\log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.61}$$

The above estimate together with the bound for large scales (3.67) gives us (3.54). In order to prove (3.61) we define auxiliary multipliers

$$\Lambda_S^n(\xi):=\sum_{a/q\in\Sigma_S}\big(\Phi_{2^{(n+1)^\tau}}(\xi-a/q)-\Phi_{2^{n^\tau}}(\xi-a/q)\big)\eta\big(2^{n^\tau(A-\chi I)}(\xi-a/q)\big),$$

$$\Delta_S^n(\xi):=\sum_{a/q\in\Sigma_S}\Phi_{2^{(n+1)^\tau}}(\xi-a/q)\Big(\eta\big(2^{(n+1)^\tau(A-\chi I)}(\xi-a/q)\big)-\eta\big(2^{n^\tau(A-\chi I)}(\xi-a/q)\big)\Big).$$

By applying the inequality (2.38) to the left hand side of (3.61) we see that

$$\mathrm{LHS}(3.61) \lesssim \sum_{i=0}^{\kappa_S+1} \Big\| \Big(\sum_j \big| \sum_{n\in I_j^i} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((v_S^{n+1} - v_S^n)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

where $I_j^i = [j2^i, (j+1)2^i) \cap [S^{1/u}, 2^{\kappa_S+1}]$ (since the inner sum telescopes). Summation with respect to j runs over $j \in \mathbb{N}$ such that $I_j^i \neq \emptyset$. Now, by the fact that $v_S^{n+1} - v_S^n = \Lambda_S^n + \Delta_S^n$ and by the triangle inequality to obtain (3.61) it suffices to prove that for every $i \leq \kappa_S$ we have

$$\Big\| \Big(\sum_j \Big| \sum_{n\in I_j^i} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Lambda_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f)\Big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.62}$$

$$\Big\| \Big(\sum_j \Big| \sum_{n\in I_j^i} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f)\Big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.63}$$

Clearly, the estimate (3.63) will follow if we show that for every subset $I \subseteq [S^{1/u}, 2^{\kappa_S+1}] \cap \mathbb{N}$ we have

$$\sum_{n\in I} \|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.64}$$

By Theorem 2.71 for every $p \in (1,\infty)$ we get an ℓ^p-estimate for the n-th term with $\log S$ gain. Since multiplier Δ_S^n is non-zero when $|2^{n^\tau A}(\xi - a/q)|_\infty \gtrsim 2^{n^\tau \chi}$ by the van der Corput estimate (2.64) we obtain ℓ^2-estimates for n-th term with $2^{-n^\tau\chi/|\Gamma|}$ loss. By complex interpolation

$$\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_S^n \mathcal{F}_{\mathbb{Z}^\Gamma} f)\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S) 2^{-n^\tau\chi/(\alpha|\Gamma|)}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which implies (3.64).

Now we can handle the estimate (3.62). By Theorem 2.71, the estimate (3.62) is a consequence of its continuous counterpart

$$\Big\| \Big(\sum_j \big| \sum_{n\in I_j^i} \mathcal{F}_{\mathbb{R}^\Gamma}^{-1}\big((\Phi_{2^{(n+1)^\tau}} - \Phi_{2^{n^\tau}})\eta\big(2^{n^\tau(A-\chi I)}\cdot\big)\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

The above estimate follows from the square function bound

$$\Bigg\| \Bigg(\sum_j \Big| \sum_{n\in I_j^i} \mathcal{F}_{\mathbb{R}^\Gamma}^{-1}\big((\Phi_{2^{(n+1)^\tau}} - \Phi_{2^{n^\tau}})\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\Big|^2\Bigg)^{1/2}\Bigg\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{3.65}$$

since for every $p \in (1,\infty)$ the error term is controlled by

$$\sum_{n=0}^{\infty} \big\|\mathcal{F}_{\mathbb{R}^\Gamma}^{-1}\big((\Phi_{2^{(n+1)^\tau}} - \Phi_{2^{n^\tau}})\big(1 - \eta(2^{n^\tau(A-\chi I)}\cdot)\big)\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

Indeed, we have a uniform L^p-bound for the n-th term. Moreover, since the function $1 - \eta(2^{n^\tau(A-\chi I)}\cdot)$ is non-zero when $|2^{k^\tau A}\xi|_\infty \gtrsim 2^{k^\tau\chi}$, by the van der Corput estimate (2.64) we obtain an L^2-estimate for the n-th term with $2^{-n^\tau\chi/|\Gamma|}$ gain. Thus, the desired bound for the error term follows by complex interpolation.

The square function estimate (3.65) can be deduced from the following inequality for the operator $\mathcal{M}_t f(x) = \mathcal{F}_{\mathbb{R}^\Gamma}^{-1}(\Phi_t \mathcal{F}_{\mathbb{R}^\Gamma} f)(x)$,

$$\Big\| \Big(\sum_{k\in\mathbb{N}} \big|(\mathcal{M}_{t_{k+1}} - \mathcal{M}_{t_k})f\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \leq C_p\|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{3.66}$$

which holds for every increasing sequence $0 < t_1 \leq t_2 \leq \cdots$ and where constant $C_p > 0$ is independent of the chosen sequence. The inequality (3.66) may be seen of as some weak form of r-variational inequality with $r = 2$, and one can prove it by using results from [42]. Indeed, the square function estimate (3.66) follows from the Khintchine-type bound

$$\Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n(\mathcal{M}_{2^{n+1}} - \mathcal{M}_{2^n})f\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

which holds uniformly for every sequence $(\varepsilon_n)_{n\in\mathbb{Z}}$ bounded by 1 due to [42, Theorem 2.28], and from the short 2-variation inequality

$$\Big\|\Big(\sum_{n\in\mathbb{Z}} V^2(\mathcal{M}_{2^t}f : t \in [n, n+1])^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

which is a direct consequence of [42, Theorem 2.39].

Estimates for large scales

The last thing to show is the estimate for large scales,

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(v^n_S\mathcal{F}_{\mathbb{Z}^\Gamma}f) : n^\tau \geq 2^{\kappa_S})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.67}$$

In order to prove (3.67) we appeal to the continuous oscillation inequality

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)} \big\|O^2_{I,N}(\mathcal{M}_tf : t \in \mathbb{R}_+)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)} \tag{3.68}$$

which will be proven in the next section. As we know, the multiplier v^n_S is localized around fractions from the set Σ_S. Let $Q_S := \operatorname{lcm}(q : a/q \in \Sigma_S)$. By property (iv) from Theorem 2.71 one has $Q_S \leq 3^S$. If we have $n^\tau \geq 2^{\kappa_S}$ then we may write

$$v^n_S(\xi) = \Pi_S(\xi) \sum_{b\in\mathbb{Z}^\Gamma} \widetilde{\Phi}_{2^{n^\tau}}(\xi - b/Q_S)$$

where

$$\Pi_S(\xi) := \sum_{a/q\in\Sigma_S} \widetilde{\eta}\big(2^{2^{\kappa_S}(A-\chi I)}(\xi - a/q)\big) \text{ and } \widetilde{\Phi}_{2^{n^\tau}}(\xi) := \Phi_{2^{n^\tau}}(\xi)\eta\big(2^{n^\tau(A-\chi I)}\xi\big), \quad \xi \in \mathbb{T}^\Gamma.$$

Therefore the inequality (3.67) will follow if we show that the inequalities

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})} \Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{b\in\mathbb{Z}^\Gamma} \widetilde{\Phi}_{2^{n^\tau}}(\cdot - b/Q_S)\mathcal{F}_{\mathbb{Z}^\Gamma}f\big) : n^\tau \geq 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.69}$$

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Pi_S\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.70}$$

hold for any $p \in (1,\infty)$. The inequality (3.70) is a straightforward consequence of Theorem 2.71. In order to prove (3.69) we use Proposition 2.70. We note that a suitable limiting argument is needed, as to apply Proposition 2.70 one needs a finite dimensional Banach spaces. At first, let $M \in \mathbb{R}$ be a fixed large positive number and let us denote $M_S := M - \lceil 2^{\kappa_S/\tau}\rceil$. We consider the following Banach spaces:

$$B_1 := (\mathbb{C}, |\cdot|) \text{ and } B_2 := \big(\mathbb{C}^{M_S}/\sim, O^2_{I,N}(\ \cdot\ : n \in [\lceil 2^{\kappa_S/\tau}\rceil, M])\big)$$

where we have quotiented out the space of constant sequences. The function $\widetilde{\Phi}_{2^{n^\tau}}$ is supported on $[-\frac{1}{4Q_S}, \frac{1}{4Q_S}]$ for large $S \in 2^{u\mathbb{N}}$ because, on the support of $\eta_{2^{\kappa_S}}$, we have $|\xi_\gamma| \leq 2^{-2^{\kappa_S}} \leq (4Q_S)^{-1}$ for all $\gamma \in \Gamma$ and large S. We may apply Proposition 2.70 with the Banach spaces B_1 and B_2 to see that the estimate

$$\big\|O^2_{I,N}\big(\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}\big(\widetilde{\Phi}_{2^{n^\tau}}\mathcal{F}_{\mathbb{R}^\Gamma} f\big) : M^\tau \geq n^\tau \geq 2^{\kappa_S}\big)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}$$

implies

$$\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{b\in\mathbb{Z}^\Gamma}\widetilde{\Phi}_{2^{n^\tau}}(\cdot - b/Q_S)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : M^\tau \geq n^\tau \geq 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

However, since the constant $C_{|\Gamma|}$ in Proposition 2.70 is independent of the choices of Banach spaces, we see that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}\big(\widetilde{\Phi}_{2^{n^\tau}}\mathcal{F}_{\mathbb{R}^\Gamma} f\big) : n^\tau \geq 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{R}^\Gamma)} \tag{3.71}$$

implies

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{b\in\mathbb{Z}^\Gamma}\widetilde{\Phi}_{2^{n^\tau}}(\cdot - b/Q_S)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \geq 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

The estimate (3.71) follows from the oscillation inequality for continuous Radon averages (3.68) since the error term is estimated by

$$\sum_{n=0}^{\infty}\big\|\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}\big(\Phi_{2^{n^\tau}}\big(1-\eta(2^{n^\tau(A-\chi I)}\cdot)\big)\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.72}$$

Again, we have a uniform L^p-bound for the n-th term and since the function $1-\eta(2^{n^\tau(A-\chi I)}\cdot)$ is non-zero when $|2^{n^\tau A}\xi|_\infty \gtrsim 2^{n^\tau\chi}$, by the van der Corput estimate (2.64) we obtain an L^2-estimate for the n-th term with $2^{-n^\tau\chi/|\Gamma|}$ gain and complex interpolation yield the result. This ends the proof of the estimate for the large scales (3.67) and consequently the proof of the oscillation inequality for discrete Radon averages.

3.2.2 Continuous Radon averages

In this section we prove the inequality (3.6). Assume that $p \in (1,\infty)$ and let $f \in C^\infty_c(\mathbb{R}^\Gamma)$. Our aim is to prove that there is a constant $C_{p,k,|\Gamma|}$ such that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(\mathcal{M}_t f : t>0)\|_{L^p(\mathbb{R}^\Gamma)} \leq C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.73}$$

Let $D>1$ be a fixed real number which will be specified later – this is the number D from Lemma 3.78 which is stated below. By Proposition 2.33 (to be more precise, by its D-dyadic counterpart) we split (3.73) into long oscillations and short variations,

$$\begin{aligned}\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(\mathcal{M}_t f : t>0)\|_{L^p(\mathbb{R}^\Gamma)} \lesssim{}& \sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,N}(\mathcal{M}_{D^n} f : n\in\mathbb{Z})\big\|_{L^p(\mathbb{R}^\Gamma)}\\ &+\Big\|\Big(\sum_{n\in\mathbb{Z}}V^2\big(\mathcal{M}_t f : t\in[D^n,D^{n+1})\big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}.\end{aligned} \tag{3.74}$$

The estimates for the short variations

$$\Big\|\Big(\sum_{n\in\mathbb{Z}}V^2\big(\mathcal{M}_t f : t\in[D^n,D^{n+1})\big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \leq C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}$$

were obtained by Jones, Seeger and Wright [32] by using the Littlewood–Paley theory. The proof is rather long and we do not present it here. We refer to [32, Theorem 1.1, Theorem 1.2] and [40, Section 9.2] for more details. In this section we focus on showing the estimate for the long oscillations

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,N}(\mathcal{M}_{D^n} f : n\in\mathbb{Z})\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f\in C^\infty_c(\mathbb{R}^\Gamma). \tag{3.75}$$

The proof of the above estimate is based on the approach of Jones, Seeger and Wright [32] taken in the context of r-variational and jump inequalities. Namely the estimate for long oscillations is obtained by approximation with a suitable dyadic martingale and here the key ingredient is the oscillation inequality for Christ's dyadic martingales which follows from the result of Jones, Kaufman, Rosenblatt and Wierdl [28, Theorem 6.4, p. 930] (see also [D1, Proposition 2.8]).

Dyadic martingales on the homogeneous spaces

In order to prove the inequality (3.75) we need to introduce the notion of dyadic martingales on the homogeneous spaces. In this the context we will follow the notation introduced in [32]. Let A be a $d \times d$ matrix whose eigenvalues have positive real parts. For any $t > 0$ we consider the dilation given by

$$t^A := \exp(A \log t). \tag{3.76}$$

We say that a quasi-norm $\rho\colon \mathbb{R}^d \to [0, \infty)$ is homogeneous with respect to the group of dilations $(t^A : t > 0)$ if $\rho(t^A x) = t\rho(x)$ for any $x \in \mathbb{R}^d$ and $t > 0$. Recall, that for a given group of dilations $(t^A : t > 0)$ by [59, Proposition 1.7, Definition 1.8] there exists a quasi-norm ρ which is homogeneous with respect to that group. Let us state some properties of quasi-norms which will be useful later on.

Proposition 3.77 ([59, Proposition 1.9]). *Let ρ be a quasi-norm which is homogeneous with respect to the group of dilations $(t^A : t > 0)$. Then:*

(a) there are constants $\alpha, \beta, \vartheta, \delta > 0$ such that

$$|x|^\alpha \lesssim \rho(x) \lesssim |x|^\beta \text{ when } \rho(x) \geq 1 \quad \text{and} \quad |x|^\delta \lesssim \rho(x) \lesssim |x|^\vartheta \text{ when } \rho(x) \leq 1;$$

(b) let us coordinatize $\mathbb{R}^d$ by ρ and ω where $\rho = \rho(x)$ and $\omega = \rho^{-A}x$. Then the volume element in $\mathbb{R}^d$ is given by

$$\mathrm{d}x = \rho^{\mathrm{tr}(A)-1} \mathrm{d}\omega \mathrm{d}\rho,$$

where $\mathrm{d}\omega$ is C^∞-measure on the ellipsoid $\langle B\omega, \omega\rangle = 1$ and B is some real positive definite symmetric matrix related to A (there is an explicit formula for B which is not given here since we will not use it).

We note that $\mathbb{R}^d$ equipped with a homogeneous quasi-norm ρ and Lebesgue measure is a space of homogeneous type with the quasi-metric induced by ρ. As it was shown by Christ [14] for any given space of homogeneous type there exists a system of dyadic cubes. We state this result in the context of $\mathbb{R}^d$ below.

Lemma 3.78 ([14, Theorem 11]). *There exist a collection of open sets $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ and constants $D > 1$, $\delta, \eta > 0$ and $C_1, C_2 < \infty$ such that*

(i) $\big|\mathbb{R}^d \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha\big| = 0$ for all $k \in \mathbb{Z}$;

(ii) if $l \leq k$ then either $Q^l_\beta \subseteq Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;

(iii) for each (l, β) and $l \leq k$, there exists a unique α such that $Q^l_\beta \subseteq Q^k_\alpha$;

(iv) each Q^k_α contains some ball $B(z^k_\alpha, \delta D^k)$ and $\mathrm{diam}(Q^k_\alpha) \leq C_1 D^k$;

(v) for each (α, k) and $t > 0$ we have $|\{x \in Q^k_\alpha : \mathrm{dist}(x, \mathbb{R}^d \setminus Q^k_\alpha) \leq tD^k\}| \leq C_2 t^\eta |Q^k_\alpha|$.

The set I_k denotes some (possibly finite) index set, depending on k.

Remark 3.79. Some comments are needed:

(a) the functions $\mathrm{diam}(A)$ and $\mathrm{dist}(x, A)$ are calculated by using a quasi-metric induced by ρ;

(b) each cube Q_α^k contains a ball and is contained in some ball, each with radius $\simeq D^k$;

(c) since the quasi-metric is translation invariant, for each (α, k) we have $|Q_\alpha^k| \simeq D^{\mathrm{tr}(A)k}$.

The martingale sequence associated with the system of dyadic cubes $\{Q_\alpha^k\}$ is of the form $\mathbb{E}_k f = \mathbb{E}[f|\mathcal{F}_k]$ where $\mathcal{F}_k$ is the σ-algebra generated by the sets Q_α^k. To be more precise, for a locally integrable function f we set

$$\mathbb{E}_k f(x) := \mathbb{E}[f|\mathcal{F}_k](x) := \frac{1}{|Q_\alpha^k|}\int_{Q_\alpha^k} f(y)\,\mathrm{d}y, \tag{3.80}$$

where Q_α^k is the unique dyadic cube from k-th generation containing $x \in \mathbb{R}^d$. The martingale difference operator is denoted by $\mathbb{D}_k f := \mathbb{E}_k f - \mathbb{E}_{k-1} f$. By the work of Jones, Kaufman, Rosenblatt and Wierdl [28] we know that for any martingale sequence the uniform oscillation inequality holds. Namely, we have the following result.

Theorem 3.81 ([28, Theorem 6.4, p. 930]). *For every $p \in (1,\infty)$ there exists a constant $C_p > 0$ such that*

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \big\|O_{I,N}^2(\mathbb{E}_n f : n \in \mathbb{Z})\big\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}.$$

The next two results which follows from [32] concern the approximation by martingales associated with Christ's dyadic cubes. We have the following.

Proposition 3.82 ([32, Lemma 3.2]). *Let ϕ be a Schwartz function such that $\int\phi = 1$ and let $\phi_{D^k}(x) = D^{-\mathrm{tr}(A)k}\phi(D^{-kA}x)$ where $D > 1$ is from Lemma 3.78. Then*

$$\|\phi_{D^{k+m}} * \mathbb{D}_m f - \mathbb{E}_{k+m}(\mathbb{D}_m f)\|_{L^2(\mathbb{R}^d)} \lesssim D^{-\varepsilon|k|}\|f\|_{L^2(\mathbb{R}^d)} \tag{3.83}$$

for some $\varepsilon > 0$.

Lemma 3.84 (cf. [32, Theorem 1.1]). *Let ϕ be a Schwartz function such that $\int\phi = 1$ and let $\phi_{D^k}(x) = D^{-\mathrm{tr}(A)k}\phi(D^{-kA}x)$ where $D > 1$ is from Lemma 3.78. Then the operator*

$$\mathcal{S}f(x) := \Big(\sum_{k\in\mathbb{Z}} |\phi_{D^k} * f(x) - \mathbb{E}_k f(x)|^2\Big)^{1/2}$$

is bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$. Moreover, for $p = 1$ the operator $\mathcal{S}$ is of weak type (1,1).

Proof. The proof is a repetition of arguments presented during the proof of [32, Theorem 1.1] but since we stated this result as a separate lemma we give the proof. At first we will show that $\mathcal{S}$ is bounded on $L^2(\mathbb{R}^d)$ which is a consequence of Proposition 3.82. Indeed, let $f \in L^1 \cap L^2$. Then one may write the following decomposition

$$f = \sum_{m\in\mathbb{Z}} \mathbb{D}_m f,$$

where the series is convergent in L^2-norm. The proof follows the same steeps as in [20, Theorem 5.4.6]. Hence we may estimate

$$\begin{aligned}
\|\mathcal{S}f\|_{L^2(\mathbb{R}^d)} &\le \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} \|\phi_{D^k} * \mathbb{D}_m f - \mathbb{E}_k(\mathbb{D}_m f)\|_{L^2(\mathbb{R}^d)}\Big)^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}} D^{-\varepsilon|k-m|}\|\mathbb{D}_m f\|_{L^2(\mathbb{R}^d)}\Big)^2\Big)^{1/2} \\
&\lesssim \Big(\sum_{m\in\mathbb{Z}} \|\mathbb{D}_m f\|_{L^2(\mathbb{R}^d)}^2\Big)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where in the second inequality we have used Lemma 3.82. Now we want to prove that $\mathcal{S}$ is of weak type (1,1), that is

$$|\{x \in \mathbb{R}^d : \mathcal{S}f(x) \geq \lambda\}| \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}, \text{ uniformly in } \lambda > 0. \tag{3.85}$$

Then by interpolation with the known bound for $p = 2$ we get that $\mathcal{S}$ is bounded on L^p with $p \in (1, 2)$, and by duality we obtain that $\mathcal{S}$ is bounded for all $p \in (1, \infty)$. We may assume that $f \geq 0$. In order to show (3.85) we apply the Calderón–Zygmund decomposition of f at height λ by using Christ's dyadic cubes from Lemma 3.78. As a consequence there is a disjoint collection of dyadic cubes $\{Q^j_\alpha : (j, \alpha) \in \Lambda\}$ such that the following conditions are fulfilled:

1. $\sum_{(j,\alpha)\in\Lambda} |Q^j_\alpha| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}$;
2. for any $(j, \alpha) \in \Lambda$ we have $\|f\|_{L^1(Q^j_\alpha)} \simeq \lambda|Q^j_\alpha|$;
3. for a.e. $x \notin \bigcup_{(j,\alpha)\in\Lambda} Q^j_\alpha$ one has $f(x) \leq \lambda$.

Given this decomposition of $\mathbb{R}^d$, we now decompose f as the sum of two functions, g and b, defined by

$$g(x) := \begin{cases} \frac{1}{|Q^j_\alpha|}\int_{Q^j_\alpha} f(y)\mathrm{d}y, & \text{if } x \in Q^j_\alpha \text{ for some } (j, \alpha) \in \Lambda, \\ f(x), & \text{if } x \notin \bigcup_{(j,\alpha)\in\Lambda} Q^j_\alpha, \end{cases}$$

and

$$b(x) := \sum_{(j,\alpha)\in\Lambda} b_{j,\alpha}(x), \quad \text{where} \quad b_{j,\alpha}(x) := \begin{cases} f(x) - \frac{1}{|Q^j_\alpha|}\int_{Q^j_\alpha} f(y)\mathrm{d}y, & \text{if } x \in Q^j_\alpha \\ 0, & \text{if } x \notin Q^j_\alpha. \end{cases}$$

Now, it is easy to see that $\sum_{(j,\alpha)\in\Lambda} \|b_{j,\alpha}\|_{L^1(\mathbb{R}^d)} \leq 2\|f\|_{L^1(\mathbb{R}^d)}$. Moreover, one has $\int_{\mathbb{R}^d} b_{j,\alpha}(x)\mathrm{d}x = 0$ and $\mathbb{E}_k b_{j,\alpha} = 0$ for $k \geq j$.

We handle the "good" function g in the usual way – by exploiting the known L^2 bounds:

$$|\{x : \mathcal{S}g(x) \geq \lambda\}| \leq \frac{1}{\lambda^2}\|\mathcal{S}g\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{\lambda^2}\|g\|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^d)},$$

where the last inequality follows because $|g(x)| \lesssim \lambda$ for a.e. $x \in \mathbb{R}^d$. Next, we will estimate the part with the "bad" function b. Let $\widetilde{Q}^j_\alpha := 2^A Q^j_\alpha$, in other words $\widetilde{Q}^j_\alpha$ is an enlarged cube with the same same center as Q^j_α. We note that by the property 1 stated above it is enough to estimate

$$\begin{aligned} \lambda\Big|\Big\{x \notin \bigcup_{(j,\alpha)\in\Lambda} \widetilde{Q}^j_\alpha : \mathcal{S}b(x) \geq \lambda\Big\}\Big| &\leq \sum_{(j,\alpha)\in\Lambda}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d\setminus\widetilde{Q}^j_\alpha} |\phi_{D^k} * b_{j,\alpha}(x) - \mathbb{E}_k b_{j,\alpha}(x)|\mathrm{d}x \\ &= \sum_{(j,\alpha)\in\Lambda}\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d\setminus\widetilde{Q}^j_\alpha} |\phi_{D^k} * b_{j,\alpha}(x)|\mathrm{d}x, \end{aligned}$$

where the last equality follows by the fact that $\mathbb{E}_k b_{j,\alpha}$ is supported in Q^j_α when $k < j$, whereas for $k \geq j$ the expression $\mathbb{E}_k b_{j,\alpha}$ equals zero. Now we will consider two separate cases: $k < j$ and $k \geq j$. In the case $k < j$ for any $N \in \mathbb{N}$ we have the following estimate

$$\int_{\mathbb{R}^d\setminus\widetilde{Q}^j_\alpha} |\phi_{D^k} * b_{j,\alpha}(x)|\mathrm{d}x \lesssim_N \int_{Q^j_\alpha} |b_{j,\alpha}(y)| \int_{\{x:\rho(x-z^j_\alpha)\geq 2C_1D^j\}} \frac{D^{-k\operatorname{tr}(A)}}{[D^{-k}\rho(x-y)]^N}\mathrm{d}x\mathrm{d}y \tag{3.86}$$

which is a consequence of the property (iv) from Lemma 3.78, the property (a) from Proposition 3.77 and the fact that ϕ is a Schwartz function. Now, since one has the following inclusion

$$\bigcup_{y\in Q^j_\alpha} \{x : \rho(x-y) \leq C_1D^j\} \subseteq \{x : \rho(x-z^j_\alpha) \leq 2C_1D^j\},$$

it is possible to estimate the inner integral in the right hand side of (3.86) by

$$\int_{\{x:\rho(x-y)\geq C_1 D^j\}} \frac{1}{[\rho(x-y)]^N} \mathrm{d}x \lesssim \int_{r\geq C_1 D^j} r^{\mathrm{tr}(A)-1-N} \mathrm{d}r \lesssim_N D^{j(\mathrm{tr}(A)-N)},$$

provided that N is large enough. In the above inequality we have used an analogue of the spherical coordinates from Proposition 3.77. Hence we estimate the left hand side of (3.86) by

$$\|b_{j,\alpha}\|_{L^1(\mathbb{R}^d)} D^{-(j-k)(N-\mathrm{tr}(A))} \lesssim \|b_{j,\alpha}\|_{L^1(\mathbb{R}^d)} D^{-\varepsilon(j-k)}, \tag{3.87}$$

for some $\varepsilon > 0$.

If $k \geq j$, then we write

$$\phi_{D^k} * b_{j,\alpha}(x) = \int_{\mathbb{R}^d} [\phi_{D^k}(x-y) - \phi(x - z^j_\alpha)_{D^k}] b_{j,\alpha}(y) \mathrm{d}y.$$

We see that if $y, z^j_\alpha \in Q^j_\alpha$ and $x \in \mathbb{R}^d \setminus \widetilde{Q}^j_\alpha$, then by the mean value theorem

$$|\phi_{D^k}(x-y) - \phi_{D^k}(x - z^j_\alpha)| \lesssim_N [D^{-k}\rho(y - z^j_\alpha)]^{1/\delta} \frac{D^{-k\mathrm{tr}(A)}}{[1 + D^{-k}\rho(x-y)]^N}$$

holds for any $N \in \mathbb{N}$, since ϕ is a Schwartz function. Consequently, one may estimate

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\widetilde{Q}^j_\alpha} |\phi_{D^k} * b_{j,\alpha}(x)| \mathrm{d}x &\lesssim_N A^{-(k-j)/\delta} \int_{Q^j_\alpha} |b_{j,\alpha}(y)| \int_{\mathbb{R}^d} \frac{D^{-k\mathrm{tr}(A)}}{[1 + D^{-k}\rho(x-y)]^N} \mathrm{d}x\mathrm{d}y \\ &\lesssim_N D^{-(k-j)/\delta} \|b_{j,\alpha}\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

where in the last inequality we again used the spherical coordinates. Combining the above estimate with (3.87) yields

$$\lambda |\{x \notin \bigcup_{(j,\alpha)\in\Lambda} \widetilde{Q}^j_\alpha : \mathcal{S}b(x) \geq \lambda\}| \lesssim \sum_{(j,\alpha)\in\Lambda} \|b_{j,\alpha}\|_{L^1(\mathbb{R}^d)} \lesssim \|f\|_{L^1(\mathbb{R}^d)},$$

which shows that the operator $\mathcal{S}$ is of weak type (1,1). □

The next result is a counterpart of [32, Theorem 1.1] in the context of the oscillation seminorm. Let σ be a compactly supported finite Borel measure on $\mathbb{R}^d$. Let us consider dilates of σ defined by

$$\langle \sigma_t, f\rangle = \int_{\mathbb{R}^d} f(t^A x) \mathrm{d}\sigma(x) \tag{3.88}$$

where t^A is as in (3.76). We additionally assume that the Fourier multiplier satisfies the following size condition

$$|\hat{\sigma}(\xi)| \lesssim |\xi|^{-a} \tag{3.89}$$

for some $a > 0$.

Theorem 3.90. *Assume that σ is a compactly supported finite Borel measure on $\mathbb{R}^d$ satisfying (3.89) for some $a > 0$. Let σ_t be as in (3.88). Then for $p \in (1, \infty)$ one has*

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \|O^2_{I,N}(f * \sigma_{D^k} : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}, \qquad f \in L^p(\mathbb{R}^d), \tag{3.91}$$

where $D > 1$ is associated with the system of D-dyadic cubes $\{Q^k_\alpha\}$.

