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Fractional Sobolev Spaces and Hardy inequalities

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Ułamkowe przestrzenie Sobolewa i nierówności Hardy'ego

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Contents

1 Introduction

This dissertation is based on the following four articles:

[I] Dyda, B., and Kijaczko, M. On density of smooth functions in weighted fractional Sobolev spaces. *Nonlinear Anal. 205* (2021), Paper No. 112231, 10.

[II] Dyda, B., and Kijaczko, M. On density of compactly supported smooth functions in fractional Sobolev spaces. *Ann. Mat. Pura Appl. (4) 201*, 4 (2022), 1855–1867.

[III] Kijaczko, M. Fractional Sobolev spaces with power weights. *Accepted in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*. DOI: 10.2422/2036-2145.202112_002 (2023).

[IV] Kijaczko, M., and Lenczewska, J. Sharp Hardy inequalities for Sobolev–Bregman forms. *Accepted in Mathematische Nachrichten* (2023).

These papers are attached at the end of this document. Moreover, during the PhD studies, the author was also involved in the following three articles related to the main theme of the thesis. When writing this dissertation, these articles are still under review, so formally they cannot be included in academic achievements. The results of this work are briefly mentioned at the end of this summary.

[V] Dyda, B. and Kijaczko, M. Sharp weighted fractional Hardy inequalities. *arXiv e-prints* (2022).

[VI] Dyda, B. and Kijaczko, M. Sharp fractional Hardy inequalities with a remainder for $1 < p < 2$. *arXiv e-prints* (2023).

[VII] Kijaczko, M. Asymptotics of weighted Gagliardo seminorms. *arXiv e-prints* (2023).

The concept of **fractional Sobolev spaces** (in the literature also called Aronszajn or Slobodeckii spaces) appeared in the 20th century, with the development of all widely understood problems of fractional nature. These problems refer to, for example, differential equations for nonlocal operators, or probability theory — more precisely, stable Lévy processes or diffusions. Compared to classical Sobolev spaces, fractional Sobolev spaces play a similar role in nonlocal-type issues. Nonlocal problems were invented to fill a gap, where the classical differential methods are no longer applicable, for example, for describing the behaviour of a phenomenon which varies in a nonsmooth, or even discontinuous way. The divergence between the application of classical and fractional methods is clearly visible on a microscopic or macroscopic scale.

Perhaps the most significant application of fractional Sobolev spaces comes from the study of integro-differential operators, such as the **fractional Laplacian** $(-\Delta)^s$. For the classical Laplace operator

$$\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2},$$

the second order Poisson partial differential equation $-\Delta f = u$, (which for the right-hand side being zero is the condition for harmonicity) plays a crucial role in physics — especially in potential theory. The nonlocal equation $(-\Delta)^s f = u$ is a fractional counterpart of this problem. While the classical Sobolev spaces serve as spaces of (weak) solutions for partial differential equations, so do the fractional ones for nonlocal problems.

Another intriguing interplay between the classical and fractional theory comes from the theory of stochastic processes, which is nowadays probably one of the most rapidly developing branches of modern mathematics, both theoretical and applied. It is well known that the Laplace operator is deeply connected with the **Brownian motion** — it is an infinitesimal generator of this process. Analogously, the fractional Laplace operator is the infinitesimal generator of a **symmetric, stable Lévy process**. Fractional Sobolev spaces connect with stochastic processes through the notion of **Dirichlet forms** — the concept well known also in the classical setting.

In this dissertation, we describe results obtained while studying the properties of fractional Sobolev spaces. The articles [I], [II] and [III] are devoted to investigation of the form of the closure of smooth, or smooth and compactly supported functions, in fractional Sobolev spaces — weighted or unweighted. We will present obtained results, most of them being conditions for which both mentioned classes of functions are or are not dense in the corresponding fractional Sobolev space. Such density properties are important for applications, because usually any computations are easier for smooth functions. The article [IV] deals with Hardy inequalities for Sobolev–Bregman forms — the latter turns out to be an important object of study in recent years. In [IV] we generalize the results for fractional Hardy inequalities on halfspaces and convex domains. The works [V] and [VI] focus on a different problem, that is fractional Hardy inequalities. We prove there various results for this topic, such as weighted fractional Hardy inequalities, fractional Hardy–Sobolev–Maz'ya inequalities, or fractional Hardy inequalities with additional terms. Hardy inequalities are an important tool in the analysis and theory of partial differential equations. Finally, the paper [VII] is devoted to asymptotics of fractional Sobolev spaces endowed with power-type weights. The results obtained there can also be applied to classical weighted Sobolev spaces.

2 Sobolev spaces and fractional Laplacian

In this chapter we will introduce and briefly discuss the most important objects and notions, which will appear later in this dissertation. Throughout this thesis, we will always assume that Ω is an open subset of the Euclidean space $\mathbb{R}^d$, $d \geq 1$. Any additional properties of the domain will be specified, if needed.

Let us start with recalling the definition of classical Sobolev spaces $W^{k,p}$.

Definition 1. Let $k \in \mathbb{N}$ and $p \geq 1$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index and let $|\alpha| = \alpha_1 + \ldots + \alpha_d$ denote its length. The **Sobolev space** $W^{k,p}(\Omega)$ is defined as the set of all functions $f \in L^p(\Omega)$ such that the mixed (weak) partial derivatives of order $|\alpha| \leq k$, that is

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} f,$$

exist and belong to $L^p(\Omega)$.

The norm in the Sobolev space is given by

$$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}$$

and makes $W^{k,p}(\Omega)$ a Banach space.

It is worth to mention here that for a given **weight**, i.e. a measurable, nonnegative function on Ω, one can also define the **weighted Sobolev space** $W^{k,p}(\Omega, w)$ through the norm

$$\|f\|_{W^{k,p}(\Omega,w)} = \left(\sum_{|\alpha|\le k} \|D^\alpha f\|^p_{L^p(\Omega,w)} \right)^{\frac{1}{p}},$$

where $L^p(\Omega, w)$ is the weighted Lebesgue space. For the survey of weighted Sobolev spaces, we recommend the book [42] by Kufner. As we have already mentioned in the introduction, weighted and unweighted Sobolev spaces are natural spaces for weak solutions of various kinds of partial differential equations.

We now turn to the most important object of this thesis, the fractional Sobolev space.

Definition 2. Let $0 < s < 1$ and $1 \le p < \infty$. Then, the **fractional Sobolev space** is defined as

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dy\, dx < \infty \right\}.$$

This is a Banach space endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Omega)},$$

where the expression

$$[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dy\, dx \right)^{\frac{1}{p}}$$

is called the **Gagliardo seminorm**.

For the most important case $p = 2$, the space $W^{2,s}(\Omega)$ becomes a Hilbert space and is usually denoted by $H^s(\Omega)$.

A very good source of information about fractional Sobolev spaces is the paper [19] of Di Nezza, Palatucci and Valdinoci. In this dissertation, we will mostly study weighted fractional Sobolev spaces, but we will define them later, as we focus on different kinds of them, which leads to different notation.

The connection between classical and fractional Sobolev spaces is not easily seen on first look. However, it turns out that the space $W^{s,p}(\Omega)$ is an intermediary space between $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in the sense of complex interpolation. The latter is a complicated issue and we will not discuss it here. In the celebrated paper [12], Bourgain, Brezis and Mironescu established the following asymptotic of the fractional Gagliardo seminorm:

$$\lim_{s\to 1^-} (1-s) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dy\, dx = K_{d,p} \int_\Omega |\nabla f(x)|^p\, dx, \qquad (1)$$

where the constant $K_{d,p}$ is given by

$$K_{d,p} = \frac{2\pi^{\frac{d-1}{2}}}{p} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+d}{2}\right)}.$$

Here Γ is the Euler Gamma function. The relation (1) holds for $p > 1$ and $f \in L^p(\Omega)$, if Ω is an extension domain for the space $W^{1,p}(\Omega)$, which, roughly speaking, means that any function from $W^{1,p}(\Omega)$ can be extended to a function from $W^{1,p}(\mathbb{R}^d)$. For $p = 1$, (1) also holds, but the assumptions are slightly different — see the paper [18] of Dávila. Thus, one may think that the fractional Sobolev spaces $W^{s,p}(\Omega)$ in some way "converge" to the classical Sobolev space $W^{1,p}(\Omega)$, as $s \to 1^-$.

Another famous result of this kind was established by Maz'ya and Shaposhnikova [51]. They proved that

$$\lim_{s \to 0^+} s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \, dy \, dx = \frac{2}{p} \left|\mathbb{S}^{d-1}\right| \int_{\mathbb{R}^d} |f(x)|^p \, dx. \tag{2}$$

Here $\left|\mathbb{S}^{d-1}\right| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface measure of the unit sphere in $\mathbb{R}^d$ and it is assumed that $f \in \bigcup_{0<s<1} W^{s,p}(\mathbb{R}^d)$. This limit has also been investigated in the magnetic setting, see [55]. Noteworthy, the relation (1) can be used to obtain a nonlocal characterization of the classical Sobolev space $W^{1,p}(\Omega)$ ([12, Theorem 2]). In the paper [VII] the author of this thesis obtained Bourgain–Brezis–Mironescu and Maz'ya–Shaposhnikova formulae for Gagliardo seminorms equipped with power-type weights and, as an application, a nonlocal characterization of weighted Sobolev spaces.

We will now shortly discuss the connection between fractional Sobolev spaces and the fractional Laplace operator. Let us start with defining the latter.

Definition 3. For $0 < s < 1$ and $f \in C_c^2(\Omega)$ (twice differentiable in a continuous way functions with compact support), the **fractional Laplace operator** is defined as

$$(-\Delta)^s f(x) = C(d,s) \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(x) - f(y)}{|x-y|^{d+2s}} \, dy, \tag{3}$$

where $C(d,s)$ is the normalizing constant given by

$$C(d,s) = \frac{\pi^{\frac{d}{2}} \Gamma(-s)}{4^s \Gamma\left(\frac{d}{2} + s\right)}.$$

Let us notice that, in order to compute $\Delta f(x)$, we only need to know the values of the function f in a neighbourhood of x. Conversely, according to the formula (3), to find the value of $(-\Delta)^s f(x)$, we need to have information about the behaviour of f in the whole domain $\mathbb{R}^d$. This is what we usually call a **nonlocality**. A good source of knowledge about the fractional Laplace operator, including its various definitions, is contained in the survey [43] of Kwaśnicki.

The connection between fractional Sobolev spaces and the fractional Laplace operator is easily visible, once we establish an elementary relation

$$\|(-\Delta f)^{\frac{s}{2}}\|^2_{L^2(\mathbb{R}^d)} = \frac{1}{2} C(d,s) [f]^2_{H^s(\mathbb{R}^d)}.$$

The above explains why the space $H^s(\mathbb{R}^d)$ is a natural object of investigation in the context of the fractional Laplacian.

Another, perhaps more mysterious and less studied operator, is the so-called **regional fractional Laplacian**, which is defined for $f \in C_c^2(\Omega)$ as

$$(-\Delta)^s_\Omega f(x) = C(d,s) \lim_{\varepsilon \to 0^+} \int_{\{|x-y|>\varepsilon\}\cap\Omega} \frac{f(x)-f(y)}{|x-y|^{d+2s}}\, dy.$$

It is easy to check that

$$\int_\Omega f(x)(-\Delta)^s_\Omega f(x)\, dx = \frac{1}{2}[f]^2_{H^s(\Omega)},$$

which again leads us to the fractional Sobolev space, this time it is $H^s(\Omega)$. This operator is related, through the notion of an infinitesimal generator, to the **censored stable process** in Ω, which, informally speaking, is a stable process "forced" to stay inside Ω, see Bogdan, Burdzy and Chen [7]. For additional information on the regional fractional Laplacian, we also refer to [31, 32].

3 Density of smooth functions in weighted fractional Sobolev spaces

In this chapter, we will present the aim and scope of the article [I]. This work is devoted to the problem of density of the set of smooth functions on Ω, denoted by $C^\infty(\Omega)$, in fractional Sobolev spaces with weights. At first, let us mention that for classical (unweighted) Sobolev spaces, the celebrated **Meyers–Serrin theorem** [53] (the famous work "$H = W$") states that $C^\infty(\Omega)$ is always dense in $W^{k,p}(\Omega)$, without assuming any regularity of the domain. A version of Meyers–Serrin theorem for weighted Sobolev spaces with Muckenhoupt weights [40, Theorem 2.5] also holds. Moreover, the fractional counterpart of the Meyers–Serrin theorem is also known [52, Theorem 3.25], that is $C^\infty(\Omega)$ is always dense in $W^{s,p}(\Omega)$. The similar question for weighted fractional Sobolev spaces is more complicated and we will present our results in this field.

3.1 Whitney decomposition

In order to prove our results, we will need a Whitney decomposition of an open set into cubes, a very important and useful tool in geometric analysis.

Definition 4. Let Ω be an open, proper subset of $\mathbb{R}^d$. A **Whitney decomposition** of Ω is a countable family of cubes $\mathcal{W} = \{Q_n\}_{n\in\mathbb{N}}$ with sides parallel to the axis and satisfying the following properties,

- $\bigcup_{n=1}^{\infty} Q_n = \Omega$;
- the cubes Q_i have disjoint interiors;
- $\operatorname{diam} Q \le \operatorname{dist}(Q, \partial\Omega) \le 4 \operatorname{diam} Q$, for any $Q \in \mathcal{W}$;
- if Q_i and Q_j intersect, then $\frac{1}{4}\operatorname{diam}(Q_i) \le \operatorname{diam} Q_j \le 4 \operatorname{diam} Q_i$;
- for any $Q \in \mathcal{W}$ there exist at most 12^d of cubes in $\mathcal{W}$, which intersect Q.

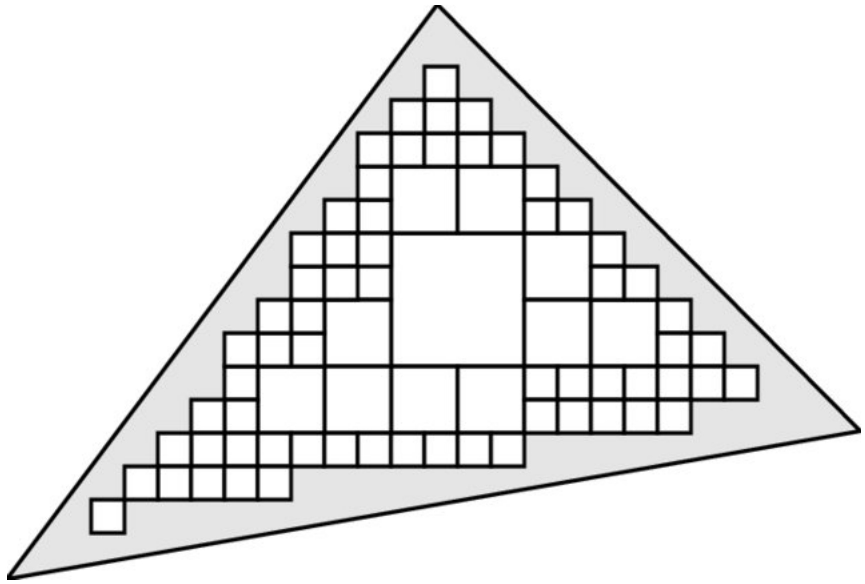

Figure 1: Whitney decomposition of a triangle. Source: [33]

In the definition above, $\partial\Omega$ is the boundary of Ω, $\operatorname{diam} A = \sup_{x,y\in A}|x-y|$ denotes the diameter of A, and $\operatorname{dist}(x,A) = \inf_{y\in A}|x-y|$ is the standard distance function. It turns out that the Whitney decomposition always exists; the details are contained in the book [59] by Elias M. Stein. Intuitively thinking, the cubes in the Whitney decomposition are smaller when reaching the boundary of Ω, and the sizes of two adjacent cubes are comparable.

3.2 Operator P^η

We want to construct a "smoothing" operator, which will serve as an approximation of an arbitrary function from the appropriate Sobolev space by smooth functions. In order to do so, we will use a Whitney decomposition.

Let $\Omega \subset \mathbb{R}^d$ be any open set and $\mathcal{W} = \{Q_n\}_{n\in\mathbb{N}}$ be a Whitney decomposition of Ω into cubes. Choose $\varepsilon > 0$ such that $(1+\varepsilon)^2 < \frac{5}{4}$, that is $\varepsilon < \frac{\sqrt{5}}{2} - 1$. We define "blown-up" cubes Q_n^* as cubes with the same center as Q_n, but the length of the side $1+\varepsilon$ times longer. Analogously, the cube Q_n^{**} is a cube with the same center as Q_n, but the length of the side $(1+\varepsilon)^2$ times longer. Thanks to our choice of ε, any point $x \in \Omega$ belongs to at most 12^d cubes Q_n^{**}.

We will need the so-called **approximation of unity**, that is a family $\{\psi_n : n \in \mathbb{N}\}$ of functions of a class $C_c^\infty(\Omega)$ (smooth functions with compact support in Ω) such that $0 \le \psi_n \le 1$ and ψ_n vanishes outside Q_n^*. Moreover, $\sum_n \psi_n = 1$ pointwise, and, for some $C > 0$ independent of n it holds

$$|\psi_n(x) - \psi_n(y)| \le \min\left\{\frac{C|x-y|}{l(Q_n)}, 1\right\},$$

where $l(Q_n)$ denotes the length of the side of Q_n.

