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Report on the doctoral dissertation
Aspects of discrete harmonic analysis
by mgr Wojciech Słomian

Introduction. The doctoral dissertation is concerned with studying bounds on singular integrals and maximal functions. This is one of the central directions in modern harmonic analysis with many applications in differential equations, probability and ergodic theory. The most known classes of such singular integrals are so called Calderón-Zygmund operators given by $\mathcal{H}_{CZ}f(x) := \int_{\mathbb{R}^k} f(x-y)K(y)dy$, where $K : \mathbb{R}^k \to \mathbb{R}$ is a non-integrable function satisfying certain natural conditions. One can also consider a discrete analog of the above operator in which the variable y is in $\mathbb{Z}^k$. These operators where studied by Calderón and Zygmund in their fundamental paper. Related to that one is often interested in bounds for the so called Hardy-Littlewood maximal function given by $\mathcal{M}_{HL}f(x) := \sup_{t>0} \frac{1}{vol(B(0,t))} \int_{B(0,t)} |f(x-y)|dy$ with an analogous definition in the discrete setting. Hardy and Littlewood have shown boundedness in $L^p(\mathbb{R}^k)$ of the maximal function $\mathcal{M}_{HL}$, also boundedness in $L^p(\mathbb{Z}^k)$ of the discrete setting follows from boundedness in the continuous case. The main object studied in the dissertation are singular integrals of Radon type: $\mathcal{H}^{\mathcal{P}}f(x) = p.v. \int_{\mathbb{R}^k} f(x - P(y))K(y)dy$, where $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_d)$ and each $\mathcal{P}_j : \mathbb{R}^k \to \mathbb{R}$ is a polynomial. One analogously defines the discrete Radon operator. The Hardy-Littlewood maximal function is also naturally generalized to this setting. These operators became central in the theory due to the groundbreaking work of J. Bourgain who studied the case of the discrete maximal function with $k = d = 1$ in relation to the Below-Furstenberg problem on pointwise a.e. convergence of ergodic averages $\frac{1}{N}\sum_{n\leq N} f(T^{\mathcal{P}(n)}y)$ for a measure preserving dynamical system (T, X, μ). It is important to point out that in this setting the boundedness of the discrete maximal function can not be deduced from boundedness of the continuous one. Therefore in his work Bourgain developed a set of tools that allowed to study the discrete problem. In particular he introduced so called *oscillation semi-norm* and studied it using the Hardy Littlewood circle method. Let $I = \{I_j : j \in \mathbb{N}\}$ be a strictly increasing sequence in $\mathbb{R}_+$. For $f : \mathbb{R}_+ \to \mathbb{C}$ one defines the oscillation seminorm by

$$O^2_{I,N}(f(t) : t > 0) := \left(\sum_{j=1}^{N} \sup_{I_j<t<I_{j+1}} |f(t) - f(I_j)|^2\right)^{1/2}$$

Let $T^b_N f(x) := \frac{1}{2N+1}\sum_{-N}^{N} f(T^{n^b}x)$. In his work Bourgain showed that for any $\lambda > 1$ and any sequence $I = (I_j)$ with $I_{j+1} > \lambda I_j$ we have

$$\|O^2_{I,N}(T^b_{\lambda^n} f : n \in \mathbb{N})\|_{L^2(X,\mu)} \leq C_{I,\lambda}(N)\|f\|_{L^2(X,\mu)}, \tag{0.1}$$

where $C_{I,\lambda}(N) = (N^{1/2})$. This is enough to give a bound on the maximal function and show pointwise a.e. convergence of the corresponding ergodic averages. Shortly after Bourgain's work it became important to study the dependence of the constant $C_{I,\lambda}(N)$ on the parameters.

The main goal studied in the doctoral dissertation is studying boundedness of operators as in (0.1) with the following generalizations: (a) one considers L^p spaces for $p > 1$, (b) one considers Radon operators (discrete and continuous) instead of only operators (T_N^n) and (c) (uniform bounds) one is interested in showing that the constant on the RHS can be chosen to depend on $p, k, d, deg(\mathcal{P})$ only (and not on I, λ, N). We will now describe main results of the dissertation.

Main results. The dissertation is divided into 4 Chapters. Chapter 1 offers an introduction to the subject and describes main results of the thesis. Chapter 2 introduces some basic tools that are used throughout. In particular it contains a proof of the Calderón transference principle which guarantees that the bounds obtained for the Radon operators will have an implication on bounds for pointwise convergence of the corresponding ergodic averages. Main results of the thesis are contained in Chapters 3 and 4. Results of Chapter 3 are based on two published papers by the author one of them written in collaboration with M. Mirek and T. Z. Szarek and the second one being a solo paper. The main results of this chapter are uniform oscillation inequalities for Radon type operators. More precisely let Ω be a non-empty bounded, open convex set in $\mathbb{R}^k$ which contains a neighborhood of 0. Let Ω_t be a dilation of Ω with scale $t \in \mathbb{R}_+$. Let $\mathcal{P} = (\mathcal{P}_1, \dots, \mathcal{P}_d) : \mathbb{R}^d \to R^k$ where each $\mathcal{P}_j$ is a polynomial. Define