Proof. The proof is based on the repetition of arguments given during the proof of [32, Theorem 1.1] but we include it for the sake of completeness. We can assume that $\int \mathrm{d}\sigma \neq 0$ since by (2.28) we see that the left hand side of (3.91) is controlled by the square function $\big(\sum_{k\in\mathbb{Z}} |f * \sigma_{D^k}(x)|^2\big)^{1/2}$, and if $\hat{\sigma}(0) = 0$ then the known estimates from [18, Theorem A, Theorem B] can be used to control it on $L^p(\mathbb{R}^d)$. Without loss of generality assume that $\int \mathrm{d}\sigma = 1$. Let $\phi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ be such that $\int \phi = 1$. Then one may write the following decomposition

$$\sigma = \phi * \sigma + (\delta_0 - \phi) * \sigma,$$

where δ_0 is the Dirac measure at 0. Therefore one can write

$$f * \sigma_{D^k}(x) = \mathcal{L}_k f(x) + \mathcal{H}_k f(x),$$

where

$$\mathcal{L}_k f(x) := f * (\phi * \sigma)_{D^k}(x) \quad \text{and} \quad \mathcal{H}_k f(x) := f * \big((\delta_0 - \phi) * \sigma\big)_{D^k}(x).$$

Hence, by the triangle inequality it is enough to prove

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \|O_{I,N}^2(\mathcal{L}_k f : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{3.92}$$

and

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \|O_{I,N}^2(\mathcal{H}_k f : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.93}$$

At first we handle the estimate (3.93). By (2.28) it is enough to prove

$$\big\|\big(\sum_{k\in\mathbb{Z}} |\mathcal{H}_k f|^2\big)\big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.94}$$

We note that $(\delta_0 - \phi) * \sigma$ is a compactly supported measure with the vanishing mean value which satisfies condition (3.89). Hence by the known results from [18] we see that (3.94) holds. The estimates for the low frequency part $\mathcal{L}_k f$ will follow from the martingale estimates. Let $(\mathbb{E}_k)_{k\in\mathbb{Z}}$ be the dyadic martingale sequence associated with the dyadic cubes related to the dilation t^A. Then we may write

$$\begin{aligned}\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}(\mathbb{Z})} \|O_{I,N}^2(\mathcal{L}_k f : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)} &\lesssim \sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}(\mathbb{Z})} \|O_{I,N}^2(\mathcal{D}_k f : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)} \\ &\quad + \sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}(\mathbb{Z})} \|O_{I,N}^2(\mathbb{E}_k f : k \in \mathbb{Z})\|_{L^p(\mathbb{R}^d)},\end{aligned}$$

where

$$\mathcal{D}_k f(x) := f * (\phi * \sigma)_{D^k}(x) - \mathbb{E}_k f(x).$$

Since by Theorem 3.81 we know that the oscillation inequality holds for the martingale $\mathbb{E}_k f$, we only need to handle the part with $\mathcal{D}_k f$. Again, by the inequality (2.28) we are reduced to show that for any $p \in (1, \infty)$ one has

$$\Big\|\big(\sum_{k\in\mathbb{Z}} |\mathcal{D}_k f|^2\big)^{1/2}\Big\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)},$$

which follows by Lemma 3.84 since $\phi * \sigma$ is a Schwartz function such that $\int \phi * \sigma = 1$. □

Proof of the estimate for the long oscillations for $\mathcal{M}_{D^k}$

Our aim is to show that the inequality (3.75) follows from Theorem 3.90. At first, let σ be a finite measure on $\mathbb{R}^\Gamma$ defined by

$$\int_{\mathbb{R}^\Gamma} f(x) \mathrm{d}\sigma(x) := \frac{1}{|\Omega|} \int_\Omega f\big((y)^\Gamma\big) \mathrm{d}y, \quad x \in \mathbb{R}^\Gamma, \quad f \in L^1(\mathbb{R}^\Gamma).$$

Clearly, the measure σ is compactly supported and the dilates of σ are given by

$$\int_{\mathbb{R}^\Gamma} f(t^A x)\mathrm{d}\sigma(x) = \frac{1}{|\Omega_t|}\int_{\Omega_t} f\big((y)^\Gamma\big)\mathrm{d}y.$$

The operator $\mathcal{M}_t$ is a convolution operator satisfying

$$\mathcal{M}_t f(x) = f * \sigma_t(x).$$

Moreover, by the van der Corput estimate in (2.64) one has

$$|\widehat{\sigma}(\xi)| \lesssim |\xi|_\infty^{-1/|\Gamma|}$$

since $\widehat{\sigma}(\xi) = \Phi_1(\xi)$ where Φ_t is given by (2.61). Thus, we see that the assumptions of Theorem 3.90 are satisfied and consequently one has

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,N}(\mathcal{M}_{D^n} f : n\in\mathbb{Z})\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}$$

which proves (3.75).

3.3 Oscillation inequality for singular integrals of Radon type – proof of Theorem 3.7

In this section we give the proof of the uniform oscillation inequality for the singular integrals of Radon type. The results in this section are based on results from [D2]. As in the case of averages, at first we focus on the discrete operator

$$H_t f(x) = \sum_{y\in\Omega_t\cap\mathbb{Z}^k} f(x-(y)^\Gamma)K(y), \quad x\in\mathbb{Z}^\Gamma.$$

Next we establish the uniform oscillation inequality for the continuous averages

$$\mathcal{H}_t f(x) = \text{p.v.}\int_{\Omega_t} f(x-(y)^\Gamma)K(y)\mathrm{d}y, \quad x\in\mathbb{R}^\Gamma.$$

In both operators the function $K\colon \mathbb{R}^k\setminus\{0\}\to\mathbb{C}$ is a Calderón–Zygmund kernel which satisfy conditions (1.4), (1.5) and (1.6). By invoking the lifting procedure for the Radon averages described in Section 2.3 it is enough to prove Theorem 3.7 only for the canonical mappings.

3.3.1 Discrete singular Radon operators

The proof will proceed in a similar way as in the case of the discrete averages (see Section 3.2.1) hence some details will be omitted. Assume that $p\in(1,\infty)$ and let $f\in\ell^p(\mathbb{Z}^\Gamma)$ be a function with a compact support. Our aim is to prove that there is a constant $C_{p,k,|\Gamma|}$ such that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(H_t f : t>0)\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,k,|\Gamma|}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.95}$$

By using the monotone convergence theorem and standard density arguments to prove (3.95) it is enough to establish

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{I})}\|O^2_{I,N}(H_t f : t\in\mathbb{I})\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C_{p,k,|\Gamma|}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.96}$$

for every finite subset $\mathbb{I} \subset \mathbb{R}_+$ with a constant $C_{p,k,|\Gamma|} > 0$ that is independent of the set $\mathbb{I}$. Let us choose $p_0 > 1$, close to 1 such that $p \in (p_0, p_0')$. Then we take $\tau \in (0,1)$ which satisfies (3.12). By Proposition 2.33 we split (3.96) into long oscillations and short variations,

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{I})} \|O^2_{I,N}(H_t f : t \in \mathbb{I})\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(H_{2^{n^\tau}} f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} + \Big\|\Big(\sum_{n=0}^{\infty} V^2\big(H_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}\big)^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.97}$$

since $H_t f \equiv 0$ for $t \in (0,1)$. Again, we may separately estimate the each term on right hand side of (3.97).

Estimate for short variations

Repeating the arguments from Section 3.2.1 we see that in order to estimate short variation it is sufficient to prove that

$$\big\|V^1(H_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim n^{\tau-1}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \quad n \in \mathbb{N}. \tag{3.98}$$

By the monotonicity of the sets Ω_t and by condition (1.4) we have

$$V^1(H_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I}) \lesssim 2^{-kn^\tau} \sum_{y\in(\Omega_{2^{(n+1)^\tau}} \setminus \Omega_{2^{n^\tau}})\cap\mathbb{Z}^k} |f(x-(y)^\Gamma)|$$

which gives us

$$\big\|V^1(H_t f : t \in [2^{n^\tau}, 2^{(n+1)^\tau}) \cap \mathbb{I})\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-kn^\tau}\big|(\Omega_{2^{(n+1)^\tau}} \setminus \Omega_{2^{n^\tau}}) \cap \mathbb{Z}^k\big|\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

By Proposition 3.15 this implies (3.98).

Estimates for long oscillations

The rest of this section is devoted to proving the following estimate for long oscillations

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(H_{2^{n^\tau}} f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.99}$$

Let us observe that $H_{2^{n^\tau}} f(x) = \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(n_{2^{n^\tau}} \mathcal{F}_{\mathbb{Z}^\Gamma} f)(x)$ where $n_{2^{n^\tau}}$ is given by (2.60). We note that due to the fact that oscillation seminorm is translation invariant the estimate (3.99) is equivalent to the following estimate

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(H_{2^{n^\tau}} f - H_1 f : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

For any $x \in \mathbb{Z}^\Gamma$ and any $n \in \mathbb{N}_0$ we can express $H_{2^{n^\tau}} f(x) - H_1 f(x)$ as a telescoping sum

$$\sum_{j=1}^{n} \big(H_{2^{j^\tau}} f - H_{2^{(j-1)^\tau}} f\big)(x) = \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n} (n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big)(x).$$

Here we use convenction that for $n = 0$ the sum equals 0. Hence, instead of proving (3.99) we may focus on proving the following estimate

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n} (n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.100}$$

As in the case of discrete averages the proof of (3.100) will require several appropriately chosen parameters. Let $\chi \in (0, 1/10)$ and let $\alpha > 0$ be such that

$$\alpha > 10\left(\frac{1}{p_0} - \frac{1}{2}\right)\left(\frac{1}{p_0} - \frac{1}{\min\{p, p'\}}\right)^{-1}.$$

Let $u \in \mathbb{N}$ be a large natural number which will be specified later. We set

$$\varrho := \min\left\{\frac{1}{10u}, \frac{\delta}{8\alpha}\right\}, \tag{3.101}$$

where $\delta > 0$ is from the estimate for the Gauss sum (3.40). Let $\Pi_{\leq n^\tau, n^\tau(A-\chi I)}$ be defined as in (3.23). We can partition the multiplier $n_{2^{j^\tau}}$ and estimate the left hand side of (3.100) by

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n}(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\Pi_{\leq j^\tau, j^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.102}$$

$$+ \sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n}(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})(1-\Pi_{\leq j^\tau, j^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.103}$$

We emphasize that the expressions in (3.102) and (3.103) correspond to major and minor arcs from the Hardy–Littlewood circle method, respectively.

Minor arcs

Using the same reasoning as for the discrete averages in order to we see that (3.103) is controlled by the following estimate

$$\Big\|V^1\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n}\big(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}}\big)\big(1-\Pi_{\leq j^\tau, j^\tau(A-\chi I)}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}$$
$$\lesssim \sum_{n=0}^{\infty}\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\big(n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}\big)\big(1-\Pi_{\leq (n+1)^\tau, (n+1)^\tau(A-\chi I)}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Consequently, it is enough to show that

$$\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\big(n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}\big)\big(1-\Pi_{\leq (n+1)^\tau, (n+1)^\tau(A-\chi I)}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.104}$$

Let us note that for any $p \in (1, \infty)$ by the inequality (3.25) we obtain

$$\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\big(n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}\big)\big(1-\Pi_{\leq (n+1)^\tau, (n+1)^\tau(A-\chi I)}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.105}$$

since by the size condition (1.4) we have the pointwise estimate

$$\Big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\big(n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)(x)\Big| \lesssim M_{2^{n^\tau}}|f|(x),$$

where M_t is the discrete Radon average (2.55).

Again, in the case $p = 2$ we have a much better bound. We make use of the following result from [43], based on Weyl's inequality, that allows us to estimate exponential sums over convex sets with rough kernels. This result generalizes Theorem 3.31.

Theorem 3.106. [43, Theorem A.1] *For every $d, k \in \mathbb{N}$ there exists $\varepsilon > 0$ such that for every polynomial*

$$P(x) = \sum_{\substack{\gamma \in \mathbb{N}_0^k,\\ 0<|\gamma|\leq d}} \xi_\gamma x^\gamma, \quad \text{with} \quad P(0) = 0, \quad \xi_\gamma \in \mathbb{R},$$

every $N > 1$, convex set $\Omega \subseteq B(0, N)$, function $\phi\colon \Omega \cap \mathbb{Z}^k \to \mathbb{C}$, multi-index $\gamma_0 \in \Gamma$, and integers $0 \leq a \leq q$ with $\gcd(a, q) = 1$ and

$$\left|\xi_{\gamma_0} - \frac{a}{q}\right| \leq \frac{1}{q^2}, \tag{3.107}$$

we have

$$\Big|\sum_{y\in\Omega\cap\mathbb{Z}^k} e(P(y))\phi(y)\Big| \lesssim_{d,k} N^k \kappa^{-\varepsilon} \log(N+1)\|\phi\|_{L^\infty(\Omega)} + N^k \sup_{\substack{|x-y|\leq N\kappa^{-\varepsilon}\\ x,y\in\Omega}} |\phi(x) - \phi(y)|, \tag{3.108}$$

where

$$\kappa = \min\{q, N^{|\gamma_0|}/q\}.$$

The implied constant in (3.108) may depend on d, k but is independent of a, q, N, and the coefficients of P.

We apply Theorem 3.106 with $\Omega = \Omega_{2^{(n+1)^\tau}}$ and with $\phi = K\mathbb{1}_{\Omega_{2^{(n+1)^\tau}}\setminus\Omega_{2^{n^\tau}}}$, hence $N = 2^{(n+1)^\tau}$. By the size condition (1.4) one obtains $\|\phi\|_{L^\infty(\Omega)} \lesssim 2^{-n^\tau k}$. Furthermore, if we assume that for some $\beta > 0$ we have

$$n^{\tau\beta} \leq q \leq 2^{|\gamma_0|n^\tau} n^{-\tau\beta} \tag{3.109}$$

then by (3.108) and the continuity condition (1.6) we have the following estimate

$$|n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}| \lesssim n^{-\tau\beta\varepsilon+\tau} + n^{-\tau\beta\sigma\varepsilon} \lesssim n^{-\tau\beta\sigma\varepsilon+\tau}$$

since $\kappa \gtrsim n^{\tau\beta}$ due to (3.109). For $\beta = (\alpha\tau^{-1} + 1)(\varepsilon\sigma)^{-1} > 0$ we get

$$|n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}| \lesssim (n+1)^{-\alpha}.$$

Thus, if we show that there are ξ_{γ_0}, a, q for which the conditions (3.107) and (3.109) hold with β specified above, then by Parseval's theorem

$$\Big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\big(n_{2^{(n+1)^\tau}} - n_{2^{n^\tau}}\big)\big(1 - \Pi_{\leq(n+1)^\tau,(n+1)^\tau(A-\chi I)}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-\alpha}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

Next, we take $p = p_0$ in (3.105) and interpolate with the above inequality to obtain (3.104). It remains to show that conditions (3.107) and (3.109) hold whenever $1 - \Pi_{\leq(n+1)^\tau,(n+1)^\tau(A-\chi I)}(\xi) \neq 0$.

Let $\beta > 0$ be fixed. For each ξ_γ by Dirichlet's principle (Lemma 3.35) there exist a_γ and q_γ such that

$$\left|\xi_\gamma - \frac{a_\gamma}{q_\gamma}\right| \leq \frac{1}{q_\gamma 2^{n^\tau|\gamma|} n^{-\tau\beta}} \leq \frac{1}{q_\gamma^2}$$

with $q_\gamma \leq 2^{n^\tau|\gamma|} n^{-\tau\beta}$. We claim that if $1 - \Pi_{\leq(n+1)^\tau,(n+1)^\tau(A-\chi I)}(\xi) \neq 0$, then $q_{\gamma_0} \geq n^{\tau\beta}$ holds for some $\gamma_0 \in \Gamma$. Suppose for a contradiction that for any $\gamma \in \Gamma$ we have $1 \leq q_\gamma < n^{\tau\beta}$. Then for $q' = \operatorname{lcm}(q_\gamma\colon \gamma \in \Gamma) \leq n^{\tau\beta|\Gamma|}$ we have

$$\left|\xi_\gamma - \frac{a'_\gamma}{q'}\right| \leq \frac{1}{2^{n^\tau|\gamma|} n^{-\tau\beta}},$$

where $a'_\gamma = a_\gamma q_\gamma^{-1} q'$. We see that $\gcd(q', (a'_\gamma)_{\gamma\in\Gamma}) = 1$. Hence, taking $a' = (a'_\gamma\colon \gamma \in \Gamma)$ and u so large that $n^{\tau\beta|\Gamma|} \le \tilde{S}$ we get that $a'/q' \in \Sigma_{\le (n+1)^{\tau u}}$. On the other hand, if $1 - \Pi_{\le (n+1)^\tau, (n+1)^\tau(A-\chi I)}(\xi) \neq 0$ then for any $a/q \in \Sigma_{\le (n+1)^{\tau u}}$, there exists $\gamma \in \Gamma$ for which

$$\left|\xi_\gamma - \frac{a_\gamma}{q}\right| > \frac{1}{32|\Gamma|^{3/2} 2^{(n+1)^\tau(|\gamma|-\chi)}}.$$

As a consequence we obtain the inequality

$$32|\Gamma|^{3/2} n^{\beta\tau} > 2^{(n+1)^\tau \chi},$$

which is false for large n. Therefore, we see that there is $\gamma_0 \in \Gamma$ such that the conditions (3.107) and (3.109) are satisfied and consequently (3.104) follows. This shows that if $u > \beta|\Gamma|$, where $\beta = (\alpha\tau^{-1} + 1)(\varepsilon\sigma)^{-1}$ and ε is from Theorem 3.106, then we have

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \big\|O^2_{I,N}(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(n_{2^{n^\tau}}(1 - \Pi_{\le n^\tau, n^\tau(A-\chi I)})\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which ends the proof of estimates for minor arcs.

Major arcs and multiplier approximation

Now, our aim is to show that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)} \Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\Big(\sum_{j=1}^{n} (n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\Pi_{\le j^\tau, j^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.110}$$

For simplicity, let us denote

$$T^\chi_{j^\tau} f(x) := \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\Pi_{\le j^\tau, j^\tau(A-\chi I)}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)(x), \quad x \in \mathbb{Z}^\Gamma.$$

The operator T_{j^τ} has the Fourier symbol given by

$$\sum_{a/q\in\Sigma_{\le j^\tau u}} \big(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}}\big)(\xi)\eta(2^{j^\tau(A-\chi I)}(\xi - a/q)), \quad x \in \mathbb{T}^\Gamma. \tag{3.111}$$

We shall show that (3.111) is, up to an acceptable error term, equal to

$$\boldsymbol{n}_j(\xi) := \sum_{a/q\in\Sigma_{\le j^\tau u}} G(a/q)\big(\Psi_{2^{j^\tau}} - \Psi_{2^{(j-1)^\tau}}\big)(\xi - a/q)\eta(2^{j^\tau(A-\chi I)}(\xi - a/q)) \tag{3.112}$$

where Ψ_t is a continuous version of the multiplier n_t given by (2.62) and $G(a/q)$ is the Gauss sum (3.39). We note that singularity which occurs in $\Psi_{2^{j^\tau}}$ does not occur in $\boldsymbol{n}_j$ because it gets cancelled by subtracting $\Psi_{2^{(j-1)^\tau}}$.

In order to approximate (3.111) by $\boldsymbol{n}_j$ we make use of the previously stated Proposition 3.41. We apply this result with $\Omega := \Omega_{2^{j^\tau}} \setminus \Omega_{2^{(j-1)^\tau}}$ and $\mathcal{K} := K$. By the size condition (1.4) one has $\|\mathcal{K}\|_{L^\infty(\Omega)} \lesssim 2^{-j^\tau k}$. From the continuity condition (1.6) we get $\sup_{|x-y|\le q} |\mathcal{K}(x) - \mathcal{K}(y)| \lesssim 2^{-kj^\tau}(q2^{-j^\tau})^\sigma$. Therefore, on the support of $\eta(2^{j^\tau(A-\chi I)}(\cdot - a/q))$ we have

$$\begin{aligned}\big|\big(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}}\big)(\xi) - G(a,q)(\Psi_{2^{j^\tau}} - \Psi_{2^{(j-1)^\tau}})(\xi - a/q)\big| &\lesssim q2^{-j^\tau} + \sum_{\gamma\in\Gamma} q|\xi_\gamma - a_\gamma/q|2^{j^\tau(|\gamma|-1)} + (q2^{-j^\tau})^\sigma \\ &\lesssim 2^{-j^\tau\sigma/2}\end{aligned}$$

for $\chi \in (0, 1/10)$, since $q \lesssim e^{j^\tau/10}$ and for any $\gamma \in \Gamma$ we have $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-j^\tau(|\gamma|-\chi)}$. By the disjointness of the supports of $\eta(2^{(j+1)^\tau(A-\chi I)}(\xi - a/q))$ we have

$$\sum_{a/q\in\Sigma_{\leq j^\tau u}} \big(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}}\big)(\xi)\eta(2^{j^\tau(A-\chi I)}(\xi - a/q)) = \boldsymbol{n}_j(\xi) + \mathcal{O}(2^{-j^\tau\sigma/2}). \tag{3.113}$$

For simplicity, let us denote

$$\mathcal{T}_j f(x) := \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}_j \mathcal{F}_{\mathbb{Z}^\Gamma} f)(x), \quad x \in \mathbb{Z}^\Gamma.$$

Then for $p \in (1, \infty)$ we have a simple estimate

$$\big\|(T_j^\chi - \mathcal{T}_j)f\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \big|\Sigma_{\leq j^\tau u}\big|\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)j^\tau/10}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.114}$$

since by [36, Proposition 2.1] each term in (3.111) and (3.112) defines a bounded multiplier on ℓ^p. For $p = 2$, by using (3.113) and by Parseval's equality, we obtain a much stronger estimate

$$\big\|(T_j^\chi - \mathcal{T}_j)f\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim 2^{-j^\tau\sigma/2}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{3.115}$$

Now, if we take $p = p_0$ in (3.114) and interpolate it with (3.115) we get

$$\big\|(T_j^\chi - \mathcal{T}_j)f\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-j^\tau\sigma/4}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.116}$$

Therefore we can replace in (3.110) the multiplier $(n_{2^{j^\tau}} - n_{2^{(j-1)^\tau}})\Pi_{\leq j^\tau, j^\tau(A-\chi I)}$ by its continuous counterpart $\boldsymbol{n}_j$ since the error term can be handled by the following estimate

$$\begin{aligned}\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\Big\|O^2_{I,N}\big(\sum_{j=1}^n (T_j^\chi - \mathcal{T}_j)f : n \in \mathbb{N}_0\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} &\lesssim \Big\|V^2\big(\sum_{j=1}^n (T_j^\chi - \mathcal{T}_j)f : n \in \mathbb{N}_0\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &\lesssim \sum_{n=1}^\infty \|(T_n^\chi - \mathcal{T}_n)f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where the last inequality follows from (3.116). Consequently, to show (3.110) it is enough to prove that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\big\|O^2_{I,N}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{j=1}^n \boldsymbol{n}_j\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n \in \mathbb{N}_0\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.117}$$

Now, we split our projection multiplier $\Pi_{\leq n^\tau, n^\tau(A-\chi I)}$ into the sum of annulus projections (3.24). By (3.22) we see that

$$\boldsymbol{n}_j(\xi) = \sum_{\substack{S\leq j^\tau u,\\ S\in 2^{u\mathbb{N}}}} \boldsymbol{n}_S^j(\xi), \tag{3.118}$$

where $\boldsymbol{n}_S^j$ is defined as

$$\boldsymbol{n}_S^j(\xi) := \sum_{a/q\in\Sigma_S} G(a/q)\big(\Psi_{2^{j^\tau}} - \Psi_{2^{(j-1)^\tau}}\big)(\xi - a/q)\eta\big(2^{j^\tau(A-\chi I)}(\xi - a/q)\big). \tag{3.119}$$

By using the decomposition (3.118) combined with triangle's inequality from Fact 2.29 and with the cut-off Proposition 2.32 we obtain

$$\begin{aligned}&\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{N}_0)}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{j=1}^n \boldsymbol{n}_j\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n \in \mathbb{N}_0\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &\quad\leq \sum_{S\in 2^{u\mathbb{N}}}\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau^S)}\Big\|O^2_{I,N}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{\substack{1\leq j\leq n\\ S^{1/u}\leq j^\tau}} \boldsymbol{n}_S^j\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \geq S^{1/u}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} + \big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\boldsymbol{n}_S^{S^{1/(\tau u)}}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where $\mathbb{D}_\tau^S = \{n \in \mathbb{N} : n^\tau \ge S^{1/u}\}$. Thus, as in the case of discrete averages, it is sufficient to show that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau^S)} \Big\| O_{I,N}^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\le j\le n\\ S^{1/u}\le j^\tau}} \boldsymbol{n}_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \ge S^{1/u}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-4\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.120}$$

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\boldsymbol{n}_S^{S^{1/(\tau u)}} \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-6\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.121}$$

since both $S^{-4\varrho}$ and $S^{-6\varrho}$ are summable in $S \in 2^{u\mathbb{N}}$.

Gaussian multiplier and scale distinction

In order to prove estimates (3.120) and (3.121) we repeat arguments used in the case of the discrete averages in Section 3.2.1. Again, the Gaussian part $G(a/q)$ in the multiplier (3.119) prevents us from applying Theorem 2.71. Let $\widetilde{\eta} := \eta(x/2)$ and set

$$v_S^j(\xi) := \sum_{a/q\in\Sigma_S} (\Psi_{2^{j^\tau}} - \Psi_{2^{(j-1)^\tau}})(\xi - a/q)\eta\big(2^{j^\tau(A-\chi I)}(\xi - a/q)\big),$$

$$\mu_S(\xi) := \sum_{a/q\in\Sigma_S} G(a/q)\widetilde{\eta}\big(2^{S^{1/u}(A-\chi I)}(\xi - a/q)\big).$$

We see that estimates (3.120) and (3.121) will follow if we show that for every $p \in (1,\infty)$ one has

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\mu_S \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{-7\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.122}$$

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^{S^{1/(\tau u)}} \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{3.123}$$

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_\tau^S)} \Big\| O_{I,N}^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\le j\le n\\ S^{1/u}\le j^\tau}} v_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \ge S^{1/u}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim S^{3\varrho}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.124}$$

The estimate (3.122) was proven in (3.52) and the estimate (3.123) is a consequence of Theorem 2.71. It remains to prove (3.124). We follow the same approach as taken in the case of (3.54). We set $\kappa_S := \lceil S^{2\varrho}\rceil$ and by Proposition 2.30 we split the left hand side of (3.124) at point 2^{κ_S}

$$\begin{aligned}\mathrm{LHS}(3.124) \lesssim &\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_{\le S}^\tau)} \Big\| O_{I,N}^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\le j\le n\\ S^{1/u}\le j^\tau}} v_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \in [S^{1/u}, 2^{\kappa_S+1}]\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &+ \sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_{\ge S}^\tau)} \Big\| O_{I,N}^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\le j\le n\\ 2^{\kappa_S}<j^\tau}} v_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau > 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where $\mathbb{D}_{\le S}^\tau := \{n \in \mathbb{N} : n^\tau \in [S^{1/u}, 2^{\kappa_S+1}]\}$ and $\mathbb{D}_{\ge S}^\tau := \{n \in \mathbb{N} : n^\tau \ge 2^{\kappa_S}\}$. Again, we separately estimate the each term of the above inequality.

Estimates for small scales

Our aim is to show that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}_{\le S}^\tau)} \Big\| O_{I,N}^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\le j\le n\\ S^{1/u}\le j^\tau}} v_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau \in [S^{1/u}, 2^{\kappa_S+1}]\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \kappa_S \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.125}$$

We apply the Rademacher–Menshov inequality (2.38) to the left hand side of the above estimate and we get that

$$\mathrm{LHS}(3.125) \lesssim \sum_{i=0}^{\kappa_S+1} \Big\| \Big(\sum_j \big|\sum_{k\in I_j^i} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^k \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

where $I_j^i = [j2^i, (j+1)2^i) \cap [S^{1/(\tau u)}, 2^{\kappa_S+1}]$. Here we are summing over $j \in \mathbb{N}$ for which $I_j^i \neq \emptyset$. One can easily see that now it is sufficient to show that for every $i \leq \kappa_S + 1$ we have

$$\Big\|\Big(\sum_j \big|\sum_{k\in I_j^i} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(v_S^k \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.126}$$

By Theorem 2.71, the estimate (3.126) is a consequence of its continuous counterpart

$$\Big\|\Big(\sum_j \big|\sum_{k\in I_j^i} \mathcal{F}_{\mathbb{R}^\Gamma}^{-1}((\Psi_{2^{k^\tau}} - \Psi_{2^{(k-1)^\tau}})\eta\big(2^{k^\tau(A-\chi I)}\,\cdot\,\big)\mathcal{F}_{\mathbb{R}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

The above estimate will follow from the square function bound

$$\Big\|\Big(\sum_j \Big|\sum_{k\in I_j^i} \mathcal{F}_{\mathbb{R}^\Gamma}^{-1}((\Psi_{2^{k^\tau}} - \Phi_{2^{(k-1)^\tau}})\mathcal{F}_{\mathbb{R}^\Gamma} f)\Big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{3.127}$$

since for every $p \in (1,\infty)$ the error term

$$\sum_{k=1}^{\infty} \big\|\mathcal{F}_{\mathbb{R}^\Gamma}^{-1}\big((\Psi_{2^{k^\tau}} - \Psi_{2^{(k-1)^\tau}})\big(1-\eta(2^{k^\tau(A-\chi I)}\cdot)\big)\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\big\|_{L^p(\mathbb{R}^\Gamma)}$$

is controlled by a constant multiple of $\|f\|_{\ell^p(\mathbb{R}^\Gamma)}$. Indeed, we have uniform L^p bounds for the k-th term. Moreover, since the function $1-\eta(2^{k^\tau(A-\chi I)}\cdot)$ is non-zero when $|2^{k^\tau A}\xi|_\infty \gtrsim 2^{k^\tau\chi}$, by the van der Corput estimate (2.67) we obtain an L^2 estimate for the k-th term with $2^{-k^\tau\chi\sigma/|\Gamma|}$ loss. Thus, the desired bound for the error term follows by complex interpolation.