Let $h\colon \mathbb{R}^d \to \mathbb{R}$ be a nonnegative smooth function supported in the unit ball $B(0,1)$

and integrating to one. For $\delta > 0$ we define its **dilation** by

$$h_\delta(x) = \delta^{-d} h\left(\frac{x}{\delta}\right).$$

It is clear that h_δ belongs to $C_c^\infty(\mathbb{R}^d)$ and is supported in $B(0,\delta)$. Moreover, we have $\int_{\mathbb{R}^d} h_\delta(x)\,dx = 1$ for any δ.

Let $\eta\colon \mathcal{W} \to (0,\infty)$ be any function on Whitney collection of cubes, which fulfills the condition $\eta(Q) < \frac{\varepsilon}{2} l(Q)$ for any $Q \in \mathcal{W}$. The simplest example is of course $\eta(Q) = cl(Q)$ for any $c < \frac{\varepsilon}{2}$.

Definition 5. For $f \in L^1_{loc}(\Omega)$, the operator P^η if defined by the following formula,

$$P^\eta f = \sum_{n=1}^{\infty} (f\psi_n) * h_{\eta(Q_n)}. \tag{4}$$

Here $f * g(x) = \int_{R^d} f(y)g(x-y)\,dy$ is the standard convolution operation. Observe first that the summation in the definition of $P^\eta f$ is at each point $x \in \Omega$ finite and has at most 12^d terms — this is a consequence of the corresponding Whitney decomposition property and the choice of ε.

We prove in [I] that P^η is well defined, $P^\eta f \in C^\infty(\Omega)$ and P^η maps the space of all compactly supported, measurable functions in Ω into $C_c^\infty(\Omega)$ (see [I], Propositions 1 and 2). These are basically the consequences of Whitney decomposition properties and the fact that the convolution with a smooth function makes a locally integrable function also smooth. The operator P^η will serve as a smooth approximation in an appropriate (semi)norm. However, there are technical difficulties that we need to face to establish the density results.

3.3 Main results

The first observation is a convergence of $P^\eta f$ to f in $L^p(\Omega)$.

Theorem 6. ([I, Theorem 3]) *Let $p \in [1,\infty)$ and $f \in L^p(\Omega)$. Then*

$$\lim_{k\to\infty} \|P^{\eta_k} f - f\|_{L^p(\Omega)} = 0,$$

provided that $\lim_{k\to\infty} \eta_k(Q) = 0$ for all $Q \in \mathcal{W}$.

The proof of this fact is quite standard — it uses basic properties of the operator P^η together with Jensen inequality combined with Whitney decomposition properties.

Next, we need to establish convergence in Gagliardo seminorms. In [I, Theorem 8], using our methods we prove a fractional counterpart of Meyers–Serrin theorem, that is we prove that $C^\infty(\Omega) \cap W^{s,p}(\Omega)$ is always dense in $W^{s,p}(\Omega)$, without any additional assumptions on the domain Ω, for $p \in [1,\infty)$ and $0 < s < 1$. Although this fact is well known, we prove it anyway to have an excuse to modify it later, when dealing with weighted Gagliardo seminorms.

Now, we will discuss how to extend the density results to weighted fractional Sobolev spaces. Recall that by **weight** we mean a nonnegative, measurable function w on Ω.

Denote

$$\widetilde{W}^{s,p}(\Omega, w) = \left\{ f\colon \Omega \to \mathbb{R}^d \textit{ measurable} : \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\, w(x)\, w(y)\, dx\, dy < \infty \right\}.$$

We understand $\widetilde{W}^{s,p}(\Omega, w)$ as a seminormed space with the weighted Gagliardo seminorm defined as

$$[f]_{W^{s,p}(\Omega,w)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, w(x) \, w(y) \, dx \, dy \right)^{\frac{1}{p}}.$$

Of course, the weighted L^p norm is

$$\|f\|_{L^p(\Omega,w)} = \left(\int_\Omega |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}}.$$

The weighted fractional Sobolev space $W^{s,p}(\Omega, w)$ is defined analogously as the unweighted one, that is we equip it with a norm which is a sum of the weighted L^p norm and the weighted Gagliardo seminorm.

In order to establish the density result for weighted Gagliardo seminorms, we need to limit ourselves to a special class of weights — continuous or locally bounded. Noteworthy, these are not very restrictive assumptions.

Definition 7. A weight $w \colon \Omega \to \mathbb{R}$ is **locally comparable to a constant** if for every compact subset $K \subset \Omega$ there exists $C_K > 0$ such that $\frac{1}{C_K} \leq w(x) \leq C_K$ for almost all $x \in K$.

The first important observation is that when w is locally comparable to a constant, then $\widetilde{W}^{s,p}(\Omega, w) \subset L^p_{loc}(\Omega)$ ([I, Proposition 9]). This ensures us that all integrals appearing in our proofs are finite. Finally, we are in a position to formulate the main result of the work [I], that is the density of smooth functions in weighted fractional Sobolev spaces.

Theorem 8. ([I, Theorem 12]) *Suppose that w is locally comparable to a constant or continuous and satisfies the condition*

$$\int_\Omega \frac{w(x)}{(1 + |x|)^{d+sp}} \, dx < \infty. \tag{5}$$

Then $C^\infty \cap \widetilde{W}^{s,p}(\Omega, w)$ is dense in $\widetilde{W}^{s,p}(\Omega, w)$.

Together with [I, Theorem 13], which establishes a convergence in the weighted L^p space, we obtain that $C^\infty \cap W^{s,p}(\Omega, w)$ is dense in $W^{s,p}(\Omega, w)$. The proofs are based on careful estimation of the norms of functions $P^\eta f$ and are quite technical. In another paper [III], which is devoted to a slightly different topic to be discussed later, the author of this dissertation proved a version of [I, Theorem 12] and [I, Theorem 13] for fractional Sobolev spaces with two different weights w and v, which are locally bounded and satisfy (5). See [III, Theorem 19] for details.

The last result of the work [I] is a density result for fractional Sobolev-type spaces for kernels different than $|x|^{-d-sp}$. We formulate this theorem below. The proof of this fact relies on arguments similar to those for previous results.

Theorem 9. ([I, Theorem 15]) *Let $p \in [1, \infty)$ and let $K \colon [0, \infty) \to [0, \infty)$ be a measurable function such that*

$$\int_0^\infty \min\{x^p, 1\} K(x) x^{d-1} \, dx < \infty. \tag{6}$$

Denote

$$[f]_K = \left(\int_\Omega \int_\Omega |f(x) - f(y)|^p K(|x - y|) \, dy \, dx \right)^{\frac{1}{p}}$$

and consider the space

$$X(\Omega) = \{f \in L^p(\Omega) : [f]_K < \infty\}$$

with the norm

$$\|f\|_{X(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_K.$$

Then $C^\infty(\Omega) \cap X(\Omega)$ *is dense in* $X(\Omega)$.

Notice that for $p = 2$, the condition (6) states that $K(|x|)$ is the density of a Lévy measure, which gives a wide perspective of applications to stochastic processes.

To summarize, in the paper [I], which is the first object of this dissertation, we established the density results for smooth functions in fractional Sobolev spaces with weights that satisfy mild assumptions and also for spaces with kernels different from $|x|^{-d-sp}$. A research directly related to [I] is contained, for example, in papers [46], [25] and [5].

4 Density of compactly supported smooth functions in fractional Sobolev spaces

In this chapter we will present results of the second article of this thesis, that is the work [II]. In this paper we describe some sufficient conditions, under which smooth and compactly supported functions are or are not dense in the fractional Sobolev space $W^{s,p}(\Omega)$ for an open, bounded set $\Omega \subset \mathbb{R}^d$. The density property is closely related to the lower and upper Assouad codimension of the boundary of Ω. We also describe explicitly the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$ under some mild assumptions about the geometry of Ω. It is well known that $C_c^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$, when Ω is a bounded Lipschitz domain and $sp \leq 1$ [30, Theorem 1.4.2.4], [60, Theorem 3.4.3]. In our research we go far beyond the Lipschitz regularity of the boundary of the domain. Notice that in contrast to the Meyers–Serrin theorem for smooth functions without compact support, the density of $C_c^\infty(\Omega)$ functions is a much more complicated problem and the answer to this depends on the properties of the domain. Noteworthy, it is quite easy to show that $C_c^\infty(\mathbb{R}^d)$ is always dense in $W^{s,p}(\mathbb{R}^d)$ — see [4, Theorem 7.38].

Throughout this chapter we use a notation

$$W_0^{s,p}(\Omega)$$

for the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$ with respect to the fractional Sobolev norm.

4.1 Geometrical notions and definitons

In this section we present geometrical notions, which will appear throughout this chapter.

Definition 10. Let $r > 0$. For open sets $\Omega \subset \mathbb{R}^d$ we define the **inner tubular neighbourhood** of Ω as

$$\Omega_r = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leq r\},$$

and for arbitrary sets $E \subset \mathbb{R}^d$ we define the **tubular neighbourhood** of E as

$$\widetilde{E}_r = \left\{x \in \mathbb{R}^d : \mathrm{dist}(x, E) \leq r\right\}.$$

Definition 11. [38, Section 3] Let $E \subset \mathbb{R}^d$. The **lower Assouad codimension** $\underline{\operatorname{co\,dim}}_A(E)$ is defined as the supremum of all $q \geq 0$, for which there exists a constant $C = C(q) \geq 1$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$ it holds

$$\left|\widetilde{E}_r \cap B(x,R)\right| \leq C\,|B(x,R)| \left(\frac{r}{R}\right)^q.$$

Conversely, the **upper Assouad codimension** $\overline{\operatorname{co\,dim}}_A(E)$ is defined as the infimum of all $s \geq 0$, for which there exists a constant $c = c(s) > 0$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$ it holds

$$\left|\widetilde{E}_r \cap B(x,R)\right| \geq c\,|B(x,R)| \left(\frac{r}{R}\right)^s.$$

We remark that having strict inequality $R < \operatorname{diam} E$ above makes the definitions applicable also for unbounded sets E; for bounded sets E we could have $R \leq \operatorname{diam} E$. Also, the Assouad codimensions may be defined in any metric measure space, with obvious changes.

In Euclidean space $\mathbb{R}^d$ we have $\underline{\dim}_A(E) = d - \overline{\operatorname{co\,dim}}_A(E)$, $\overline{\dim}_A(E) = d - \underline{\operatorname{co\,dim}}_A(E)$, where $\underline{\dim}_A(E)$ and $\overline{\dim}_A(E)$ denote, respectively, the well known lower and upper Assouad dimension — see for example [38, Section 2] for this result. If $\underline{\operatorname{co\,dim}}_A(E) = \overline{\operatorname{co\,dim}}_A(E)$, we simply denote it by $\operatorname{co\,dim}_A(E)$. For Lipschitz domains Ω we always have $\operatorname{co\,dim}_A(\partial\Omega) = 1$ and $\dim_A(\partial\Omega) = d - 1$.

We recall a geometric notion from [61].

Definition 12. A set $E \subset \mathbb{R}^d$ is **κ-plump** with $\kappa \in (0,1)$ if, for each $0 < r < \operatorname{diam}(E)$ and each $x \in \overline{E}$, there is $z \in \overline{B}(x,r)$ such that $B(z,\kappa r) \subset E$.

Following [47, Theorem A.12], we define a notion of σ-homogenity.

Definition 13. Let $E \subset \mathbb{R}^d$ and let $V(E,x,\lambda,r) = \{y \in \mathbb{R}^d : \operatorname{dist}(y,E) \leq r, |x-y| \leq \lambda r\}$. We say that E is **$\sigma$-homogeneous**, if there exists a constant L such that

$$|V(E,x,\lambda,r)| \leq L r^d \lambda^\sigma$$

for all $x \in E$, $\lambda \geq 1$ and $r > 0$.

If $0 < r < R < \operatorname{diam}(E)$, then taking $\lambda = R/r$ in the definition gives

$$\left|\widetilde{E}_r \cap B(x,R)\right| = \left|V\Big(E,x,\frac{R}{r},r\Big)\right| \leq C\,|B(x,R)| \left(\frac{r}{R}\right)^{d-\sigma},$$

where $C = C(d,E)$ is a constant. This means that if $\underline{\operatorname{co\,dim}}_A(E) = s$, then $(d-s)$-homogeneous sets are precisely these sets E, for which the supremum in the definition of the lower Assouad codimension is attained. For the definition of the concept of homogenity from a different point of view the Reader may also see [47, Definition 3.2].

To give an example of the introduced definitions, consider the set $\Omega \subset \mathbb{R}^2$ bounded by the Koch curve — that is the **Koch snowflake**. It may be shown that Ω is plump, σ-homogeneous for $\sigma = \log_3 4$ and the Assouad codimension of its boundary is $2 - \log_3 4$, while the Assouad dimension is $\log_3 4$. Despite that $\partial\Omega$ is an example of a fractal and it is a very irregular object, all results from this chapter can be applied to an appropriate space $W^{s,p}(\Omega)$.

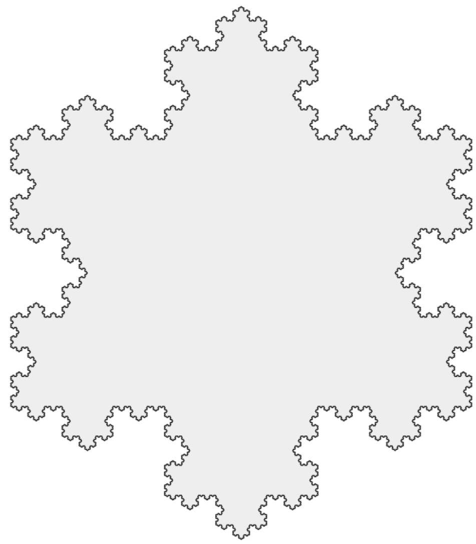

Figure 2: Koch snowflake. Source: Wikipedia

4.2 Main results

Let us describe our methods. As we already have mentioned, the operator $P^\eta f$ defined by (4) maps the space of compactly supported functions into $C_c^\infty(\Omega)$. That means that the closures of both these classes of functions in the space $W^{s,p}(\Omega)$ coincide [II, Proposition 12]. According to this, for given $f \in W^{s,p}(\Omega)$, we can construct explicitly a sequence of compactly supported functions approximating f in $W^{s,p}(\Omega)$ and we do not need to care about its smoothness. As a first step, we also show that if the measure of Ω is finite, then $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$ if and only if the nonzero constant function is in $W_0^{s,p}(\Omega)$ — [II, Lemma 13]. That reduces the problem of density for finding the sequence of compactly supported functions, which approximate in $W^{s,p}(\Omega)$ the function constantly equal to one. In order to do so, we use functions v_n defined by

$$v_n(x) = \max\{\min\{2 - nd_\Omega(x), 1\}, 0\} = \begin{cases} 1 & \text{when } d_\Omega(x) \leq 1/n, \\ 2 - nd_\Omega(x) & \text{when } 1/n < d_\Omega(x) \leq 2/n, \\ 0 & \text{when } d_\Omega(x) > 2/n. \end{cases}$$

We denote here $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$. Notice that v_n are continuous, but clearly nonsmooth, which is not a problem, as the appropriate closures are the same. In [II, Lemma 10] we derive an inequality

$$[fv_n]^p_{W^{s,p}(\Omega)} \leq Cn^{sp} \int_{\Omega_{\frac{3}{n}}} |f(x)|^p \, dx + C \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x) - f(y)|^p}{|x - y|^{d+sp}} \, dy \, dx, \tag{7}$$

which is a key to further computations. Of course, when f is constant, the second term in (7) vanishes.

Let us see that it is relatively easy to state when $C_c^\infty(\Omega)$ is not dense in $W^{s,p}(\Omega)$. If Ω is bounded and plump and $u_n \to 1$ in $W^{s,p}(\Omega)$ (in particular in $L^p(\Omega)$), there is a subsequence u_{n_k} convergent to 1 almost everywhere. Therefore, if $sp > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, by the fractional Hardy inequality from [24, Corollary 3]

$$[u_{n_k}-1]^p_{W^{s,p}(\Omega)} = [u_{n_k}]^p_{W^{s,p}(\Omega)} = \int_\Omega\int_\Omega \frac{|u_{n_k}(x)-u_{n_k}(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \geq c\int_\Omega \frac{|u_{n_k}(x)|^p}{d_\Omega(x)^{sp}}\,dx,$$

and by Fatou's lemma,

$$0=\lim_{k\to\infty}[u_{n_k}]^p_{W^{s,p}(\Omega)} \geq c\int_\Omega \liminf_{k\to\infty}\frac{|u_{n_k}(x)|^p}{d_\Omega(x)^{sp}}\,dx = c\int_\Omega \frac{dx}{d_\Omega(x)^{sp}} > 0.$$

That leads to a contradiction. Therefore, if $sp > \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W^{s,p}_0(\Omega) \neq W^{s,p}(\Omega)$.

We now present main results of the work [II]. They are collected in three theorems. The first one gives necessary and sufficient conditions under which smooth compactly supported functions are or are not dense in fractional Sobolev space. The second theorem describes explicitly the form of the space $W^{s,p}_0(\Omega)$, and the third provides the embedding of the weighted Lebesgue space into the fractional Sobolev space, that is $L^p(\Omega, \operatorname{dist}(x,\partial\Omega)^{-sp}) \subset W^{s,p}(\Omega)$.

Theorem 14. ([II, Theorem 2]) *Let $\Omega \subset \mathbb{R}^d$ be a nonempty bounded open set, let $0<s<1$ and $1 \leq p < \infty$.*
(I) If $sp < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W^{s,p}_0(\Omega) = W^{s,p}(\Omega)$.
(II) If Ω is a $(d-sp)$-homogeneous set, $sp = \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ and $p>1$, then $W^{s,p}_0(\Omega) = W^{s,p}(\Omega)$.
(III) If Ω is κ-plump and $sp > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W^{s,p}_0(\Omega) \neq W^{s,p}(\Omega)$.