$$M_N^{\mathcal{P}} f(x) = \frac{1}{|\Omega_t \cap \mathbb{Z}^k|} \sum_{n \in \Omega_t \cap \mathbb{Z}^k} f(x - \mathcal{P}(n)) \tag{0.2}$$

with an analogous definition in the continuous case (which we denote by $\mathcal{M}_N^{\mathcal{P}}$). The first main result of Chapter 3 (based on the paper with Mirek and Szarek) is the following uniform bound:

$$\sup_{N \in \mathbb{N}} \sup_{I \in \sigma_N(\mathbb{R}_+)} \|O_{I,N}^2(M_t^{\mathcal{P}} f \ : \ t > 0)\|_{\ell^p(\mathbb{Z}^d)} \le C_{p,d,k,deg\mathcal{P}} \|f\|_{\ell^p(\mathbb{Z}^d)}, \tag{0.3}$$

with an analogous result in the continuous case. This result yields the first uniform bounds on this type of Radon operators. It is important to point out that in all the previous results a small power of N appeared on the RHS. It is a significant advancement in the theory with strong implications. Moreover according to the author, the above theorem is entirely up to him which is quite impressive. The other main result of Chapter 3 (based on a single authored paper by the author) is a generalization of (0.3) to singular Radon averages and integrals, i.e. operators of the form

$$H_N^{\mathcal{P}} f(x) = \sum_{n \in \Omega_t \cap \mathbb{Z}^k} f(x - \mathcal{P}(n)) K(n),$$

where K satisfies some standard regularity conditions. This result is also a significant advancement as it is the first time one has such uniform bounds on these operators. Moreover one should also mention that previously this problem was open even for a single linear polynomial. Let me now shortly describe the methods involved in proving both of the theorems from Chapter III. The estimates naturally split into bounding *short variations* and *long oscillations*. Bounds on short variations follow from earlier work of Mirek, Stein and Zorin-Kranich (also in the case of singular Radon operators). The main work in both cases goes into obtaining bounds on long oscillations. The bounds on long oscillations are obtained by the use of the circle method. The use of this method requires obtaining bounds on minor and major arcs. In the setting of the theorems bounds for minor arcs follow from Weyl bounds on exponential sums along polynomials. Estimates for major arcs are more involved. Here the author uses a variety of techniques: reductions established by Mirek, Stein and Zorin-Kranich, Ionescu-Waigner theory, Fourier multipliers and the method of martingale on homogeneous spaces. Additionally the case of general Radon operators uses the extra idea of representing the operator as a telescoping sum of operators. Proofs of the main results use a wide spectrum of techniques developed previously by different authors and additionally require new ideas (especially for the large scale situation).

Let me now pass to a short description of Chapter 4 which is based on a single authored paper by the author. The content of this chapter is the use of a different approach to study oscillation inequalities first invented in the work of Mirek, Stein and Zorin-Kranich (bootstrap approach). This approach in various forms is used in different areas of mathematics and has been also successfully used in harmonic analysis. In rough terms it is based on obtaining better bounds by a clever use of already known bounds. The author uses the bootstrap argument to prove uniform oscillation inequalities for some singular Radon operators (as those in (0.2)). This way the author reproves some of the main results from Chapter 3 using different approach. Even though in the thesis the author uses the bootstrap approach to reprove results obtained earlier by Mirek, Stein, Zorin-Kranich and also by the author, it seems that this new technique has the potential to be used for other problems in oscillation bounds for singular Radon operators. The proof presented by the author is more self contained and the two main ingredients in the proof is the discrete Littlewood-Paley theory of Mirek and a bootstrapping lemma of Duondikoetxea, de Francia. The main novelty of the proof is the bootstrap approach for bounding the seminorms $S^p_{\mathbb{Z}^\Gamma}(M_{2^n}f \, : \, n \in \mathbb{N})$.

Summary. Main results of the dissertation are obtained in Chapters 3 and 4. The results of Chapter 3 offer first *uniform* estimates on oscillation type inequalities for singular Radon operators. Obtaining such uniform bounds was one of the open problems in the theory (all previous bound had a dependence on N). To obtain the bounds the author had to master a variety of techniques from harmonic analysis and also significantly develop existing machinery. I consider these results as an important step in understanding bounds for Radon operators. The novelty of the techniques lies in sharp bounds in the large scale regime. Chapter 4 is of slightly different nature. It offers a new novel approach to classical problems in the theory based on the bootstrap approach. In the thesis the author uses it to reprove some known results but the technique seems to be adaptable to more general setting and has the potential to be used for other open problems in the field. For this reason I consider Chapter 4 as an important step to a new approach to study oscillation inequalities. The thesis is generally well written even though there are some misprints that I will omit commenting on.

In my opinion the dissertation of Wojciech Słomian „Aspects of discrete harmonic analysis" easily fulfills all requirements put forward for doctoral dissertations. The results in the thesis are central in the theory of Radon operators and are of central interest to the community. Moreover the obtained results prove that the author is very talented, knows a wide variety of techniques from harmonic analysis and has deep mathematical understanding of the fundamental concepts in the field. I **strongly recommend** the approval of the doctoral dissertation of mgr Wojciech Słomian to the Scientific Council and to confer him the corresponding titles. I also recommend a **distinction** of the thesis.

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