As in the case of the continuous averages $\mathcal{M}_t$, the square function bound (3.127) can be deduced from the following inequality for the operator $\mathcal{H}_t f(x)$,

$$\Big\|\Big(\sum_{k\in\mathbb{N}} \big|(\mathcal{H}_{t_{k+1}} - \mathcal{H}_{t_k})f\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \leq C_p\|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{3.128}$$

which holds for every increasing sequence $0 < t_1 \leq t_2 \leq \cdots$. The constant $C_p > 0$ is independent of the chosen sequence. The proof of (3.128) follows the same lines as the proof of (3.66).

Estimates for large scales

The proof of the oscillation estimate for the discrete singular operators will be completed if we show that

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})} \Big\|O^2_{I,N}\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\sum_{\substack{1\leq j\leq n\\ 2^{\kappa_S}<j^\tau}} v_S^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n^\tau > 2^{\kappa_S}\Big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.129}$$

We would like to exploit the almost telescoping nature of the multipliers appearing in (3.129). We do this by introducing new approximating multipliers. Let

$$\widetilde{v}_S^j(\xi) := \sum_{a/q\in\Sigma_S} (\Psi_{2^{j^\tau}} - \Psi_{2^{(j-1)^\tau}})(\xi - a/q)\eta(2^{2^{\tau\kappa_S}(A-\chi)}(\xi - a/q)).$$

Since $j^\tau \geq 2^{\kappa_S}$, the expression

$$\eta(2^{j^\tau(A-\chi I)}(\xi - a/q)) - \eta(2^{2^{\kappa_S}(A-\chi I)}(\xi - a/q))$$

is nonzero only when $|\xi_\gamma - a_\gamma/q| \gtrsim 2^{-j^\tau(|\gamma|-\chi)}$ for some $\gamma \in \Gamma$. Hence, by the van der Corput estimate in (2.67) we get

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((v_S^j - \widetilde{v}_S^j)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim 2^{-j^\tau \chi\sigma/|\Gamma|}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)},$$

whereas for any $p \neq 2$, by property (i) from Theorem 2.71, we have

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((v_S^j - \widetilde{v}_S^j)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \big|\Sigma_{\leq j^{\tau u}}\big|\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)j^{\tau/10}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

By interpolating the above inequalities we get

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((v_S^j - \widetilde{v}_S^j)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-j^\tau \varepsilon}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{3.130}$$

with some $\varepsilon > 0$. This estimate allows us to replace v_S^j by $\widetilde{v}_S^j$ in (3.128) since the error term can be estimated by

$$\sum_{j\in\mathbb{N}}\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((v_S^j - \widetilde{v}_S^j)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which by (3.130) is bounded by a constant multiple of $\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$. We note that the telescoping gives us

$$\sum_{\substack{1\leq j\leq n\\ 2^{\kappa_S}\leq j^\tau}} \widetilde{v}_S^j = \sum_{a/q\in\Sigma_S} \big(\Psi_{2^{n^\tau}} - \Psi_{2^{2^{\kappa_S}}}\big)(\xi - a/q)\eta(2^{2^{\tau\kappa_S}(A-\chi I)}(\xi - a/q)).$$

Since the oscillation seminorm is translation invariant, the inequality (3.129) will follow if we show

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{D}^\tau_{\geq S})}\big\|O^2_{I,N}\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_S^n\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n^\tau > 2^{\kappa_S}\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(S)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{3.131}$$

where Δ_S^n is defined as

$$\Delta_S^n(\xi) := \sum_{a/q\in\Sigma_S} \Psi_{2^{n^\tau}}(\xi - a/q)\eta(2^{2^{\tau\kappa_S}(A-\chi I)}(\xi - a/q)), \quad \xi \in \mathbb{T}^\Gamma.$$

The inequality (3.131) is proven in the same way as the estimate (3.67), by appealing to Magyar–Stein–Wainger sampling principle (Proposition 2.70) and the uniform oscillation inequality for the continuous singular integral

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\big\|O^2_{I,N}(\mathcal{H}_t f : t \in \mathbb{R}_+)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)} \tag{3.132}$$

which is proven in the next section.

3.3.2 Continuous singular Radon operators

In this section we prove the inequality (3.9). Assume that $p \in (1,\infty)$ and let $f \in C_c^\infty(\mathbb{R}^\Gamma)$. Our aim is to prove that there is a constant $C_{p,k,|\Gamma|}$ such that

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(\mathcal{H}_t f : t > 0)\|_{L^p(\mathbb{R}^\Gamma)} \leq C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.133}$$

Let $D > 1$ as in Lemma 3.78. By Proposition 2.33 we split (3.133) into long oscillations and short variations,

$$\begin{aligned}\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{R}_+)}\|O^2_{I,N}(\mathcal{H}_t f : t > 0)\|_{L^p(\mathbb{R}^\Gamma)} &\lesssim \sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,N}(\mathcal{H}_{D^n} f : n \in \mathbb{Z})\big\|_{L^p(\mathbb{R}^\Gamma)}\\ &\quad + \Big\|\Big(\sum_{n\in\mathbb{Z}} V^2\big(\mathcal{H}_t f : t \in [D^n, D^{n+1})\big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}.\end{aligned} \tag{3.134}$$

Again, we focus only on the estimate for the long oscillations

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\big\|O^2_{I,N}(\mathcal{H}_{D^n}f : n\in\mathbb{Z})\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f\in C^\infty_{\mathrm{c}}(\mathbb{R}^\Gamma), \tag{3.135}$$

since the estimates for the short variations

$$\Big\|\Big(\sum_{n\in\mathbb{Z}} V^2\big(\mathcal{H}_t f : t\in[D^n, D^{n+1})\big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \le C_{p,k,|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}$$

were obtained by Jones, Seeger and Wright [32] by using the Littlewood–Paley theory.

The proof of the inequality (3.135) is based on the Duoandikoetxea–Rubio de Francia decomposition (3.136) and the oscillation inequality for compactly supported measures (3.91).

Oscillation inequality for the operator $\mathcal{H}_t$

Due to the differential nature of the oscillation seminorm it is enough to proof (3.135) for the "complement" Radon transform given by

$$\widetilde{\mathcal{H}}_t f(x) := \int_{\Omega_t^c} f(x-(y)^\Gamma)K(y)\mathrm{d}y, \quad x\in\mathbb{R}^\Gamma.$$

The presented approach is known, see [18], and was used in the context of r-variations estimates [42] and jump inequalities [32]. At the beginning, we see that we can express $\widetilde{\mathcal{H}}_{D^k}$ as a telescoping sum

$$\widetilde{\mathcal{H}}_{D^k} f(x) = \sum_{j\ge k}\mu_{D^j} * f(x),$$

where

$$\mu_{D^k} * f(x) := \int_{\Omega_{D^{k+1}}\setminus\Omega_{D^k}} f(x-(y)^\Gamma)K(y)\mathrm{d}y.$$

Now, let φ be a smooth compactly supported function such that $\hat{\varphi}(0)=1$.

$$\varphi_{D^k}(x) := D^{-k\mathrm{tr}(A)}\varphi(D^{-kA}x), \quad x\in\mathbb{R}^\Gamma,$$

where A is the matrix of the form (2.63). We employ the following decomposition (cf. [18, Theorem E])

$$\widetilde{\mathcal{H}}_{D^k} f = \varphi_{D^k} * \mathcal{H}f - \big(\varphi * \sum_{j<0}\mu_{D^j}\big)_{D^k} * f + \big(\sum_{j\ge0}(\delta_0-\varphi)*\mu_{D^j}\big)_{D^k} * f, \tag{3.136}$$

where

$$\mathcal{H}f(x) := \mathrm{p.v.}\int_{\mathbb{R}^k} f(x-(y)^\Gamma)K(y)\mathrm{d}y$$

is the full Radon transform. The oscillation inequality for the term $\varphi_{D^k} * \mathcal{H}f$ follows by Theorem 3.90 and by the fact that $\mathcal{H}$ is bounded on L^p. For the second term in (3.136) we use estimate (2.28) to get

$$\sup_{N\in\mathbb{N}}\sup_{I\in\mathfrak{S}_N(\mathbb{Z})}\Big\|O^2_{I,N}\Big(f * \big(\varphi * \sum_{j<0}\mu_{D^j}\big)_{D^k} : k\in\mathbb{Z}\Big)\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \Big\|\Big(\sum_{k\in\mathbb{Z}}\big|\big(\varphi * \sum_{j<0}\mu_{D^j}\big)_{D^k} * f\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}.$$

We note that we have

$$\varphi * \sum_{j<0}\mu_{D^j}(x) = \mathrm{p.v.}\int_{\Omega}\varphi(x-(y)^\Gamma)K(y)\mathrm{d}y, \quad x\in\mathbb{R}^\Gamma,$$

and we see that $\varphi * \sum_{j<0} \mu_{D^j}$ is a convolution with a compactly supported distribution. Hence by [20, Theorem 2.3.20] the function $\varphi*\sum_{j<0} \mu_{D^j}$ is a Schwartz function with mean value zero, by the cancellation condition (1.5). As a consequence, for any $A \in \mathbb{N}$, one has the following estimate

$$\big|\varphi * \sum_{j<0} \mu_{D^j}(x)\big| \lesssim_A \frac{1}{1+|x|^A}, \quad x \in \mathbb{R}^\Gamma.$$

The above estimate implies that for any $p \in (1,\infty)$ we have

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \big(\varphi * \sum_{j<0} \mu_{D^j}\big)_{D^k} \Big| * f \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f \in L^p(\mathbb{R}^\Gamma).$$

Moreover, since $\varphi*\sum_{j<0} \mu_{D^j}$ is a Schwartz function with mean value zero, we have the following estimate for the Fourier transform

$$\big| \big(\widehat{\varphi * \sum_{j<0} \mu_{D^j}}\big)_{D^k}(\xi) \big| \lesssim \min\big\{ |D^k\xi|_\infty, |D^k\xi|_\infty^{-1} \big\}, \quad \xi \in \mathbb{R}^\Gamma.$$

Hence by [18, Theorem B] it is easy to see that one has the following square function estimate

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big| \big(\varphi * \sum_{j<0} \mu_{D^j}\big)_{D^k} * f \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

which proves the oscillation inequality for the second term.

It remains to estimate the third term occurring in (3.136). By using (2.28) and by the triangle inequality we obtain

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \Big\| O^2_{I,N}\Big(f*\big(\sum_{j\geq 0} (\delta_0-\varphi)*\mu_{D^j}\big)_{D^k} : k\in\mathbb{Z} \Big) \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \sum_{j\geq 0} \Big\| \Big(\sum_{k\in\mathbb{Z}} \big| f*\big((\delta_0-\varphi)*\mu_{D^j}\big)_{D^k} \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)}.$$

Therefore it is enough to show that for some positive constant c_p one has

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big| f * \big((\delta_0-\varphi) * \mu_{D^j}\big)_{D^k} \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim D^{-c_p j} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.137}$$

The uniform L^p estimates of the above square function follows from the results from [18]. Indeed, by the size condition (1.4) one has

$$\big| f * \big((\delta_0-\varphi) * \mu_{D^j}\big)_{D^k} \big| \lesssim \mathcal{M}_{D^{j+k}} * |f * (\delta_0 - \varphi)_{D^k}|,$$

where $\mathcal{M}_t$ is the Radon averaging operator given by (2.57). It is known that for any $p \in (1,\infty)$ one has

$$\Big\| \sup_{t>0} |\mathcal{M}_t f| \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.138}$$

This follows by [56, Chapter 9, Proposition 2]. The above estimate can also be derived from Theorem 3.4 and Proposition 2.7. The inequality (3.138) implies that the maximal function associated with $\big| f * \big((\delta_0 - \varphi) * \mu_{D^j}\big)_{D^k} \big|$ is uniformly bounded (with respect to j) on $L^p(\mathbb{R}^\Gamma)$ for any $p \in (1,\infty)$. Moreover, since $j \leq 0$, by the mean value theorem and the van der Corput from in (2.67) we have

$$\big| (1-\widehat{\phi}(\xi)) \widehat{\mu_{D^j}}(\xi) \big| \lesssim \min\big\{ |\xi|_\infty^{\sigma/|\Gamma|}, |D^{jA}\xi|_\infty^{-\sigma/|\Gamma|} \big\} \lesssim \min\big\{ |\xi|_\infty^{\sigma/|\Gamma|}, |\xi|_\infty^{-\sigma/|\Gamma|} \big\}, \quad \xi \in \mathbb{T}^\Gamma.$$

Therefore, by [18, Theorem B] we get the uniform L^p estimate

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \big| f * \big((\delta_0-\varphi) * \mu_{D^j}\big)_{D^k} \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{3.139}$$

In the case of $p = 2$, we get a rapidly decreasing estimate as $j \to \infty$. This is based on Plancherel's theorem. Again, by the mean value theorem one has

$$|1 - \hat{\varphi}(D^{kA}\xi)| \lesssim |D^{kA}\xi|_\infty^{\sigma/|\Gamma|} \quad \text{when} \quad |D^{kA}\xi| \le 1,$$

and since $\hat{\varphi}$ is a bounded function we get that

$$|1 - \hat{\varphi}(D^{kA}\xi)| \lesssim |D^{kA}\xi|_\infty^{\sigma/|\Gamma|} \quad \text{when} \quad |D^{kA}\xi| \ge 1.$$

Moreover, by the estimates in (2.67) we obtain

$$|\widehat{\mu}_{D^j}(D^{kA}\xi)| \lesssim \min\big\{|D^{(k+j)A}\xi|_\infty^{\sigma/|\Gamma|}, |D^{(k+j)A}\xi|_\infty^{-\sigma/|\Gamma|}\big\}.$$

Taking into account the above inequalities one obtains the following bound for the Fourier transform of the square function,

$$\sum_{k\in\mathbb{Z}} |(1 - \widehat{\varphi}(D^{kA}\xi))\widehat{\mu}_{2^j}(D^{kA}\xi)|^2$$
$$\lesssim \sum_{k\in\mathbb{Z}} |D^{kA}\xi|_\infty^{\sigma/|\Gamma|} |D^{(k+j)A}\xi|_\infty^{-\sigma/|\Gamma|} \min\big\{|D^{(k+j)A}\xi|_\infty^{\sigma/|\Gamma|}, |D^{(k+j)A}\xi|_\infty^{-\sigma/|\Gamma|}\big\} \lesssim D^{-\varepsilon j}$$

for some $\varepsilon > 0$. Hence, we get the L^2-estimate with rapidly decreasing factor

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} \big|f * \big((\delta_0 - \varphi) * \mu_{D^j}\big)_{D^k}\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim D^{-\varepsilon j}\|f\|_{L^2(\mathbb{R}^\Gamma)}. \tag{3.140}$$

Interpolating (3.139) with (3.140) yields (3.137) and as a consequence we get

$$\sup_{N\in\mathbb{N}} \sup_{I\in\mathfrak{S}_N(\mathbb{Z})} \Big\|O_{I,N}^2\big(\widetilde{\mathcal{H}}_{D^k} f : k \in \mathbb{Z}\big)\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

which ends the proof of Theorem 3.7 in the continuous case.

Chapter 4

Bootstrap approach to Radon operators

In this chapter we present the idea of bootstrap in harmonic analysis. At the beginning of this chapter we formulate its core idea and later we present a set of examples situated in different settings. We are particularly interested in the bootstrap proof of the jump inequality for continuous Radon averages $\mathcal{M}_t$ which was given by Mirek, Stein and Zorin-Kranich. Their approach is presented in Section 4.2. Finally, in Section 4.3 we present the bootstrapping proof of Theorem 1.51 which was the main result of [D3].

4.1 The idea of bootstrap in harmonic analysis

According to Cambridge English dictionary[1] the verb *bootstrap* means "to improve your situation or become more successful, without help from others or without advantages that others have". This definition captures the essence of the bootstrap approach in proving inequalities in harmonic analysis. Roughly speaking, the bootstrap method of proving some inequality consists of estimating the left hand side of the inequality, say L, by the expression of the form $C \cdot L^\theta$ with $\theta \in [0, 1)$ and $C > 0$ being independent of L. This leads to the following relation

$$L \le CL^\theta. \tag{4.1}$$

Dividing both sides by L^θ we get $L^{1-\theta} \le C$ and since $\theta \in [0, 1)$ this gives us

$$L \le C^{\frac{1}{1-\theta}}$$

which provide us with a non-trivial bound for L. The name bootstrap for this procedure refers to operating only with the quantity L which is given at the beginning. In order to better illustrate this procedure we give the bootstrap proof of Hölder's inequality.

Proposition 4.2. *[Hölder's inequality] Let $(X, \mathcal{B}(X), \mu)$ be a measure space and let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then for all functions $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ we have*

$$\|fg\|_{L^1(X,\mu)} \le \|f\|_{L^p(X,\mu)} \|g\|_{L^q(X,\mu)}.$$

There are many proofs of Holder's inequality – the standard proof uses Young's inequality for products which states that for $a, b \ge 0$ we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

whenever $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. In our proof we do not use any non-trivial additional results.

[1] https://dictionary.cambridge.org/dictionary/english/bootstrap

Proof of Proposition 4.2. The case when $p = 1$ and $q = \infty$ (and vice versa) is easy to establish so we restrict ourselves to the case when $p, q \in (1, \infty)$. Let $C > 0$ denote the smallest constant for which we have

$$\|fg\|_{L^1(X,\mu)} \leq C\|f\|_{L^p(X,\mu)}\|g\|_{L^q(X,\mu)} \tag{4.3}$$

for all measure spaces $(X, \mathcal{B}(X), \mu)$. At first, we will show that $C < \infty$. Without loss of generality we can assume that $\|f\|_{L^p(X,\mu)} = \|g\|_{L^q(X,\mu)} = 1$. We have

$$\begin{aligned}\|fg\|_{L^1(X,\mu)} &= \int_X |f(x)g(x)|\mathrm{d}\mu(x) = \int_X \left(|f(x)|^p\right)^{1/p}\left(|g(x)|^q\right)^{1/q}\mathrm{d}\mu(x) \\ &\leq \int_X \max\left\{|f(x)|^p, |g(x)|^q\right\}^{\frac{1}{p}+\frac{1}{q}}\mathrm{d}\mu(x) \leq \int_X |f(x)|^p\mathrm{d}\mu(x) + \int_X |g(x)|^q\mathrm{d}\mu(x) = 2.\end{aligned}$$

The above argument shows that (4.3) holds with the constant $C \leq 2$. Now we use the *tensor power trick*[2] to bootstrap the inequality (4.3) to obtain the relation of the form (4.1) for the constant C. Let us define the tensor powers of f and g by setting, for any $x, y \in X$,

$$F(x, y) := f(x)f(y) \quad \text{and} \quad G(x, y) := g(x)g(y)$$

The new functions act on the product measure space $(X \times X, \mathcal{B}(X) \otimes \mathcal{B}(X), \mu \otimes \mu)$. It is easy to note that

$$\|F\|_{L^p(X\times X,\mu\otimes\mu)} = \|f\|^2_{L^p(X,\mu)} \tag{4.4}$$

for any $p \in (1, \infty)$ and any function $f \in L^p(X, \mu)$. Now, let us write

$$\|fg\|^2_{L^1(X,\mu)} = \int_{X\times X} |F(x, y)G(x, y)|\mathrm{d}(\mu \otimes \mu)(x, y) \leq C\|F\|_{L^p(X\times X,\mu\otimes\mu)}\|G\|_{L^q(X\times X,\mu\otimes\mu)},$$

where in the last inequality we used (4.3). By the equality (4.4) and by taking the square root of both sides we get

$$\|fg\|_{L^1(X,\mu)} \leq C^{1/2}\|f\|_{L^p(X,\mu)}\|g\|_{L^q(X,\mu)}. \tag{4.5}$$

Since C is the smallest constant for which the inequality (4.3) holds the inequality (4.5) implies that

$$C \leq C^{1/2}.$$

Since we know that $C < \infty$ the above relation implies that $C \leq 1$ which ends the proof. □

As we just saw we operate only with the constant C and some clever tricks. We do not need any auxiliary results. As we will see in the sequel, usually we do not have that comfort and for more sophisticated results we need additional tools. However, the number and complexity of required tools is considerably less than in standard approaches. The main problem with using bootstrap proofs is the hardness of inventing them because, as we just saw, it usually require an idea that differs from an approach that is imposed at the first glance when we face the problem.

It is difficult to say where the idea of bootstrap first appeared and who first came up with it. It seems that the first bootstrap proof, without calling it by this name, of the non-trivial result was given by Bochner. In 1959 Bochner [2] gave a new proof of M. Riesz theorem about the L^p-boundedness of the conjugate Fourier series of f given by

$$-i\sum_{n=-\infty}^{\infty} \mathrm{sgn}(n)\hat{f}(n)e^{2\pi inx}.$$

[2] An interesting article about tensor power trick (and other "tricks") can be found on Terence Tao blog https://terrytao.wordpress.com/2007/09/05/amplification-arbitrage-and-the-tensor-power-trick/.

Bochner in a clever way uses the binomial theorem to establish the relation of the form (4.1). We present Bochner's proof of the M. Riesz theorem in the next section.

The next footprint of the bootstrap approach can be found in the work of Nagel, Stein and Wainger [46], from 1978, where they studied the problem of differentiation in lacunary directions. Although the name bootstrap does not appear there either the authors were aware their method can be called by that name. Roughly speaking, their argument based on the following observation. Let $(M_k)_{k\in\mathbb{N}}$ be a family of linear operators with uniformly bounded L^1-norm and assume that for some $p \in (1, 2]$ we have the following maximal estimate

$$\|\sup_{k\in\mathbb{N}} |M_k f|\|_{L^p(X,\mu)} \leq C_p \|f\|_{L^p(X,\mu)}. \tag{4.6}$$

Then for $\frac{1}{r} < \frac{1}{2}(1 + \frac{1}{p})$ we have the vector-valued estimate

$$\Big\|\big(\sum_{k\in\mathbb{N}} |M_k f_k|^2\big)^{1/2}\Big\|_{L^r(X,\mu)} \leq C_r \Big\|\big(\sum_{k\in\mathbb{N}} |f_k|^2\big)^{1/2}\Big\|_{L^r(X,\mu)}. \tag{4.7}$$

Now, if we know that (4.6) holds for $p = 2$, then we know that (4.7) holds for $r > 4/3$. Next, if the operator M_k has "good-behavior", we can use the vector-valued estimate (4.7) to prove the maximal estimate (4.6) for $p > 4/3$. Consequently, from the inequality (4.6) for $p = 2$ we obtained the same inequality for $p > 4/3$. We we can apply the same procedure, this time with $p = 4/3$, to get that (4.6) holds for $p > 8/7$. We successively apply this procedure for

$$p > 4/3,\ p > 8/7,\ \ldots,\ p > 2^j/(2^j - 1) \to 1$$

which shows that (4.6) holds for all $p \in (1, 2]$. Although not apparent at the first glance, it can be shown that the above-described procedure corresponds to the inequality

$$C_p(N) \leq D_p C_p(N)^{\frac{2-p}{2}} \tag{4.8}$$

where $C_p(N)$ is the constant from the truncated maximal estimate

$$\|\sup_{k\leq N} |M_k f|\|_{L^p(X,\mu)} \leq C_p(N)\|f\|_{L^p(X,\mu)}, \quad N \in \mathbb{N},$$

and $D_p > 0$ is some absolute constant independent of $N \in \mathbb{N}$. The inequality (4.8) implies the bound $C_p(N) \lesssim_p 1$ and in consequence, by the monotone convergence theorem, the L^p-bounds for the complete maximal function.

Argument of Nagel, Stein and Wainger relied heavily on some geometrical considerations. Later, Duoandikoetxea and Rubio de Francia [18] used the ideas of bootstrap from [46] (see Lemma 4.43 below) to prove L^p bounds for maximal Radon transform. At the same time, Christ formulated the bootstrap argument from [46] in a fairly abstract way, which was used and published by Carbery [11, 12]. Finally, Mirek, Stein and Zorin-Kranich [42] managed to use the bootstrap argument to establish jump inequalities in a very abstract setting.

The boundedness of the conjugate Fourier series

The question about the L^p-bounds for the conjugate Fourier series is related to the question of the convergence of Fourier series in L^p. By Plancherel's theorem this result easily holds for $p = 2$. However the case $p \neq 2$ is more problematic. It turns out that the question about the convergence in L^p norm of the Fourier series is equivalent to the boundedness on L^p of the conjugate Fourier series. This observation was used by M. Riesz [53] who proved the convergence of the Fourier series in ℓ^p norm by establishing the boundedness of the conjugate function. More details about the conjugate Fourier series and its relations to the convergence of Fourier series can be found in [20, Chapter 3.5].

The first proof of the boundedness of the conjugate Fourier series was given by M. Riesz [53] in 1927. Riesz's original proof was long and rather non-elementary. A few decades later the problem of the boundedness of the conjugate function was studied by Bochner [2]. In 1959 Bochner gave a new proof of M. Riesz's theorem – much shorter and more elementary than the original proof. In this proof Bochner uses binominal theorem and bootstrap ideas to prove the boundedness of the conjugate function on $L^p(\mathbb{T})$ for $p = 2k$, $k \in \mathbb{N}$ and then uses interpolation and duality. We present Bochner's proof in this section. At the beginning let us state the definition of the conjugate Fourier series.

Definition 4.9. For $f \in \mathcal{C}^\infty(\mathbb{T})$ we define the conjugate function f by

$$\overline{\mathcal{S}}f(x) := -i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n)\hat{f}(n)\boldsymbol{e}(nx)$$

where $\hat{f}(n)$ are the Fourier coefficients of the function f and $\operatorname{sgn}(n)$ is the sign function defined by

$$\operatorname{sgn}(x) := \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -1, & \text{for } x < 0. \end{cases}$$

We note that for $f \in \mathcal{C}^\infty(\mathbb{T})$ the series defining $\overline{\mathcal{S}}f$ is absolutely convergent and therefore it is a well-defined function.

Theorem 4.10 (M. Riesz Theorem)**.** *For any $p \in (1, \infty)$ there is a constant $C_p > 0$ such that for every $f \in \mathcal{C}^\infty(\mathbb{T})$ we have*

$$\big\|\overline{\mathcal{S}}f\big\|_{L^p(\mathbb{T})} \le C_p \|f\|_{L^p(\mathbb{T})}.$$

As a consequence the operator $\overline{\mathcal{S}}$ has a bounded extension on $L^p(\mathbb{T})$.

Below we give the proof of Riesz's theorem due to Bochner [2] which can be also found in the book of Grafakos [20, Theorem 3.5.6]. We slightly modified the presentation to be more in line with our theme but the main ideas remain unchanged.

Proof of Theorem 4.10. Let us consider the truncated conjugate series given by

$$\overline{\mathcal{S}}_N f(t) := -i \sum_{n=-N}^{N} \operatorname{sgn}(n)\hat{f}(n)\boldsymbol{e}(nx)$$

and let $C_p(N) > 0$ denote the smallest constant for which the inequality

$$\|\overline{\mathcal{S}}_N f\|_{L^p(\mathbb{T})} \le C_p(N)\|f\|_{L^p(\mathbb{T})} \tag{4.11}$$

holds for $p \in (1, \infty)$. At first we show that the constant $C_p(N)$ is finite for each $N \in \mathbb{N}$. Let us note that for each $n \in \mathbb{N}$ we have the trivial estimate $|\hat{f}(n)| \le \|f\|_{L^1(\mathbb{T})}$ and in consequence we have

$$\big\|\overline{\mathcal{S}}_N f\|_{L^p(\mathbb{T})} \lesssim_p N\|f\|_{L^1(\mathbb{T})} \lesssim_p N\|f\|_{L^p(\mathbb{T})},$$

where the last inequality follows by Hölder's inequality. This shows that $C_p(N) \lesssim_p N < \infty$. However, we will show that for any $p \in (1, \infty)$ there exists a constant $C_p > 0$ such that $C_p(N) \le C_p$ for any $N \in \mathbb{N}$ and then by Fatou's lemma we get

$$\big\|\overline{\mathcal{S}}f\|_{L^p(\mathbb{T})} \le \liminf_{N\to\infty} \big\|\overline{\mathcal{S}}_N f\|_{L^p(\mathbb{T})} \lesssim_k \|f\|_{L^p(\mathbb{T})}$$

which ends the proof of (4.10). Since $C_p(N)$ are non-decreasing in N, without loss of generality we can assume that $C_p(N) > 1$ for large $N \in \mathbb{N}$, otherwise the proof is done.