Theorem 15. ([II, Theorem 3]) *Let $0<s<1$ and $1\leq p<\infty$. Suppose that $\Omega \neq \emptyset$ is a bounded, open κ-plump set. If $\overline{\operatorname{co\,dim}}_A(\partial\Omega) < sp$, then*

$$W^{s,p}_0(\Omega) = \left\{ f \in W^{s,p}(\Omega) : \int_\Omega \frac{|f(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp}}\,dx < \infty \right\}.$$

Theorem 16. ([II, Theorem 4]) *Let $0<s<1$ and $1\leq p<\infty$. Suppose that $\Omega \neq \emptyset$ is a bounded, open κ-plump set. If $\underline{\operatorname{co\,dim}}_A(\partial\Omega) > sp$, then there exists a constant c such that*

$$\int_\Omega \frac{|f(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp}}\,dx \leq c\|f\|^p_{W^{s,p}(\Omega)} < \infty, \quad \textit{for all } f \in W^{s,p}(\Omega).$$

The statement of the part (I) of Theorem 14 remains true if we assume that $sp < d - \overline{\dim}_M(\partial\Omega)$, where $\overline{\dim}_M(\partial\Omega)$ is the upper Minkowski dimension introduced in the Definition 20. Our results have classical (non-fractional) counterparts, see [42, Example 9.11] or [41]. Moreover, similar issues were investigated in [20], [15], [13], [25], [20] and [5].

The last theorem of the article [II] is the following version of a fractional Hardy inequality in the case (T'). Although it is an interesting result itself, we need it in the proof of Theorem 16. A special case of (T') for $p=2$ can be found in [52, Lemma 3.32] and [17].

Theorem 17 ([24] in cases (T) and (F))**.** *Let $0<p<\infty$, $H\in(0,1]$ and $\eta\in\mathbb{R}$. Suppose $\Omega\neq\emptyset$ is a proper κ-plump open set in $\mathbb{R}^d$ and $\phi:(0,\infty)\to(0,\infty)$ is a function so that either condition (T), or condition (T'), or condition (F) holds*

(T) $\eta + \overline{\dim}_A(\partial\Omega) - d < 0$, *$\Omega$ is unbounded,* $\phi \in \mathrm{WUSC}(\eta, 0, H^{-1})$,

(T') $\eta + \overline{\dim}_A(\partial\Omega) - d < 0$, *$\Omega$ is bounded,* $\phi \in \mathrm{WUSC}(\eta, 0, H^{-1})$,

(F) $\eta + \underline{\dim}_A(\partial\Omega) - d > 0$, *$\Omega$ is bounded or $\partial\Omega$ is unbounded, and* $\phi \in \mathrm{WLSC}(\eta, 0, H)$.

Then there exist constants $c = c(d, s, p, \Omega, \phi)$ and R such that the following inequality

$$\int_\Omega \frac{|u(x)|^p}{\phi(d_\Omega(x))}\, dx \le c \int_\Omega \int_{\Omega \cap B(x, R d_\Omega(x))} \frac{|u(x) - u(y)|^p}{\phi(d_\Omega(x)) d_\Omega(x)^d}\, dy\, dx + c\xi \|u\|^p_{L^p(\Omega)}, \tag{8}$$

holds for all measurable functions u for which the left hand side is finite, with $\xi = 0$ in the cases (T) and (F) and $\xi = 1$ in the case (T').

Recall that a function $\phi : (0, \infty) \to (0, \infty)$ satisfies $\mathrm{WLSC}(\eta, 0, H)$ (respectively, $\mathrm{WUSC}(\eta, 0, H^{-1})$) and write $\phi \in \mathrm{WLSC}(\eta, 0, H)$ ($\phi \in \mathrm{WUSC}(\eta, 0, H^{-1})$), if

$$\phi(st) \ge H t^\eta \phi(s), \quad s > 0\,,$$

for every $t \ge 1$ (respectively, for every $t \in (0, 1]$). The inequality (8) is an example of the fractional Hardy-type inequality. We describe such inequalities wider in Section 6.

To summarize, in the work [II] we focus on bounded domains Ω and give necessary and sufficient conditions for which $C_c^\infty(\Omega)$ functions are or are not dense in the fractional Sobolev space $W^{s,p}(\Omega)$. The conditions are given in terms of geometric properties of the domain Ω, such as plumpness and the Assouad codimension of its boundary, as well as the values of parameters s and p. We also prove a variant of a fractional Hardy inequality. Our results generalize some partially known before, as well as an analogy to local phenomena, and shed light on a structure of fractional Sobolev spaces on irregular domains.

5 Fractional Sobolev spaces with power weights

This chapter is devoted to the description of the article [III]. In this work we investigate the form of the closure of the smooth, compactly supported functions in the weighted fractional Sobolev space for bounded Ω. We focus on the weights w, v being powers of the distance to the boundary of the domain. Our results depend on the lower and upper Assouad codimension of the boundary of Ω, similarly as for the unweighted case. For such weights we also prove the comparability between the full weighted fractional Gagliardo seminorm and the truncated one, which is an interesting and nontrivial result itself.

5.1 Uniform domains and comparablity

Dealing with the closure of $C_c^\infty(\Omega)$ in weighted fractional Sobolev spaces combines techniques from both previously described here works [I] and [II] — it is also more involved. The main difference is that we need to establish a **comparability** between the full and truncated weighted fractional Gagliardo seminorms.

In this chapter we will use the notation $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$. The **fractional Sobolev space with power weights** is the space

$$W^{s,p;\,\alpha,\beta}(\Omega) = \left\{ f \in L^p(\Omega) : [f]_{W^{s,p;\,\alpha,\beta}(\Omega)} < \infty \right\},$$

where

$$[f]_{W^{s,p;\,\alpha,\beta}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx\right)^{\frac{1}{p}} \tag{9}$$

is the corresponding weighted Gagliardo seminorm. One can also exchange the L^p space appearing above to any weighted L^p space for continuous or locally bounded weight — the L^p component is essentially irrelevant for our purposes, as the fractional seminorms are the most important. Weighted fractional Sobolev spaces like ours appeared in the literature during past decades — it is worth to quote here the papers [20], [3] or [2].

One of our main results is the comparability between the full and truncated Gagliardo seminorms. For $0 < \theta \le 1$ the truncated seminorm is

$$\left(\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dy\, dx\right)^{\frac{1}{p}}, \tag{10}$$

where $B(x,R)$ is the Euclidean ball centered at x with radius R. Observe that the inner integral in (10) is localized around x, therefore it is obvious that the truncated Gagliardo seminorm is strictly smaller than the full seminorm (9). The comparability asserts that, in some cases, the converse inequality also holds for all $f \in L^1_{loc}(\Omega)$, with a global constant depending on $\alpha, \beta, \theta, d, \Omega, s$ and p. This is a very nontrivial property. The unweighted cases have been studied before by many authors — for example Dyda [22] proved the comparability for cones instead of balls in the inner integral. Prats and Saksman [56] and Rutkowski [57] studied Triebel–Lizorkin spaces, generalizing fractional Sobolev spaces for two exponents p and q. Some versions of the reduction of the integration theorems can also be found in [14], [16], [39] and [58].

The comparability will hold for a special class of domains, usually called **uniform domains**. We present the definition below.

Definition 18. A domain (i.e. connected, open set) $\Omega \subset \mathbb{R}^d$ is **uniform**, if there exists a constant $C \ge 1$ such that for all points $x, y \in \Omega$ there is a curve $\gamma\colon [0,l] \to \Omega$ joining them, parameterized by arc length and satisfying $l \le C|x-y|$ and $\operatorname{dist}(z,\partial\Omega) \ge \frac{1}{C}\min\{|z-x|,|z-y|\}$ for all $z \in \gamma$.

One can also define the uniformity property using Whitney decomposition and chains of cubes — see Prats and Saksman [56] for details. Uniform domains and various reformulations of the definitions above appear also in [28], [49] and [50]. To give a concrete, nontrivial example, we remark here that the Koch snowflake is known to be uniform, despite the highly irregular behaviour of its boundary. Loosely speaking, a domain is uniform if any two points can be connected by a curve with length in some way proportional to the distance between them.

As the first main result of the article [III], we present the comparability theorem.

Theorem 19. ([III, Theorem 2]) *Let Ω be a nonempty, bounded, uniform domain, let $0 < s < 1$ and $1 \le p < \infty$. Moreover, let $0 < \theta \le 1$. Suppose that $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. Then the full seminorm $[f]_{W^{s,p;\,\alpha,\beta}(\Omega)}$ and the truncated seminorm*

$$\left(\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dy\, dx\right)^{\frac{1}{p}}$$

are comparable, that is there exists a constant $C = C(\theta, d, s, p, \alpha, \beta, \Omega) > 0$ such that

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \frac{dy}{d_\Omega(y)^\beta} \frac{dx}{d_\Omega(x)^\alpha} \le C \int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \frac{dy}{d_\Omega(y)^\beta} \frac{dx}{d_\Omega(x)^\alpha}\, dy\, dx,$$

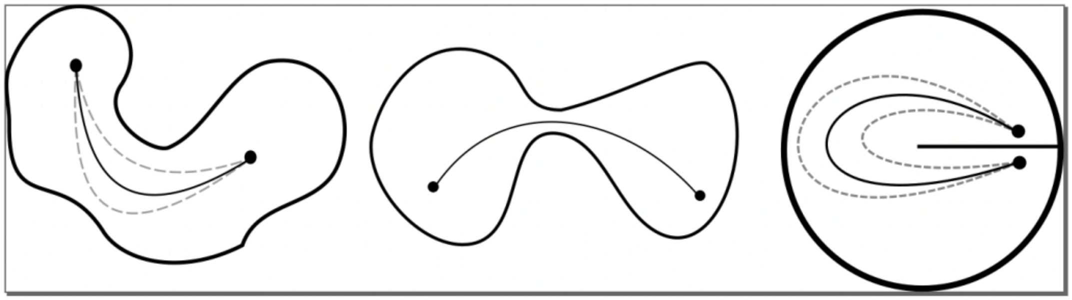

Figure 3: The first and the second set are uniform, even despite some narrowness around the middle. The third domain is not uniform — taking two points from the opposite sides of the excluded segment and bringing them closer together, we will not find a curve joining them and satisfying the demanded properties. Source: maths.ed.ac.uk

for all $f \in L^1_{loc}(\Omega)$.

Our proof is very technical and relies on methods introduced by Prats and Saksman in [56] for the unweighted case — it uses chains of cubes and other complicated notions connected with Whitney decomposition. The difference that weights give is that we need a Muckenhoupt A_1 property of the function $d_\Omega^{-\alpha}$ to obtain some maximal–type estimates appearing in the proof. Recall that a weight w belongs to the class A_1 if there exists a constant $A > 0$ such that for all cubes $Q \subset \mathbb{R}^d$ it holds

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le A \inf_{y\in Q} w(y).$$

It turns out that $d_\Omega^{-\alpha} \in A_1$ if and only if $0 \le \alpha < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, see [23, Theorem 1.1 (B)]. This result allows as to obtain our comparability theorem for $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$.

5.2 Results on density

In the work [III], together with the Assouad codimension we use the Minkowski dimension. We present the definition below. Recall that $\widetilde{E}_r$ is defined in Section 4.1.

Definition 20. The **upper Minkowski dimension** of a set $E \subset \mathbb{R}^d$ is defined as

$$\overline{\dim}_M(E) = \inf\{s \ge 0 : \limsup_{r\to 0} \left|\widetilde{E}_r\right| r^{d-s} = 0\},$$

see for example [35, Section 2].

It is not hard to see that $\underline{\operatorname{co\,dim}}_A(E) \le d - \overline{\dim}_M(E)$ and the equality holds if E is $(d - \underline{\operatorname{co\,dim}}_A(E))$- homogeneous. Moreover (considering again open, bounded sets Ω), the **distance zeta function**

$$\zeta_\Omega(q) := \int_\Omega \frac{dx}{d_\Omega(x)^q}$$

is finite if $q < d - \overline{\dim}_M(\partial\Omega)$ and infinite if $q > d - \overline{\dim}_M(\partial\Omega)$ (see [35, Lemma 3.3 and Lemma 3.5]).

At first sight, it is not obvious that the space $W^{s,p;\alpha,\beta}(\Omega)$ is nontrivial and contains $C_c^\infty(\Omega)$ as a subset. We prove in [III, Lemma 16], using the comparability, that for bounded, uniform domains we indeed have $C_c^\infty(\Omega) \subset W^{s,p;\,\alpha,\beta}(\Omega)$ if $0 < s < 1$, $1 \le p < \infty$, $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ and $\alpha + \beta < d - \overline{\dim}_M(\partial\Omega) + p(1-s)$. When we abandon the assumption about the uniformity, then the same holds for $\alpha + \beta < d - \overline{\dim}_M(\partial\Omega)$. That ensures as that all considered spaces are nontrivial.

Similarly as before, by

$$W_0^{s,p;\,\alpha,\beta}(\Omega)$$

we denote the closure of $C_c^\infty(\Omega)$ in $W^{s,p;\,\alpha,\beta}(\Omega)$ with respect to the weighted Sobolev-type norm. We are interested in describing the form of the space $W_0^{s,p;\,\alpha,\beta}(\Omega)$, as for the unweighted case.

Finally, we are in position to list the main results from the paper [III]. They generalize the previous results from [II] to power weights.

Theorem 21. ([III, Theorem 3]) *Let $\Omega \subset \mathbb{R}^d$ be a nonempty, bounded, open set, let $0 < s < 1$, $1 \le p < \infty$ and $\alpha, \beta \ge 0$.*
(I) If $sp + \alpha + \beta < d - \overline{\dim}_M(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) = W^{s,p;\,\alpha,\beta}(\Omega)$.
(II) If Ω is $(d - sp - \alpha - \beta)$-homogeneous, $p > 1$ and $sp + \alpha + \beta = \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) = W^{s,p;\,\alpha,\beta}(\Omega)$.
(III) If Ω is κ-plump and $sp + \alpha + \beta > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) \neq W^{s,p;\,\alpha,\beta}(\Omega)$.

Theorem 22. ([III, Theorem 4]) *Let $\Omega \subset \mathbb{R}^d$ be a nonempty, bounded, uniform and open set, let $0 < s < 1$, $1 \le p < \infty$ and $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. If $sp + \alpha + \beta > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, then*

$$W_0^{s,p;\,\alpha,\beta}(\Omega) = \left\{ f \in W^{s,p;\,\alpha,\beta}(\Omega) : \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\,dx < \infty \right\}.$$

Theorem 23. ([III, Theorem 7]) *Let $1 \le p < \infty$ and $0 < s < 1$. Suppose that $\Omega \neq \emptyset$ is an open, uniform, bounded set such that $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ and $sp + \alpha + \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. Then there exists a constant c such that*

$$\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\,dx \le c\|f\|^p_{W^{s,p;\,\alpha,\beta}(\Omega)} < \infty,$$

for all $f \in W^{s,p;\,\alpha,\beta}(\Omega)$.

Our proofs are based on similar methods as in previous two works. The comparability is a completely new ingredient. Notice that in Theorems 22 and 23 we have the assumption about the uniformity of the domain Ω, which was not needed in the unweighted case. This is because in the proofs we need a comparability from Theorem 19. We do not know if these results can be extended to some non-uniform domains.

To summarize, in the article [III] we extended previously obtained results for unweighted fractional Sobolev spaces to the weighted ones with weights being powers of the distance to the boundary of the domain. The main difference is that in the weighted case we need the comparability between the full and truncated weighted Gagliardo seminorms, which is itself an interesting result.

6 Sharp Hardy inequalities for Sobolev–Bregman forms

In this section we describe results obtained in [IV]. We now drift away from fractional Sobolev spaces and focus on a related topic — **fractional Hardy inequalities**. Recall that the classical Hardy inequality in $\mathbb{R}^d$ has a form

$$\int_{\mathbb{R}^d} |\nabla f(x)|^p \, dx \geq \left(\frac{|d-p|}{p}\right)^p \int_{\mathbb{R}^d} \frac{|f(x)|^p}{|x|^p} \, dx, \tag{11}$$

where $f \in C_c^1(\mathbb{R}^d)$ for $p < d$ and $f \in C_c^1(\mathbb{R}^d \setminus \{0\})$, when $p > d$. The constant $\left(\frac{|d-p|}{p}\right)^p$ appearing in (11) is **optimal**, i.e. it cannot be replaced by a bigger one.

We say that an open set $\Omega \subset \mathbb{R}^d$ and parameters p and d **admit the Hardy inequality**, if

$$\int_{\Omega} |\nabla f(x)|^p \, dx \geq C(d, p, \Omega) \int_{\Omega} \frac{|f(x)|^p}{d_\Omega(x)^p} \, dx, \qquad f \in C_c^1(\Omega). \tag{12}$$

It is a classical fact that (12) holds when Ω is convex and $p > 1$. The optimal constant is then independent on Ω and given by $\left(\frac{p-1}{p}\right)^p$. Hardy inequalities were also investigated in abstract metric measure spaces, see the article [44] by Lehrbäck and the refereneces therein. The role of Hardy inequalities in analysis and PDE's is not to be underestimated. This inequality allows to control the behaviour of a function by its derivative. A good survey for Hardy inequalities is the book [54] by Kufner and Opic.

In the context of this thesis, we are interested in fractional Hardy inequalities. A general form of these inequalities is

$$\int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \geq C(d, s, p, \Omega) \int_{\Omega} \frac{|f(x)|^p}{d_\Omega(x)^{sp}} \, dx, \; f \in C_c^1(\Omega). \tag{13}$$

Let us briefly sketch the history of the fractional Hardy inequalites. One-dimensional versions of (13) appeared in the '60s in the works of Jakovlev [37] and Grisvard [29]. For $p = 2$ and $\Omega = \mathbb{R}^d$ (then we put $d_\Omega(x) = |x|$), the optimal constant in (13) has been computed by Herbst [36] and independently by Yafaev [62] and Beckner [6]. Multidimensional fractional Hardy inequalities became an object of interest in the '90s — for example in the articles of Heinig, Kufner and Persson [34], Mamedov [48], and a bit later of Chen and Song [17] (for $p = 2$). A big step forward in this theory was Dyda's work [21], where it is shown that (13) is satisfied, among others, when Ω is a Lipschitz domain and $sp > 1$. In [8] Bogdan and Dyda computed the optimal constant for $p = 2$ and the halfspace $D = \{(x_1 \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ and conjected that the same optimal constant is valid for all convex domains, similarly to the classical case. The optimal constant for the halfspace is given by

$$\kappa_{d,\alpha} = \frac{\pi^{\frac{d-1}{2}} \Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)} \frac{\mathrm{B}\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right) - 2^\alpha}{\alpha 2^\alpha}, \tag{14}$$

where B is the Euler Beta function. This conjecture was proved a few years later by Loss and Sloane [45]. In 2008 Frank and Seiringer published a celebrated paper [26] presenting a new abstract approach to fractional Hardy inequalities, which they use to compute the sharp constant in (13) for $\mathbb{R}^d$ and general $p \geq 1$, and later also for the halfspace [27].

Our contribution [IV] to the fractional Hardy inequalities is devoted to the parallel topic, related to **Bregman divergence** and **Sobolev–Bregman** forms. The right-hand side of (13) is replaced with the following form:

$$\mathcal{E}_p[u] := \frac{1}{2}\int_D\int_D (u(x)-u(y))(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle})|x-y|^{-d-\alpha}\,dy\,dx, \tag{15}$$

defined for $p \in (1,\infty)$ and $u\colon \mathbb{R}^d \to \mathbb{R}$, where

$$a^{\langle k\rangle} := |a|^k \operatorname{sgn} a, \quad a,k \in \mathbb{R}$$

is the **French power**. The parameter α plays the role of sp, so the notation is not consistent to previous works, but fits better for stochastic processes and is taken straightly from the work [8]. We call such integral forms the **Sobolev-Bregman forms**. In [IV] we extend the results of Bogdan and Dyda [8] and Loss and Sloane [45]. Also, our work is motivated by a recent paper of Bogdan, Jakubowski, Lenczewska and Pietruska-Pałuba [11], where similar ideas on $\mathbb{R}^d$ lead to results about the contractivity of the Feynman–Kac semigroup generated by fractional Laplacian with Hardy potential. Forms related to (15) and Bregman divergence appeared recently in [10] or [9].

For $\alpha \neq 1$ let

$$\kappa_{d,p,\alpha} = -\frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\left(\mathrm{B}\left(\tfrac{\alpha-1}{p}+1,-\alpha\right) + \mathrm{B}\left(\alpha-\tfrac{\alpha-1}{p},-\alpha\right) + \tfrac{1}{\alpha}\right) \geq 0. \tag{16}$$

Recall that $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $1/\Gamma$ can be extended analytically to the whole of $\mathbb{R}$, hence $\mathrm{B}(x,y)$ is well defined for all $x,y \neq 0,-1,-2,\ldots$. Noteworthy, $\kappa_{d,p,1} = 0$ (understood as the limit of $\kappa_{d,p,\alpha}$ as $\alpha \to 1$).

The following is the first main result of [IV]. It is an analogue of the fractional Hardy inequalities for Sobolev–Bregman form.

Theorem 24. ([IV, Theorem 2]) *Let $0<\alpha<2$, $d=1,2,\ldots$ and $1<p<\infty$. For every $u \in C_c(D)$,*

$$\mathcal{E}_p[u] \geq \kappa_{d,p,\alpha}\int_D \frac{|u(x)|^p}{x_d^\alpha}dx\,, \tag{17}$$

and the constant in (17) is the best possible, i.e. it cannot be replaced by a bigger one.

In [11], a similar inequality for the whole space $\mathbb{R}^d$ instead of D is proved. Our second main result is a direct generalization of Loss–Sloane work [45] — we prove a variant of fractional Hardy inequality for general domains and Sobolev–Bregman forms.

Theorem 25. ([IV, Theorem 2]) *Let Ω be an open, proper subset of $\mathbb{R}^d$ and let $1<\alpha<2$. Then, for $u \in C_c(\Omega)$,*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,p,\alpha}\int_\Omega \frac{|u(x)|^p}{m_\alpha(x)^\alpha}\,dx, \tag{18}$$

where

$$m_\alpha(x)^\alpha = \frac{\int_{\mathbb{S}^{d-1}}|\omega_d|^\alpha\,d\omega}{\int_{\mathbb{S}^{d-1}} d_{\omega,\Omega}(x)^{-\alpha}\,d\omega}, \quad d_{\omega,\Omega}(x) = \min\{|t| : x+t\omega \notin \Omega\}.$$

In particular, if Ω is convex, then

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,p,\alpha}\int_\Omega \frac{|u(x)|^p}{\operatorname{dist}(x,\partial\Omega)^\alpha}\,dx. \tag{19}$$

The constant in (19) is optimal.

Our proofs rely on methods from [8], [11], [45]. We also provide an abstract form of the Hardy inequality form Sobolev–Bregman forms in the spirit of Frank and Seiringer [26] and [27], see [IV, Lemma 2]. This is required to obtain Hardy inequality for general domains. The most difficult thing to prove is the verification of the optimality of the constant $\kappa_{d,p,\alpha}$ in the inequalities (17) and (19).

To summarize, in the work [IV] we focus on fractional Hardy inequalites for Sobolev–Bregman forms defined via Bregman divergence. Our main results are the analogues of fractional Hardy inequality for the halfspace and general convex domains.

7 Information about other research

In this chapter we briefly describe other results obtained during the PhD studies, which has not been published yet. When writing these words, the articles based on them are still under review and are available on the arXiv webpage.

7.1 Sharp weighted fractional Hardy inequalities

In the work [V] we focus on **weighted fractional Hardy inequalities**. A basic form of such inequality is

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} \operatorname{dist}(x,\partial\Omega)^{-\alpha} \operatorname{dist}(y,\partial\Omega)^{-\beta}\, dy\, dx \geq C \int_\Omega \frac{|f(x)|^p}{\operatorname{dist}(x,\partial\Omega)^{sp+\alpha+\beta}}\, dx, \tag{20}$$

where $u \in C_c(\Omega)$. We compute explicitly the sharp constant C in (20) when $\Omega = \mathbb{R}^d$, Ω is a halfspace or a convex domain. Sharpness of the obtained constants is proved. Some partial results for $\mathbb{R}^d$ were known before, see Abdellaoui and Bentifour [1]. We also present a limit of the inequality (20) for $s \to 0^+$, which is not possible in the unweighted case. Finally, assuming among others that $p \geq 2$, we derive a **weighted fractional Hardy–Sobolev–Maz'ya** inequality

$$\int_{\mathbb{R}^d_+} \int_{\mathbb{R}^d_+} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}} x_d^\alpha\, y_d^\beta\, dy\, dx - \mathcal{D} \int_{\mathbb{R}^d_+} \frac{|u(x)|^p}{x_d^{sp-\alpha-\beta}}\, dx \geq C \left(\int_{\mathbb{R}^d_+} |u(x)|^q x_d^{\frac{q}{p}(\alpha+\beta)}\, dx \right)^{\frac{p}{q}},$$

where $C = C(\alpha,\beta,d,s,p) > 0$ is a constant, $\mathcal{D}$ is an optimal constant for the halfspace $\mathbb{R}^d_+$ in the inequality (20) and $q = \frac{dp}{d-sp}$. Hardy–Sobolev–Maz'ya-type inequalities are interesting itself, because they combine both Hardy and Sobolev inequalities in one result.

7.2 Sharp fractional Hardy inequalities with a remainder for $1 < p < 2$

The paper [VI] is devoted to fractional Hardy inequalities like (12), but with the additional term on the left-hand side, which we call a **remainder**. Such inequalities were proved before by Frank and Seiringer in [26] and [27], but only for $p \geq 2$. We propose a different form of the remainder for this case, which is a new result. As an application, we prove that the fractional Hardy–Sobolev–Maz'ya inequality for the halfspace and convex domains is valid also in the range of $1 < p < 2$, which is a new result even for the unweighted case.

7.3 Asymptotics of weighted Gagliardo seminorms

In the paper [VII] we establish limits involving Gagliardo seminorms with power weights, as $s \to 1^-$ or $s \to 0^+$. Under appropriate assumptions on parameters and functions, we prove the relations

$$\lim_{s\to 1^-}(1-s)\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx = K_{d,p}\int_\Omega \frac{|\nabla f(x)|^p}{d_\Omega(x)^{\alpha+\beta}}\,dx,$$

$$\lim_{\alpha\to 0^+}\alpha\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^d|x|^\alpha|y|^\alpha}\,dy\,dx = 2\left|\mathbb{S}^{d-1}\right|\int_{\mathbb{R}^d}|f(x)|^p\,dx,$$

and

$$\lim_{\alpha\to d^-}(d-\alpha)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(x)-f(y)|^p}{|x-y|^d|x|^\alpha|y|^\alpha}\,dy\,dx = 2\left|\mathbb{S}^{d-1}\right|\int_{\mathbb{R}^d}\frac{|f(x)|^p}{|x|^{2d}}\,dx.$$

Moreover, we provide a nonlocal characterisation of classical Sobolev spaces with power weights. All results from [VII] generalize the well known papers of Bourgain, Brezis and Mironescu [12] and Maz'ya and Shaposhnikova [51].

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On density of smooth functions in weighted fractional Sobolev spaces

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ABSTRACT

We prove that smooth C^∞ functions are dense in weighted fractional Sobolev spaces on an arbitrary open set, under some mild conditions on the weight. We also obtain a similar result in non-weighted spaces defined by some kernel similar to $x \mapsto |x|^{-d-sp}$. One may consider the results to be a version of the Meyers–Serrin theorem.

1. Introduction

We discuss the problem of density of smooth functions in the fractional Sobolev space $W^{s,p}(\Omega)$, as well as in the weighted fractional Sobolev space $W^{s,p}(\Omega, w)$; for the definition of the latter we refer the reader to Section 4. It turns out that for weights w which are locally comparable to a constant on Ω or continuous, and which satisfy certain integrability property (9), smooth functions $C^\infty(\Omega)$ are dense in $W^{s,p}(\Omega, w)$, see Theorem 12.

Our strategy of the proof follows the approach of [5, proof of Theorem 3.25], in that we first decompose the function f being approximated into the sum of functions f_n supported on the (enlarged) Whitney cubes, which is done by using a partition of unity. Then we convolve each f_n with a dilation of a fixed smooth function. In the non-weighted case, the scale of the dilation is dependent on the size of the Whitney cube, to make sure that the support of the convolution does not grow too much. That way we obtain a family of linear operators P^{η_k}, each mapping the function to a smooth approximating function, with the error of approximation going to zero when η_k are sufficiently small. In the weighted case, the scale of the dilation

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is dependent also on the function being approximated, and the resulting approximating operators are no longer linear.

The proof works for general open sets $\Omega \subset \mathbb{R}^d$, and the result seems to be new in the case of the weighted Sobolev spaces, see Theorem 12, or a more general kernel, see Theorem 15. The other standard approach to prove such a density result is to use the extension theorem [8], however it does not hold for all open sets Ω.

Our paper is motivated by the article [2], where the authors consider a similar problem for weights $w(x) = |x|^{-a}$ in $\mathbb{R}^d$ (or, translated to our setting, in $\mathbb{R}^d \setminus \{0\}$). They consider however the density of the compactly supported smooth functions, the problem that we do not address. We note here that if one knows that the compactly supported functions (not necessarily smooth) are dense in $W^{s,p}(\Omega, w)$, then our result immediately gives the density of the space $\mathcal{C}_C^\infty(\Omega)$, see Proposition 2.

Let us also mention other articles on similar topics. In [4] Luiro and Vähäkangas considered slightly different fractional Sobolev spaces, that are equipped with the seminorm

$$|f|_{W^{s,p,K}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{sp}} K(x-y)\, dx\, dy \right)^{\frac{1}{p}},$$

where the kernel K does not have to be radial. The authors find some condition which is sufficient for the space $\mathcal{C}^\infty(\mathbb{R}^d) \cap W^{s,p,K}(\mathbb{R}^d)$ to be dense in $W^{s,p,K}(\mathbb{R}^d)$ (see [4], (3.8) and Lemma 3.4). We obtain a similar result, Theorem 15, with more general sets Ω, but less general kernels K.

In [3] Fiscella, Servadei and Valdinoci considered similar Sobolev space $X_0^{s,p}(\Omega)$ of functions f with the finite norm

$$\|f\|_{L^p(\mathbb{R}^d)} + \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x) - f(y)|^p K(x-y)\, dx\, dy \right)^{1/p},$$

but vanishing outside Ω, with some assumptions on the kernel K. The authors proved that the space $\mathcal{C}_C^\infty(\Omega)$ of smooth functions that are compactly supported in Ω, is dense in $X_0^{s,p}(\Omega)$, when Ω is either a hypograph or a domain with continuous boundary (see [3], Theorems 2 and 6).

In [1] Baalal and Berghout considered fractional Sobolev spaces with variable exponents $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ and proved that under certain conditions for the functions p and q, compactly supported, smooth functions are dense in $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$.

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2. Operator P^η

2.1. Definition

Let $\Omega \subset \mathbb{R}^d$ be an open set and $\mathcal{W} = \{Q_1, Q_2, \dots\}$ be a Whitney decomposition of Ω into cubes, like in [6]. Choose ε such that $(1+\varepsilon)^2 < \frac{5}{4}$. Let also $\{\psi_n : n \in \mathbb{N}\}$ be a partition of unity, that is $\psi_n(x) = 1$, when $x \in Q_n$, $\psi_n = 0$ outside Q_n^*, where Q_n^* is the cube Q_n "blown up" $1+\varepsilon$ times (the cube with the same center, but the length of the edge $1+\varepsilon$ times longer), ψ_n is a class of $\mathcal{C}_C^\infty$ and $\sum_{n=1}^\infty \psi_n = 1$. Let $p \in [1, \infty)$ and $f \in L^p(\Omega)$.

We note that $|\psi_n(x) - \psi_n(y)| \le \frac{C|x-y|}{l(Q_n)} \wedge 1$ for some constant $C > 0$.

Let us fix a function $h \colon \mathbb{R}^d \to \mathbb{R}$ such that $h \ge 0$, $\int_{\mathbb{R}^d} h(x)\, dx = 1$, $\operatorname{supp} h = B(0,1)$ and $h \in C^\infty(\mathbb{R}^d)$. For $\delta > 0$ we define the dilation

$$h_\delta(x) = \frac{1}{\delta^d} h\left(\frac{x}{\delta}\right), \quad (x \in \mathbb{R}^d).$$

The function h_δ is a class of $\mathcal{C}_C^\infty(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} h_\delta(x)\, dx = 1$ for every $\delta > 0$.

For a function $g : \mathbb{R}^d \to \mathbb{R}$ and $t \in \mathbb{R}^d$ we define its translation $\tau_t g$ by the formula

$$\tau_t g(x) = g(x - t), \quad (x \in \mathbb{R}^d).$$

Let $\eta : \mathcal{W} \to (0, \infty)$ be a function such that $\eta(Q) < \frac{\varepsilon}{2} l(Q)$ for every $Q \in \mathcal{W}$, where $l(Q)$ denotes the length of the edge of the cube Q. In particular, we may take $\eta = \delta l$, where $\delta \in (0, \varepsilon/2)$. For such a function η we define the operator P^η as

$$P^\eta f = \sum_{n=1}^{\infty} (f\psi_n) * h_{\eta(Q_n)}, \quad f \in L^1_{loc}(\Omega), \tag{1}$$

where we put $f = 0$ on $\mathbb{R}^d \setminus \Omega$.

Proposition 1. *The operator P^η is well defined and $P^\eta f \in \mathcal{C}^\infty(\Omega)$ for $f \in L^1_{loc}(\Omega)$.*

Proof. We observe that the function $(f\psi_n) * h_{\eta(Q_n)}$,

$$(f\psi_n) * h_{\eta(Q_n)}(x) = \int_{\mathbb{R}^d} f(x-y)\psi_n(x-y)h_{\eta(Q_n)}(y)\, dy,$$

vanishes outside Q_n^{**}. Indeed, if $x \notin Q_n^{**}$, then either $x - y \notin Q_n^*$, which implies $\psi_n(x-y) = 0$, or $y \notin B(0, \eta(Q_n))$, which implies $h_{\eta(Q_n)}(y) = 0$, because if $x - y \in Q_n^*$, then $x \in Q_n^* + y \subset Q_n^{**}$ for $|y| < \eta(Q_n)$, thanks to our choice of ε.