Now our aim is to show that there is a constant $C_p > 0$ such that

$$\|\overline{\mathcal{S}}_N f\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}, \quad f \in \mathcal{C}^\infty(\mathbb{T}). \tag{4.12}$$

We note that it is enough to proof (4.12) only for trigonometric polynomials since if $f \in \mathcal{C}^\infty(\mathbb{T})$ there is a sequence of trigonometric polynomials f_n which is uniformly convergent to f and by Fatou's lemma we have

$$\|\overline{\mathcal{S}}_N(f)\|_{L^p(\mathbb{T})} \leq \liminf_{n\to\infty} \|\overline{\mathcal{S}}_N(f_n)\|_{L^p(\mathbb{T})} \leq C_p \liminf_{n\to\infty} \|f_n\|_{L^p(\mathbb{T})} = C_p\|f\|_{L^p(\mathbb{T})}.$$

Let f be a trigonometric polynomial on $\mathbb{T}$ given by

$$f(t) = \sum_{n=-M}^{M} a_m \boldsymbol{e}(nt), \quad t \in [0,1),$$

for some $M \in \mathbb{N}$ and some complex coefficients a_n. We note that one can write

$$f(t) = \Big[\sum_{n=-M}^{M} \frac{a_n + \overline{a}_{-n}}{2}\boldsymbol{e}(nt)\Big] + i\Big[\sum_{n=-M}^{M} \frac{a_n - \overline{a}_{-n}}{2i}\boldsymbol{e}(nt)\Big]$$

where the expressions in the brackets are real-valued trigonometric polynomials. Therefore we may assume that f is a real-valued and by subtracting the constant term we may assume that $\hat{f}(0) = 0$. Since f is real-valued polynomial we have $\hat{f}(-n) = \overline{\hat{f}(n)}$ and as a consequence we may write

$$\overline{\mathcal{S}}_N f(t) = -i\sum_{n=1}^{\min\{M,N\}} \hat{f}(n)\boldsymbol{e}(nt) + i\sum_{n=1}^{\min\{M,N\}} \hat{f}(n)\boldsymbol{e}(-nt) = 2\mathrm{Re}\Big[-i\sum_{n=1}^{\min\{M,N\}} \hat{f}(n)\boldsymbol{e}(nt)\Big].$$

Hence, we see that $\overline{\mathcal{S}}_N f$ is also a real-valued polynomial without the constant term. Consequently, the polynomial $f + i\overline{\mathcal{S}}_N f$ contains only positive frequencies $\sin(nt)$ and $\cos(nt)$, for $n \in \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$ we have

$$\int_{\mathbb{T}} (f(t) + i\overline{\mathcal{S}}_N f(t))^{2k} \mathrm{d}t = 0$$

since there is no constant term and the polynomial $f + i\overline{\mathcal{S}}_N f$ contains only positive frequencies. By using the binomial theorem and taking the real parts, we get

$$\sum_{m=0}^{k} (-1)^{k-m}\binom{2k}{2m}\int_{\mathbb{T}} f(t)^{2m}\overline{\mathcal{S}_N} f(t)^{2k-2m}\mathrm{d}t = 0.$$

Next, since f and $\overline{\mathcal{S}}_N f$ are real-valued we obtain

$$\|\overline{\mathcal{S}}_N f\|_{L^{2k}(\mathbb{T})}^{2k} \leq \sum_{m=1}^{k}\binom{2k}{2m}\int_{\mathbb{T}} f(t)^{2m}\overline{\mathcal{S}_N} f(t)^{2k-2m}\mathrm{d}t.$$

Now we may apply Hölder's inequality with exponents $2k/(2k-2m)$ and $2k/(2m)$ to integral under the sum to get

$$\|\overline{\mathcal{S}}_N f\|_{L^{2k}(\mathbb{T})}^{2k} \leq \sum_{m=1}^{k}\binom{2k}{2m}\|\overline{\mathcal{S}_N} f\|_{L^{2k}(\mathbb{T})}^{2k-2m}\|f\|_{L^{2k}(\mathbb{T})}^{2m} \lesssim_k \sum_{j=1}^{k} C_{2k}(N)^{2k-2m}\|f\|_{L^{2k}(\mathbb{T})}^{2k},$$

where in the last inequality we used (4.11). Consequently, we write

$$\|\overline{\mathcal{S}}_N f\|_{L^{2k}(\mathbb{T})} \lesssim_k \sum_{j=1}^{k} C_{2k}(N)^{1-m/k} \|f\|_{L^{2k}(\mathbb{T})}. \tag{4.13}$$

Now, since $C_{2k}(N) \geq 1$ and $C_{2k}(N)$ is the smallest constant for which (4.11) holds, the inequality (4.13) implies

$$C_{2k}(N) \lesssim_k C_{2k}(N)^{1-1/k}$$

which in turn implies that $C_{2k}(N) \lesssim_k 1$ for any $N \in \mathbb{N}$. This shows that $\overline{\mathcal{S}}_N$ is L^p-bounded on the class of real-valued polynomials with $\hat{f}(0) = 0$. We can easily remove this assumption by noting that the conjugate function of the constant function is equal to zero. Then we write

$$\|\overline{\mathcal{S}}_N f\|_{L^{2k}(\mathbb{T})} = \|\overline{\mathcal{S}}_N\big(f - \hat{f}(0)\big)\|_{L^{2k}(\mathbb{T})} \lesssim_k \|f - \hat{f}(0)\|_{L^{2k}(\mathbb{T})} \lesssim_k \|f\|_{L^{2k}(\mathbb{T})}$$

and in consequence $\overline{\mathcal{S}}_N$ is L^p-bounded on the class of real-valued polynomials. Since a general trigonometric polynomial may be written as $P+iQ$ where P and Q are real-valued trigonometric polynomials by linearity we get that (4.12) holds for any trigonometric polynomial and $p = 2k$ with $k \in \mathbb{N}$. By interpolation we obtain that (4.12) holds for any $p \in [2, \infty)$. Finally, we observe that the adjoint operator of $\overline{\mathcal{S}}_N f$ is $-\overline{\mathcal{S}}_N f$. By duality, estimate (4.12) is also valid for $p \in (1, 2)$. □

The boundedness of the Hardy–Littlewood maximal function via bootstrap approach

The next example of the bootstrapping proof in harmonic analysis in the boundedness of the Hardy–Littlewood maximal function on $L^2(\mathbb{R}^d)$. In the proof we use the technique called the TT^*-method to obtain some form of the bootstrap inequality (4.1).

Before we start let us remind the definition of the Hardy–Littlewood maximal function. For any $r > 0$ and any locally integrable function f on $\mathbb{R}^d$ we define the average operator $A_r f$ by setting

$$A_r f(x) := \frac{1}{B(x,r)} \int_{B(x,r)} f(y)\mathrm{d}y, \quad x \in \mathbb{R}^d.$$

By using the TT^* method we will give the bootstrap proof of the L^2-estimate for the maximal function corresponding to the operators A_r.

Theorem 4.14 (L^2-estimate for the Hardy–Littlewood maximal function)**.** *Let $f \in L^2(\mathbb{R}^d)$ be a positive function. Then there is a constant $C_d > 0$ such that*

$$\|\sup_{r>0} A_r f\|_{L^2(\mathbb{R}^d)} \leq C_d \|f\|_{L^2(\mathbb{R}^d)}. \tag{4.15}$$

If we use the Marcinkiewicz interpolation theorem to interpolate the estimate (4.15) with the trivial L^∞-estimate

$$\|\sup_{r>0} A_r f\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\mathbb{R}^d)}$$

we get that for any $p \in [2, \infty]$ there is a constant $C_{p,d} > 0$ such that

$$\|\sup_{r>0} A_r f\|_{L^p(\mathbb{R}^d)} \leq C_{p,d} \|f\|_{L^p(\mathbb{R}^d)}, \quad f \in L^p(\mathbb{R}^d).$$

Obviously, the above result does not cover the case when $p \in (1, 2)$ and the weak type estimate when $p = 1$ unlike the standard approach which uses the Vitali-type covering lemma. However, despite a weaker result the TT^* is interesting on its own since it is a perfect tool to handle problems in the Hilbert space setting.

As one can easily guess, the TT^* method is based on the concept of the adjont operator T^*, the definition of which is given below.

Definition 4.16. Let $T\colon \mathcal{H} \to \mathcal{H}$ be linear operator from a Hilbert space $\mathcal{H}$ to itself. We say that $T^*\colon \mathcal{H} \to \mathcal{H}$ is the *adjoint operator* of T if

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad \text{for all } x, y, \in \mathcal{H}.$$

The existence and uniqueness of the adjoint operator follows from the Riesz representation theorem for Hilbert spaces.

The next lemma despite its simple formulation and easy proof is the core principle of the TT^* method.

Lemma 4.17. *Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bouned linear mapping from the Hilbert space $\mathcal{H}$ to itself and let $T^*\colon \mathcal{H} \to \mathcal{H}$ be its adjoint. Then we have the following norm equalities*

$$\|T\|_{\mathcal{H}\to\mathcal{H}} = \|T^*\|_{\mathcal{H}\to\mathcal{H}} = \|TT^*\|^{1/2}_{\mathcal{H}\to\mathcal{H}}. \tag{4.18}$$

Proof. We start by proving the first inequality which is a consequence of duality. Let $\|\cdot\|_{\mathcal{H}}$ denote the norm in the Hilbert space $\mathcal{H}$. Then we have the following equalities

$$\begin{aligned}\|T\|_{\mathcal{H}\to\mathcal{H}} = \sup_{\|x\|_{\mathcal{H}}\le 1} \|Tx\|_{\mathcal{H}} &\overset{\text{duality}}{=} \sup_{\|x\|_{\mathcal{H}}\le 1} \sup_{\|y\|_{\mathcal{H}}\le 1} |\langle Tx, y\rangle| \overset{\text{definition of } T^*}{=} \sup_{\|x\|_{\mathcal{H}}\le 1} \sup_{\|y\|_{\mathcal{H}}\le 1} |\langle x, T^*y\rangle| \\ &= \sup_{\|x\|_{\mathcal{H}}\le 1} \sup_{\|y\|_{\mathcal{H}}\le 1} |\langle T^*y, x\rangle| = \sup_{\|y\|_{\mathcal{H}}\le 1} \|T^*y\|_{\mathcal{H}} = \|T^*\|_{\mathcal{H}\to\mathcal{H}}.\end{aligned}$$

This shows that the first equality in (4.18) holds. Now our aim is to show the second equality in (4.18). At first we note that

$$\|TT^*\|_{\mathcal{H}\to\mathcal{H}} \le \|T\|_{\mathcal{H}\to\mathcal{H}}\|T^*\|_{\mathcal{H}\to\mathcal{H}} = \|T^*\|^2_{\mathcal{H}\to\mathcal{H}}$$

where the last equality follows by the first part of the proof. This shows the inequality in one direction. On the other hand, for any $x \in \mathcal{H}$, we have

$$\|T^*x\|^2_{\mathcal{H}} = \langle x, TT^*x\rangle \le \|x\|_{\mathcal{H}}\|TT^*x\|_{\mathcal{H}} \le \|x\|^2_{\mathcal{H}}\|TT^*\|_{\mathcal{H}\to\mathcal{H}}$$

which implies

$$\|T^*\|^2_{\mathcal{H}\to\mathcal{H}} \le \|TT^*\|_{\mathcal{H}\to\mathcal{H}}$$

and completes the proof. □

The idea of the TT^*-method relies on the self-cancellation properties of T which may occur if we consider the operator TT^* instead of T or T^* alone. The following simple example, based on matrices, will help us catch the idea of TT^*-method.

Example 4.19. Let us consider the space $\mathbb{C}^2$ endowed with the standard Euclidean norm given by

$$\|(x, y)\|_2 := \sqrt{|x|^2 + |y|^2}, \quad x, y \in \mathbb{C}.$$

Then the pair $(\mathbb{C}^2, \|\cdot\|_2)$ is a complex-valued Hilbert space. For $\mathbf{x} \in \mathbb{C}^2$ we consider the linear operator $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(\mathbf{x}) := A\mathbf{x}$ where

$$A := \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

Let us calculate the operator norm of the transformation T. By the definition

$$\|T\|_{\mathbb{C}^2\to\mathbb{C}^2} := \sup_{\|\mathbf{x}\|_2\le 1} \|A\mathbf{x}\|_2.$$

We have $A\mathbf{x} = (x - iy, x + iy)$ for $\mathbf{x} = (x, y)$ with $x, y \in \mathbb{C}$. Since x and y are complex numbers we can write $x = x_1 + ix_2$ and $y = y_1 + iy_2$ for some real numbers x_1, x_2, y_1, y_2. Consequently,

$$A\mathbf{x} = \big(x_1 + y_2 + i(x_2 - y_1), x_1 - y_2 + i(x_2 + y_1)\big).$$

Now, if we calculate the Euclidean norm of the vector $A\mathbf{x}$ we get

$$\|A\mathbf{x}\|_2 = \sqrt{2(x_1^2 + x_2^2 + y_1^2 + y_2^2)} = \sqrt{2}\|\mathbf{x}\|_2$$

which shows that $\|T\|_{\mathbb{C}^2\to\mathbb{C}^2} = \sqrt{2}$. The above calculation is simple but needs a bit of work and requires introducing the new variables.

Now let us use the TT^*-method to calculate the operator norm of T. We note that $T^*\colon \mathbb{C}^2 \to \mathbb{C}^2$ is of the form $T^*(\mathbf{x}) = A^*\mathbf{x}$ where A^* is the Hermitian conjugate of A given by

$$A^* = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}.$$

As a result, the operator $TT^*\colon \mathbb{C}^2 \to \mathbb{C}^2$ can be written as $TT^*(\mathbf{x}) = AA^*\mathbf{x}$ where

$$AA^* = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{4.20}$$

Since the matrix AA^* is diagonal it is easy to see that $\|A\mathbf{x}\|_2 = 2\|x\|_2$ which shows that $\|TT^*\|_{\mathbb{C}^2\to\mathbb{C}^2} = 2$. By Lemma 4.17 we obtain that $\|T\|_{\mathbb{C}^2\to\mathbb{C}^2} = \sqrt{2}$.

In the above example when we calculated the AA^* matrix the self-cancellation has occurred which resulted in simplification of the matrix AA^* to the diagonal form. This allows us to immediately calculate the norm of TT^* and consequently the norm of T. However, not every operator exhibits the obvious self-cancellation – this applies especially to operators acting on the infinite-dimensional Hilbert spaces. Fortunately, this approach works in the case of the Hardy–Littlewood maximal function.

Proof of Theorem 4.15. Let $\in L^2(\mathbb{R}^d)$. By Hölder's inequality we can see that f is locally integrable. Further, we note that the function $(0, \infty) \ni r \mapsto A_r f(x)$ is continuous for every $x \in \mathbb{R}^d$. Consequently, we may restrict the supremum in (4.15) to positive rational numbers. Further, by using the monotone convergence theorem, we may restrict the supremum to a finite set $\mathcal{R} \subset \mathbb{Q}$. Hence, it is enough to show that

$$\big\|\sup_{r\in\mathcal{R}} A_r f\|_{L^2(\mathbb{R}^d)} \le C_d\|f\|_{L^2(\mathbb{R}^d)}$$

where the constant $C_d > 0$ is independent of the set $\mathcal{R} \subset \mathbb{Q}$. Now, let $C_d(\mathcal{R})$ denote the smallest constant for which we have

$$\big\|\sup_{r\in\mathcal{R}} A_r f\|_{L^2(\mathbb{R}^d)} \le C_d(\mathcal{R})\|f\|_{L^2(\mathbb{R}^d)}. \tag{4.21}$$

Clearly, the constant $C_d(\mathcal{R})$ is finite for each set $\mathcal{R}$ since we have

$$\big\|\sup_{r\in\mathcal{R}} A_r f\|_{L^2(\mathbb{R}^d)} \le \#\mathcal{R}\|f\|_{L^2(\mathbb{R}^d)}$$

which proves that $C_d(\mathcal{R}) \le \#\mathcal{R} < \infty$. Observe that, by linearization of the supremum, the inequality (4.21) is equivalent to the estimate

$$\big\|A_{r(x)}f(x)\big\|_{L^2(\mathbb{R}^d,\mathrm{d}x)} \le C_d(\mathcal{R})\|f\|_{L^2(\mathbb{R}^d)},$$

for all measurable functions $r\colon \mathbb{R}^d \to \mathcal{R}$.

Now let us fix the function r and let us define the operator

$$T_r f(x) := A_{r(x)} f(x) = \frac{1}{|B(x,r(x))|} \int_{\mathbb{R}^d} f(y) \mathbb{1}_{B(x,r(x))}(y) \mathrm{d}y, \quad x \in \mathbb{R}^d.$$

Then we may interpret $C_d(\mathcal{R})$ as follows

$$C_d(\mathcal{R}) := \sup_r \|T_r\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}$$

where the supremum is taken over all measurable functions $r : \mathbb{R}^d \to \mathcal{R}$, i.e. $C_d(\mathcal{R})$ is the largest L^2-norm of the operators T_r. It can be easily seen that the operator T_r is an integral operator with kernel

$$K_r(x,y) := \frac{1}{|B(x,r(x))|} \mathbb{1}_{B(x,r(x))}(y), \quad x, y \in \mathbb{R}^d.$$

Now we calculate its adjoint. Let $f, g \in L^2(\mathbb{R}^d)$. We have

$$\langle T_r f, g \rangle = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} f(y) K_r(x,y) \mathrm{d}y \Big) \overline{g(x)} \mathrm{d}x = \int_{\mathbb{R}^d} f(y) \overline{\int_{\mathbb{R}^d} K_r(x,y) g(x) \mathrm{d}x} \mathrm{d}y$$

and consequently the adjoint operator of T_r is given by

$$T^* g(y) = \int_{\mathbb{R}^d} g(x) \frac{1}{|B(x,r(x))|} \mathbb{1}_{B(x,r(x))}(y) \mathrm{d}x, \quad y \in \mathbb{R}^d.$$

Therefore, the operator TT^* is given by

$$TT^* f(z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \frac{1}{|B(x,r(x))||B(z,r(z))|} \mathbb{1}_{B(x,r(x))}(y) \mathbb{1}_{B(z,r(z))}(y) \mathrm{d}x \mathrm{d}y, \quad z \in \mathbb{R}^d.$$

The integral in y can be easily computed since it is nonzero only when $y \in B(x,r(x)) \cap B(z,r(z)) \neq \emptyset$ and since for any $r > 0$ we have $|B(x,r)| \approx_d r^d$ we may write

$$\int_{\mathbb{R}^d} \mathbb{1}_{B(x,r(x))}(y) \mathbb{1}_{B(z,r(z))}(y) \mathrm{d}y \lesssim_d \min\{r(x), r(z)\}^d.$$

The condition that $B(x,r(x)) \cap B(z,r(z)) \neq \emptyset$ can be translated into a condition that binds together x and z,

$$|x - z| \leq r(x) + r(z)$$

which means that $x \in B(z, r(x) + r(z))$.

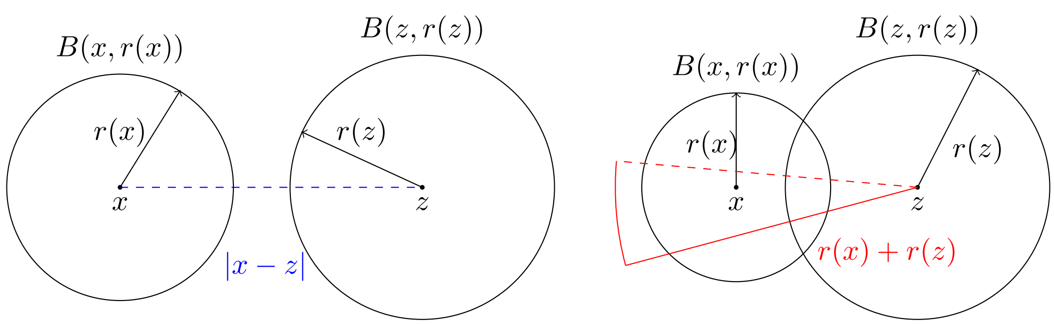

Figure 4.1: In the first picture we have $|x - z| > r(x) + r(z)$ so the balls do not intersect. In the picture on the right by red color we marked the part of the ball $B(z, r(x) + r(z))$ which contains x.

Therefore we can estimate

$$\begin{aligned}|TT^*f(z)| &\lesssim_d \int_{\mathbb{R}^d} |f(x)|\mathbb{1}_{B(z,r(x)+r(z))}(x)\frac{1}{[r(z)r(x)]^d}\min\{r(x),r(z)\}^d \mathrm{d}x\\ &= \int_{\mathbb{R}^d} \mathbb{1}_{B(z,r(x)+r(z))}(x)\frac{|f(x)|}{\max\{r(x),r(z)\}^d}\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d} |f(x)|\mathbb{1}_{B(z,2r(x))}(x)\frac{1}{r(x)^d}\mathrm{d}x + \int_{\mathbb{R}^d} |f(x)|\mathbb{1}_{B(z,2r(z))}(x)\frac{1}{r(z)^d}\mathrm{d}x\\ &\lesssim_d T^*_{2r}|f|(z) + T_{2r}|f|(z)\end{aligned}$$

where in the penultimate estimate we used the fact that

$$\mathbb{1}_{B(z,r(x)+r(z))}(x)\frac{1}{\max\{r(x),r(z)\}^d} \leq \mathbb{1}_{B(z,2r(x))}(x)\frac{1}{r(x)^d} + \mathbb{1}_{B(z,2r(z))}(x)\frac{1}{r(z)^d}.$$

Consequently, we obtain the following inequality

$$|T_rT_r^*f(z)| \lesssim_d T^*_{2r}|f|(z) + T_{2r}|f|(z), \quad z \in \mathbb{R}^d.$$

By using the scaling properties we see that

$$T_{2r}f(z) = T_{\tilde{r}}\tilde{f}(z/2), \quad z \in \mathbb{R}^d,$$

where $\tilde{r}(z) = r(2z)$ and $\tilde{f}(z) = f(2z)$ and similarly for T^*_{2r}. This fact together with Lemma 4.17 implies that

$$\|T^*_{2r}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} + \|T_{2r}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim C_d(\mathcal{R}).$$

This leads to the following estimate

$$\sup_r \|T_rT_r^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim_d C_d(\mathcal{R}). \tag{4.22}$$

On the other hand, by Lemma 4.17, we get

$$\sup_r \|T_rT_r^*\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = \sup_r \|T_r\|^2_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = C_d(\mathcal{R})^2.$$

The above inequality together with (4.22) give us

$$C_d(\mathcal{R}) \lesssim_d C_d(\mathcal{R})^{1/2}$$

and since the constant $C_d(\mathcal{R})$ is finite this shows that $C_d(\mathcal{R}) \lesssim_d 1$ with the implicit constant independent of the set $\mathcal{R}$. □

The idea of the TT^* argument can be traced back to Kolgomorov and Seliverstov [33]. It was further elaborated and popularized by Stein [57] and collaborators [47].

Remark 4.23. By exploiting the same ideas as in the proof of Lemma 4.17 one could prove that

$$\|T\|_{\mathcal{H}\to\mathcal{H}} = \|T^*\|_{\mathcal{H}\to\mathcal{H}} = \|T^*T\|^{1/2}_{\mathcal{H}\to\mathcal{H}}$$

which can be used to consider the T^*T variant of the TT^*-method. Sometimes using T^*T can do better than TT^*. However, in our case of maximal function this approach leads to

$$T^*Tf(z) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\frac{1}{|B(x,r(x))|^2}\mathbb{1}_{B(x,r(x))}(y)\mathbb{1}_{B(x,r(x))}(z)\mathrm{d}y\mathrm{d}x$$

where one does not see any obvious cancellation to exploit.

4.2 Jump inequalities for continuous Radon averages

In 2020 Mirek, Stein and Zorin-Kranich [42] managed to use the bootstrap argument to establish jump inequalities in a very abstract setting. In particular, , they have used this approach to establish the jump inequality for the continuous Radon averages

$$\mathcal{M}_t f(x) = \frac{1}{|\Omega_t|} \int_{\Omega_t} f(x - (y)^\Gamma) \mathrm{d}y, \quad x \in \mathbb{R}^\Gamma. \tag{4.24}$$

The aim of this section is to present the bootstrapping proof of the jump inequality for $\mathcal{M}_t$ which is due to Mirek, Stein and Zorin-Kranich. To be more precise we prove the following result.

Theorem 4.25. *([42, Theorem 1.22] and [32, Theorem 1.5]) Let $p \in (1, \infty)$. Then for any $f \in L^p(\mathbb{R}^\Gamma)$ we have*

$$J^2_{L^p(\mathbb{R}^\Gamma)}(\mathcal{M}_t f : t > 0) \lesssim_{p,k,|\Gamma|} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.26}$$

The first proof of Theorem 4.25 was given by Jones–Seeger–Wright [32, Theorem 1.5] and it was given for the averages $\mathcal{M}_t$ over Euclidean balls, $\Omega_t = B(0,t)$. The general case of the convex bodies Ω_t was proven by Mirek, Stein and Zorin-Kranich [42, Theorem 1.22].

Let $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^\Gamma)$ and let $\mathbb{U} := \bigcup_{n\in\mathbb{Z}} 2^n\mathbb{N}$ be the set of non-negative rational numbers whose denominators in reduced form are powers of 2. By standard density arguments it suffices to show that

$$J^2_{L^p(\mathbb{R}^\Gamma)}(\mathcal{M}_t f : t \in \mathbb{U}) \lesssim_{p,k,|\Gamma|} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.27}$$

By using Proposition 2.33 we may split (4.27) into long jumps and short variations,

$$\begin{aligned} J^2_{L^p(\mathbb{R}^\Gamma)}(\mathcal{M}_t f : t \in \mathbb{U}) &\lesssim J^2_{L^p(\mathbb{R}^\Gamma)}(\mathcal{M}_{2^n} f : n \in \mathbb{Z}) \\ &\quad + \Big\| \Big(\sum_{n\in\mathbb{Z}} V^2\big(\mathcal{M}_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)}. \end{aligned} \tag{4.28}$$

Now, we separately estimate the each term on right hand side of (4.28).

Estimates for the long jumps

Let $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^\Gamma)$. Here we present the bootstrapping proof of the inequality

$$J^2_{L^p(\mathbb{R}^\Gamma)}\big(\mathcal{M}_{2^n} f : n \in \mathbb{Z}\big) \lesssim_{p,d,|\Gamma|} \|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{4.29}$$

which was given by Mirek, Stein and Zorin-Kranich [42, Theorem 2.14.]. The presentation has been adjusted to the particular setting of the operator of (4.24) since the original version is written in a more general context. Roughly speaking, the main idea of proving (4.29) is to approximate $\mathcal{M}_{2^n}$ by a suitably chosen family of smooth functions and then use the Littlewood–Paley theory to estimate the approximation error.

The operator $\mathcal{M}_t$ is related to the following group of dilations

$$\delta_t(x) := \big(t^{|\gamma|} x_\gamma : \gamma \in \Gamma\big), \quad x \in \mathbb{R}^\Gamma. \tag{4.30}$$

In order to construct a suitable approximation family we need to take into account the relation between $\mathcal{M}_t$ and the dilations (4.30). For any $\xi \in \mathbb{R}^\Gamma$ we define the quasi-norm associated with (4.30) by setting

$$q(\xi) := \max_{\gamma\in\Gamma}(|\xi_\gamma|^{\frac{1}{|\gamma|}}).$$

Then $q\colon \mathbb{R}^\Gamma \to [0,\infty)$ is a smooth function on $\mathbb{R}^\Gamma \setminus \{0\}$. Let $\Theta\colon \mathbb{R} \to [0,\infty]$ be given by

$$\Theta(x) := c_{|\Gamma|}^{-1} e^{-x^{2|\Gamma|}},$$

where $c_{|\Gamma|} := \int_{\mathbb{R}} e^{-x^{2|\Gamma|}} \mathrm{d}x$. Then Θ is a non-negative Schwartz function with integral one. We define the family of Schwartz functions on $\mathbb{R}^\Gamma$, related to Θ, by setting

$$\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_t)(\xi) := \mathcal{F}_{\mathbb{R}}\big(\Theta\big)(tq(\xi)), \quad t>0, \quad \xi \in \mathbb{R}^\Gamma. \tag{4.31}$$

Then by [32, Theorem 1.1] we know that for every $1<p<\infty$ we have

$$J^2_{L^p(\mathbb{R}^\Gamma)}\big(\Theta_{2^n} * f : n \in \mathbb{Z}\big) \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f \in L^p(\mathbb{R}^\Gamma). \tag{4.32}$$

Moreover, by the results from Section 2.1 this estimate implies the r-variational inequality, which in turn implies that the maximal estimates,

$$\Big\|\sup_{n\in\mathbb{Z}} \big|\Theta_{2^n} * f\big|\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)}, \tag{4.33}$$

hold for all $p \in (1,\infty)$. The inequality (4.33) has been known for a long time and can be deduced from the Hardy–Littlewood maximal theorem [56, Proposition on p. 486]. We use the convolution family $\big(\Theta_{2^n} * f\big)_{n\in\mathbb{Z}}$ to approximate the operators $\mathcal{M}_{2^n} f$. By Proposition 2.29 we have

$$\begin{aligned} J^2_{L^p(\mathbb{R}^\Gamma)}\big(\mathcal{M}_{2^n} f : n \in \mathbb{Z}\big) &\lesssim J^2_{L^p(\mathbb{R}^\Gamma)}\big(\Theta_{2^n} * f : n \in \mathbb{Z}\big) + J^2_{L^p(\mathbb{R}^\Gamma)}\big(\mathcal{M}_{2^n} f - \Theta_{2^n} * f : n \in \mathbb{Z}\big) \\ &\lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)} + \Big\|\Big(\sum_{n\in\mathbb{Z}} |\mathcal{M}_{2^n} f - \Theta_{2^n} * f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}, \end{aligned}$$

where in the last inequality we used (4.32) and (2.28).