Since $Q_n^{**} \subset \frac{5}{4} Q_n$ by our choice of ε, each point $x \in \Omega$ belongs to at most 12^d cubes Q_n^{**} (see [6], chapter VI). Therefore the sum (1) has at each point only finitely many nonzero terms, thus the result follows. □

Proposition 2. *If $f \in L^1_{loc}(\Omega)$ satisfies $f = 0$ outside a compact set $K \subset \Omega$, then also $P^\eta(f) = 0$ outside some compact set $K' \subset \Omega$.*

Proof. We observe that only finitely many of the functions $f\psi_n$ are not identically zero. Since $\operatorname{supp}(f\psi_n) * h_{\eta(Q_n)} \subset Q_n^{**}$, it follows that $\operatorname{supp} P^\eta(f)$ is contained in a finite union of cubes Q_n^{**}, which is a compact subset of Ω. □

2.2. Convergence of the operator P^η in $L^p(\Omega)$

Theorem 3. *Let $p \in [1, \infty)$ and $f \in L^p(\Omega)$. Then*

$$\lim_{k \to \infty} \|P^{\eta_k} f - f\|_{L^p(\Omega)} = 0,$$

provided $\lim_{k\to\infty} \eta_k(Q) = 0$ for every $Q \in \mathcal{W}$.

Proof. We have

$$\begin{aligned}
\|P^{\eta_k} f - f\|^p_{L^p(\Omega)} &= \int_\Omega |P^{\eta_k} f(x) - f(x)|^p dx \\
&= \int_\Omega \left| \sum_{n=1}^{\infty} (f\psi_n) * h_{\eta_k(Q_n)}(x) - \sum_{n=1}^{\infty} f(x)\psi_n(x) \right|^p dx \\
&\le \int_\Omega \left(\sum_{n=1}^{\infty} \left| (f\psi_n) * h_{\eta_k(Q_n)}(x) - f(x)\psi_n(x) \right| \right)^p dx.
\end{aligned}$$

The sum above is finite at each point x and has at most 12^d nonzero terms. Thus, recalling that $\int_{\mathbb{R}^d} h_t(x)dx = 1$ for every $t > 0$ and using Jensen inequality we obtain

$$\begin{aligned}
&\int_\Omega \left(\sum_{n=1}^{\infty} \left| (f\psi_n) * h_{\eta_k(Q_n)}(x) - f(x)\psi_n(x) \right| \right)^p dx \\
&\leq M \int_\Omega \sum_{n=1}^{\infty} \left| (f\psi_n) * h_{\eta_k(Q_n)}(x) - f(x)\psi_n(x) \right|^p dx \\
&= M \int_\Omega \sum_{n=1}^{\infty} \left| \int_{\mathbb{R}^d} \left(f(x-y)\psi_n(x-y) - f(x)\psi_n(x) \right) h_{\eta_k(Q_n)}(y) dy \right|^p dx \\
&\leq M \sum_{n=1}^{\infty} \int_\Omega \int_{\mathbb{R}^d} |f(x-y)\psi_n(x-y) - f(x)\psi_n(x)|^p h_{\eta_k(Q_n)}(y) \, dy \, dx \\
&= M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_y (f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} h_{\eta_k(Q_n)}(y) \, dy \\
&= M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u} (f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} h(u) \, du, \qquad (2)
\end{aligned}$$

where $M = 12^{d(p-1)}$. Furthermore,

$$\int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u} (f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} h(u) \, du \leq 2^p \|f\psi_n\|^p_{L^p(\mathbb{R}^d)},$$

and

$$\sum_{n=1}^{\infty} \|f\psi_n\|^p_{L^p(\mathbb{R}^d)} = \sum_{n=1}^{\infty} \int_{Q_n^*} |f(x)\psi_n(x)|^p \, dx \leq 12^d \|f\|^p_{L^p(\mathbb{R}^d)} < \infty.$$

Since $\lim_{k\to\infty} \|\tau_{\eta_k(Q_n)u} (f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} = 0$, using Lebesgue dominated convergence theorem twice in (2) we get the assertion of the theorem. □

3. Sobolev Spaces

For a measurable function f defined on $\Omega \subset \mathbb{R}^d$, we define its *Gagliardo seminorm* by

$$[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \right)^{1/p}.$$

For $0 < s < 1$ and $1 \leq p < \infty$ we define the *fractional Sobolev space* $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) = \{f \in L^p(\Omega) : [f]_{W^{s,p}(\Omega)} < \infty\}.$$

3.1. Convergence of the operator P^η in Gagliardo seminorm

Lemma 4. *Suppose that $\Omega \subset \mathbb{R}^d$ and $f \in W^{s,p}(\Omega)$. Then*

$$[P^{\eta_k} f - f]^p_{W^{s,p}(\Omega)} \leq M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u} (g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} h(u) \, du, \qquad (3)$$

where $M = 12^{d(p-1)}$, and

$$g_n(x,y) = \begin{cases} \dfrac{f(x)\psi_n(x) - f(y)\psi_n(y)}{|x-y|^{\frac{d}{p}+s}}, & x, y \in \Omega; \\ 0, & (x,y) \in (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega \times \Omega). \end{cases} \qquad (4)$$

Furthermore,

$$\|g_n\|^p_{L^p(\mathbb{R}^{2d})} \le c(p,d,s)\left([f]^p_{W^{s,p}(Q_n^*)} + \|f\|^p_{L^p(Q_n^*)} l(Q_n)^{-sp}\right) < \infty \tag{5}$$

for some constant $c(p,d,s)$ depending only on p, d, s.

Proof. By arguments similar to that from the proof of Theorem 3,

$$[P^{\eta_k} f - f]^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|P^{\eta_k} f(x) - f(x) - P^{\eta_k} f(y) + f(y)|^p}{|x-y|^{d+sp}}\, dx\, dy$$

$$\le M \sum_{n=1}^\infty \int_\Omega \int_\Omega \int_{\mathbb{R}^d} \frac{|(f\psi_n)(x-t) - (f\psi_n)(x) - (f\psi_n)(y-t) + (f\psi_n)(y)|^p}{|x-y|^{d+sp}}$$

$$\times h_{\eta_k(Q_n)}(t)\, dt\, dx\, dy$$

$$\le M \sum_{n=1}^\infty \int_{\mathbb{R}^d} \|\tau_t(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} h_{\eta_k(Q_n)}(t)\, dt \tag{6}$$

$$= M \sum_{n=1}^\infty \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u}(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} h(u)\, du, \tag{7}$$

which proves the first part of the Lemma. To prove the remaining part, we observe that

$$\begin{aligned}|f(x)\psi_n(x) - f(y)\psi_n(y)|^p &= |f(x)\psi_n(x) - f(x)\psi_n(y) + f(x)\psi_n(y) - f(y)\psi_n(y)|^p \\ &\le 2^{p-1}(|f(x)|^p|\psi_n(x) - \psi_n(y)|^p + |\psi_n(y)|^p|f(x) - f(y)|^p).\end{aligned}$$

Since $\operatorname{supp}\psi_n \subset Q_n^*$,

$$\|g_n\|^p_{L^p(\mathbb{R}^{2d})} = \int_\Omega \int_\Omega \frac{|f(x)\psi_n(x) - f(y)\psi_n(y)|^p}{|x-y|^{d+sp}}\, dx\, dy$$

$$\le 2 \int_\Omega \int_{Q_n^*} \frac{|f(x)\psi_n(x) - f(y)\psi_n(y)|^p}{|x-y|^{d+sp}}\, dx\, dy$$

$$\le 2^p \int_\Omega \int_{Q_n^*} \frac{|f(x)|^p|\psi_n(x) - \psi_n(y)|^p}{|x-y|^{d+sp}}\, dx\, dy + 2^p \int_\Omega \int_{Q_n^*} \frac{|\psi_n(y)|^p|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dx\, dy$$

$$=: 2^p(I_1 + I_2).$$

We have $|\psi_n(y)| \le 1$, thus

$$I_2 \le \int_{Q_n^*} \int_{Q_n^*} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dx\, dy = [f]^p_{W^{s,p}(Q_n^*)} < \infty.$$

Since $|\psi_n(x) - \psi_n(x+w)| \le \frac{C|w|}{l(Q_n)} \wedge 1$, therefore

$$\begin{aligned}I_1 &= \int_{Q_n^*} \int_{\Omega - x} \frac{|f(x)|^p|\psi_n(x) - \psi_n(x+w)|^p}{|w|^{d+sp}}\, dw\, dx \\ &\le \int_{Q_n^*} |f(x)|^p \int_{\Omega - x} \left(\frac{C^p|w|^p}{l(Q_n)^p} \wedge 1\right) |w|^{-d-sp}\, dw\, dx \\ &\le C^{sp} \int_{Q_n^*} |f(x)|^p \int_{\mathbb{R}^d} (|z|^p \wedge 1)\, |z|^{-d-sp} l(Q_n)^{-sp}\, dz\, dx \\ &= C' \|f\|^p_{L^p(Q_n^*)} l(Q_n)^{-sp},\end{aligned}$$

with C' depending on s, d, p only. $\square$

Theorem 5. *Suppose that* $\Omega \subset \mathbb{R}^d$, $f \in W^{s,p}(\Omega)$ *and*

$$\int_\Omega \frac{|f(x)|^p}{\gamma(x)^{sp}}\,dx < \infty, \tag{8}$$

where $\gamma(x) = \operatorname{dist}(x, \Omega^c)$. *Then*

$$\lim_{k\to\infty} [P^{\eta_k} f - f]_{W^{s,p}(\Omega)} = 0,$$

provided $\lim_{k\to\infty} \eta_k(Q) = 0$ *for every* $Q \in \mathcal{W}$.

Proof. By (3) and $\lim\limits_{k\to\infty} \|\tau_{\eta_k(Q_n)u}(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} = 0$, it is enough to justify applications of Lebesgue dominated convergence theorem in Lemma 4. To this end, we observe that

$$\int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u}(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} h(u)\,du \le 2^p \|g_n\|^p_{L^p(\mathbb{R}^{2d})}.$$

Furthermore,

$$\sum_{n=1}^\infty [f]^p_{W^{s,p}(Q_n^*)} \le 12^d [f]^p_{W^{s,p}(\Omega)} < \infty$$

and, by Whitney decomposition properties, $l(Q_n) \ge \frac{\gamma(x)}{(5+\varepsilon)\sqrt{d}} \ge \frac{\gamma(x)}{6\sqrt{d}}$ for $x \in Q_n^*$, thus,

$$\sum_{n=1}^\infty \|f\|^p_{L^p(Q_n^*)} l(Q_n)^{-sp} \le \left(6\sqrt{d}\right)^{sp} \sum_{n=1}^\infty \int_{Q_n^*} \frac{|f(x)|^p}{\gamma(x)^{sp}}\,dx \le \left(6\sqrt{d}\right)^{sp} 12^d \int_\Omega \frac{|f(x)|^p}{\gamma(x)^{sp}}\,dx < \infty. \quad \square$$

We recall a geometric notion from [7].

Definition 6. A set $A \subset \mathbb{R}^d$ is *κ-plump* with $\kappa \in (0,1)$ if, for each $0 < r < \operatorname{diam}(A)$ and each $x \in \bar{A}$, there is $z \in \bar{B}(x,r)$ such that $B(z, \kappa r) \subset A$.

Corollary 7. *Suppose that* $\Omega \subset \mathbb{R}^d$ *is an open set such that its complement* Ω^c *is κ-plump with some* $\kappa \in (0,1)$, *and* $|\partial\Omega| = 0$. *Let* $f \in W^{s,p}(\mathbb{R}^d)$ *with* $f = 0$ *on* Ω^c. *Then*

$$\lim_{k\to\infty} [P^{\eta_k} f - f]_{W^{s,p}(\mathbb{R}^d)} = 0,$$

provided $\lim_{k\to\infty} \eta_k(Q) = 0$ *for every* $Q \in \mathcal{W}$.

Proof. We will show that such a function f satisfies the assumptions of Theorem 5 with the set $\mathbb{R}^d \setminus \partial\Omega$ in place of Ω. Indeed, thanks to our assumptions we have $\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} |f(x)|^p |x-y|^{-d-sp}\,dy\,dx < \infty$. Fix $R < \operatorname{diam}(\Omega^c)$ and let $x \in \Omega$ with $\gamma(x) = \operatorname{dist}(x, \Omega^c) < R$. Then

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\Omega} \frac{dy}{|x-y|^{d+sp}} &\ge \int_{B(x,2\gamma(x))\cap\Omega^c} \frac{dy}{|x-y|^{d+sp}} \\ &\ge C\gamma(x)^{-d-sp} |B(x, 2\gamma(x)) \cap \Omega^c| \ge C'\gamma(x)^{-sp},\end{aligned}$$

where the last inequality follows from the κ-plumpness of Ω^c. Thus

$$\int_{\{x\in\Omega:\,\gamma(x)<R\}} |f(x)|^p \gamma(x)^{-sp}\,dx < \infty.$$

Since $f \in L^p(\mathbb{R}^d)$ and $f = 0$ on Ω^c, it follows that $\int_{\mathbb{R}^d\setminus\partial\Omega} \frac{|f(x)|^p}{\gamma(x)^{sp}}\,dx < \infty$. $\square$

The next result is essentially a fractional counterpart of the Meyers–Serrin theorem. The proof may be found for example in [5, Theorem 3.25]. We nevertheless provide the proof using our notation, as it is going to be modified in the next section.

Theorem 8 *([5]). Let $p \in [1, \infty)$ and $s \in (0, 1)$. Then the functions of a class $\mathcal{C}^\infty(\Omega) \cap W^{s,p}(\Omega)$ are dense in $W^{s,p}(\Omega)$.*

Proof. Let us fix a function $f \in W^{s,p}(\Omega)$. Using notation (4) from Lemma 4, for all natural numbers k and n, we choose $\eta_k(Q_n) < \frac{\varepsilon}{2k} l(Q_n)$ small enough so that the following inequality holds,

$$\|\tau_t (g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} < \frac{1}{k\,2^n}, \quad 0 < t < \eta_k(Q_n).$$

Then from Lemma 4 it follows that

$$\begin{aligned}[P^{\eta_k} f - f]^p_{W^{s,p}(\Omega)} &= M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u} (g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} h(u)\, du \\ &\le M \sum_{n=1}^{\infty} \frac{1}{k\,2^n} = \frac{M}{k} \to 0,\end{aligned}$$

when $k \to \infty$. The convergence $P^{\eta_k} f \to f$ in $L^p(\Omega)$ follows from Theorem 3, because $\eta_k(Q) \to 0$ for each $Q \in \mathcal{W}$. Finally, $P^{\eta_k} f \in C^\infty(\Omega)$ by Proposition 1. □

4. Convergence in weighted spaces

In this section we extend our results to the case of weighted Sobolev spaces. Namely, for a *weight* w, i.e., a nonnegative measurable function, we define the seminorm

$$[f]_{W^{s,p}(\Omega,w)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} w(y)w(x)\, dy\, dx \right)^{\frac{1}{p}},$$

and the weighted L^p norm

$$\|f\|_{L^p(\Omega,w)} = \left(\int_\Omega |f(x)|^p w(x)\, dx \right)^{1/p}.$$

We also denote

$$\widetilde{W}^{s,p}(\Omega, w) = \left\{ f\colon \Omega \to \mathbb{R} : f \text{ measurable, } [f]_{W^{s,p}(\Omega,w)} < \infty \right\}.$$

Proposition 9. *If w is locally comparable to a constant, that is for every compact $K \subset \Omega$ there is a constant $C_K > 0$ such that $\frac{1}{C_K} \le w(x) \le C_K$ for all $x \in K$, then $\widetilde{W}^{s,p}(\Omega, w) \subset L^p_{loc}(\Omega)$.*

Proof. Fix two compact sets $K, L \subset \Omega$ of positive measure and let $C = \sup_{x\in K} \sup_{y\in L} |x - y| < \infty$. To prove the inclusion, let us see that

$$\infty > \int_L \int_K \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} w(x)w(y)\, dx\, dy \ge C^{-d-sp} \int_L \int_K |f(x) - f(y)|^p w(x)w(y)\, dx\, dy.$$

By Fubini–Tonelli theorem, the inner integral $\int_K |f(x) - f(y)|^p w(x)\, dx$ is finite for almost all $f(y)$. Hence, for such $f(y)$, using the triangle inequality and the local boundedness of w, we have

$$\int_K |f(x)|^p w(x)\, dx \le 2^{p-1} \left(\int_K |f(x) - f(y)|^p w(x)\, dx + |f(y)|^p \int_K w(x)\, dx \right) < \infty.$$

Now,

$$\int_K |f(x)|^p\,dx \le C_K \int_K |f(x)|^p w(x)\,dx < \infty. \quad \square$$

Remark 10. If w is continuous, then we can change Ω to $\Omega' = \Omega \setminus \{x : w(x) = 0\}$. The set Ω' is still open and w is locally comparable to a constant on Ω', so we can consider the space $W^{s,p}(\Omega', w)$ instead of $W^{s,p}(\Omega, w)$.

Lemma 11. *If $y \in Q_n^*$ and $x \notin Q_n^{**}$, then $|x-y| \ge \frac{\varepsilon}{\varepsilon+\sqrt{d}}|x-x_n|$, where x_n is the center of cube Q_n.*

Proof. We have $|x-y| \ge \frac{(1+\varepsilon)^2 l(Q_n)-(1+\varepsilon)l(Q_n)}{2} = \frac{\varepsilon(1+\varepsilon)l(Q_n)}{2}$ and $|y-x_n| \le \operatorname{diam} Q_n^*/2 = (1+\varepsilon)l(Q_n)\sqrt{d}/2$. Hence, $|x-y| \ge \frac{\varepsilon(1+\varepsilon)}{2}\frac{2}{(1+\varepsilon)\sqrt{d}}|y-x_n| = \frac{\varepsilon}{\sqrt{d}}|y-x_n|$. The assertion of the lemma follows from triangle inequality $|x-x_n| \le |x-y| + |y-x_n|$. $\square$

Theorem 12. *Suppose that w is locally comparable to a constant or continuous and satisfies the condition*

$$\int_\Omega \frac{w(x)}{(1+|x|)^{d+sp}}\,dx < \infty \tag{9}$$

Then $C^\infty(\Omega) \cap \widetilde{W}^{s,p}(\Omega, w)$ is dense in $\widetilde{W}^{s,p}(\Omega, w)$.