Now our aim is to establish the following bound

$$\Big\|\Big(\sum_{n\in\mathbb{Z}} |(\mathcal{M}_{2^n} f - \Theta_{2^n} * f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_{p,d,|\Gamma|} \|f\|_{L^p(\mathbb{R}^\Gamma)} \tag{4.34}$$

for any $p \in (1,\infty)$. Here we use the bootstrap argument. Let $N \in \mathbb{N}$ and let $C_p(N) > 0$ denote the smallest constant $C>0$ for which

$$\Big\|\Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} f - \Theta_{2^n} * f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \le C\|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

Clearly, the constant $C_p(N) > 0$ is finite since we have $C_p(N) \lesssim N$. Without loss of generality we can assume that $C_p(N) > 1$ and $N \in \mathbb{N}$ is large. Now, our aim is to show that $C_p(N) \lesssim_{p,d,|\Gamma|} 1$ with the implicit constant being independent of N. In order to do so we apply the Littlewood–Paley theory. Let $\phi_0\colon \mathbb{R} \to [0,\infty)$ be a smooth function such that $0 \le \phi_0 \le \mathbb{1}_{[1/2,2]}$ and its dilates $\phi_j(\xi) := \phi_0(2^j\xi)$ satisfy

$$\sum_{j\in\mathbb{Z}} \phi_j(x) = \mathbb{1}_{(0,\infty)}(x), \quad x \in \mathbb{R}. \tag{4.35}$$

For each $j \in \mathbb{Z}$, by using functions ϕ_j and the quasi-norm (4.31), we define the Littlewood–Paley operators S_j by

$$\mathcal{F}_{\mathbb{R}^\Gamma}(S_j f)(\xi) := \phi_j(q(\xi))\mathcal{F}_{\mathbb{R}^\Gamma}(f)(\xi), \quad \xi \in \mathbb{R}^\Gamma. \tag{4.36}$$

Then for any $f \in L^2(\mathbb{R}^\Gamma)$ one has

$$\sum_{j\in\mathbb{Z}} S_j f = f, \tag{4.37}$$

where the above equality holds in the L^2-norm. Indeed, by Plancherel's theorem one has

$$\lim_{N,M\to\infty}\Big\|\sum_{j=-N}^{M}S_jf-f\Big\|^2_{L^2(\mathbb{R}^\Gamma)}=\lim_{N,M\to\infty}\Big\|\mathcal{F}_{\mathbb{R}^\Gamma}f\Big(\sum_{j=-N}^{M}\phi_j(q(\cdot))-1\Big)\Big\|^2_{L^2(\mathbb{R}^\Gamma)}=0,$$

where the last inequality follows by (4.35) and the dominated convergence theorem with the dominant $2\mathcal{F}_{\mathbb{R}^\Gamma}f\in L^2(\mathbb{R}^\Gamma)$. Moreover, by [54, Theorem II.1.5] we obtain that for any $p\in(1,\infty)$ we have the Littlewood–Paley inequality

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_jf|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}\lesssim\|f\|_{L^p(\mathbb{R}^\Gamma)},\quad f\in L^p(\mathbb{R}^\Gamma). \tag{4.38}$$

Now, we use the above Littlewood–Paley operators S_j and for each $p\in(1,\infty)$, by (4.37), we estimate

$$\Big\|\Big(\sum_{|n|\le N}|\mathcal{M}_{2^n}f-\Theta_{2^n}*f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}\le\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{|n|\le N}|\mathcal{M}_{2^n}S_{n+j}f-\Theta_{2^n}*S_{n+j}f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.39}$$

Next our aim is to estimate the inner terms. At first we handle the case $p=2$ since it will give us a nice decay. Namely, we shall show that

$$\Big\|\Big(\sum_{n\in\mathbb{Z}}|\mathcal{M}_{2^n}S_{n+j}f-\Theta_{2^n}*S_{n+j}f|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^\Gamma)}\lesssim_{k,|\Gamma|}2^{-c|j|}\|f\|_{L^2(\mathbb{R}^\Gamma)}, \tag{4.40}$$

for some $c>0$. Since by (4.31) the function Θ_{2^n} is related to the Schwartz function with integral one has the following estimates for the Fourier transform of Θ_{2^n},

$$\big|\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_{2^n})(\xi)\big|\lesssim|2^{nA}\xi|_\infty^{-1/|\Gamma|},\quad\big|\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_{2^n})(\xi)-1\big|\lesssim|2^{nA}\xi|_\infty,\quad\xi\in\mathbb{R}^\Gamma. \tag{4.41}$$

Moreover, we know that $\mathcal{M}_tf=\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}(\Phi_t\mathcal{F}_{\mathbb{R}^\Gamma}f)$ where Φ_t is given by (2.61). Therefore, if we combine estimates (4.41) with the estimates for the function Φ_t given in (2.64) we obtain

$$\big|\Phi_{2^n}(\xi)-\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_{2^n})(\xi)\big|\lesssim\min\big\{|2^{nA}\xi|_\infty^{-1/|\Gamma|},|2^{nA}\xi|_\infty^{1/|\Gamma|}\big\}. \tag{4.42}$$

Hence, by Plancherel's theorem one gets

$$\Big\|\Big(\sum_{n\in\mathbb{Z}}|\mathcal{M}_{2^n}S_{n+j}f-\Theta_{2^n}*S_{n+j}f|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^\Gamma)}=\int_{\mathbb{R}^\Gamma}\sum_{n\in\mathbb{Z}}\big|\big(\Phi_{2^n}-\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_{2^n})\big)\phi_{j+n}(q(\cdot))\mathcal{F}_{\mathbb{R}^\Gamma}f\big|^2\mathrm{d}\xi.$$

We note that on the support of $\phi_{n+j}\big(q(\xi)\big)$ one has $|2^{nA}\xi|_\infty\simeq2^{-|j|}$ hence, by (4.42) we get

$$\int_{\mathbb{R}^\Gamma}\sum_{n\in\mathbb{Z}}\big|(\Phi_{2^n}-\mathcal{F}_{\mathbb{R}^\Gamma}(\Theta_{2^n}))\phi_{j+n}(q(\xi))\mathcal{F}_{\mathbb{R}^\Gamma}f\big|^2\mathrm{d}\xi\lesssim2^{-|j|/|\Gamma|}\|f\|_{L^p(\mathbb{R}^\Gamma)}$$

which proves (4.40).

In order to handle the case $p\neq2$ in (4.39) we make use of the "bootstrap lemma" that allows us to deduce a vector-valued inequality from a maximal one. This lemma originates in the work of Duoandikoetxea and Rubio de Francia [18, Lemma on p. 544]. The version presented here is its improvement due to Mirek, Stein and Zorin-Kranich [42].

Lemma 4.43. [42, Lemma 2.8] *Suppose that $(X,\mathcal{B},\mu)$ is a σ-finite measure space and $(B_n)_{n\in\mathbb{J}}$ is a sequence of linear operators on $L^1(X)+L^\infty(X)$ indexed by a countable set $\mathbb{J}$. The corresponding maximal operator is defined by*

$$B_{*,\mathbb{J}}f:=\sup_{n\in\mathbb{J}}\sup_{|g|\le|f|}|B_ng|,$$

where the supremum is taken in the lattice sense. Let $q_0, q_1 \in [1,\infty]$ *and* $0 \le \vartheta \le 1$ *with* $\frac{1}{2} = \frac{1-\vartheta}{q_0}$ *and* $q_0 \le q_1$. *Let* $q_\vartheta \in [q_0, q_1]$ *be given by* $\frac{1}{q_\vartheta} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1} = \frac{1}{2} + \frac{1-q_0/2}{q_1}$. *Then*

$$\Big\|\Big(\sum_{n\in\mathbb{J}}|B_n g_n|^2\Big)^{1/2}\Big\|_{L^{q_\vartheta}} \le \big(\sup_{n\in\mathbb{J}}\|B_n\|_{L^{q_0}\to L^{q_0}}\big)^{1-\vartheta}\|B_{*,\mathbb{J}}\|^{\vartheta}_{L^{q_1}\to L^{q_1}}\Big\|\Big(\sum_{n\in\mathbb{J}}|g_n|^2\Big)^{1/2}\Big\|_{L^{q_\vartheta}}.$$

Proof. Let us consider the operator $\widetilde{B}g := (B_n g_n)_{n\in\mathbb{J}}$ acting on sequences of functions $g = (g_n)_{n\in\mathbb{J}}$ in $L^1(X) + L^\infty(X)$. Then by Fubini's theorem one has

$$\begin{aligned}\|\widetilde{B}g\|_{L^{q_0}(X;\ell^{q_0}(\mathbb{J}))} = \big\|\|B_n g_n\|_{L^{q_0}(X)}\big\|_{\ell^{q_0}(\mathbb{J})} &\le \big(\sup_{n\in\mathbb{J}}\|B_n\|_{L^{q_0}\to L^{q_0}}\big)\big\|\|g_n\|_{L^{q_0}(X)}\big\|_{\ell^{q_0}(\mathbb{J})}\\ &= \big(\sup_{n\in\mathbb{J}}\|B_n\|_{L^{q_0}\to L^{q_0}}\big)\|g\|_{L^{q_0}(X;\ell^{q_0}(\mathbb{J}))}.\end{aligned}$$

On the other hand, by definition of the maximal operator $B_{*,\mathbb{J}}$, we may write

$$\begin{aligned}\|\widetilde{B}g\|_{L^{q_1}(X;\ell^\infty(\mathbb{J}))} &= \big\|\sup_{n\in\mathbb{J}}|B_n g_n|\big\|_{L^{q_1}(X)} \le \big\|B_{*,\mathbb{J}}(\sup_{n\in\mathbb{J}}|g_n|)\big\|_{L^{q_1}(X)}\\ &\le \|B_{*,\mathbb{J}}\|_{L^{q_1}\to L^{q_1}}\big\|\sup_{n\in\mathbb{J}}|g_n|\big\|_{L^{q_1}(X)} = \|B_{*,\mathbb{J}}\|_{L^{q_1}\to L^{q_1}}\|g\|_{L^{q_1}(X;\ell^\infty(\mathbb{J}))}.\end{aligned}$$

The claim for $q_\theta \in [q_0, q_1]$ follows by the Riesz interpolation theorem [20, Exrecise 4.5.2.] for vector-valued spaces $L^{q_0}(X;\ell^{q_0}(\mathbb{J}))$ and $L^{q_1}(X;\ell^\infty(\mathbb{J}))$. □

The next result is a counterpart of the above lemma related to the Littlewood–Paley operators.

Lemma 4.44. [42, Lemma 2.9] *Suppose that* $(X, \mathcal{B}, \mu)$ *is a* σ*-finite measure space with a sequence of operators* $(S_j)_{j\in\mathbb{Z}}$ *that satisfy the Littlewood–Paley inequality* (4.38). *Let* $1 \le q_0 \le q_1 \le 2$ *and* $L \in \mathbb{N}$ *be a positive integer and let* $\mathbb{V}_L = \{(n,m)\in\mathbb{Z}^2 : 0 \le m \le L-1\}$. *Let* $(M_{n,m})_{(n,m)\in\mathbb{V}_L}$ *be a sequence of linear operators bounded on* $L^{q_1}(X)$ *such that*

$$\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{m=0}^{L-1}\big|M_{n,m}S_{n+j}f\big|^2\Big)^{1/2}\Big\|_{L^2(X)} \le a_j\|f\|_{L^2(X)}, \qquad f \in L^2(X) \tag{4.45}$$

for some positive numbers $(a_j)_{j\in\mathbb{Z}}$. *Then for all* $f \in L^{q_1}(X)$ *we have*

$$\begin{aligned}&\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{m=0}^{L-1}\big|M_{n,m}S_{n+j}f\big|^2\Big)^{1/2}\Big\|_{L^{q_1}(X)}\\ &\quad\lesssim L^{\frac{2-q_1}{2-q_0}\frac{1}{2}}\big(\sup_{(n,m)\in\mathbb{V}_L}\|M_{n,m}\|^{\frac{2-q_1}{2-q_0}\frac{q_0}{2}}_{L^{q_0}\to L^{q_0}}\big)\|M_{*,\mathbb{V}_L}\|^{\frac{2-q_1}{2}}_{L^{q_1}\to L^{q_1}}a_j^{\frac{q_1-q_0}{2-q_0}}\|f\|_{L^{q_1}(X)}.\end{aligned} \tag{4.46}$$

Proof. When $q_1 = 2$ then this case is identical to the hypothesis (4.45) and we are done, so suppose $q_1 < 2$. Let ϑ and $q_\vartheta \in [q_0, q_1]$ be as in Lemma 4.43, then by using that lemma we write

$$\begin{aligned}&\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{m=0}^{L-1}\big|M_{n,m}S_{n+j}f\big|^2\Big)^{1/2}\Big\|_{L^{q_\vartheta}(X)}\\ &\quad\lesssim \big(\sup_{(n,m)\in\mathbb{V}_L}\|M_{n,m}\|^{1-\vartheta}_{L^{q_0}\to L^{q_0}}\big)\|M_{*,\mathbb{V}_L}\|^{\vartheta}_{L^{q_1}\to L^{q_1}}\Big\|\Big(\sum_{n\in\mathbb{Z}}\sum_{m=0}^{L-1}\big|S_{n+j}f\big|^2\Big)^{1/2}\Big\|_{L^{q_\vartheta}(X)}\\ &\quad\lesssim L^{1/2}\big(\sup_{(n,m)\in\mathbb{V}_L}\|M_{n,m}\|^{1-\vartheta}_{L^{q_0}\to L^{q_0}}\big)\|M_{*,\mathbb{V}_L}\|^{\vartheta}_{L^{q_1}\to L^{q_1}}\|f\|_{L^{q_\vartheta}},\end{aligned} \tag{4.47}$$

where in the last inequality we used the Littlewood–Paley inequality (4.38). Since $q_\vartheta \le q_1 < 2$, there is a unique $\nu \in (0,1]$ such that $\frac{1}{q_1} = \frac{\nu}{q_\vartheta} + \frac{1-\nu}{2}$. Substituting the definition of q_ϑ we obtain $\frac{1}{q_1} = \frac{\nu\vartheta}{q_1} + \frac{1}{2}$. It follows that

$$1-\vartheta = \frac{q_0}{2}, \qquad \vartheta = \frac{2-q_0}{2}, \qquad \nu\vartheta = \frac{2-q_1}{2},$$
$$\nu = \frac{2-q_1}{2-q_0}, \qquad \nu(1-\vartheta) = \frac{2-q_1}{2-q_0}\frac{q_0}{2}, \qquad 1-\nu = \frac{q_1-q_0}{2-q_0}.$$

Interpolating (4.47) with the hypothesis (4.45) gives the claim (4.46) for q_1. □

We make use of the above result to estimate the L^p-norm in (4.39). Now, let us consider the case $p \in (1,2]$ only. We have already showed that $C_2(N) \lesssim 1$ which follows by (4.39) and (4.39), so we may assume that $p \in (1,2)$. We apply Lemma 4.44 to the L^p-norm term in (4.40) since by (4.40) we know that the condition (4.45) is satisfied. We apply it with the operator $M_{n,0}f := \mathcal{M}_{2^n} f - \Theta_{2^n} * f$, the parameters $L = 1$, $q_0 = 1$ and $q_1 = p$. Consequently, one can write the following inequality

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} S_{n+j} f - \Theta_{2^n} * S_{n+j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \sup_{n\in\mathbb{Z}} \big\| M_{n,0} \big\|_{L^1\to L^1}^{\frac{2-p}{2}} \big\| M_{*,N} \big\|_{L^p\to L^p}^{\frac{2-p}{2}} 2^{-c_p|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

with $c_p > 0$, where the operator $M_{*,N}$ is defined as

$$M_{*,N} f := \sup_{|n|\le N} \sup_{|g|\le|f|} \big| \mathcal{M}_{2^n} f - \Theta_{2^n} * f \big|.$$

It can be easily seen that one has $\big\| M_{n,0} \big\|_{L^1\to L^1} \lesssim 1$. Moreover, we have the following pointwise estimate

$$|M_{*,\mathbb{Z}} f| \le \sup_{|n|\le N} \Theta_{2^n} * |f| + \sup_{|n|\le N} \mathcal{M}_{2^n} |f| \le 2 \sup_{|n|\le N} \Theta_{2^n} * |f| + \Big(\sum_{|n|\le N} \big| \mathcal{M}_{2^n} |f| - \Theta_{2^n} * |f| \big|^2 \Big)^{1/2}$$

which, by the definition of constant $C_p(N)$ and the maximal inequality (4.33), implies

$$\big\| M_{*,N} \big\|_{L^p\to L^p} \lesssim_p 1 + C_p(N) \lesssim C_p(N)$$

since $C_p(N) > 1$. Hence, we may write

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} S_{n+j} f - \Theta_{2^n} * S_{n+j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_{p,k,|\Gamma|} C_p(N)^{\frac{2-p}{2}} 2^{-c_p|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

for some $c_p > 0$. By (4.39) the above estimate implies that

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} f - \Theta_{2^n} * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_p(N)^{\frac{2-p}{2}} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.48}$$

By the definition of the constant $C_p(N)$ the above inequality shows that

$$C_p(N) \lesssim_{p,k,|\Gamma|} C_p(N)^{\frac{2-p}{2}}$$

which implies $C_p(N) \lesssim_{p,k,|\Gamma|} 1$ and this ends the proof of (4.34) in the case when $p \in (1,2]$.

In the case $p \in (2,\infty)$ we use duality. If $p > 2$ then its dual exponent p' satisfies $p' < 2$. Hence, we may repeat the above arguments for p' and obtain that

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} S_{n+j} f - \Theta_{2^n} * S_{n+j} f|^2 \Big)^{1/2} \Big\|_{L^{p'}(\mathbb{R}^\Gamma)} \lesssim_{p',k,|\Gamma|} C_{p'}(N)^{\frac{2-p'}{2}} 2^{-c_{p'}|j|} \|f\|_{L^{p'}(\mathbb{R}^\Gamma)}, \tag{4.49}$$

for some $c_{p'} > 0$. Since $\mathcal{M}_{2^n} S_{n+j} f - \Theta_{2^n} * S_{n+j} f$ are convolution operators, the inequality (4.49) holds also for p. Namely, one has

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} S_{n+j} f - \Theta_{2^n} * S_{n+j} f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_{p',k,|\Gamma|} C_{p'}(N)^{\frac{2-p'}{2}} 2^{-c_{p'}|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

The above inequality, together with (4.39), implies

$$\Big\| \Big(\sum_{|n|\le N} |\mathcal{M}_{2^n} f - \Theta_{2^n} * f|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_{p'}(N)^{\frac{2-p'}{2}} \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

which in turn implies

$$C_p(N) \lesssim C_{p'}(N)^{\frac{2-p'}{2}}.$$

By the first part we get that $C_{p'}(N) \lesssim_{p',k,|\Gamma|} 1$ which shows that $C_p(N) \lesssim_{p',k,|\Gamma|} 1$ for $p \in (2,\infty)$. This ends the proof of (4.34) for $p \in (2,\infty)$ and consequently the proof of the estimates for the long jumps.

Estimates for the short variations of the continuous Radon operators

Let $f \in C_c^\infty(\mathbb{R}^\Gamma)$. We need to show the L^p-estimates for the short variations in (4.28). Namely, we will show that

$$\Big\| \Big(\sum_{n\in\mathbb{Z}} V^2\big(\mathcal{M}_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_{p,k,|\Gamma|} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.50}$$

Here we present the bootstrap proof of (4.50) which is due to Mirek, Stein and Zorin-Kranich [42, Theorem 2.39 – case (3)]. In the proof of (4.50) we use some tools introduced during the proof of the estimate for the long jumps (4.29) – see the previous section for more details.

Let $N \in \mathbb{N}$ and let $C_p(N) > 0$ denote the smallest constant $C > 0$ for which

$$\Big\| \Big(\sum_{|n|\le N} V^2\big(\mathcal{M}_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \le C \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.51}$$

By the square function estimate (2.28) we know that for each $N \in \mathbb{N}$ we have $C_p(N) \lesssim_{N,p} 1$. Now, our aim is to show that $C_p(N) \lesssim_{p,d,|\Gamma|} 1$ with the implicit constant being independent of N. Without loss of generality we can assume that $C_p(N) > 1$ and $N \in \mathbb{N}$ is large.

In order to show that the constant $C_p(N)$ is finte we make use of the Rademacher–Menshov inequality for the short variations. Namely, for any $N \in \mathbb{N}$ and for any function $g \colon \mathbb{U} \to \mathbb{C}$ one has

$$\begin{aligned} &\Big(\sum_{|n|\le N} \big(V^2(g(t) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \\ &\qquad \lesssim \sum_{l\ge 0} \Big(\sum_{|n|\le N} \sum_{m=0}^{2^l-1} \big| g(2^n + 2^{n-l}(m+1)) - g(2^n + 2^{n-l} m) \big|^2 \Big)^{1/2}. \end{aligned} \tag{4.52}$$

The above inequality follows from the Rademacher–Menshov inequality (2.36). Indeed, the inequality (4.52) is a consequence of the estimate

$$V^2\big(g(t) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big) \lesssim \sum_{l\ge 0} \Big(\sum_{m=0}^{2^l-1} \big| g(2^n + 2^{n-l}(m+1)) - g(2^n + 2^{n-l} m) \big|^2 \Big)^{1/2}. \tag{4.53}$$

At first we note that it is enough to prove

$$V^2\big(f(t) : t \in [0, 2^n] \cap \mathbb{U}\big) \lesssim \sum_{l\ge 0} \Big(\sum_{m=0}^{2^l-1} \big| f(2^{n-l}(m+1)) - f(2^{n-l} m) \big|^2 \Big)^{1/2} \tag{4.54}$$

for $f(t) := g(2^n + t)$. Let $M \in \mathbb{N}$ be a large natural number and let us consider the set

$$\mathbb{U}_M := \{u/2^M : u \in \mathbb{N} \text{ and } 0 \le u \le 2^{n+M}\}.$$

Then one has $V^2(g(t) : t \in \mathbb{U}_M) = V^2\big(g(t/2^M) : t \in [0, 2^{n+M}] \cap \mathbb{Z}\big)$ and by the Rademacher–Menshov inequality (2.36) we may write

$$\begin{aligned} V^2(g(t) : t \in \mathbb{U}_M) &\lesssim \sum_{l=0}^{n+M} \Big(\sum_{m=0}^{2^{n+M-l}-1} \big|f(2^{l-M}(m+1)) - f(2^{l-M}m)\big|^2\Big)^{1/2} \\ &= \sum_{l=0}^{n+M} \Big(\sum_{m=0}^{2^{l}-1} \big|f(2^{n-l}(m+1)) - f(2^{n-l}m)\big|^2\Big)^{1/2}. \end{aligned}$$

By taking $M \to \infty$ in the above estimate we obtain (4.54) which ends to proof of (4.52). By using (4.52) we may estimate the left hand side of (4.51),

$$\text{LHS}(4.51) \lesssim \sum_{l \ge 0} \Big\| \Big(\sum_{|n| \le N} \sum_{m=0}^{2^l-1} \big|\mathcal{M}_{2^n+2^{n-l}(m+1)} f - \mathcal{M}_{2^n+2^{n-l}m} f\big|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \tag{4.55}$$

$$\lesssim \sum_{l \ge 0} \sum_{j \in \mathbb{Z}} \Big\| \Big(\sum_{|n| \le N} \sum_{m=0}^{2^l-1} \big|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f\big|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)}, \tag{4.56}$$

where in the lest inequality we used the Littlewood–Paley operators S_j defined in (4.36). Now our aim is to estimate the inner terms with the L^p-norm. At first we handle the case of $p = 2$. By Plancherel's theorem and the estimate (2.64) we get

$$\big\|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f\big\|_{L^2(\mathbb{R}^\Gamma)} \lesssim 2^{-c|j|} \|S_{n+j} f\|_{L^2(\mathbb{R}^\Gamma)}, \tag{4.57}$$

for some $c > 0$, since on the support of $\mathcal{F}_{\mathbb{R}^\Gamma}(S_{j+n})(\xi)$ one has $|2^{nA}\xi|_\infty \simeq 2^{-|j|}$. On the other hand, by Minkowski's integral inequality, for any $p \in [1, \infty)$ and any $g \in L^p(\mathbb{R}^\Gamma)$ one has

$$\big\|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) g\big\|_{L^p(\mathbb{R}^\Gamma)} \le \frac{\big|\Omega_{2^n+2^{n-l}(m+1)} \setminus \Omega_{2^n+2^{n-l}m}\big|}{\big|\Omega_{2^n+2^{n-l}(m+1)}\big|} \|g\|_{L^p(\mathbb{R}^\Gamma)}.$$

By Proposition 2.66 we have

$$\frac{\big|\Omega_{2^n+2^{n-l}(m+1)} \setminus \Omega_{2^n+2^{n-l}m}\big|}{\big|\Omega_{2^n+2^{n-l}(m+1)}\big|} \lesssim 2^{-l}$$

which implies that

$$\big\|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) g\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim 2^{-l} \|g\|_{L^p(\mathbb{R}^\Gamma)}, \tag{4.58}$$

for any $p \in [1, \infty)$ and any $g \in L^p(\mathbb{R}^\Gamma)$. In particular we may apply the above bound for $p = 2$ and with $g = S_{n+j} f$ to get

$$\big\|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f\big\|_{L^2(\mathbb{R}^\Gamma)} \lesssim 2^{-l} \|S_{n+j} f\|_{L^2(\mathbb{R}^\Gamma)}. \tag{4.59}$$

Therefore, by Plancherel's theorem, (4.57) and (4.59) we get

$$\Big\| \Big(\sum_{|n| \le N} \sum_{m=0}^{2^l-1} \big|(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f\big|^2\Big)^{1/2} \Big\|_{L^2(\mathbb{R}^\Gamma)} \lesssim 2^{-l/4} 2^{-c/4|j|} \|f\|_{L^2(\mathbb{R}^\Gamma)} \tag{4.60}$$

for some $c > 0$. We note that the above estimate is summable in $l \geq 0$ and $j \in \mathbb{Z}$ which, by the inequality (4.55), shows that $C_2(N) \lesssim 1$ with the implicit constant being independent of $N \in \mathbb{N}$. This ends the proof of (4.50) for $p = 2$.

Now we consider the case of $p \in (1, 2)$. In order to get the L^p bounds in terms of $C_p(N)$ we use Lemma 4.44 with $L = 2^l$, the set $\mathbb{V}_{N,l} := \{(n, m) \in \mathbb{Z}^2 : |n| \leq N, 0 \leq m \leq 2^l - 1\}$, the operators $M_{n,m} := \mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}$ and the parameters $q_0 = 1$, $q_1 = p$. By (4.60) we know that the condition (4.45) is satisfied hence we write

$$\begin{aligned} &\Big\| \Big(\sum_{|n| \leq N} \sum_{m=0}^{2^l-1} \big| (\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \\ &\qquad \lesssim 2^{l\frac{2-p}{2}} \sup_{(n,m)\in\mathbb{V}_{N,l}} \big\| M_{n,m} \big\|_{L^1\to L^1}^{\frac{2-p}{2}} \big\| M_{*,\mathbb{V}_{N,l}} \big\|_{L^p\to L^p}^{\frac{2-p}{2}} 2^{-l\frac{p-1}{2}} 2^{-c_p'|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)}, \end{aligned} \tag{4.61}$$

where $c_p' > 0$ and the maximal function $M_{*,\mathbb{V}_{N,l}}$ is given by

$$M_{*,\mathbb{V}_{N,l}} f := \sup_{\substack{|n|\leq N \\ 0\leq m\leq 2^l-1}} \sup_{|g|\leq|f|} \big| (\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) f \big|.$$

It can be easily seen that by (4.58) applied with $p = 1$ we have $\sup_{(n,m)\in\mathbb{V}_{N,l}} \big\| M_{n,m} \big\|_{L^1\to L^1} \lesssim 2^{-l}$ hence (4.61) can be reduced to

$$\begin{aligned} &\Big\| \Big(\sum_{|n| \leq N} \sum_{m=0}^{2^l-1} \big| (\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \\ &\qquad \lesssim \big\| M_{*,\mathbb{V}_{N,l}} \big\|_{L^p\to L^p}^{\frac{2-p}{2}} 2^{-l\frac{p-1}{2}} 2^{-c_p'|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)}. \end{aligned} \tag{4.62}$$

Further, we handle the quantity $\big\| M_{*,\mathbb{V}_{N,l}} \big\|_{L^p\to L^p}$. At first, we note that one has the following pointwise estimate

$$\big| (\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) f \big| \lesssim \sup_{n\in\mathbb{Z}} \mathcal{M}_{2^n} |f| + \Big(\sum_{|n|\leq N} V^2\big(\mathcal{M}_t|f| : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2}.$$

In the previous section we have showed the jump inequality for the dyadic scales (4.29) which, by the results from Section 2.1, implies that for any $r \in (2, \infty)$ and for any $p \in (1, \infty)$

$$\big\| V^r(\mathcal{M}_{2^n} f : n \in \mathbb{Z}) \big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f \in L^p(\mathbb{R}^\Gamma).$$

In turn, this implies the maximal function estimate,

$$\big\| \sup_{n\in\mathbb{Z}} |\mathcal{M}_{2^n} f| \big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)}, \quad f \in L^p(\mathbb{R}^\Gamma).$$

If we combine together the above observations then we obtain that $\big\| M_{*,\mathbb{V}_{N,l}} \big\| \lesssim_p C_p(N)$. Therefore, we have

$$\Big\| \Big(\sum_{|n| \leq N} \sum_{m=0}^{2^l-1} \big| (\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m}) S_{n+j} f \big|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_p(N)^{\frac{2-p}{2}} 2^{-lc_p''} 2^{-c_p'|j|} \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

for some constants $c_p', c_p'' > 0$. By (4.55) this implies

$$\Big\| \Big(\sum_{|n|\leq N} V^2\big(\mathcal{M}_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_p(N)^{\frac{2-p}{2}} \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

By the definition of the constant $C_p(N)$, one has

$$C_p(N) \lesssim_{p,k,|\Gamma|} C_p(N)^{\frac{2-p}{2}}$$

which proves $C_p(N) \lesssim_{p,k,|\Gamma|} 1$ and this ends the proof of (4.50) in the case when $p \in (1,2)$.