Proof. We extend w to be 0 outside Ω. If w is continuous, then we use Remark 10 and change Ω to Ω' in all the computations below. Similarly as in the previous cases, using the notations (4) from Lemma 4, we have

$$[P^{\eta_k} f - f]^p_{W^{s,p}(\Omega,w)} \le M \sum_{n=1}^\infty \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u}(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d}, w\times w)} h(u)\,du.$$

We obtain for $t < \eta_k(Q_n)$,

$$\begin{aligned}
&\|\tau_t(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d}, w\times w)} \\
&\le \int_{Q_n^*}\int_{Q_n^{**}} \frac{|f(x-t)\psi_n(x-t) - f(y-t)\psi_n(y-t) - f(x)\psi_n(x) + f(y)\psi_n(y)|^p}{|x-y|^{d+sp}} w(y)w(x)\,dy\,dx \\
&+ 2\int_{Q_n^*}\int_{\Omega\setminus Q_n^{**}} \frac{|f(x)\psi_n(x) - f(x-t)\psi_n(x-t)|^p}{|x-y|^{d+sp}} w(y)w(x)\,dy\,dx \\
&=: I_1 + 2I_2.
\end{aligned}$$

For the integral I_1 we have the following estimate

$$\begin{aligned}
I_1 &\le C_n^2 \int_{Q_n^*}\int_{Q_n^{**}} \frac{|f(x-t)\psi_n(x-t) - f(y-t)\psi_n(y-t) - f(x)\psi_n(x) + f(y)\psi_n(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&\le C_n^2 \|\tau_t(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})},
\end{aligned}$$

where $C_n = \sup_{x\in Q_n^{**}} w(x)$. Let us now focus on the integral I_2. Using Lemma 11, if $x \in Q_n^*$ and $y \notin Q_n^{**}$, then $|x-y| \ge c|y-x_n|$ for $c = \varepsilon/(\varepsilon+\sqrt{d})$, when x_n is the center of the cube Q_n. Thus, we obtain

$$\begin{aligned}
I_2 &\le c^{-d-sp} \int_{Q_n^*}\int_{\Omega\setminus Q_n^{**}} \frac{|f(x)\psi_n(x) - f(x-t)\psi_n(x-t)|^p}{|y-x_n|^{d+sp}} w(y)w(x)\,dy\,dx \\
&\le C_n c^{-d-sp} \int_{Q_n^*} |f(x)\psi_n(x) - f(x-t)\psi_n(x-t)|^p\,dx \int_{\Omega\setminus Q_n^{**}} \frac{w(y)}{|y-x_n|^{d+sp}}\,dy \\
&\le D_n \|\tau_t(f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)},
\end{aligned}$$

where, thanks to Proposition 9 the norm above is finite and

$$D_n = C_n c^{-d-sp} \int_{\Omega \setminus Q_n^{**}} \frac{w(y)}{|y - x_n|^{d+sp}} \, dy.$$

The integral above is finite, because for $y \notin Q_n^{**}$ it holds $|y - x_n| \geq l(Q_n)/2$ and $|y - x_n| \geq |y| - |x_n|$, therefore $|y - x_n|$ is bounded from below by a constant multiple of $1 + |y|$.

Now we need to repeat the proof of Theorem 8: for all natural numbers k and n, we choose $\eta_k(Q_n) < \frac{\varepsilon}{2k} l(Q_n)$ such that

$$\|\tau_t(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d})} < \frac{1}{k 2^{n+1} C_n^2}$$

and

$$\|\tau_t(f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} < \frac{1}{k 2^{n+2} D_n},$$

for $0 < t < \eta_k(Q_n)$. Hence,

$$\begin{aligned} [P^{\eta_k} f - f]^p_{W^{s,p}(\Omega, w)} &\leq M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u}(g_n) - g_n\|^p_{L^p(\mathbb{R}^{2d}, w \times w)} h(u) \, du \\ &\leq \frac{M}{k} \to 0, \end{aligned}$$

when $k \to \infty$. $\square$

Theorem 13. *Suppose that w is locally comparable to a constant or continuous. Then $C^\infty(\Omega) \cap L^p(\Omega, w)$ is dense in $L^p(\Omega, w)$.*

Proof. If w is continuous, then, according to Remark 10 we should replace Ω by Ω'. Analogously as in the proof of Theorem 3 we obtain that

$$\|P^{\eta_k} f - f\|^p_{L^p(\Omega, w)} \leq M \sum_{n=1}^{\infty} C_n \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u}(f\psi_n) - f\psi_n\|^p_{L^p(\Omega, w)} h(u) \, du.$$

Since the function $\tau_{\eta_k(Q_n)u}(f\psi_n)$ has support in Q_n^{**} for $u \in \operatorname{supp} h$, taking $C_n = \sup_{x \in Q_n^{**}} w(x)$ we obtain

$$\|\tau_{\eta_k(Q_n)u}(f\psi_n) - f\psi_n\|^p_{L^p(\Omega, w)} \leq C_n \|\tau_{\eta_k(Q_n)u}(f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)}.$$

We proceed as in the proof of Theorem 12 by choosing $\eta_k(Q_n) < \frac{\varepsilon}{2k} l(Q_n)$ such that $\|\tau_t(f\psi_n) - f\psi_n\|^p_{L^p(\mathbb{R}^d)} < \frac{1}{k 2^{n+1} C_n}$ for $0 < t < \eta_k(Q_n)$ and we obtain the desired result. $\square$

Remark 14. Suppose that $\Omega = \mathbb{R}^d \setminus \{0\}$ and $w(x) = |x|^{-a}$. The condition (9) becomes

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{dx}{|x|^a (1 + |x|)^{d+sp}} < \infty,$$

which is equivalent to

$$a \in (-sp, d).$$

Analogous, but slightly different weighted Sobolev spaces were considered in [2]. Dipierro and Valdinoci considered density of compactly supported smooth functions in weighted Sobolev space $\dot{W}^{s,p}(\mathbb{R}^d) = \widetilde{W}^{s,p}(\mathbb{R}^d, w) \cap L^{p_s^*}(\mathbb{R}^d, |\cdot|^{-2a/p})$ for $a \in [0, \frac{d-sp}{2})$ and $p_s^* = \frac{dp}{d-sp}$. Notice that however we do not have density of compactly supported functions, Theorems 12 and 13 combined provide a larger scale of the parameter a and a general exponent q instead of p_s^*. We can also change $\mathbb{R}^d \setminus \{0\}$ for any open set Ω.

Appendix

In this section we show how to generalise the results to the case of Sobolev spaces defined by some kernel K, see below.

Theorem 15. *Let $p \in [1,\infty)$, $\Omega \subset \mathbb{R}^d$ be an open set and let $K\colon [0,\infty) \to [0,\infty)$ be a measurable function such that*

$$\int_0^\infty (x^p \wedge 1)\, K(x) x^{d-1}\, dx < \infty.$$

Denote

$$[f]_K := \left(\int_\Omega \int_\Omega |f(x) - f(y)|^p K(|x-y|)\, dy\, dx \right)^{1/p}$$

and consider the space

$$X(\Omega) = \{ f \in L^p(\Omega) : [f]_K < \infty \}$$

with the norm

$$\|f\|_{X(\Omega)} = \left(\int_\Omega |f(x)|^p\, dx + [f]_K^p \right)^{\frac{1}{p}}.$$

Then the functions of a class $\mathcal{C}^\infty(\Omega) \cap X(\Omega)$ are dense in $(X(\Omega), \|\cdot\|_{X(\Omega)})$.

Proof. First we go through the proof of Lemma 4, where we now estimate the seminorm $[P^{\eta_k} f - f]_K^p$ and take

$$g_n(x,y) = (f(x)\psi_n(x) - f(y)\psi_n(y))\, K(|x-y|)^{\frac{1}{p}}, \quad \text{for } x, y \in \Omega.$$

The only part of the proof that essentially changes is the estimate of I_1, which becomes

$$\begin{aligned} I_1 &= \int_{Q_n^*} \int_{\Omega - x} |f(x)|^p |\psi_n(x) - \psi_n(x+w)|^p K(|w|)\, dw\, dx \\ &= C' \|f\|^p_{L^p(Q_n^*)} \int_{\mathbb{R}^d} \left(\frac{C^p |w|^p}{l(Q_n)^p} \wedge 1 \right) K(|w|)\, dw\, dx. \end{aligned}$$

We observe that

$$\int_{\mathbb{R}^d} \left(\frac{C^p |w|^p}{l(Q_n)^p} \wedge 1 \right) K(|w|)\, dw \le \left(\frac{C^p}{l(Q_n)^p} \vee 1 \right) \int_{\mathbb{R}^d} (|w|^p \wedge 1)\, K(|w|)\, dw < \infty.$$

Having established an analogous version of Lemma 4, we proceed as in the proof of Theorem 8 and obtain the desired result. □

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On density of compactly supported smooth functions in fractional Sobolev spaces

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Abstract
We describe some sufficient conditions, under which smooth and compactly supported functions are or are not dense in the fractional Sobolev space $W^{s,p}(\Omega)$ for an open, bounded set $\Omega \subset \mathbb{R}^d$. The density property is closely related to the lower and upper Assouad codimension of the boundary of Ω. We also describe explicitly the closure of $C_c^\infty(\Omega)$ in $W^{s,p}(\Omega)$ under some mild assumptions about the geometry of Ω. Finally, we prove a variant of a fractional order Hardy inequality.

Keywords Fractional Sobolev spaces · Smooth functions · Density · Assouad codimension · Assouad dimension · Fractional Hardy inequality

Mathematics Subject Classification Primary 46E35 · Secondary 35A15 · 26D15

1 Introduction

We discuss the problem of density of compactly supported smooth functions in the fractional Sobolev space $W^{s,p}(\Omega)$, which is well known to hold when Ω is a bounded Lipschitz domain and $sp \leq 1$ [14, Theorem 1.4.2.4],[26, Theorem 3.4.3]. We extend this result to bounded, plump open sets with a dimension of the boundary satisfying certain inequalities. To this end, we use the Assouad dimensions and codimensions. We also describe explicitly the closure of $C_c^\infty(\Omega)$ in the fractional Sobolev space, provided that Ω satisfies the fractional Hardy inequality.

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $0 < s < 1$ and $1 \leq p < \infty$. We recall that the *fractional Sobolev space* is defined as

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$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) \, : \, \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} \, dy \, dx < \infty \right\}.$$

This is a Banach space endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Omega)},$$

where $[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \, dy \, dx \right)^{1/p}$ is called the *Gagliardo seminorm*. Throughout the paper we consider only real-valued functions, but we note that all results are clearly valid also for complex-valued functions, by means of decomposing them into a sum of real and imaginary part.

Definition 1 By $W_0^{s,p}(\Omega)$ we denote the closure of $C_c^\infty(\Omega)$ (the space of all smooth functions with compact support in Ω) in $W^{s,p}(\Omega)$ with respect to the Sobolev norm.

The following theorem is our main result on the connection between $W_0^{s,p}(\Omega)$ and $W^{s,p}(\Omega)$. For the relevant geometric definitions, we refer the Reader to Sect. 2. Here we only note that for bounded Lipschitz domains one has $\underline{\operatorname{co\,dim}}_A(\partial\Omega) = \overline{\operatorname{co\,dim}}_A(\partial\Omega) = 1$ and the other geometrical assumptions of Theorem 2 do hold (that is, bounded Lipschitz domains are $(d-1)$-homogeneous and κ-plump), hence the classical case is included.

Theorem 2 *Let* $\Omega \subset \mathbb{R}^d$ *be a nonempty bounded open set, let* $0 < s < 1$ *and* $1 \le p < \infty$.

(I) *If* $sp < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, *then* $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$.
(II) *If* Ω *is a* $(d-sp)$*-homogeneous set*, $sp = \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ *and* $p > 1$, *then* $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$.
(*III*) *If* Ω *is* κ*-plump and* $sp > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, *then* $W_0^{s,p}(\Omega) \neq W^{s,p}(\Omega)$.

We remark that a result similar to the part (I) and (III) in the Theorem 2 was obtained by Caetano in [6] in the context of Besov spaces and Triebel–Lizorkin spaces, but with the Minkowski dimension instead of Assouad dimension. That result is not directly comparable with ours, as for less regular domains spaces $W^{s,p}$ do not necessarily coincide with the appropriate Triebel–Lizorkin spaces. We refer the Reader to [5] for a discussion on the space $W_0^{s,p}$ and different similarly defined spaces. We also want to mention that analogous, but slightly different problems were considered in [12] (spaces of functions vanishing outside Ω), [8] (the weighted case) and [1] (spaces with variable exponents).

In the case (III) above, we also obtain the following characterization of the space $W_0^{s,p}(\Omega)$. For the proof, see Sect. 5.

Theorem 3 *Let* $0 < s < 1$ *and* $1 \le p < \infty$. *Suppose that* $\Omega \neq \emptyset$ *is a bounded, open* κ*-plump set. If* $\overline{\operatorname{co\,dim}}_A(\partial\Omega) < sp$, *then*

$$W_0^{s,p}(\Omega) = \left\{ f \in W^{s,p}(\Omega) \, : \, \int_\Omega \frac{|f(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}} \, dx < \infty \right\}. \tag{1}$$

In the case (I) of Theorem 2 equality (1) also holds, or in other words, we have an inclusion between the Sobolev and weighted L^p space, $W^{s,p}(\Omega) \subset L^p(\Omega, \operatorname{dist}(x, \partial\Omega)^{-sp})$. This fact is made quantitative in the next theorem; for its proof, see Sect. 5 as well.

Theorem 4 *Let* $0 < s < 1$ *and* $1 \le p < \infty$. *Suppose that* $\Omega \neq \emptyset$ *is a bounded, open* κ*-plump set. If* $\underline{\operatorname{co\,dim}}_A(\partial\Omega) > sp$, *then there exists a constant* c *such that*

$$\int_\Omega \frac{|f(x)|^p}{\operatorname{dist}(x, \partial\Omega)^{sp}}\, dx \le c\|f\|^p_{W^{s,p}(\Omega)} < \infty, \quad \text{for all } f \in W^{s,p}(\Omega). \tag{2}$$

Theorem 3 and 4 have classical (non-fractional) counterparts, see [20, Example 9.11] or [19].

Finally, we extend the results of [11, Theorem 1, Corollary 3]. Namely, we prove the case (T') in the following version of the fractional Hardy inequality. For the definitions of the conditions WLSC and WUSC , we refer the reader to the Appendix, while the plumpness and Assouad dimensions are defined in Sect. 2. We would also like to note that a special case of (T') (assuming in particular $p = 2$) was proved in [25, Lemma 3.32] and [7].

Theorem 5 ([11] in cases (T) *and* (F)) *Let* $0 < p < \infty$, $H \in (0, 1]$ *and* $\eta \in \mathbb{R}$. *Suppose* $\Omega \neq \emptyset$ *is a proper* κ*-plump open set in* $\mathbb{R}^d$ *and* $\phi : (0, \infty) \to (0, \infty)$ *is a function so that either condition* (*T*), *or condition* (*T'*), *or condition* (*F*) *holds*

(T) $\eta + \overline{\dim}_A(\partial\Omega) - d < 0$, Ω is unbounded, $\phi \in$ WUSC $(\eta, 0, H^{-1})$,
(T') $\eta + \dim_A(\partial\Omega) - d < 0$, Ω is bounded, $\phi \in$ WUSC $(\eta, 0, H^{-1})$,
(F) $\eta + \underline{\dim}_A(\partial\Omega) - d > 0$, Ω is bounded or $\partial\Omega$ is unbounded, and $\phi \in$ WLSC $(\eta, 0, H)$.

Then there exist constants $c = c(d, s, p, \Omega, \phi)$ and R such that the following inequality

$$\int_\Omega \frac{|u(x)|^p}{\phi(d_\Omega(x))}\, dx \le c \int_\Omega \int_{\Omega \cap B(x, R d_\Omega(x))} \frac{|u(x) - u(y)|^p}{\phi(d_\Omega(x)) d_\Omega(x)^d}\, dy\, dx + c\xi \|u\|^p_{L^p(\Omega)}, \tag{3}$$

holds for all measurable functions u *for which the left hand side is finite, with* $\xi = 0$ *in the cases* (*T*) *and* (*F*) *and* $\xi = 1$ *in the case* (*T'*).

There is a huge literature about fractional Hardy inequalities; we refer the Reader to [9, 11, 17] and the references therein. We would also like to draw Reader's attention to a paper [23] from 1999 by Farman Mamedov. This not very well-known paper is one of the first to deal with multidimensional fractional order Hardy inequalities.