As in the case of long jumps, for $p \in (2,\infty)$ we make use of duality. We repeat the above arguments for p' and obtain that

$$\Big\|\Big(\sum_{|n|\le N}\sum_{m=0}^{2^l-1}\big|(\mathcal{M}_{2^n+2^{n-l}(m+1)}-\mathcal{M}_{2^n+2^{n-l}m})S_{n+j}f\big|^2\Big)^{1/2}\Big\|_{L^{p'}(\mathbb{R}^\Gamma)} \lesssim C_{p'}(N)^{\frac{2-p'}{2}}2^{-lc''_p}2^{-c'_p|j|}\|f\|_{L^{p'}(\mathbb{R}^\Gamma)}, \tag{4.63}$$

for some constants $c'_p, c''_p > 0$. Since $(\mathcal{M}_{2^n+2^{n-l}(m+1)} - \mathcal{M}_{2^n+2^{n-l}m})S_{n+j}$ are the convolution operators, we see that (4.63) holds also for p. Hence, by (4.55) we get

$$\Big\|\Big(\sum_{|n|\le N} V^2\big(\mathcal{M}_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim C_{p'}(N)^{\frac{2-p'}{2}}\|f\|_{L^p(\mathbb{R}^\Gamma)}$$

which shows that

$$C_p(N) \lesssim_{p,k,|\Gamma|} C_{p'}(N)^{\frac{2-p'}{2}}.$$

Since $p' < 2$, by the first part of the proof, we have $C_{p'}(N)^{\frac{2-p'}{2}} \lesssim_{p',k,|\Gamma|} 1$ which gives $C_p(N) \lesssim_{p,k,|\Gamma|} 1$. This ends the proof of (4.50) in the case when $p \in (2,\infty)$ and therefore the proof of Theorem 4.25.

4.3 Seminorm estimates for Radon type operators on $\mathbb{Z}^d$ – proof of Theorem 1.51

As we have seen in the previous section, an appropriate usage of the Littlewood–Paley theory allows us to establish jumps inequalities (4.26) almost without using other tools while maintaining clarity of the presentation. In this context, the problem of interest is whether a similar approach can be used in the discrete setting. This question is particularly interesting since we know that most "continuous" methods do not apply in the discrete setting and usually discrete problems require a totally different approach. Surprisingly, the bootstrap approach with a few changes can be used in the context of the discrete Radon averages. We managed to do this in [D2] where the seminorm inequalities for the discrete Radon averages where established. The paper [D2] was motivated by the work of Mirek, Stein and Zorin-Kranich [42] which was partially recalled in the previous section and by the work of Mirek [37] where the discrete Littewood–Paley theory was established.

The exposition of this section is based on the paper [D2]. Some tools and methods used are very similar (if not the same) as in Section 3.2 hence sometimes we make a reference to a relevant result. By the lifting procedure (Lemma 2.54) it is sufficient to prove Theorem 1.51 for the averages M_t given by (2.55).

Theorem 4.64. *Let $p \in (1,\infty)$. Then for every $f \in \ell^p(\mathbb{Z}^\Gamma)$ we have*

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_t f : t > 0) \lesssim_{\mathcal{S}^p,p,k,|\Gamma|} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.65}$$

where the implicit constant may depend on the choice of the seminorm $\mathcal{S}^p$.

The rest of this chapter is devoted to proving Theorem 4.64.

Assume that $p \in (1,\infty)$ and let $f \in \ell^p(\mathbb{Z}^\Gamma)$ be a compactly supported function. Let $\mathbb{U} := \bigcup_{n\in\mathbb{Z}} 2^n\mathbb{N}$. Let us note that it is enough to establish the following inequality

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_t f : t \in \mathbb{U}) \lesssim_{\mathcal{S}^p} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.66}$$

where the implied constant may depend on the seminorm $\mathcal{S}_p$ and $p \in (1,\infty)$ but is independent of f. Let us choose $p_0 \in (1,2)$, close to 1 such that $p \in (p_0, p_0')$. Note that $M_t f \equiv f$ for $t \in (0,1)$. By Proposition 2.33 we can split (4.66) into dyadic scales (long "jumps") and short variations

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_t f : t \in \mathbb{U}) \lesssim \mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_{2^n} f : n \in \mathbb{N}_0) + \Big\| \Big(\sum_{n=0}^{\infty} V^2\big(M_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.67}$$

We will estimate separately each part of the right hand side of (4.67).

4.3.1 Estimates for the dyadic scales

The aim of this subsection is to give a proof of the estimate for the dyadic scales,

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_{2^n} f : n \in \mathbb{N}_0) \lesssim_{\mathcal{S}^p} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.68}$$

where the implicit constant may only depend on the seminorm $\mathcal{S}_p$, but is independent of f. For this purpose we will exploit the following bootstrap argument. For $N \in \mathbb{N}$ let us consider the following cut-off seminorms

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_{2^n} f : n \in [0,N] \cap \mathbb{N}_0).$$

By $C_p(N)$ we denote the smallest constant $C > 0$ for which the following estimate holds

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}(M_{2^n} f : n \in [0,N] \cap \mathbb{N}_0) \le C\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \quad f \in \ell^p(\mathbb{Z}^\Gamma).$$

Clearly, the constant $C_p(N)$ is finite for each $N \in \mathbb{N}$ since by (2.28) one has

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(M_{2^n} f : n \in [0,N] \cap \mathbb{N}_0\big) \lesssim \Big\| \Big(\sum_{n=0}^{N} |M_{2^n} f|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim N\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

and hence $C_p(N) \lesssim N < \infty$. However, we will show that there exist a constant $C_p > 0$ such that $C_p(N) \lesssim_{\mathcal{S}^p} 1$ with the implicit constant being independent of $N \in \mathbb{N}$. If such a constant exists, then by taking limit as $N \to \infty$ and by using the monotone convergence theorem one easily obtains (4.68). Without loss of generality we can assume that $R_p(N) > 1$ and $N \in \mathbb{N}$ is large.

We make use of the Hardy–Littlewood circle method related to the Ionescu–Wainger fractions from Theorem 2.71. We start by noting that the operator M_{2^n} is a Fourier multiplier operator with multiplier m_{2^n} given by (2.59). Similar to the proof of Theorem 3.4 (Section 3.2) the proof of (4.68) require several appropriately chosen parameters. Let $\alpha > 0$ be such that

$$\alpha > 100\left(\frac{1}{p_0} - \frac{1}{2}\right)\left(\frac{1}{p_0} - \frac{1}{\min\{p,p'\}}\right)^{-1}. \tag{4.69}$$

Fix $\chi \in (0,1/10)$ and let $u \in \mathbb{N}$ be a large natural number which will be specified later. Let $\eta\colon \mathbb{R}^\Gamma \to [0,1]$ be a smooth function such that

$$\eta(x) = \begin{cases} 1, & |x| \le 1/(16|\Gamma|), \\ 0, & |x| \ge 1/(8|\Gamma|). \end{cases} \tag{4.70}$$

Let us set $\varrho := (10u)^{-1}$ and recall the family of rational fractions $\Sigma_{\leq n^u}$ related to the parameter ϱ described in Theorem 2.71. For each $n \in \mathbb{N}$ we define the following function

$$\Xi_n(\xi) := \sum_{a/q \in \Sigma_{\leq n^u}} \eta^2\big(2^{n(A-\chi I)}(\xi - a/q)\big), \tag{4.71}$$

where I is the $|\Gamma| \times |\Gamma|$ identity matrix and A is a matrix defined in (2.63). We note that the functions (4.71) corresponds to the functions $\Pi_{\leq n^\tau, n^\tau(A-\chi I)}$ with $\tau = 1$ defined in (3.23). We decided to use the different symbol to make an appropriate distinction between them since we will be using them in a different way. By Theorem 2.71 we have that

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Xi_n \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim_{u,p} \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.72}$$

which follows by the fact that for large n one has $\varepsilon_\gamma := 2^{-n(|\gamma|-\chi)} \leq e^{-n^{1/5}} = e^{-(n^u)^{2\varrho}}$. We use projections defined in (4.71) to partition the multiplier m_{2^n},

$$\begin{aligned}\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big) &\lesssim \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big)\\ &\quad + \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big).\end{aligned}$$

Now, just as in Section 3.2 our aim is to estimate the each term separately.

Estimates for the minor arcs

Now, our aim is to prove that

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big) \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.73}$$

The proof of (4.73) is a straightforward repetition of the arguments presented during the proof of (3.28). Hence, we do not present the exact details here. We note that by (2.28) the seminorm $\mathcal{S}^p$ is bounded by the 2-variation seminorm. Moreover, since the r-variation seminorms are non-increasing in r we may estimate V^2 by the V^1 and consequently

$$\text{LHS(4.73)} \leq \big\|V^1(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \sum_{n=0}^{\infty}\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Therefore, it is enough to show

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.74}$$

For any $p \in (1,\infty)$ by the inequality (4.72) we have

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim_{u,p} \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.75}$$

In the case of $p = 2$ we use Weyl's inequality (Theorem 3.31) to obtain a rapidly decreasing bound. Thus, if we show that there are ξ_{γ_0}, a, q for which the conditions (3.32) and (3.33) hold, then

$$|m_{2^n}(\xi)| \lesssim_{|\Gamma|,k} \frac{2^{nk}}{|\Omega_{2^n}\cap\mathbb{Z}^k|}(n+1)^{-\alpha} \lesssim_\Omega (n+1)^{-\alpha}$$

with $\alpha > 0$ from (4.69), since $|\Omega_{2^n}\cap\mathbb{Z}^k| \gtrsim_\Omega 2^{nk}$. Therefore, by Parseval's theorem one may write

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((1-\Xi_n)m_{2^n}\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-\alpha}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}$$

and by interpolating the above inequality with (4.75) for $p = p_0$ we obtain (4.74). Let $u > \beta|\Gamma|$, where β is from Theorem 3.31. In order to verify conditions (3.32) and (3.33) one uses Dirichlet's principle (Lemma 3.35) and repeats exactly the same steps as in the proof (3.29) but with $\tau = 1$. We omit the details.

Major arcs and scale distinction

We can now turn our attention to the major arcs. Let $\widetilde{\eta}(x) := \eta(x/2)$. We define new multipliers by setting

$$\Xi_n^s(\xi) := \sum_{a/q\in\Sigma_{s^u}} \eta^2\big(2^{n(A-\chi I)}(\xi - a/q)\big)\widetilde{\eta}^2\big(2^{s(A-\chi I)}(\xi - a/q)\big),$$

where $\Sigma_{s^u} = \Sigma_{\leq (s+1)^u} \setminus \Sigma_{\leq s^u}$ for $s \in \mathbb{N}$ and $\Sigma_{0^u} = \Sigma_{\leq 1}$. It is easy to see that one has

$$\Xi_n(\xi) = \sum_{s=0}^{n-1} \Xi_n^s(\xi).$$

Since By Proposition 2.29 we know that $\mathcal{S}^p$ satisfies the "trinagle ineqlaity" we may write that

$$\begin{aligned}\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big) &\lesssim \sum_{s=0}^{N} \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [0,N]\cap\mathbb{N}_0\big)\\ &\lesssim \sum_{s=0}^{N} \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [s,N]\cap\mathbb{N}_0\big)\\ &+ \big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^s}\Xi_s^s\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)},\end{aligned}$$

where the last inequality follows by Proposition 2.32. Now, for $s \in \mathbb{N}_0$ we set $\kappa_s := 20|\Gamma|\lceil (s+1)^{1/10}\rceil$ and by Proposition 2.30 we see that the expression under the sum is bounded by

$$\begin{aligned}\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [s,N]\cap\mathbb{N}_0, n \leq 2^{\kappa_s+1}\big) + \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0, n > 2^{\kappa_s}\big)\\ + \big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^s}\Xi_s^s\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}.\end{aligned}$$

The first term corresponds to small scales and the second one to large scales. For $p \in (1,\infty)$ we will show the following bounds:

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^s}\Xi_s^s\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.76}$$

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in \mathbb{N}_0, n > 2^{\kappa_s}\big) \lesssim (s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.77}$$

Moreover, in the case of $p \in (1,2]$, we prove that the inequality

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [s,N]\cap\mathbb{N}_0, n \leq 2^{\kappa_s+1}\big) \lesssim C_p(N)^{\beta(p)}(s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.78}$$

holds with some $\beta(p) \in [0,1)$. If we show the above inequalities and combine them with the estimates for the minor arcs (4.73), then for $p \in (1,2]$ we obtain

$$C_p(N) \lesssim 1 + \sum_{s=0}^{\infty}(s+1)^{-3}(C_p(N)^{\beta(p)} + 1) \lesssim C_p(N)^{\beta(p)},$$

since $C_p(N) \geq 1$ and $\beta(p) \in [0,1)$. This gives $C_p(N) \lesssim_p 1$ and thus the proof is complete in the case of $p \in (1,2]$. When $p \in (2,\infty)$ a minor change is required for the estimate (4.78). Namely, in this case we show that

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [s,N]\cap\mathbb{N}_0, n \leq 2^{\kappa_s+1}\big) \lesssim C_{p'}(N)^{\beta'(p)}(s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.79}$$

where $1/p + 1/p' = 1$ and $\beta'(p) \in (0,1)$. Since $p' \in (1,2)$, by the first part one has that $C_{p'}(N) \lesssim_{p'} 1$ and consequently

$$C_p(N) \lesssim 1 + \sum_{s=0}^{\infty}(s+1)^{-3}(C_{p'}(N)^{\beta'(p)} + 1) \lesssim_p 1$$

which finishes the proof in the case of $p \in (2,\infty)$.

Multiplier approximation and estimate for (4.76)

At first we prove the estimate (4.76) which is relatively easy. We note that $\eta\widetilde{\eta} = \eta$ and as a consequence we see that

$$\Xi_s^s(\xi) = \sum_{a/q \in \Sigma_{s^u}} \eta^2\big(2^{s(A-\chi)}(\xi - a/q)\big).$$

By Theorem 2.71, for any $p \in (1, \infty)$, we obtain the estimate

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_{2^s}\Xi_s^s\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.80}$$

In the case of $p = 2$ we will approximate the multiplier $m_{2^s}\Xi_s^s$ by a suitably chosen integral and we show that it is equal, up to reasonable error, to

$$\boldsymbol{m}_s(\xi) := \sum_{a/q \in \Sigma_{s^u}} G(a/q)\Phi_{2^s}(\xi - a/q)\eta^2(2^{s(A-\chi I)}(\xi - a/q)) \tag{4.81}$$

where Φ_t is the continuous counterpart of the multiplier m_t given by (2.61) and $G(a/q)$ is the Gauss sum given by (3.39).

In order to approximate $m_{2^s}\Xi_s^s$ by $\boldsymbol{m}_s$ we make use of Proposition 3.41. At first we use it with $\mathcal{K} \equiv 1$, $\Omega_{2^s} \subseteq B(0, 2^s)$ and $\xi = a/q = 1$ and as a result we get

$$\big||\Omega_{2^s} \cap \mathbb{Z}^k| - |\Omega_{2^s}|\big| \lesssim 2^{s(k-1)}. \tag{4.82}$$

In the next step we define an auxiliary multiplier

$$\widetilde{m}_{2^s}(\xi) := \frac{1}{|\Omega_{2^s}|} \sum_{y \in \Omega_{2^s} \cap \mathbb{Z}^k} e(\xi \cdot (y)^\Gamma).$$

By (4.82) we obtain

$$|m_{2^s}(\xi) - \tilde{m}_{2^s}(\xi)| \leq \frac{\big||\Omega_{2^s}| - |\Omega_{2^s} \cap \mathbb{Z}^k|\big|}{|\Omega_{2^s}||\Omega_{2^s} \cap \mathbb{Z}^k|}|\Omega_{2^s} \cap \mathbb{Z}^k| \lesssim \frac{2^{s(k-1)}}{2^{sk}} = 2^{-s}. \tag{4.83}$$

Now, we again use Proposition 3.41 this time with $\Omega_{2^s} \subseteq B(0, 2^s)$, $\mathcal{K} = |\Omega_{2^s}|^{-1}\mathbb{1}_{\Omega_{2^s}}$ and $\varepsilon_\gamma = 1$. Hence, on the support of Ξ_s^s,

$$\big|\widetilde{m}_{2^s}(\xi) - G(a/q)\Phi_{2^s}(\xi - a/q)\big| \lesssim q2^{-s} + \sum_{\gamma \in \Gamma} q|\xi_\gamma - a_\gamma/q|2^{s(|\gamma|-1)} \lesssim 2^{-s/2}, \tag{4.84}$$

since $q \leq e^{s^{1/10}}$ and for any $\gamma \in \Gamma$ we have $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-s(|\gamma|-\chi)}$. Consequently, by (4.83) and (4.84) one has

$$\big|m_{2^s}(\xi) - G(a/q)\Phi_{2^s}(\xi - a/q)\big| \lesssim 2^{-s/2},$$

which, by the disjointness of the supports of $\eta\big(2^{s(A-\chi I)}(\xi - a/q)\big)$, shows that

$$(m_{2^s}\Xi_s^s)(\xi) = \boldsymbol{m}_s(\xi) + \mathcal{O}(2^{-s/2}).$$

Observe that due to the estimate (3.40) we have $\max_{\xi \in \mathbb{T}^\Gamma} |v_s(\xi)| \lesssim (s+1)^{-\delta u}$ and by Plancherel's theorem

$$\begin{aligned} \big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_{2^s}\Xi_s^s\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} &\leq \big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^s}\Xi_s^s - \boldsymbol{m}_s)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} + \big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\boldsymbol{m}_s\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \\ &\lesssim (2^{-s/2} + (s+1)^{-\delta u})\|f\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-\alpha}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)} \end{aligned} \tag{4.85}$$

provided that $u \in \mathbb{N}$ satisfies $u > \alpha\delta^{-1}$. Interpolation (4.80) for $p = p_0$ with (4.85) shows that (4.76).

Estimates for the large scales

Now we focus on proving the estimate for the large scales,

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi^s_n\mathcal{F}_{\mathbb{Z}^\Gamma}f) : n\in\mathbb{N}_0, n>2^{\kappa_s}\big) \lesssim (s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.86}$$

The proof of the above inequality is similar in spirit to the proof of (3.67). However we need to establish some additional approximations. By Proposition 3.41 (compare with (4.84)) we have

$$m_{2^n}\Xi^s_n(\xi) = \boldsymbol{m}^n_s(\xi) + \mathcal{O}(2^{-n/2}),$$

where

$$\boldsymbol{m}^n_s(\xi) := \sum_{a/q\in\Sigma_{s^u}} G(a,q)\Phi_{2^n}(\xi - a/q)\eta^2(2^{n(A-\chi I)}(\xi - a/q))\tilde{\eta}^2(2^{s(A-\chi I)}(\xi - a/q)).$$

Now, by Theorem 2.71 one has

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi^s_n\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

and by the estimate property (i) from Theorem 2.71 we get

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}^n_s\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)(s+1)^{1/10}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)(n+1)^{1/10}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

As a result, for every $p\in(1,\infty)$, we obtain

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^n}\Xi^s_n - \boldsymbol{m}^n_s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)(n+1)^{1/10}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.87}$$

On the other hand, by Plancherel's theorem

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^n}\Xi^s_n - \boldsymbol{m}^n_s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim 2^{-n/2}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

Interpolating the above estimate with (4.87) for $p=p_0$ yields

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^n}\Xi^s_n - \boldsymbol{m}^n_s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim 2^{-c_p n}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

for some $c_p>0$ and since $n\geq s$ we may write

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^n}\Xi^s_n - \boldsymbol{m}^n_s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-3}2^{-c_p n/2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Therefore, since by (2.27) we know that $\mathcal{S}^p$ is bounded by the ℓ^2-norm, we may write

$$\begin{aligned}\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi^s_n\mathcal{F}_{\mathbb{Z}^\Gamma}f) : n\in\mathbb{N}_0, n>2^{\kappa_s}\big) &\lesssim \sum_{n=2^{\kappa_s}}^{\infty}\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^n}\Xi^s_n - \boldsymbol{m}^n_s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\ &\quad + \mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}^n_s\mathcal{F}_{\mathbb{Z}^\Gamma}f) : n\in\mathbb{N}_0, n>2^{\kappa_s}\big),\end{aligned}$$

which shows that it sufficient to show

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\boldsymbol{m}^n_s\mathcal{F}_{\mathbb{Z}^\Gamma}f) : n\in\mathbb{N}_0, n>2^{\kappa_s}\big) \lesssim (s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.88}$$

instead of (4.86).

In order to show (4.88) we follow the approach introduced during the proof of (3.67). The multiplier $\boldsymbol{m}^n_s$ is localized around fractions from the set Σ_{s^u}. Let $Q_{s^u} := \operatorname{lcm}(q : a/q\in\Sigma_{s^u})$. By property (iv) from Theorem 2.71 one has $Q_S\leq 3^{s^u}$. If we have $n\geq 2^{\kappa_s}$ then we may write

$$\boldsymbol{m}^n_s(\xi) = \Pi_s(\xi)\sum_{b\in\mathbb{Z}^\Gamma}\widetilde{\Phi}_{2^n}(\xi - b/Q_{s^u}) \tag{4.89}$$

where

$$\Pi_s(\xi) := \sum_{a/q\in\Sigma_{s^u}} \widetilde{\eta}\big(2^{2^{\kappa_S}(A-\chi I)}(\xi - a/q)\big), \text{ and } \widetilde{\Phi}_{2^n}(\xi) := \Phi_{2^n}(\xi)\eta\big(2^{n(A-\chi I)}\xi\big), \quad \xi \in \mathbb{T}^\Gamma.$$

In view of (4.89) it is enough to show that for every $p \in (1, \infty)$ one has

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\Big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\sum_{b\in\mathbb{Z}^\Gamma} \widetilde{\Phi}_{2^n}(\xi - b/Q_{s^u})\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : n \in \mathbb{N}_0, n > 2^{\kappa_s}\Big) \lesssim \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.90}$$

and

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Pi_s \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.91}$$

In the case of the oscillation and r-variational seminorm, by Proposition 2.70, the inequality (4.90) follows from the continuous counterpart

$$\mathcal{S}^p_{\mathbb{R}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}(\widetilde{\Phi}_{2^n}\mathcal{F}_{\mathbb{R}^\Gamma} f\big) : n \in \mathbb{N}_0, n > 2^{\kappa_s}\big) \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

Indeed, the function $\widetilde{\Phi}_{2^n}$ is supported on the cube $Q^{-1}_{s^u}[-1/2, 1/2]^\Gamma$, because for $n > 2^{\kappa_s}$ one has $2^{n(|\gamma|-\chi)} \geq 4Q_{s^u}$. In fact, the same type of the limiting-quotient argument which is presented during the proof (3.68), should also be used, as to apply Proposition 2.70 one need a finite dimensional Banach spaces. We omit the details. On the other hand, the estimate

$$\mathcal{S}^p_{\mathbb{R}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}(\widetilde{\Phi}_{2^n}\mathcal{F}_{\mathbb{R}^\Gamma} f\big) : n \in \mathbb{N}_0, n > 2^{\kappa_s}\big) \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.92}$$

follows by the seminorm estimates for the continuous Radon averages $\mathcal{M}_t$,

$$\mathcal{S}^p_{\mathbb{R}^\Gamma}\big(\mathcal{M}_{2^n} f : n \in \mathbb{N}_0, n > 2^{\kappa_s}\big) \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.93}$$

since the error term is estimated by

$$\sum_{n=0}^{\infty}\big\|\mathcal{F}^{-1}_{\mathbb{R}^\Gamma}\big(\Phi_{2^n}\big(1 - \eta(2^{n(A-\chi I)}\cdot)\big)\mathcal{F}_{\mathbb{R}^\Gamma} f\big)\big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim \|f\|_{L^p(\mathbb{R}^\Gamma)}. \tag{4.94}$$

For the proof of (4.94) put $\tau = 1$ in (3.72). The inequality (4.93) in the case when $\mathcal{S}^p$ is the oscillation seminorm was proven in Theorem 3.4 and when $\mathcal{S}^p$ is the r-variation seminorm this was proven by Mirek, Stein and Trojan [40, Theorem A].

In the case of the jump quasi-seminorm in order to deduce the discrete inequality (4.90) from the continuous one (4.92) we use [41, Theorem 1.3] which is counterpart of the Magyar–Stein–Wainger sampling principle in the context of the jump inequality. The jump inequality for $\mathcal{M}_t$ was proven in Theorem 4.25. This ends the proof of (4.90).

Now we show (4.91). At first, for $p = 2$, we see that by (3.40) one has

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Pi_s \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-u\delta}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.95}$$

For $p \neq 2$ let us define a new multiplier

$$\mu_{J,s}(\xi) := m_J(\xi) \sum_{a/q\in\Sigma_{s^u}} \widetilde{\eta}^2\big(2^{s(A-\chi I)}(\xi - a/q)\big)$$

where $J = \lfloor e^{(s+1)^{1/2}} \rfloor$ and m_J is the multiplier given by (2.59). By Theorem 2.71 we get

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\mu_{J,s}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.96}$$

If $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-s(|\gamma|-\chi)}$ for any $\gamma \in \Gamma$, then by Proposition 3.41

$$m_J(\xi) = G(a,q)\Phi_J(\xi - a/q) + \mathcal{O}(e^{-1/2(s+1)^{1/2}}) \tag{4.97}$$

which can be shown in the same way as (4.84) was. Observe that one may write

$$|\Pi_s(\xi) - \mu_{J,s}(\xi)| \lesssim e^{-1/2(s+1)^{1/2}},$$

since by the mean value theorem

$$|1 - \Phi_J(\xi - a/q)| \lesssim |J^A(\xi - a/q)|_\infty \lesssim e^{-1/2(s+1)^{1/2}}$$

and hence, by Plancherel's theorem,

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((\Pi_s - \mu_{J,s})\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim e^{-1/2(s+1)^{1/2}} \|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{4.98}$$

Moreover, for any $p \in (1,\infty)$, by property (i) from Theorem 2.71 one has

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((\Pi_s - \mu_{J,s})\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim e^{(|\Gamma|+1)(s+1)^{1/10}} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.99}$$

Interpolating (4.98) with (4.99) leads to

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((\Pi_s - \mu_{J,s})\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

This, together with (4.96), gives

$$\big\|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Pi_s \mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.100}$$

For $u > \alpha\delta^{-1}$, by interpolating the above inequality for $p = p_0$ with (4.95), we receive (4.91) which ends the proof of (4.91) and consequently the estimates for the large scales.

Small scales and the discrete Littlewood–Paley theory: estimates for (4.78) and (4.79)

We begin with writing inequalities (4.78) and (4.79) in a more convenient form, namely

$$\mathcal{S}^p_{\mathbb{Z}^\Gamma}\big(\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(m_{2^n}\Xi^s_n \mathcal{F}_{\mathbb{Z}^\Gamma} f) : n \in [s,N] \cap \mathbb{N}_0, n \le 2^{\kappa_s+1}\big) \lesssim B_p(N)(s+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.101}$$

where for $p \in (1,2]$ the constant $B_p(N) = C_p(N)^{\beta(p)}$ and for $p \in (2,\infty)$ we have $B_p(N) = C_{p'}(N)^{\beta'(p)}$ with $\beta(p), \beta'(p) \in [0,1)$. Next, we apply the Rademacher–Menshov inequality (2.38) and estimate

$$\mathrm{LHS}(4.101) \lesssim \sum_{i=0}^{\kappa_s+1} \bigg\|\bigg(\sum_{j=0}^{2^{\kappa_s+1-i}-1}\Big|\sum_{n\in I^i_j} \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((m_{2^{n+1}}\Xi^s_{n+1} - m_{2^n}\Xi^s_n)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big|^2\bigg)^{1/2}\bigg\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.102}$$

where $I^i_j = [j2^i, (j+1)2^i) \cap [s, \min\{N, 2^{\kappa+1}\}) \cap \mathbb{N}$ since the inner sum telescopes. Now, by triangle's inequality one has

$$\mathrm{RHS}(4.102) \le \sum_{i=0}^{\kappa_s+1} \bigg\|\bigg(\sum_j\Big|\sum_{n\in I^i_j} \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((m_{2^{n+1}} - m_{2^n})\Xi^s_n \mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big|^2\bigg)^{1/2}\bigg\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.103}$$

$$+ \sum_{i=0}^{\kappa_s+1} \bigg\|\bigg(\sum_j\Big|\sum_{n\in I^i_j} \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(m_{2^{n+1}}(\Xi^s_{n+1} - \Xi^s_n)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big|^2\bigg)^{1/2}\bigg\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.104}$$

Here and later on we will omit the limits of summation in j for the sake of clarity. Now we invoke Khintchine's inequality [20, Apendix C] to (4.103) and (4.104) and as a consequence we see that the estimate (4.101) will follow if we show that inequalities

$$\Big\|\sum_{j}\sum_{n\in I_j^i}\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_j(m_{2^{n+1}}-m_{2^n})\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim B_p(N)(s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.105}$$

$$\Big\|\sum_{j}\sum_{n\in I_j^i}\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_j m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim (s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.106}$$

hold for every $i\le\kappa_s+1$ and any sequence $(\varepsilon_j\colon j\le 2^{\kappa_s+1-i}-1)\subseteq\{-1,1\}$. Finally, for estimates (4.105) and (4.106) it is enough to show that for any interval $I\subseteq[s,\min\{N,2^{\kappa_s+1}\})\cap\mathbb{N}$ and for any sequence $(\varepsilon_n\colon n\in I)\subseteq\{-1,1\}$ one has

$$\Big\|\sum_{n\in I}\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n(m_{2^{n+1}}-m_{2^n})\Xi_n^s\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim B_p(N)(s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.107}$$

$$\Big\|\sum_{n\in I}\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim (s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.108}$$

At first we will prove estimate (4.108) since it is relatively easy. By triangle's inequality it is enough to establish

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim (s+1)^{-5}(n+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.109}$$

and then (4.108) will follow. For any $p\in(1,\infty)$ by Theorem 2.71 we have

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}\lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.110}$$

Again, in the case of $p=2$ we will approximate appropriate multiplier to get a more precise estimate. By Proposition 3.41 (compare with (4.84), the only difference is the error term which is a consequence of the inequality $2^{-n/2}\lesssim 2^{-(n+s)/4}$ since $n\ge s$) one has

$$m_{2^{n+1}}(\xi)=G(a/q)\Phi_{2^{n+1}}(\xi-a/q)+\mathcal{O}(2^{-(n+s)/4}), \tag{4.111}$$

where a/q is the rational approximation of ξ such that for every $\gamma\in\Gamma$ holds $|\xi_\gamma-a_\gamma/q|\lesssim 2^{-n(|\gamma|-\chi)}$. Next, we note that the expression

$$\eta^2(2^{(n+1)(A-\chi I)}(\xi-a/q))-\eta^2(2^{n(A-\chi I)}(\xi-a/q))$$

is nonzero only for ξ such that $|2^{(n+1)(A-\chi I)}(\xi-a/q)|_\infty\gtrsim 1$ and $|2^{n(A-\chi I)}(\xi-a/q)|_\infty\lesssim 1$. Hence, by the estimate (3.40) and by the van der Corput estimate in (2.64) we have that $|G(a/q)\Phi_{2^{n+1}}(\xi-a/q)|\lesssim (s+1)^{-u\delta}2^{-n\chi/|\Gamma|}$. Consequently, one has

$$|m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)(\xi)|\lesssim (s+1)^{-u\delta}2^{-n\chi/|\Gamma|}+2^{-(n+s)/4}\lesssim (s+1)^{-\alpha}2^{-n\chi/|\Gamma|}$$

provided that $u>\alpha\delta^{-1}$. Therefore, by Plancherel's theorem

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_{2^{n+1}}(\Xi_{n+1}^s-\Xi_n^s)\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big\|_{\ell^2(\mathbb{Z}^\Gamma)}\lesssim (s+1)^{-\alpha}2^{-n\chi/|\Gamma|}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

Interpolating the above inequality with (4.110) for $p=p_0$ yields (4.109).

Now we focus our attention on the proof of the estimate (4.107). For this purpose we introduce new multipliers of the form

$$\Xi_n^{s,j}(\xi):=\sum_{a/q\in\Sigma_{s^u}}\eta^2\big(2^{nA+jI}(\xi-a/q)\big)\tilde{\eta}^2\big(2^{s(A-\chi I)}(\xi-a/q)\big),\quad j\in\mathbb{Z}.$$

We have the following decomposition

$$\Xi_n^s(\xi) = \sum_{-\lfloor \chi n \rfloor \le j < n} \big(\Xi_n^{s,j}(\xi) - \Xi_n^{s,j+1}(\xi)\big) + \big(\Xi_n^{s,-\chi n}(\xi) - \Xi_n^{s,-\lfloor \chi n \rfloor}(\xi)\big) + \Xi_n^{s,n}(\xi),$$

since the sum above telescopes. By using the new multipliers one may write

$$\begin{aligned}\mathrm{LHS}(4.107) \le \Big\| \sum_{n\in I} \sum_{-\lfloor \chi n \rfloor \le j < n} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n (m_{2^{n+1}} - m_{2^n})(\Xi_n^{s,j} - \Xi_n^{s,j+1})\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ + \Big\| \sum_{n\in I} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n (m_{2^{n+1}} - m_{2^n})\big((\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor \chi n \rfloor}) + \Xi_n^{s,n}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}.\end{aligned}$$

Consequently, to obtain (4.107) it is enough to show two inequalities:

$$\Big\| \sum_{n\in I} \sum_{-\lfloor \chi n \rfloor \le j < n} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n (m_{2^{n+1}} - m_{2^n})(\Xi_n^{s,j} - \Xi_n^{s,j+1})\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim B_p(N)(s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.112}$$

and

$$\Big\| \sum_{n\in I} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n (m_{2^{n+1}} - m_{2^n})\big((\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor \chi n \rfloor}) + \Xi_n^{s,n}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-5}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.113}$$

We start with showing that (4.113) holds. By triangle's inequality it will follow from

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^{n+1}} - m_{2^n})\big((\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor \chi n \rfloor}) + \Xi_n^{s,n}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-5}(n+1)^{-3}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.114}$$

For any $p \in (1,\infty)$ by Theorem 2.71 one has

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^{n+1}} - m_{2^n})\big((\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor \chi n \rfloor}) + \Xi_n^{s,n}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.115}$$

Again, in the case of $p = 2$ we have much better estimate. Let us denote

$$\psi_n^{a/q}(\xi) := \eta^2\big(2^{n(A-\chi I)}(\xi - a/q)\big) - \eta^2\big(2^{nA - \lfloor \chi n \rfloor I}(\xi - a/q)\big).$$

Observe that $\psi_n^{a/q} \ne 0$ for ξ such that $|2^{nA-\lfloor \chi n \rfloor I}(\xi - a/q)|_\infty \gtrsim 1$ and $|2^{n(A-\chi I)}(\xi - a/q)|_\infty \lesssim 1$. By using Proposition 3.41 one can show that

$$(m_{2^{n+1}} - m_{2^n})(\xi)\psi_n^{a/q}(\xi) = G(a/q)(\Phi_{2^{n+1}} - \Phi_{2^n})(\xi - a/q)\psi_n(\xi) + \mathcal{O}(2^{-(n+s)/4}), \tag{4.116}$$

where a/q is some rational approximation of ξ such that $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma| - \chi)}$ for every $\gamma \in \Gamma$. Hence, by the van der Corput estimate from (2.64) and by the estimate (3.40) we obtain

$$(m_{2^{n+1}} - m_{2^n})(\xi)(\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor \chi n \rfloor})(\xi) \lesssim (s+1)^{-\delta u}2^{-n\chi/|\Gamma|} + \mathcal{O}(2^{-(n+s)/4}) \lesssim (s+1)^{-\alpha}2^{-n\chi/|\Gamma|}, \tag{4.117}$$

provided that $u > \alpha\delta^{-1}$. Analogously, we have

$$\begin{aligned}(m_{2^{n+1}} - m_{2^n})(\xi)\eta^2\big(2^{n(A+I)}(\xi - a/q)\big) = G(a/q)(\Phi_{2^{n+1}} - \Phi_{2^n})(\xi - a/q)\eta^2\big(2^{n(A+I)}(\xi - a/q)\big) \\ + \mathcal{O}(2^{-(n+s)/4}),\end{aligned}$$

with a/q such that $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|+1)}$ for each $\gamma \in \Gamma$. Observe that, by the first inequality in (2.64) we get $|\Phi_{2^{n+1}} - \Phi_{2^n}| \lesssim |2^{nA}\xi|_\infty \lesssim 2^{-n}$ and by the estimate (3.40) one obtains

$$(m_{2^{n+1}} - m_{2^n})(\xi)\Xi_n^{s,n}(\xi) \lesssim (s+1)^{-\delta u}2^{-n} + \mathcal{O}(2^{-(n+s)/4}) \lesssim (s+1)^{-\alpha}2^{-n\chi/|\Gamma|}, \tag{4.118}$$

provided that $u > \alpha\delta^{-1}$. As a result of (4.117) and (4.118) we may write

$$\big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^{n+1}} - m_{2^n})\big((\Xi_n^{s,-\chi n} - \Xi_n^{s,-\lfloor\chi n\rfloor}) + \Xi_n^{s,n}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-\alpha} 2^{-n\chi/|\Gamma|}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

By interpolating the above with (4.115) for $p = p_0$ we see that (4.114) holds.

Now we may return to (4.112). If we change the order of summation, we see that the left hand side of (4.112) is bounded by

$$\sum_{j\in\mathbb{Z}}\Big\|\sum_{\substack{n\in I,\\ -\lfloor\chi n\rfloor\le j<n}} \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n(m_{2^{n+1}} - m_{2^n})(\Xi_n^{s,j} - \Xi_n^{s,j+1})\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Hence, it is enough to prove that

$$\begin{aligned}\Big\|\sum_{\substack{n\in I,\\ n\ge\max\{-j/\chi,\, j-1\}}} &\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\varepsilon_n(m_{2^{n+1}} - m_{2^n})(\Xi_n^{s,j} - \Xi_n^{s,j+1})\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\ &\lesssim (s+1)^{-5} B_p(N) 2^{-|j|\beta}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}\end{aligned} \tag{4.119}$$

holds for some $\beta = \beta_p > 0$. Remark that one has

$$\eta^2\big(2^{nA+jI}\xi\big) - \eta^2\big(2^{nA+(j+1)I}\xi\big) = \big(\eta^2\big(2^{nA+jI}\xi\big) - \eta^2\big(2^{nA+(j+1)I}\xi\big)\big)\big(\eta\big(2^{nA+(j-1)I}\xi\big) - \eta\big(2^{nA+(j+2)I}\xi\big)\big)$$

and therefore

$$\big(\Xi_n^{s,j} - \Xi_n^{s,j+1}\big)(\xi) = \Delta_{n,s}^{j,1}(\xi)\Delta_{n,s}^{j,2}(\xi),$$

where

$$\Delta_{n,s}^{j,1}(\xi) := \sum_{a/q\in\Sigma_{s^u}}\Big[\eta\Big(2^{nA+(j-1)I}(\xi - a/q)\Big) - \eta\Big(2^{nA+(j+2)I}(\xi - a/q)\Big)\Big]\widetilde{\eta}(2^{s(A-\chi I)}(\xi - a/q)),$$

$$\Delta_{n,s}^{j,2}(\xi) := \sum_{a/q\in\Sigma_{s^u}}\Big[\eta^2\big(2^{nA+jI}(\xi - a/q)\big) - \eta^2\big(2^{nA+(j+1)I}(\xi - a/q)\big)\Big]\widetilde{\eta}(2^{s(A-\chi I)}(\xi - a/q)).$$

Now we will derive from the discrete Littlewood–Paley theory which originates in [37, Theorem 3.3]. Let $j, n \in \mathbb{Z}$ and let $\Phi_{j,n}(\xi) = \Phi(2^{nA+jI}\xi)$, where Φ is a Schwartz function such that $\Phi(0) = 0$. Observe that one has

$$|\Phi_{n,j}(\xi)| \lesssim \min\{|2^{nA+jI}\xi|, |2^{nA+jI}\xi|^{-1}\}.$$

Moreover, for any $p \in (1,\infty)$ there is a constant $C_p > 0$ such that

$$\big\|\sup_{n\in\mathbb{Z}}|\mathcal{F}_{\mathbb{R}^\Gamma}^{-1}(|\Phi_{n,j}|\mathcal{F}_{\mathbb{R}^\Gamma} f)|\big\|_{L^p(\mathbb{R}^\Gamma)} \le C_p\|f\|_{L^p(\mathbb{R}^\Gamma)}.$$

Hence, by [18, Theorem B] for any $-\infty \le M_1 \le M_2 \le \infty$ we have

$$\Big\|\Big(\sum_{M_1\le n\le M_2}\big|\mathcal{F}_{\mathbb{R}^\Gamma}^{-1}(\Phi_{j,n}\mathcal{F}_{\mathbb{R}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^\Gamma)} \lesssim_p \|f\|_{L^p(\mathbb{R}^\Gamma)},$$

where the implied constant is independent of j, M_1 and M_2. Therefore, by Theorem 2.71 the multiplier

$$\Omega_N^{j,n}(\xi) := \sum_{a/q\in\Sigma_{s^u}}\Phi_{j,n}(\xi - a/q)\widetilde{\eta}(2^{s(A-\chi I)}(\xi - a/q)) \tag{4.120}$$

satisfy

$$\Big\|\Big(\sum_{M_1\le n\le M_2}\big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Omega_N^{j,n}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.121}$$

Moreover, if Φ is a real valued function, then the dual version of the inequality (4.121) also holds, namely

$$\Big\| \sum_{M_1 \le n \le M_2} \big| \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Omega^{j,n}_N \mathcal{F}_{\mathbb{Z}^\Gamma} f_n) \big| \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \log(s+1) \Big\| \Big(\sum_{M_1 \le n \le M_2} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.122}$$

where $(f_n \colon M_1 \le n \le M_2)$ is a sequence of functions such that

$$\Big\| \Big(\sum_{M_1 \le n \le M_2} |f_n|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} < \infty.$$

It is easy to see that multipliers $\Delta^{j,1}_{n,s}$ and $\Delta^{j,2}_{n,s}$ can be written as (4.120). Hence, by applying the inequality (4.122) to the multiplier $\Delta^{j,1}_{n,s}$ we get

$$\begin{aligned} \mathrm{LHS}(4.119) &= \Big\| \sum_{\substack{n \in I, \\ n \ge \max\{-j/\chi, j-1\}}} \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\varepsilon_n \Delta^{j,1}_{n,s} (m_{2^{n+1}} - m_{2^n}) \Delta^{j,2}_{n,s} \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &\lesssim \log(s+1) \Big\| \Big(\sum_{\substack{n \in I, \\ n \ge \max\{-j/\chi, j-1\}}} \big| \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((m_{2^{n+1}} - m_{2^n}) \Delta^{j,2}_{n,s} \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) \big|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \end{aligned}$$

Consequently, the estimate (4.119) will follow if we prove that

$$\begin{aligned} &\Big\| \Big(\sum_{\substack{n \in I, \\ n \ge \max\{-j/\chi, j-1\}}} \big| \mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big((m_{2^{n+1}} - m_{2^n}) \Delta^{j,2}_{n,s} \mathcal{F}_{\mathbb{Z}^\Gamma} f\big) \big|^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ &\qquad\qquad \lesssim (s+1)^{-10} B_p(N) 2^{-|j|\beta} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \end{aligned} \tag{4.123}$$

for any $p \in (1, \infty)$.

Bootstrap estimates for the square function in (4.123)

We start with proving some estimates in the case of $p = 2$. For simplicity, we denote

$$\psi^{a/q,s}_{n,j}(\xi) := \big[\eta^2\big(2^{nA+jI}(\xi - a/q)\big) - \eta^2\big(2^{nA+(j+1)I}(\xi - a/q)\big)\big] \tilde{\eta}\big(2^{s(A-\chi I)}(\xi - a/q)\big).$$

Observe that $\psi^{a/q,s}_{n,j}$ is nonzero only if $|\xi_\gamma - a_\gamma/q| \le 2^{-(n|\gamma|+j)} \le 2^{-n(|\gamma|-\chi)}$ for $\gamma \in \Gamma$ since $n \ge -j/\chi$. By Proposition 3.41 we have

$$m_{2^n}(\xi) = G(a,q)\Phi_{2^n}(\xi - a/q) + \mathcal{O}(2^{-n/2}), \tag{4.124}$$

where a/q satisfy $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|-\chi)}$ for every $\gamma \in \Gamma$. By estimates from (2.64) one has

$$|\Phi_{2^{n+1}}(\xi) - \Phi_{2^n}(\xi)| \lesssim \min\{|2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/|\Gamma|}\}. \tag{4.125}$$

To simplify notation we denote $w_n(\xi) := \min\{|2^{nA}\xi|_\infty, |2^{nA}\xi|_\infty^{-1/|\Gamma|}\}$. By using estimates (4.124), (4.125) and (3.40) we conclude

$$|(m_{2^{n+1}} - m_{2^n})(\xi)| \lesssim q^{-\delta} w_n(\xi - a/q) + \mathcal{O}(2^{-n/2}).$$

In a similar spirit one obtains

$$|(m_{2^{n+1}} - m_{2^n})(\xi)| \psi^{a/q,s}_{n,j}(\xi) \lesssim q^{-\delta} 2^{-|j|/|\Gamma|} + \mathcal{O}(2^{-n/2}),$$

since the function $\psi_{n,j}^{a/q,s}$ is nonzero only if $2^{-(j+2)} \lesssim |2^{nA}(\xi - a/q)|_\infty \lesssim 2^{-j}$. Finally, because one has $q > s^u$ and $n \geq s$ we can write

$$|(m_{2^{n+1}} - m_{2^n})(\xi)| \lesssim (s+1)^{-u\delta} w_n(\xi - a/q) + (s+1)^{-u\delta}\mathcal{O}(2^{-n/4})$$
$$|(m_{2^{n+1}} - m_{2^n})(\xi)|\psi_{n,j}^{a/q,s}(\xi) \lesssim (s+1)^{-u\delta}2^{-|j|/d} + (s+1)^{-u\delta}\mathcal{O}(2^{-n/4}).$$

Hence, by using the above estimates we obtain

$$\begin{aligned}&\sum_{\substack{n\in I,\\ n\geq \max\{-j/\chi,\, j-1\}}} \sum_{a/q\in\Sigma_{s^u}} |(m_{2^{n+1}} - m_{2^n})(\xi)\psi_{n,j}^{a/q,s}(\xi)|^2\\ &\lesssim \sum_{a/q\in\Sigma_{s^u}} \sum_{\substack{n\in I,\\ n\geq \max\{-j/\chi,\, j-1\}}} (s+1)^{-2u\delta}(w_n(\xi-a/q)+2^{-n/4})(2^{-|j|/d}+2^{-n/4})\tilde{\eta}^2\big(2^{s(A-\chi I)}(\xi-a/q)\big)\\ &\lesssim (s+1)^{-2u\delta}2^{-|j|\beta},\end{aligned}$$

for some $\beta > 0$, since

$$\sum_{n\geq 0}(w_n(\xi-a/q)+2^{-n/4}) \lesssim 1 \quad \text{and} \quad \sum_{a/q\in\Sigma_{s^u}} \tilde{\eta}^2\big(2^{s(A-\chi I)}(\xi-a/q)\big) \lesssim 1.$$

Hence, by Plancherel's theorem

$$\Big\|\Big(\sum_{\substack{n\in I,\\ n\geq \max\{-j/\chi,\, j-1\}}} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^{n+1}} - m_{2^n})\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma}^{f}\big)\big|^2\Big)^{1/2}\Big\|_{\ell^2(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-\delta u}2^{-|j|\beta/2}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}. \tag{4.126}$$

Let us note that if $\delta u > 10$ the above estimate together with the estimate for the large scales (4.77) proves that $C_2(N) < \infty$ for any $N \in \mathbb{N}$ so we have proven estimate (4.68) in the case of $p = 2$. In order to handle other values of p we make use of Lemma 4.43. Here we have to make some distinction between the jump quasi-seminorm and other seminorms.

Case of the oscillation and the r-variational seminorm. At first let $p \in (1, 2)$. We will apply Lemma 4.43 with the set $\mathbb{J} := \big\{n \in \mathbb{N}_0 \colon n \in I, n \geq \max\{-j/\chi,\, j-1\}\big\} \subseteq [0, N)$, parameters $q_0 = 1$, $q_1 = p$, $\vartheta = 1/2$, operators $B_n = M_{2^{n+1}} - M_{2^n}$ and functions $g_n = \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)$. Now, since the norm of the operator M_{2^n} is uniformly bounded we see that for every $q \in (1, \infty)$ one has

$$\sup_{n\leq N}\|B_n\|_{\ell^q\to\ell^q} \lesssim 1.$$

If $\mathcal{S}_{\mathbb{Z}^\Gamma}^p$ is the oscillation seminorm,

$$\mathcal{S}_p(M_{2^n}f : n \in [0, N] \cap \mathbb{N}_0) = \sup_{K\in\mathbb{N}}\sup_{I\in\mathfrak{S}_K([0,N]\cap\mathbb{N}_0)} \big\|O_{I,K}^2(M_{2^n}f : n \in [0, N] \cap \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

then by Proposition 2.7 it is easy to check that for every $q \in (1, \infty)$ we have

$$\|B_{*,\mathbb{J}}\|_{\ell^q\to\ell^q} \lesssim C_q(N),$$

where

$$B_{*,\mathbb{J}}f := \sup_{n\leq N}\sup_{|g|\leq|f|}\big|(M_{2^{n+1}} - M_{2^n})g\big|.$$

In the case of r-variational seminorm,

$$\mathcal{S}_p(M_{2^n}f : n \in [0, N] \cap \mathbb{N}_0) = \big\|V^r(M_{2^n}f : n \in [0, N] \cap \mathbb{N}_0)\big\|_{\ell^p(\mathbb{Z}^\Gamma)},$$

by inequality (2.10) we see that for any $r \in (2, \infty)$ the estimate

$$\|B_{*,n<N}\|_{\ell^q \to \ell^q} \lesssim C_q(N)$$

also holds for $q \in (1, \infty)$. Hence, in the case of the oscillation and r-variational seminorm by Lemma 4.43 we may write

$$\begin{aligned}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^{n+1}}-m_{2^n})\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}& \\ \lesssim C_p(N)^{1/2}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}& \\ \lesssim C_p(N)^{1/2}\log(s+1)\|f\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)},&\end{aligned} \tag{4.127}$$

where in the last inequality we have used (4.121). Since $q_{1/2} < p < 2$, there exists $t \in (0, 1)$ such that $\frac{1}{p} = \frac{t}{q_{1/2}} + \frac{1-t}{2}$. If we use definition of $q_{1/2}$ from Lemma 4.43 we see that

$$t = 2 - p.$$

Hence, by interpolating (4.126) with (4.127) one has

$$\begin{aligned}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^{n+1}}-m_{2^n})\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{p}(\mathbb{Z}^\Gamma)}& \\ \lesssim (s+1)^{-u\delta(1-t)}2^{-|j|\frac{\beta}{2}(1-t)}C_p(N)^{(2-p)/2}\log(s+1)\|f\|_{\ell^{p}(\mathbb{Z}^\Gamma)}.&\end{aligned}$$

Since $u \in \mathbb{N}$ can be large, we get that (4.123) is satisfied with $B_p(N) = C_p(N)^{(2-p)/2}$. Hence, we see that for $p \in (1, 2)$ the inequality (4.78) holds with $\beta(p) := \frac{2-p}{2} \in [0, 1)$.

Now let us assume that $p \in (2, \infty)$. Then one has $p' \in (1, 2)$ and therefore by applying Lemma 4.43 with $q_0 = 1$, $q_1 = p'$ and $\vartheta = 1/2$ we obtain

$$\begin{aligned}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^{n+1}}-m_{2^n})\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}& \\ \lesssim C_{p'}(N)^{1/2}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}(\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}& \\ \lesssim C_{p'}(N)^{1/2}\log(s+1)\|f\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)},&\end{aligned} \tag{4.128}$$

where $q_{1/2} = 2p/(2p-1)$. Now, since $B_n = M_{2^{n+1}} - M_{2^n}$ is a convolution operator we see that by duality the inequality (4.128) holds for $q'_{1/2} = 2p$. Since $2 < p < q'_{1/2}$ there exists $\tau \in [0, 1)$ such that $\frac{1}{p} = \frac{\tau}{q'_{1/2}} + \frac{1-\tau}{2}$ and

$$\tau = \frac{2-p}{1-p}.$$

Hence, by interpolating (4.126) with (4.128) for $q'_{1/2}$ we may write

$$\begin{aligned}\Big\|\Big(\sum_{n\in\mathbb{J}}\big|\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}((m_{2^{n+1}}-m_{2^n})\Delta^{j,2}_{n,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f)\big|^2\Big)^{1/2}\Big\|_{\ell^{p}(\mathbb{Z}^\Gamma)}& \\ \lesssim (s+1)^{-u\delta(1-\tau)}2^{-|j|\frac{\beta}{2}(1-\tau)}C_{p'}(N)^{\frac{2-p}{2(1-p)}}\log(s+1)\|f\|_{\ell^{p}(\mathbb{Z}^\Gamma)}.&\end{aligned}$$

Since $u \in \mathbb{N}$ can be large, we get that (4.123) is satisfied with $B_p(N) = C_{p'}(N)^{\frac{2-p}{2(1-p)}}$. Hence, we see that for $p \in (2, \infty)$ the inequality (4.79) holds with $\beta'(p) := \frac{2-p}{2(1-p)} = \frac{2-p'}{2} \in [0, 1)$.

Case of the jump quasi-seminorm In the context of the jump quasi-seminorm we need to proceed in a slightly different way since in this case we do not have a pointwise estimate of the form (2.10) or even an ℓ^p-estimate like in Proposition 2.7. Fortunately, for $r > 2$ one has "weak ℓ^p"-estimate (2.23) for the r-variation which we will use at this moment. As mentioned before we have already proved that $C_2(N) \lesssim 1$. Hence, we may assume that $p \in (1, 2)$. Let us consider $\lambda \in (0, 1)$ such that

$$\lambda > \max\Big\{0, \frac{4-3p}{(p-2)^2}\Big\}. \tag{4.129}$$

We are going to apply Lemma 4.43 with parameters $q_0 = 1$, $q_1 = \lambda p + (1-\lambda)2$, $\vartheta = 1/2$, operators $B_n = M_{2^{n+1}} - M_{2^n}$ and functions $g_n = \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)$. If λ satisfy condition (4.129), then one has $q_{1/2} < p < q_1 < 2$. Furthermore, for any $q \in (1, \infty)$

$$\sup_{n<N} \|B_n\|_{\ell^q \to \ell^q} \lesssim 1 \quad \text{and} \quad \|B_{*,\mathbb{J}}\|_{\ell^q \to \ell^q} \lesssim \|V^3(M_{2^n} : n \in [0, N] \cap \mathbb{N}_0)\|_{\ell^q \to \ell^q},$$

where the last inequality follows by (2.10). By using the inequality (2.23) we get weak type estimates

$$\begin{aligned}
\big\|V^3(M_{2^n} f : n \in [0, N] \cap \mathbb{N}_0)\big\|_{\ell^{2,\infty}(\mathbb{Z}^\Gamma)} &\lesssim \|f\|_{\ell^2(\mathbb{Z}^\Gamma)},\\
\big\|V^3(M_{2^n} f : n \in [0, N] \cap \mathbb{N}_0)\big\|_{\ell^{p,\infty}(\mathbb{Z}^\Gamma)} &\lesssim C_p(N)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.
\end{aligned}$$

Since $p < q_1 < 2$, one may use Marcinkiewicz's interpolation theorem to get

$$\|B_{*,\mathbb{J}}\|_{\ell^{q_1} \to \ell^{q_1}} \lesssim_p C_p(N)^{\frac{p(2-q_1)}{q_1(2-p)}}.$$

Therefore,

$$\begin{aligned}
&\Big\|\Big(\sum_{n\in\mathbb{J}} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((m_{2^{n+1}} - m_{2^n})\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}\\
&\qquad\lesssim C_p(N)^{\frac{p(2-q_1)}{2q_1(2-p)}} \Big\|\Big(\sum_{n\in\mathbb{J}} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}\\
&\qquad\lesssim C_p(N)^{\frac{p(2-q_1)}{2q_1(2-p)}} \log(s+1)\|f\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)},
\end{aligned} \tag{4.130}$$

where the last inequality again follows by (4.121). Since $q_{1/2} < p < 2$, there exists $t \in (0, 1)$ such that $\frac{1}{p} = \frac{t}{q_{1/2}} + \frac{1-t}{2}$. Hence, by the definition of $q_{1/2}$ one has

$$t\frac{p(2-q_1)}{2q_1(2-p)} = \frac{2-q_1}{2}.$$

Interpolating (4.126) with (4.130) leads to

$$\begin{aligned}
&\Big\|\Big(\sum_{n\in\mathbb{J}} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((m_{2^{n+1}} - m_{2^n})\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\
&\qquad\lesssim (s+1)^{-u\delta(1-t)} 2^{-|j|\beta(1-t)/2} C_p(N)^{\frac{2-q_1}{2}} \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.
\end{aligned}$$

Since $u \in \mathbb{N}$ can be large, we get that (4.123) in the case of the jump quasi-seminorm is satisfied with $B_p(N) = C_p(N)^{(2-q_1)/2}$. Hence, we see that for $p \in (1, 2]$ the inequality (4.78) holds with $\beta(p) = \frac{2-q_1}{2} \in [0, 1)$.

In the case of $p \in (2, \infty)$ we again use the duality to obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{n\in\mathbb{J}} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((m_{2^{n+1}} - m_{2^n})\Delta_{n,s}^{j,2}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)}\\
&\qquad\lesssim (s+1)^{-u\delta(1-\tau)} 2^{-|j|\frac{\beta}{2}(1-\tau)} C_{p'}(N)^{\frac{2-p}{2(1-p)}} \log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.
\end{aligned}$$

Since $u \in \mathbb{N}$ can be large, we get that (4.123) in the case of the jump quasi-seminorm is satisfied with $B_p(N) = C_{p'}(N)^{\frac{2-p}{2(1-p)}}$. As a consequence, for $p \in (2, \infty)$ the inequality (4.79) holds with $\beta'(p) := \frac{2-p}{2(1-p)} = \frac{2-p'}{2} \in [0, 1)$.