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2 Geometrical definitions

We denote the distance from $x \in \mathbb{R}^d$ to a set $E \subset \mathbb{R}^d$ by $\operatorname{dist}(x, E) = \inf_{y \in E} |x - y|$; for open sets $\Omega \subset \mathbb{R}^d$ we write $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Definition 6 Let $r > 0$. For open sets $\Omega \subset \mathbb{R}^d$, we define the *inner tubular neighbourhood* of Ω as

$$\Omega_r = \left\{ x \in \Omega : d_\Omega(x) \le r \right\},$$

and for arbitrary sets $E \subset \mathbb{R}^d$, we define the *tubular neighbourhood* of E as

$$\widetilde{E}_r = \left\{ x \in \mathbb{R}^d : \operatorname{dist}(x, E) \le r \right\}.$$

Definition 7 [18, Section 3] Let $E \subset \mathbb{R}^d$. The *lower Assouad codimension* $\underline{\operatorname{co\,dim}}_A(E)$ is defined as the supremum of all $q \ge 0$, for which there exists a constant $C = C(q) \ge 1$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$, it holds

$$\left| \widetilde{E}_r \cap B(x,R) \right| \le C |B(x,R)| \left(\frac{r}{R} \right)^q.$$

Conversely, the *upper Assouad codimension* $\overline{\operatorname{co\,dim}}_A(E)$ is defined as the infimum of all $s \ge 0$, for which there exists a constant $c = c(s) > 0$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$, it holds

$$\left| \widetilde{E}_r \cap B(x,R) \right| \ge c |B(x,R)| \left(\frac{r}{R} \right)^s.$$

We remark that having strict inequality $R < \operatorname{diam} E$ above makes the definitions applicable also for unbounded sets E; for bounded sets E we could have $R \le \operatorname{diam} E$.

In Euclidean space $\mathbb{R}^d$, we have $\underline{\dim}_A(E) = d - \overline{\operatorname{co\,dim}}_A(E)$, $\overline{\dim}_A(E) = d - \underline{\operatorname{co\,dim}}_A(E)$, where $\underline{\dim}_A(E)$ and $\overline{\dim}_A(E)$ denote, respectively, the well known lower and upper Assouad dimension – see for example [18, Section 2] for this result. Recall that the upper Assouad dimension of a given set E is defined as the infimum of all exponents $s \ge 0$ for which there exists a constant $C = C(s) \ge 1$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$ the ball $B(x,R) \cap E$ can be covered by at most $C(R/r)^s$ balls with radius r, centered at E. Analogously, the lower Assouad dimension is characterized by the supremum of all exponents $t \ge 0$ for which there is a constant $c = c(t) > 0$ such that the ball $B(x,R) \cap E$ can be covered by at least $\mathrm{c}(R/r)^t$ balls with radius r and centered at E. If $\underline{\operatorname{co\,dim}}_A(E) = \overline{\operatorname{co\,dim}}_A(E)$, we simply denote it by $\operatorname{co\,dim}_A(E)$.

We recall a geometric notion from [27].

Definition 8 A set $E \subset \mathbb{R}^d$ is κ *-plump* with $\kappa \in (0,1)$ if, for each $0 < r < \operatorname{diam}(E)$ and each $x \in \overline{E}$, there is $z \in \overline{B}(x,r)$ such that $B(z, \kappa r) \subset E$.

Following [22, Theorem A.12], we define a notion of σ-homogenity.

Definition 9 Let $E \subset \mathbb{R}^d$ and let $V(E,x,\lambda,r) = \{ y \in \mathbb{R}^d : \operatorname{dist}(y,E) \le r, |x-y| \le \lambda r \}$. We say that E is *$\sigma$-homogeneous*, if there exists a constant L such that

$$|V(E,x,\lambda,r)| \le L r^d \lambda^\sigma$$

for all $x \in E$, $\lambda \ge 1$ and $r > 0$.

If $0 < r < R < \operatorname{diam}(E)$, then taking $\lambda = R/r$ in the definition gives

$$\left|\widetilde{E}_r \cap B(x,R)\right| = \left|V\left(E,x,\frac{R}{r},r\right)\right| \le C|B(x,R)|\left(\frac{r}{R}\right)^{d-\sigma},$$

where $C = C(d,E)$ is a constant. This means that if $\underline{\operatorname{co\,dim}}_A(E) = s$, then $(d-s)$-homogeneous sets are precisely these sets E, for which the supremum in the definition of the lower Assouad codimension is attained. For the definition of the concept of homogenity from a different point of view, the Reader may also see [22, Definition 3.2].

Finally, let us note that for example in part I of Theorem 2, we need the assumption $sp < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ only to obtain the bound (5). For that a slightly weaker assumption in terms of Minkowski (co)dimension would suffice, however, we need Assouad (co)dimensions for other parts of the paper, and therefore, we prefer to use only them. Let us only recall that the upper Minkowski dimension of a set $E \subset \mathbb{R}^d$ is defined as

$$\overline{\dim}_M(E) = \inf\{s \ge 0 : \limsup_{r\to 0} \left|\widetilde{E}_r\right| r^{d-s} = 0\},$$

see for example [15, Section 2]. The statement of the part (I) of Theorem 2 remains true if we assume that $sp < d - \overline{\dim}_M(\partial\Omega)$.

3 Lemmas

The following lemma is the key to our further computations. We recall that $\Omega_{\frac{3}{n}}$ appearing in (4) is the inner tubular neighbourhood of Ω, see Definition 6.

Lemma 10 *Let*

$$v_n(x) = \max\left\{\min\left\{2 - nd_\Omega(x), 1\right\}, 0\right\} = \begin{cases} 1 & \text{when } d_\Omega(x) \le 1/n, \\ 2 - nd_\Omega(x) & \text{when } 1/n < d_\Omega(x) \le 2/n, \\ 0 & \text{when } d_\Omega(x) > 2/n. \end{cases}$$

There exists a constant $C = C(d,s,p,\Omega) > 0$ *such that the following inequality holds for all functions* $f \in W^{s,p}(\Omega)$

$$[fv_n]^p_{W^{s,p}(\Omega)} \le Cn^{sp}\int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx + C\int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dy\,dx. \tag{4}$$

Proof Fix $f \in W^{s,p}(\Omega)$ and define $f_n = fv_n$. We have

$$\begin{aligned}
[f_n]^p_{W^{s,p}(\Omega)} &= \int_\Omega\int_\Omega \frac{|f(x)v_n(x)-f(y)v_n(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&= \int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)v_n(x)-f(y)v_n(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&\quad + 2\int_{\Omega_{\frac{2}{n}}}\int_{\Omega\setminus\Omega_{\frac{3}{n}}} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&=: J_1 + 2J_2.
\end{aligned}$$

First we estimate J_1,

$$
\begin{aligned}
2^{1-p}J_1 &\leq \int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p|v_n(x)-v_n(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&\quad + \int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|v_n(y)|^p|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&=: K_1 + K_2.
\end{aligned}
$$

Since $|v_n| \leq 1$, we obtain

$$
K_2 \leq \int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dy\,dx.
$$

Furthermore, $|v_n(x)-v_n(y)| \leq \min\{1, n|x-y|\}$, hence, for K_1 we can compute that

$$
\begin{aligned}
K_1 &\leq \int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p(\min\{1,n|x-y|\})^p}{|x-y|^{d+sp}}\,dy\,dx \\
&\leq \int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx \int_{|x-y|>1/n} \frac{dy}{|x-y|^{d+sp}} + n^p \int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx \int_{|x-y|<1/n} \frac{dy}{|x-y|^{d-(1-s)p}} \\
&\leq Cn^{sp}\int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx.
\end{aligned}
$$

Since $|v_n| \leq 1$, for J_2 we have

$$
\begin{aligned}
J_2 &= \int_{\Omega_{\frac{2}{n}}}\int_{\Omega\setminus\Omega_{\frac{3}{n}}} \frac{|f(x)|^p|v_n(x)|^p}{|x-y|^{d+sp}}\,dy\,dx \\
&\leq \int_{\Omega_{\frac{2}{n}}} |f(x)|^p\,dx \int_{\Omega\setminus\Omega_{\frac{3}{n}}} \frac{dy}{|x-y|^{d+sp}} \\
&\leq \int_{\Omega_{\frac{2}{n}}} |f(x)|^p\,dx \int_{B(x,1/n)^c} \frac{dy}{|x-y|^{d+sp}} \\
&\leq Cn^{sp}\int_{\Omega_{\frac{2}{n}}} |f(x)|^p\,dx.
\end{aligned}
$$

Hence, we obtain for some (new) constant C that

$$
[f_n]^p_{W^{s,p}(\Omega)} \leq Cn^{sp}\int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx + C\int_{\Omega_{\frac{3}{n}}}\int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dy\,dx.
$$

□

Definition 11 By $W^{s,p}_c(\Omega)$, we denote the closure of all compactly supported functions in $W^{s,p}(\Omega)$ (not necessarily smooth) with respect to the Sobolev norm.

The key property, which allows us to get rid of the smoothness and rely only on the compactness of the support, is the result below.

Proposition 12 *We have* $W_0^{s,p}(\Omega) = W_c^{s,p}(\Omega)$.

Proof This is a straightforward consequence of [10, Proposition 2 and proof of Theorem 8]. ☐

It turns out that to prove the density of compactly supported functions in the fractional Sobolev space, we only need to find a sequence which approximates the function $\mathbb{1}_\Omega$ (the indicator of Ω).

Lemma 13 *Let* Ω *be an open set such that* $|\Omega| < \infty$. *We have*

$$W_0^{s,p}(\Omega) = W^{s,p}(\Omega) \Longleftrightarrow \mathbb{1}_\Omega \in W_0^{s,p}(\Omega)$$

Proof Implication "$\Longrightarrow$" is obvious, therefore we proceed to prove the implication from right to left. According to Proposition 12, we need to prove that if the function $\mathbb{1}_\Omega$ can be approximated by some family of functions $g_n \in W_c^{s,p}(\Omega)$, then every function $f \in W^{s,p}(\Omega)$ can be approximated by functions from $W_c^{s,p}(\Omega)$. Since $L^\infty(\Omega) \cap W^{s,p}(\Omega)$ is dense in $W^{s,p}(\Omega)$ (because the truncated functions $f^N = \min\{\max\{f, -N\}, N\}$ tend to f in $W^{s,p}(\Omega)$, as $N \longrightarrow \infty$), we may assume that $f \in L^\infty(\Omega)$. Moreover, we may also assume that $0 \le g_n \le 1$, because if $g_n \longrightarrow \mathbb{1}_\Omega$ in $W^{s,p}(\Omega)$, then also $\widetilde{g}_n = \max\{\min\{g_n, 1\}, 0\} \longrightarrow \mathbb{1}_\Omega$, since we have $|\widetilde{g}_n(x) - \widetilde{g}_n(y)| \le |g_n(x) - g_n(y)|$.

Define $f_n = f g_n \in W_c^{s,p}(\Omega)$. Observe that

$$\begin{aligned}
[f - f_n]^p_{W^{s,p}(\Omega)} &= \int_\Omega \int_\Omega \frac{|f(x)(1 - g_n(x)) - f(y)(1 - g_n(y))|^p}{|x - y|^{d+sp}}\, dy\, dx \\
&\le 2^{p-1} \int_\Omega \int_\Omega \frac{|f(x)|^p |g_n(x) - g_n(y)|^p}{|x - y|^{d+sp}}\, dy\, dx \\
&\quad + 2^{p-1} \int_\Omega \int_\Omega \frac{|1 - g_n(y)|^p |f(x) - f(y)|^p}{|x - y|^{d+sp}}\, dy\, dx \\
&\le 2^{p-1} \|f\|_\infty^p [g_n]^p_{W^{s,p}(\Omega)} \\
&\quad + 2^{p-1} \int_\Omega \int_\Omega \frac{|1 - g_n(y)|^p |f(x) - f(y)|^p}{|x - y|^{d+sp}}\, dy\, dx.
\end{aligned}$$

Since $g_n \longrightarrow \mathbb{1}_\Omega$ in $L^p(\Omega)$, there is a subsequence $g_{n_k} \longrightarrow \mathbb{1}_\Omega$ almost everywhere. Hence, for such a subsequence we have

$$\begin{aligned}
[f - f_{n_k}]^p_{W^{s,p}(\Omega)} &\le 2^{p-1} \|f\|_\infty^p [g_{n_k}]^p_{W^{s,p}(\Omega)} \\
&\quad + 2^{p-1} \int_\Omega \int_\Omega \frac{|1 - g_{n_k}(y)|^p |f(x) - f(y)|^p}{|x - y|^{d+sp}}\, dy\, dx.
\end{aligned}$$

The first term above is convergent to 0, since $g_{n_k} \longrightarrow \mathbb{1}_\Omega$ in $W^{s,p}(\Omega)$. The convergence of the second term follows from Lebesgue dominated convergence theorem. Moreover, it is trivial to show that $f_n \longrightarrow f$ in $L^p(\Omega)$, and hence, the proof is finished. ☐

4 Proof of Theorem 2

Proof of Theorem 2, case I According to Lemma 13, we only need to prove that the function $f = \mathbb{1}_\Omega$ can be approximated by compactly supported functions. Let $f_n = fv_n$, where v_n is as in the Lemma 10 and let $\underline{d} = \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. By Lemma 10 (note that in this case the second term in inequality (4) is 0), we have

$$[f_n]^p_{W^{s,p}(\Omega} \leq Cn^{sp} \int_{\Omega_{\frac{3}{n}}} dx = Cn^{sp} \left|\Omega_{\frac{3}{n}}\right|.$$

If $sp < \underline{d}$, then, by the definition of lower Assouad codimension, for every $\varepsilon > 0$ we have

$$\left|\Omega_{\frac{3}{n}}\right| \leq C' \left(\frac{1}{n}\right)^{\underline{d}-\varepsilon}. \tag{5}$$

Hence, for some new constant C we have

$$[f_n]^p_{W^{s,p}(\Omega)} \leq Cn^{sp} n^{\varepsilon-\underline{d}} \longrightarrow 0,$$

when $n \longrightarrow \infty$, by choosing $0 < \varepsilon < \underline{d} - sp$, which is feasible thanks to our assumption. □

Proof of Theorem 2, case II We proceed like in the above proof of the first part of the Theorem 2 and obtain

$$[f_n]^p_{W^{s,p}(\Omega)} \leq Cn^{sp} \left|\Omega_{\frac{3}{n}}\right|.$$

Since Ω is $(d - sp)$-homogeneous and $\underline{\operatorname{co\,dim}}_A(\partial\Omega) = sp$, then it follows that $\left|\Omega_{\frac{3}{n}}\right| \leq C' n^{-sp}$ and, in consequence, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p}(\Omega)$.

The following argument was kindly pointed out to us by Lorenzo Brasco, see also [4, Theorem 4.4] for a similar argument. It is well known that for $p > 1$ the space $W^{s,p}(\Omega)$ is reflexive. Hence, by Banach–Alaoglu and Eberlein–Šmulian theorem, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ weakly convergent to some f. Since $W^{s,p}_0(\Omega)$ is both closed and convex subset of $W^{s,p}(\Omega)$, by [3, Theorem 2.3.6] it is also weakly closed, so we have $f \in W^{s,p}_0(\Omega)$. Then it suffices to see that $f = \mathbb{1}_\Omega$ by the uniqueness of the limit, since f_{n_k} strongly converges to $\mathbb{1}_\Omega$ in $L^p(\Omega)$. This ends the proof. □

Proof of Theorem 2, case III Let $\overline{d} = \overline{\operatorname{co\,dim}}_A(\partial\Omega)$. We will show that the indicator of Ω cannot be approximated by functions with compact support. Indeed, let u_n be any sequence of compactly supported functions such that $\|u_n - \mathbb{1}_\Omega\|_{W^{s,p}(\Omega)} \longrightarrow 0$. In particular $u_n \longrightarrow \mathbb{1}_\Omega$ in $L^p(\Omega)$, so there is a subsequence u_{n_k} convergent almost everywhere to $\mathbb{1}_\Omega$. If $sp > \overline{d}$, we can use the fractional Hardy inequality from [11, Corollary 3] in the case (F) with $\beta = 0$ to obtain

$$[u_{n_k}-]^p_{W^{s,p}(\Omega)} = [u_{n_k}]^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|u_{n_k}(x) - u_{n_k}(y)|^p}{|x-y|^{d+sp}} \, dy \, dx$$
$$\geq c \int_\Omega \frac{|u_{n_k}(x)|^p}{d_\Omega(x)^{sp}} \, dx.$$

By Fatou's lemma,

$$0 = \lim_{k\to\infty} [u_{n_k}]^p_{W^{s,p}(\Omega)} \geq c \int_\Omega \liminf_{k\to\infty} \frac{|u_{n_k}(x)|^p}{d_\Omega(x)^{sp}}\, dx$$
$$= c \int_\Omega \frac{dx}{d_\Omega(x)^{sp}} > 0.$$

We obtain a contradiction. □

Example 14 (Lipschitz domains) Let Ω be a bounded Lipschitz domain. In this case, we have $\operatorname{co\,dim}_A(\partial\Omega) = 1$ and, by the cone property, $|\Omega_r| = O(r)$, hence, Theorem 2 generalises the classical result [14, Theorem 1.4.2.4].

Example 15 (Koch snowflake) Let $\Omega \subset \mathbb{R}^2$ denote the domain bounded by the Koch snowflake. It is well known that the Hausdorff dimension of the Koch *curve* is $\frac{\log 4}{\log 3}$. Thus, also its Assouad dimension is $\frac{\log 4}{\log 3}$, since it is a self-similar set satisfying open set condition, see [13, Corollary 2.11]. The Koch snowflake is a finite union of copies of Koch curves, therefore its Assouad dimension is again $\frac{\log 4}{\log 3}$, see [13, Theorem 2.2] and [22, Theorem A.5(3)]. Hence $\operatorname{co\,dim}_A(\partial\Omega) = 2 - \frac{\log 4}{\log 3}$.