4.3.2 Estimates for short variations

Assume that $p \in (1, \infty)$ and let $f \in \ell^p(\mathbb{Z}^\Gamma)$ be a compactly supported function. In this section we focus on bounding the short variations, namely we want to establish the following estimate

$$\Big\| \Big(\sum_{n=0}^{\infty} V^2\big(M_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim_{\mathcal{S}_p} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.131}$$

For this purpose, for $N \in \mathbb{N}$, let us consider the following cut-off short variations

$$\Big(\sum_{n=0}^{N} V^2\big(M_t f : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2}.$$

Let $C_p(N)$ denote the smallest constant $C > 0$ for which the following estimate holds

$$\Big\| \Big(\sum_{n=0}^{N} V^2\big(M_t f : n \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \le C \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \quad f \in \ell^p(\mathbb{Z}^\Gamma). \tag{4.132}$$

By the estimate (2.27) we know that $C_p(N) \lesssim_{N,p} 1$. Using again the bootstrap argument we will show that $C_p(N) \lesssim_p 1$. The proof will proceed in a similar way as in the case of the dyadic scales hence we will omit some details. Without loss of generality we can assume that $C_p(N) > 1$ and $N \in \mathbb{N}$ is large. Let $\chi \in (0, 1/10)$ and let $u \in \mathbb{N}$ be a fixed large number. For each $n \in \mathbb{N}$ we define the following function

$$\Xi_n(\xi) := \sum_{a/q \in \Sigma_{\le n^u}} \eta\big(2^{n(A - \chi I)}(\xi - a/q)\big), \tag{4.133}$$

where η is a bump function of the form (4.70), I is the $|\Gamma| \times |\Gamma|$ identity matrix, A is the matrix (2.63) and $\Sigma_{\le n^u}$ is the set of the Ionescu–Wainger rational fractions related to the parameter $\varrho = (10u)^{-1}$. Recall that we may write $M_t f = \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(m_t \mathcal{F}_{\mathbb{Z}^\Gamma} f)$ where m_t is the multiplier corresponding to M_t given by (2.59). Next, we use functions (4.133) to estimate the left hand side of (4.132) by

$$\Big\| \Big(\sum_{n=0}^{N} V^2\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Xi_n m_t \mathcal{F}_{\mathbb{Z}^\Gamma} f) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.134}$$

$$+ \Big\| \Big(\sum_{n=0}^{N} V^2\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((1 - \Xi_n) m_t \mathcal{F}_{\mathbb{Z}^\Gamma} f) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.135}$$

Similar to the case of the dyadic scales, the first expression corresponds to the major arcs and the second one to the minor arcs in the Hardy–Littlewood circle method.

Minor arcs

Again, we start with the estimate for the minor arcs since it is relatively easy and follows the same rule as in the case of the dyadic scales. By the triangle inequality it is enough to show

$$\big\| V^2\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((1 - \Xi_n) m_t \mathcal{F}_{\mathbb{Z}^\Gamma} f) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big) \big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

We note that for each $n \in \mathbb{N}_0$ only a finite set of numbers from $[2^n, 2^{n+1}] \cap \mathbb{U}$ give a contribution to the above variational seminorm. Hence it is enough to prove that for some $a(n) \in \mathbb{N}$ we have

$$\big\|V^2\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((1-\Xi_n)m_{t/2^{a(n)}}\mathcal{F}_{\mathbb{Z}^\Gamma} f) : t \in [2^{n+a(n)}, 2^{n+1+a(n)}] \cap \mathbb{N}_0\big)\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.136}$$

In order to prove the above inequality we make use of the following.

Proposition 4.137 ([40, Inequality (2.8)]). *Let $1 \le r \le p$ and $(f_j : j \in \mathbb{N})$ is a sequence of functions in $\ell^p(\mathbb{Z}^\Gamma)$ and $v - u \ge 2$. Then*

$$\big\|V^r(f_j : j \in [u, v])\big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim \max\{\boldsymbol{U}_p, (v-u)^{1/r}\boldsymbol{U}_p^{1-1/r}\boldsymbol{V}_p^{1/r}\} \tag{4.138}$$

where

$$\boldsymbol{U}_p := \max_{u \le j \le v} \|f_j\|_{\ell^p(\mathbb{Z}^\Gamma)} \quad \text{and} \quad \boldsymbol{V}_p := \max_{u \le j \le v} \|f_{j+1} - f_j\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Since the variational norm is non-increasing in r we may replace 2-variation V^2 in (4.136) by V^r where $r = \min\{2, p\}$ and use Proposition 4.137 to estimate the left hand side of (4.136) by

$$\max\{\mathbf{U}_p, 2^{(n+a(n))/r}\mathbf{U}_p^{1-1/r}\mathbf{V}_p^{1/r}\}$$

where

$$\mathbf{U}_p = \max_{2^{n+a(n)} \le t \le 2^{n+a(n)+1}} \big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((1-\Xi_n)m_{t/2^{a(n)}}\mathcal{F}_{\mathbb{Z}^\Gamma} f)\big\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

and

$$\mathbf{V}_p = \max_{2^{n+a(n)} \le t \le 2^{n+a(n)+1}} \big\|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}((m_{(t+1)/2^{a(n)}} - m_{t/2^{a(n)}})(1-\Xi_n)\hat{f})\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

In order to estimate $\mathbf{V}_p$ we make use of Proposition 3.15. It follows that

$$\frac{|(\Omega_{(t+1)/2^{a(n)}} \setminus \Omega_{t/2^{a(n)}}) \cap \mathbb{Z}^k|}{|\Omega_{(t+1)/2^{a(n)}} \cap \mathbb{Z}^k|} \lesssim_\Omega t^{-1} \lesssim 2^{-(n+a(n))},$$

since $t \simeq 2^{n+a(n)}$. Hence, by the above inequality and by using Theorem 2.71 for functions Ξ_n we get that for every $p \in (1,\infty)$ one has

$$\mathbf{V}_p \lesssim 2^{-(n+a(n))}\log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.139}$$

Let us note that for any $p \in (1,\infty)$ by Theorem 2.71 we obtain

$$\mathbf{U}_p \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.140}$$

In the case of $\mathbf{U}_2$ we have a much better estimate. Again we use Theorem 3.31 to bound exponential sums over convex sets. Let

$$\alpha > 10\left(1 - \frac{1}{r}\right)^{-1}\left(\frac{1}{p_0} - \frac{1}{2}\right)\left(\frac{1}{p_0} - \frac{1}{\min\{p, p'\}}\right)^{-1}.$$

One can show, in the same way as in case of the dyadic jumps, that on the minor arcs the conditions (3.32) and (3.33) are satisfied – we omit the proof. Hence by Theorem 3.31 we get

$$\big|m_{t/2^{a(n)}}(\xi)\big| \lesssim_{|\Gamma|,k} \frac{t^k}{2^{ka(n)}|\Omega_{t/2^{a(n)}} \cap \mathbb{Z}^k|}\log(t/2^{a(n)})^{-\alpha} \lesssim_\Omega (n+1)^{-\alpha},$$

since $|\Omega_{t/2^{a(n)}} \cap \mathbb{Z}^k| \simeq_\Omega (t/2^{a(n)})^k$ and $t \simeq 2^{n+a(n)}$. Consequently, by Parseval's theorem we have

$$\mathbf{U}_2 \lesssim (n+1)^{-\alpha}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

Next, by interpolating the above inequality with (4.140) with $p = p_0$ we obtain

$$\mathbf{U}_p \lesssim (n+1)^{-10(1-1/r)^{-1}}\log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

and we see that, together with (4.139), implies

$$\max\{\mathbf{U}_p, 2^{(n+a(n))/r}\mathbf{U}_p^{1-1/r}\mathbf{V}_p^{1/r}\} \lesssim (n+1)^{-2}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$$

which in turn implies (4.136).

Major arcs

Now, our aim is to estimate (4.134). In this case we will follow the approach presented in the proof of the estimate for the long jumps. The case of the short jumps is in some way easier since there is no need to consider small and large scales. In order to estimate (4.134) we introduce new multipliers

$$\Xi_n^j(\xi) = \sum_{a/q \in \Sigma_{\leq n^u}} \eta\big(2^{nA+jI}(\xi - a/q)\big), \quad j \in \mathbb{Z}.$$

Then one may write (compare with Section 4.3.1)

$$\Xi_n(\xi) = \sum_{-\lfloor \chi n \rfloor \leq j < n} \big(\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi)\big) + \big(\Xi_n^{-\chi n}(\xi) - \Xi_n^{-\lfloor \chi n \rfloor}(\xi)\big) + \Xi_n^n(\xi).$$

Next, we use the new multipliers and estimate (4.134) by

$$\Big\| \Big(\sum_{n=0}^{N} V^2 \Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \Big(m_t \big(\sum_{-\lfloor \chi n \rfloor \leq j < n} \Xi_n^j - \Xi_n^{j+1} \big) \mathcal{F}_{\mathbb{Z}^\Gamma} f \Big) : t \in [2^n, 2^{n+1}] \cap \mathbb{U} \Big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \tag{4.141}$$

$$+ \Big\| \Big(\sum_{n=0}^{N} V^2 \Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big((m_t - m_{2^n})(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor} + \Xi_n^n) \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) : t \in [2^n, 2^{n+1}] \cap \mathbb{U} \Big)^2 \Big)^{1/2} \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.142}$$

At first we will show that (4.142)$\lesssim (n+1)^{-2} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}$. We may replace 2-variation V^2 by r-variation V^r where $r = \min\{2, p\}$. Moreover, again we note that for each $n \in \mathbb{N}_0$ only a finite set of numbers from $[2^n, 2^{n+1}] \cap \mathbb{U}$ give a contribution to the above variational seminorm. Hence it is enough to prove that for some $a(n) \in \mathbb{N}$ we have

$$\begin{aligned} \Big\| V^r \Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big((m_{t/2^{a(n)}} - m_{2^n})(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) : t \in [2^n, 2^{n+1}] \cap \mathbb{N}_0 \Big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \\ \lesssim (n+1)^{-2} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \end{aligned} \tag{4.143}$$

and

$$\Big\| V^r \Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big((m_{t/2^{a(n)}} - m_{2^n}) \Xi_n^n \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) : t \in [2^n, 2^{n+1}] \cap \mathbb{N}_0 \Big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (n+1)^{-2} \|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.144}$$

We handle (4.143) and (4.144) simultaneously. We use Proposition 4.137 to obtain that

$$\begin{aligned} \text{LHS}(4.143) &\lesssim \max\{\mathbf{U}_p, 2^{(n+a(n))/r} \mathbf{U}_p^{1-1/r} \mathbf{V}_p^{1/r}\}, \\ \text{LHS}(4.144) &\lesssim \max\{\mathbf{W}_p, 2^{(n+a(n))/r} \mathbf{W}_p^{1-1/r} \mathbf{M}_p^{1/r}\} \end{aligned}$$

where

$$\begin{aligned} \mathbf{U}_p &:= \max_{2^{n+a(n)} \leq t \leq 2^{n+a(n)+1}} \Big\| \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \Big(\big(m_{t/2^{a(n)}} - m_{2^n} \big) (\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor}) \mathcal{F}_{\mathbb{Z}^\Gamma} f \Big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}, \\ \mathbf{V}_p &:= \max_{2^{n+a(n)} \leq t \leq 2^{n+a(n)+1}} \Big\| \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big(\big(m_{(t+1)/2^{a(n)}} - m_{t/2^{a(n)}} \big) \big(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor \chi n \rfloor} \big) \mathcal{F}_{\mathbb{Z}^\Gamma} \Big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \end{aligned}$$

and

$$\begin{aligned} \mathbf{W}_p &:= \max_{2^{n+a(n)} \leq t \leq 2^{n+a(n)+1}} \Big\| \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big((m_{t/2^{a(n)}} - m_{2^n}) \Xi_n^n \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}, \\ \mathbf{M}_p &:= \max_{2^{n+a(n)} \leq t \leq 2^{n+a(n)+1}} \Big\| \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1} \big((m_{(t+1)/2^{a(n)}} - m_{t/2^{a(n)}}) \Xi_n^n \mathcal{F}_{\mathbb{Z}^\Gamma} f \big) \Big\|_{\ell^p(\mathbb{Z}^\Gamma)}. \end{aligned}$$

As before, by Theorem 2.71 we get that for any $p \in (1,\infty)$

$$\mathbf{U}_p \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \quad \text{and} \quad \mathbf{W}_p \lesssim \log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.145}$$

By using Proposition 3.15 and again Theorem 2.71 we obtain that for any $p \in (1,\infty)$ one has

$$\mathbf{V}_p \lesssim 2^{-(n+a(n))}\log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)} \quad \text{and} \quad \mathbf{M}_p \lesssim 2^{-(n+a(n))}\log(n+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

In the case of $p=2$ we will approximate the multiplier $m_{t/a(n)}$ by a suitably chosen integral. Remark, since $t/a(n) \simeq 2^n$ one may write estimates in (2.64) as

$$|\Phi_{t/a(n)}(\xi)| \lesssim |2^{nA}\xi|_\infty^{-1/|\Gamma|} \quad \text{and} \quad |\Phi_{t/a(n)}(\xi) - 1| \lesssim |2^{nA}\xi|_\infty. \tag{4.146}$$

We start by using Proposition 3.41 with $\Omega_{t/a(n)} \subseteq B(0,2^n)$, $\mathcal{K} = \mathbb{1}_{\Omega_{t/a(n)}}$, and $\xi = a/q = 1$ to obtain

$$\big||\Omega_{t/a(n)} \cap \mathbb{Z}^k| - |\Omega_{t/a(n)}|\big| \lesssim \big(t/a(n)\big)^{(k-1)}. \tag{4.147}$$

Further, we define an auxiliary multiplier

$$\widetilde{m}_{t/a(n)}(\xi) = \frac{1}{|\Omega_{t/a(n)}|}\sum_{y\in\Omega_{t/a(n)}\cap\mathbb{Z}^k} e(\xi\cdot(y)^\Gamma).$$

By (4.147) we have

$$|m_n(\xi) - \widetilde{m}_n(\xi)| \lesssim 2^{-n} \tag{4.148}$$

since $t/a(n) \simeq 2^n$. Next, we use Proposition 3.41 with $\Omega_{t/a(n)} \subseteq B(0,2^n)$, $\mathcal{K} = |\Omega_{t/a(n)}|^{-1}\mathbb{1}_{\Omega_{t/a(n)}}$ and $\varepsilon_\gamma = 1$. Note that $\|\mathcal{K}\|_{L^\infty(\Omega)} \lesssim (t/a(n))^{-k}$ and $\sup_{x,y\in\Omega:\,|x-y|\le q}|\mathcal{K}(x)-\mathcal{K}(y)| = 0$. Therefore, for $t/a(n) \simeq 2^n$ we have

$$\big|\widetilde{m}_{t/a(n)}(\xi) - G(a,q)\Phi_{t/a(n)}(\xi - a/q)\big| \lesssim q\big(t/a(n)\big)^{-1} + \sum_{\gamma\in\Gamma} q|\xi_\gamma - a_\gamma/q|\big(t/a(n)\big)^{(|\gamma|-1)} \lesssim 2^{-n/2} \tag{4.149}$$

for sufficiently small χ, since $q \le e^{n^{1/10}}$ and for any $\gamma \in \Gamma$ we have $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|-\chi)}$. Consequently, by (4.148) and (4.149) we have

$$\big|m_{t/a(n)}(\xi) - G(a,q)\Phi_{t/a(n)}(\xi - a/q)\big| \lesssim 2^{-n/2}, \tag{4.150}$$

Now we are able to estimate the quantities $\mathbf{U}_2$ and $\mathbf{W}_2$. Let

$$\psi_n^{a/q}(\xi) := \eta\big(2^{n(A-\chi I)}(\xi - a/q)\big) - \eta\big(2^{nA-\lfloor\chi n\rfloor I}(\xi - a/q)\big).$$

Observe that this function is nonzero only for ξ such that $|2^{nA-\lfloor\chi n\rfloor I}(\xi - a/q)|_\infty \gtrsim 1$ and $|2^{n(A-\chi I)}(\xi - a/q)|_\infty \lesssim 1$. By using (4.150) we may show that

$$\big(m_{t/2^{a(n)}} - m_{2^n}\big)(\xi)\psi_n^{a/q}(\xi) = G(a,q)\big(\Phi_{t/2^{a(n)}} - \Phi_{2^n}\big)(\xi - a/q)\psi_n(\xi) + \mathcal{O}(2^{-n/2}), \tag{4.151}$$

where a/q is some rational approximation of ξ with $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|-\chi)}$ for every $\gamma \in \Gamma$. We see that $\psi_n^{a/q} \neq 0$ when $|2^{nA}\xi|_\infty^{-1} \lesssim 2^{-n\chi}$, so by the first inequality in (4.146) one has

$$\Big|\big(m_{t/2^{a(n)}} - m_{2^n}\big)(\xi)(\Xi_n^{-\chi n} - \Xi_n^{-\lfloor\chi n\rfloor})(\xi)\Big| \lesssim 2^{-n\chi/|\Gamma|} + \mathcal{O}(2^{-n/2}) \lesssim 2^{-n\chi/|\Gamma|}.$$

Analogously, one can show

$$\begin{aligned}\big(m_{t/2^{a(n)}} - m_{2^n}\big)(\xi)\eta\big(2^{n(A+I)}(\xi - a/q)\big) =& G(a,q)\big(\Phi_{t/2^{a(n)}} - \Phi_{2^n}\big)(\xi - a/q)\eta\big(2^{n(A+I)}(\xi - a/q)\big)\\ &+ \mathcal{O}(2^{-n/2})\end{aligned}$$

with a/q such that $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|+1)}$ for $\gamma \in \Gamma$. Next, by the second inequality in (4.146) we get $|\Phi_{t/2^{a(n)}} - \Phi_{2^n}| \lesssim |2^{nA}\xi|_\infty$. Since $|2^{nA}\xi|_\infty \lesssim 2^{-n}$ we obtain

$$(m_n - m_{2^n})(\xi)\Xi_n^n(\xi) \lesssim 2^{-n} + \mathcal{O}(2^{-n/2}) \lesssim 2^{-n\chi/|\Gamma|}.$$

Consequently,

$$\mathbf{U}_2 \lesssim 2^{-n\chi/|\Gamma|}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)} \quad \text{and} \quad \mathbf{W}_2 \lesssim 2^{-n\chi/|\Gamma|}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}.$$

By interpolating the above with suitable inequalities from (4.145) we get that (4.143) and (4.144) hold.

Now let us go back to (4.141). For $p \in (1, 2]$ it is enough to show

$$\Big\|\Big(\sum_{n=0}^{N} V^2\Big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\Big(m_t\big(\sum_{-\lfloor\chi n\rfloor \le j < n} \Xi_n^j - \Xi_n^{j+1}\big)\mathcal{F}_{\mathbb{Z}^\Gamma} f\Big) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\Big)^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim C_p(N)^{\frac{2-p}{2}}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.152}$$

We see that this estimate implies that one has

$$C_p(N) \lesssim_p C_p(N)^{\frac{2-p}{2}}.$$

This gives $C_p(N) \lesssim_p 1$ and thus the proof is complete in the case of $p \in (1, 2]$. For $p \in (2, \infty)$ we will show that

$$\mathrm{LHS}(4.152) \lesssim C_{p'}(N)^{\frac{2-p'}{2}}\|f\|_{L^p(\mathbb{Z}^\Gamma)} \tag{4.153}$$

where $1/p + 1/p' = 1$. This gives $C_p(N) \lesssim_p C_{p'}(N)^{\frac{2-p'}{2}}$ and by the first part we know that $C_{p'} \lesssim_p 1$ which ends the proof when $p \in (2, \infty)$.

Estimates for (4.152) and discrete Littlewood–Paley theory

Now, we take a look at the left hand side of (4.152) in the case of $p \in (1, \infty)$. Let $\widetilde{\eta}(x) := \eta(x/2)$ and define a new multiplier

$$\Delta_{n,s}^j(\xi) := \sum_{a/q \in \Sigma_{s^u}} \big[\eta\big(2^{nA+jI}(\xi - a/q)\big) - \eta\big(2^{nA+(j+1)I}(\xi - a/q)\big)\big]\widetilde{\eta}\big(2^{s(A-\chi I)}(\xi - a/q)\big),$$

where $\Sigma_{s^u} := \Sigma_{\le (s+1)^u} \setminus \Sigma_{\le s^u}$ for $s \in \mathbb{N}$ and $\Sigma_{0^u} := \Sigma_{\le 1}$. We see that

$$\Xi_n^j(\xi) - \Xi_n^{j+1}(\xi) = \sum_{s=0}^{n-1} \Delta_{n,s}^j(\xi). \tag{4.154}$$

Consequently, if we use (4.154) and change the order of summation we see that the estimate (4.152) will follow if we prove that

$$\Big\|\Big(\sum_{\substack{0 \le n \le N, \\ l \ge \max\{-j/\chi, j-1, s-1\}}} V^2\big(\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(m_t\Delta_{n,s}^j\mathcal{F}_{\mathbb{Z}^\Gamma} f\big) : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-2}B_p(L)2^{-|j|\beta}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}, \tag{4.155}$$

for some $\beta = \beta_p > 0$ where for $p \in (1, 2]$ the constant $B_p(N)$ is equal to $C_p(N)^{(2-p)/2}$ and for $p \in (2, \infty)$ we have $B_p(N) = C_{p'}(N)^{(2-p')/2}$. Now, if we apply the Rademacher–Menshov inequality for the short jumps (4.52) we see it is enough to establish

$$\Big\|\Big(\sum_{\substack{0 \le n \le N, \\ n \ge \max\{-j/\chi, j-1, s-1\}}} \sum_{m=0}^{2^i-1} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big((m_{2^n+2^{n-i}(m+1)} - m_{2^n+2^{n-i}m})\Delta_{n,s}^j\mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^p(\mathbb{Z}^\Gamma)} \lesssim (s+1)^{-2}(i+1)^{-2}B_p(L)2^{-|j|\beta}\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}. \tag{4.156}$$

Estimates for square function in (4.156)

For simplicity we denote $B_{n,m} := M_{2^n+2^{n-i}(m+1)} - M_{2^n+2^{n-i}m}$. At first we will prove (4.156) in the case of $p = 2$. For simplicity we will denote

$$\psi_{n,j,s}^{a/q}(\xi) = \big[\eta\big(2^{nA+jI}(\xi - a/q)\big) - \eta\big(2^{nA+(j+1)I}(\xi - a/q)\big)\big]\widetilde{\eta}\big(2^{s(A-\chi I)}(\xi - a/q)\big).$$

Remark that the function $\psi_{n,j,s}^{a/q}$ is nonzero only if $|\xi_\gamma - a_\gamma/q| \le 2^{-(n|\gamma|+j)} \le 2^{-n(|\gamma|-\chi)}$ for $\gamma \in \Gamma$, due to the condition $n \ge -j/\chi$. Now we approximate discrete multipliers by their continuous counterparts by using Proposition 3.41 and

$$(m_{2^n+2^{n-i}(m+1)} - m_{2^n+2^{n-i}m})(\xi)\psi_{n,j,s}^{a/q}(\xi) = G(a/q)\big(\Phi_{2^n+2^{n-i}(m+1)} - \Phi_{2^n+2^{n-i}m}\big)(\xi - a/q) + \mathcal{O}(2^{-n/2}),$$

where a/q is the rational approximation of ξ such that for every $\gamma \in \Gamma$ holds $|\xi_\gamma - a_\gamma/q| \lesssim 2^{-n(|\gamma|-\chi)}$. Remark, that since $2^{-(j+2)} \lesssim |2^{nA}(\xi - a/q)|_\infty \lesssim 2^{-j}$ on the support of $\psi_{l,j,s}^{a/q}$ we can use estimates (4.146) and (3.40) to prove that on the support of $\Delta_{n,s}^j$ we have

$$|m_{2^n+2^{n-i}(m+1)} - m_{2^n+2^{n-i}m}| \lesssim (s+1)^{-u\delta}(2^{-|j|/|\Gamma|} + 2^{-n/4}). \tag{4.157}$$

On the other hand, by Proposition 3.15 (compare with (4.58)) we have

$$|m_{2^n+2^{n-i}(m+1)} - m_{2^n+2^{n-i}m}| \lesssim 2^{-i}.$$

Consequently, one has

$$|m_{2^n+2^{n-i}(m+1)} - m_{2^n+2^{n-i}m}|^2 \lesssim 2^{-3i/2}(s+1)^{-\delta u/2}(2^{-|j|/(2|\Gamma|)} + 2^{-n/8}). \tag{4.158}$$

Let $\mathbb{J} := \{(n,m) \in \mathbb{Z}^2 : n \in [n_0, N], m \in [0, 2^i - 1]\}$ where $n_0 = \max\{-j/\chi,\ j-1,\ s-1\}$. By Parseval's theorem, the inequality (4.158) implies

$$\begin{aligned}
&\Big\|\Big(\sum_{(n,m)\in\mathbb{J}} \big|B_{n,m}\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\Delta_{n,s}^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^2(\mathbb{Z}^\Gamma)} \\
&\qquad\lesssim 2^{-i/4}(s+1)^{-u\delta/4}2^{-|j|\beta}\Big\|\Big(\sum_{n_0\le n\le N} \big|\mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}\big(\Delta_{n,s}^j \mathcal{F}_{\mathbb{Z}^\Gamma} f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^2(\mathbb{Z}^\Gamma)} \\
&\qquad\lesssim 2^{-i/4}(s+1)^{-u\delta/3}2^{-|j|\beta}\|f\|_{\ell^2(\mathbb{Z}^\Gamma)}
\end{aligned} \tag{4.159}$$

where the last inequality follows by (4.121). If $u \in \mathbb{N}$ is large enough then this implies that $C_2(N) \lesssim 1$ which ends the proof in the case of $p = 2$.

For $p \in (1,2)$ we will use the bootstrap Lemma 4.43. We will apply it with parameters $q_0 = 1$, $q_1 = p$, $\vartheta = 1/2$, to a countable set $\mathbb{J}$, the operator $B_{n,m}$ and the functions $g_{n,m} = \mathcal{F}_{\mathbb{Z}^\Gamma}^{-1}(\Delta_{n,s}^j \mathcal{F}_{\mathbb{Z}^\Gamma} f)$. It is easy to check that for every $q \in (1,\infty)$ we have

$$\sup_{(n,m)\in\mathbb{J}} \|B_{n,m}\|_{L^1\to L^1} \lesssim 2^{-i} \quad \text{and} \quad \|B_{*,\mathbb{J}}\|_{L^q\to L^q} \lesssim C_q(N),$$

where

$$B_{*,\mathbb{J}} f := \sup_{(n,m)\in\mathbb{J}} \sup_{|g|\le|f|} \big|(M_{2^n+2^{n-i}(m+1)} - M_{2^n+2^{n-i}m})g\big|.$$

Indeed, the first inequality follows from Proposition 3.15. For the second inequality we observe that for any $m = 0, \ldots, 2^i - 1$ and any $n \le N$ one has the following pointwise estimate

$$|M_{2^n+2^{n-i}(m+1)}f - M_{2^n+2^{n-i}m}f| \lesssim \sup_{n\in\mathbb{Z}} M_{2^n}|f| + \Big(\sum_{n=0}^{N} V^2\big(M_t|f| : t \in [2^n, 2^{n+1}] \cap \mathbb{U}\big)^2\Big)^{1/2}.$$

By results from the previous section we know that for each $q \in (1,\infty)$ the maximal function $\sup_{n\in\mathbb{Z}} M_{2^n}|f|$ is ℓ^q-bounded. Therefore, one has $\|B_{*,\mathbb{J}}\|_{L^q\to L^q} \lesssim C_q(N)$. Hence, by applying Lemma 4.43 and inequality (4.121) we obtain

$$\Big\|\Big(\sum_{(n,m)\in\mathbb{J}}\big|B_{n,m}\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\Delta^j_{n,\,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)} \lesssim C_p(L)^{1/2}\log(s+1)\|f\|_{\ell^{q_{1/2}}(\mathbb{Z}^\Gamma)}. \tag{4.160}$$

Since $q_{1/2} < p \le 2$, we can interpolate (4.159) with (4.160) to get that for $t\in(0,1)$ such that $\frac{1}{p} = \frac{t}{q_{1/2}} + \frac{1-t}{2}$ one has

$$\Big\|\Big(\sum_{(n,m)\in\mathbb{J}}\big|B_{n,m}\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\Delta^j_{n,\,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^{p}(\mathbb{Z}^\Gamma)}$$
$$\lesssim 2^{-i/4(1-t)}(s+1)^{-u\delta(1-t)/3}2^{-(1-t)|j|\beta}C_p(N)^{\frac{2-p}{2}}\log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

Since $u\in\mathbb{N}$ can be large we see that (4.156) holds in the case of $p\in(1,2)$.

When $p\in(2,\infty)$ the desired result follows by duality since $B_{n,m}$ is a convolution operator. Indeed, since $p' < 2$we get that there is $\tau\in(0,1)$ such that $\frac{1}{p} = \frac{\tau}{q'_{1/2}} + \frac{1-\tau}{2}$ for which one has

$$\Big\|\Big(\sum_{(n,m)\in\mathbb{J}}\big|B_{n,m}\mathcal{F}^{-1}_{\mathbb{Z}^\Gamma}\big(\Delta^j_{n,\,s}\mathcal{F}_{\mathbb{Z}^\Gamma}f\big)\big|^2\Big)^{1/2}\Big\|_{\ell^{p}(\mathbb{Z}^\Gamma)}$$
$$\lesssim 2^{-i/4(1-t)}(s+1)^{-u\delta(1-t)/3}2^{-(1-t)|j|\beta}C_{p'}(N)^{\frac{2-p'}{2}}\log(s+1)\|f\|_{\ell^p(\mathbb{Z}^\Gamma)}.$$

If $u\in\mathbb{N}$ is large enough we see that the above bound is summable with $s\in\mathbb{N}$, $i\in\mathbb{N}$ and $j\in\mathbb{Z}$ which shows that (4.153) holds. This ends the proof of the estimates for short variations and therefore the proof of Theorem 4.64.

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