Moreover, by [21, Theorem 1.1] the volume of the inner tubular neighbourhood of Ω is described by the formula

$$|\Omega_r| = G_1(r) r^{2-\frac{\log 4}{\log 3}} + G_2(r) r^2,$$

where G_1 and G_2 are continuous, periodic functions (in consequence bounded). Hence, for $r < 1$ we have $|\Omega_r| = O\left(r^{2-\frac{\log 4}{\log 3}}\right)$. Since in addition Ω is κ - plump, by Theorem 2 we obtain that if $p = 1$, then $C_c^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$ if $s < 2 - \frac{\log 4}{\log 3}$ and is not dense if $s > 2 - \frac{\log 4}{\log 3}$. Moreover, if $p > 1$, then the density result holds if and only if $sp \leq 2 - \frac{\log 4}{\log 3}$. We do not know what is happening in the remaining case $p = 1$ and $s = 2 - \frac{\log 4}{\log 3}$.

5 The space $W^{s,p}_0(\Omega)$

Based on our previous results, we are able to describe explicitly the space $W^{s,p}_0(\Omega)$ in some particular cases. Namely, we can describe this space for Ω, s and p satisfying the following weak fractional Hardy inequality.

Definition 16 We say that Ω admits a weak (s, p)–fractional Hardy inequality, if there exists a constant $c = c(d, s, p, \Omega)$ such that for every $f \in C_c^\infty(\Omega)$ it holds

$$\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp}}\, dx \leq c \|f\|^p_{W^{s,p}(\Omega)}.$$

In the case when the norm $\|f\|_{W^{s,p}(\Omega)}$ above can be replaced by the seminorm $[f]_{W^{s,p}(\Omega)}$, we say that Ω admits an (s, p)–fractional Hardy inequality.

Theorem 17 *Suppose that Ω admits a weak (s,p)-fractional Hardy inequality. Then*

$$W_0^{s,p}(\Omega) = \left\{ f \in W^{s,p}(\Omega) : \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp}}\,dx < \infty \right\}.$$

Proof By Lemma 10, if $\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp}}\,dx < \infty$, then $f \in W_0^{s,p}(\Omega)$, because in this case

$$n^{sp} \int_{\Omega_{\frac{3}{n}}} |f(x)|^p\,dx \le 3^{sp} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p}{d_\Omega(x)^{sp}}\,dx \longrightarrow 0,$$

when $n \longrightarrow \infty$. In fact, for that part we do not need the assumption about Hardy inequality.

Suppose that Ω admits a weak (s,p)–Hardy inequality and $f \in W_0^{s,p}(\Omega)$. Let f_n be a sequence of smooth and compactly supported functions convergent to f in $W^{s,p}(\Omega)$. In particular, $f_n \longrightarrow f$ in $L^p(\Omega)$, so there exists a subsequence f_{n_k} convergent to f almost everywhere. We have by Fatou lemma

$$\begin{aligned}
\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp}}\,dx &= \int_\Omega \lim_{k\to\infty} \frac{|f_{n_k}(x)|^p}{d_\Omega(x)^{sp}}\,dx \\
&\le \liminf_{k\to\infty} \int_\Omega \frac{|f_{n_k}(x)|^p}{d_\Omega(x)^{sp}}\,dx \\
&\le c \liminf_{k\to\infty} \|f_{n_k}\|^p_{W^{s,p}(\Omega)} \\
&= c\|f\|^p_{W^{s,p}(\Omega)} < \infty.
\end{aligned}$$

□

Proof of Theorem 3 From part (F) of Theorem 5 with $\eta = sp$, $\varphi(t) = t^{sp}$, Ω admits an (s,p)-fractional Hardy inequality and also a weak (s,p)-fractional Hardy inequality. Thus, the result follows from Theorem 17. □

Proof of Theorem 4 From part (T') of Theorem 5, inequality (2) holds for all functions f for which the left hand side of (2) is finite. Thus by Theorem 3, it holds for all functions $f \in W_0^{s,p}(\Omega)$. However, by part (I) of Theorem 2, $W_0^{s,p}(\Omega) = W^{s,p}(\Omega)$, and the result follows. □

Appendix

We recall from [2, Section 3] the notion of a global weak lower (or upper) scaling condition (WLSC or WUSC for short). As in [11], we will use a different, but equivalent formulation. We note that in our setting the middle parameter in WLSC or WUSC is always zero, and thus, we could omit it, however we prefer to keep the notation consistent with [2, 11].

Definition 18 Let $\eta \in \mathbb{R}$ and $H \in (0,1]$. We say that a function $\phi : (0,\infty) \to (0,\infty)$ satisfies $\mathrm{WLSC}(\eta,0,H)$ (resp., $\mathrm{WUSC}(\eta,0,H^{-1})$) and write $\phi \in \mathrm{WLSC}(\eta,0,H)$ ($\phi \in \mathrm{WUSC}(\eta,0,H^{-1})$), if

$$\phi(st) \ge Ht^\eta \phi(s), \quad s > 0, \tag{6}$$

for every $t \geq 1$ (resp., for every $t \in (0, 1]$).

We begin with the following observation:

$$\text{If } \Omega \subset \mathbb{R}^d \text{is a nonempty open bounded set, then } \overline{\dim}_A(\partial\Omega) \geq d - 1. \tag{7}$$

For the proof, we will provide the following argument by the user rpotrie from [24]. Since $\partial\Omega$ disconnects $\mathbb{R}^d$, its topological dimension has to be at least $d-1$, see [16, Theorem IV.4]. But the topogical dimension does not exceed Hausdorff dimension [16, page 107], and the latter in turn does not exceed the upper Assouad dimension [22, Theorem A.5(10)], consequently (7) holds.

Proof of case (T') in Theorem 5 It seems possible to adapt the original proof for this case, however, since the proof was quite involved and technical, we prefer to choose another strategy. Namely, we will reduce (T') to the case (T). Let us assume that the general assumptions of Theorem 5 and the assumptions in (T') hold.

Let us fix $x_0 \in \Omega$ and put $M = \operatorname{diam}\Omega$. We consider an open set $\Omega_1 = \mathbb{R}^d \setminus \overline{B}(x_0, 2M)$. Let $G = \Omega \cup \Omega_1$. Observe that $\operatorname{dist}(\Omega, \Omega_1) \geq M$, hence $\partial G = \partial\Omega \cup \partial\Omega_1$. Therefore,

$$\overline{\dim}_A(\partial G) = \max\{\overline{\dim}_A(\partial\Omega), \overline{\dim}_A(\partial\Omega_1)\} = \max\{\overline{\dim}_A(\partial\Omega), d-1\} = \overline{\dim}_A(\partial\Omega),$$

by [22, Theorem A.5(3)] and (7).

We may also need to redefine the function ϕ. To this end, put $\eta_0 = \eta$ if $\eta > 0$, while in the case when $\eta \leq 0$, we choose $\eta_0 > 0$ such that

$$\eta_0 + \overline{\dim}_A(\partial\Omega) - d < 0.$$

We note that this is possible, because κ-plumpness of Ω implies that $\partial\Omega$ is porous, and that in turn by [22, Theorem 5.2] implies that $\overline{\dim}_A(\partial\Omega) < d$. We define

$$\psi(x) = \begin{cases} \phi(x), & \text{when } x \in (0, M]; \\ \phi(T)(\frac{x}{T})^{\eta_0}, & \text{when } x \in (M, \infty). \end{cases}$$

We claim that such a function ψ satisfies the condition $\mathrm{WUSC}(\eta_0, 0, H^{-1})$. We omit a straightforward check of (6) in three possible cases, when the two numbers $st \leq s$ in that equation lie in either $(0, M]$ or (M, ∞).

We apply the case (T) of the Theorem 5 (proved in [11]) to the open set G, the number η_0 and the function $\psi \in \mathrm{WUSC}(\eta_0, 0, H^{-1})$. It follows that there exist constants c and R such that

$$\int_G \frac{|u(x)|^p}{\psi(d_G(x))}\, dx \leq c \int_G \int_{G\cap B(x, R d_G(x))} \frac{|u(x) - u(y)|^p}{\psi(d_G(x)) d_G(x)^d}\, dy\, dx \tag{8}$$

holds for all measurable functions $u : G \to \mathbb{R}$ for which the left hand side is finite.

Let us consider an arbitrary measurable functions $u : \Omega \to \mathbb{R}$ for which $\int_\Omega \frac{|u(x)|^p}{\phi(d_G(x))}\, dx < \infty$, and extend it by zero on Ω_1 to obtain a function defined on the whole set G. Inequality (8) for this function u has the following form,

$$\begin{aligned}\int_\Omega \frac{|u(x)|^p}{\phi(d_G(x))}\,dx &\le c\int_\Omega\int_{\Omega\cap B(x,Rd_G(x))}\frac{|u(x)-u(y)|^p}{\phi(d_G(x))d_G(x)^d}\,dy\,dx\\ &+c\int_{\Omega_1}\int_{\Omega\cap B(x,Rd_G(x))}\frac{|u(y)|^p}{\psi(d_G(x))d_G(x)^d}\,dy\,dx\\ &+c\int_\Omega\int_{\Omega_1\cap B(x,Rd_G(x)))}\frac{|u(x)|^p}{\psi(d_G(x))d_G(x)^d}\,dy\,dx\\ &=:\ c(I_1+I_2+I_3).\end{aligned}$$

In the integral I_2, when $x\in\Omega_1$ and $y\in\Omega\cap B(x,Rd_G(x))$, then $M\le|x-y|\le Rd_G(x)$ and therefore $d_G(x)\ge M/R$. Consequently,

$$I_2\le\|u\|^p_{L^p(\Omega)}\int_{\{x\in\Omega_1:d_G(x)\ge M/R\}}\frac{dx}{\psi(d_G(x))d_G(x)^d}. \tag{9}$$

From the definition of the function ψ and the fact that $\psi\in$ WUSC $(\eta_0,0,H^{-1})$, it follows that there exists a constant $c(M/R,H,\eta_0)$ such that

$$\psi(z)\ge c(M/R,H,\eta_0)z^{\eta_0},\quad \text{for} z\ge M/R. \tag{10}$$

Therefore, the integral in (9) is convergent and so $I_2\le c'\|u\|^p_{L^p(\Omega)}$.

For the integral I_3 , we observe that when $x\in\Omega$ and $y\in\Omega_1\cap B(x,Rd_G(x))$, then $d_G(x)=d_\Omega(x)$ and $M\le|y-x|\le Rd_G(x)$, so $d_G(x)\ge M/R$. Therefore, by (10) the function $\psi(d_G(x))^{-1}d_G(x)^{-d}$ is bounded from above. Furthermore, since $|y-x_0|\le M+|y-x|\le M+Rd_G(x)\le M(1+R)$, the following inclusion $\Omega_1\cap B(x,Rd_G(x))\subset B(x_0,M(1+R))$ holds for all $x\in\Omega$. Thus also in this case $I_3\le c'\|u\|^p_{L^p(\Omega)}$.

Consequently, I_1 is equal to the first term on the right side of (3), while I_2 and I_3 are bounded by the second term. □

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FRACTIONAL SOBOLEV SPACES WITH POWER WEIGHTS

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Abstract. We investigate the form of the closure of the smooth, compactly supported functions $C_c^\infty(\Omega)$ in the weighted fractional Sobolev space $W^{s,p;\,w,v}(\Omega)$ for bounded Ω. We focus on the weights w, v being powers of the distance to the boundary of the domain. Our results depend on the lower and upper Assouad codimension of the boundary of Ω. For such weights we also prove the comparability between the full weighted fractional Gagliardo seminorm and the truncated one.

1. Introduction and preliminaries

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $0 < s < 1$ and $1 \le p < \infty$. We recall that the *fractional Sobolev space* is defined as

$$W^{s,p}(\Omega) = \left\{ f \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}}\, dy\, dx < \infty \right\}.$$

This is a Banach space endowed with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p}(\Omega)},$$

where $[f]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\, dy\, dx \right)^{1/p}$ is called the *Gagliardo seminorm.*

In this paper we consider weighted fractional Sobolev spaces. For *weights* w, v (i.e. measurable nonnegative functions on Ω) we define the *weighted Gagliardo seminorm* as

$$[f]_{W^{s,p;\,w,v}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} w(y) v(x)\, dy\, dx \right)^{\frac{1}{p}}$$

and the *weighted fractional Sobolev space* as

$$W^{s,p;\,w,v}(\Omega) = \left\{ f \in L^p(\Omega) : [f]_{W^{s,p;\,w,v}(\Omega)} < \infty \right\}.$$

For bounded Ω the space defined above is always nonempty, because it contains constant functions. Moreover, if $w_\alpha(x) = \operatorname{dist}(x, \partial\Omega)^{-\alpha}$ and $v_\beta(y) = \operatorname{dist}(y, \partial\Omega)^{-\beta}$ for $\alpha, \beta \in \mathbb{R}$, we denote

$$W^{s,p;\,w_\alpha, v_\beta}(\Omega) =: W^{s,p;\,\alpha,\beta}(\Omega).$$

The space $W^{s,p;\,w,v}(\Omega)$ is equipped with the natural norm

$$\|f\|_{W^{s,p;\,w,v}(\Omega)} = \|f\|_{L^p(\Omega)} + [f]_{W^{s,p;\,w,v}(\Omega)}.$$

We remark here that all results of the paper remain true if we replace the space $L^p(\Omega)$ appearing in the definition of $W^{s,p;\,w,v}(\Omega)$ by the weighted analogue $L^p(\Omega, W)$ for any almost everywhere positive weight W, which is locally comparable to a constant (see Definition 18) or continuous and satisfies $\int_\Omega W(x)\, dx < \infty$. Notice that the last condition ensures that the constant function $\mathbb{1}_\Omega$ is in $L^p(\Omega, W)$. However, for simplicity we consider only the unweighted case.

For an open set Ω we use the notation $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

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Definition 1. By $W_0^{s,p;\,w,v}(\Omega)$ we denote the closure of $C_c^\infty(\Omega) \cap W^{s,p;\,w,v}(\Omega)$ (smooth functions with compact support in Ω) in $W^{s,p;\,w,v}(\Omega)$ with respect to the weighted fractional Sobolev norm and by $W_c^{s,p;\,w,v}(\Omega)$ we denote the closure of all compactly supported, measurable functions in Ω (not necessarily smooth) in $W^{s,p;\,w,v}(\Omega)$ with respect to the weighted fractional Sobolev norm. We also denote $W_0^{s,p;\,w_\alpha,v_\beta}(\Omega) =: W_0^{s,p;\,\alpha,\beta}(\Omega)$, $W_c^{s,p;\,w_\alpha,v_\beta}(\Omega) =: W_c^{s,p;\,\alpha,\beta}(\Omega)$.

We refer to Section 3 for a discussion on the cases when $C_c^\infty(\Omega)$ is or is not a subset of $W^{s,p;\,\alpha,\beta}(\Omega)$. In general, it may occure that the space $W_0^{s,p;\,\alpha,\beta}(\Omega)$ is empty.

The main result of this paper is a generalization of the density result for unweighted fractional Sobolev spaces, which can be found in [9, Theorem 2]. We present some necessary and sufficient conditions for the space $C_c^\infty(\Omega)$ to be dense in $W^{s,p;\,\alpha,\beta}(\Omega)$. In the negative case, under some additional assumptions we also find explicitly the form of the space $W_0^{s,p;\,\alpha,\beta}(\Omega)$. The necessary geometrical and technical definitions are contained in Section 2. In Section 3 we present Lemmas, most of them being generalization of these from [9] and [10] for the weighted case.

Let us remark that the weighted fractional Sobolev spaces related to the weighted Sobolev-type norm $[\,\cdot\,]_{W^{s,p;\,\alpha,\beta}(\Omega)} + \|\cdot\|_{L^{p^*}(\Omega,W)}$ and the problem of density of $C_c^\infty(\Omega)$ were investigated before by Dipierro and Valdinoci in [6] for the case $\Omega = \mathbb{R}^d \setminus \{0\}$, $\alpha = \beta \in [0,(d-sp)/2)$, $p^* = dp/(d-sp)$ and $W(x) = |x|^{-\frac{2\alpha d}{d-sp}}$. However, this problem is not directly comparable to ours, because we consider only bounded sets Ω. Similar weighted fractional Sobolev spaces were an object of study in [1] in connection with weighted Caffarelli–Kohn–Nirenberg and fractional Hardy inequalities. Moreover, related results for unweighted Sobolev-type spaces can be found for example in [12], where the authors considered spaces of functions vanishing on $\mathbb{R}^d \setminus \Omega$ and in [2], where the problem of density of $C_c^\infty(\Omega)$ functions was investigated in the context of the fractional Sobolev spaces with variable exponents.

Section 4 is devoted to the comparability result between the full weighted seminorm and the truncated one in the space $W^{s,p;\,w,v}(\Omega)$. This comparability is important to us in proving our main results (to be more specific - in Lemma 16 and Lemma 22). However, it is also very interesting and nontrivial property itself. Similar results were obtained before by Dyda [7] for Gagliardo-type seminorms with the additional homogeneous kernels (like indicators of cones), by Prats and Saksman [23] in a more general context of Triebel-Lizorkin spaces and generalized later by Rutkowski [24] for the kernels of the form $|x-y|^{-d}\varphi(|x-y|)^{-q}$, with φ satisfying certain technical assumptions. Some versions of the reduction of the integration theorems can also be found in [4], [5] and [16]. We want to point here that a variant of comparability is nonexplicitly contained in the early work of Seeger [25]. We prove a weighted analogue of the reduction of the integration theorem for the space $W^{s,p;\,\alpha,\beta}(\Omega)$, provided that $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. This result is stated below.

Theorem 2. *Let Ω be a nonempty, bounded, uniform domain, let $0 < s < 1$ and $1 \le p < \infty$. Moreover, let $0 < \theta \le 1$. Suppose that $0 \le \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial(\Omega)$. Then the full seminorm $[f]_{W^{s,p;\,\alpha,\,\beta}(\Omega)}$ and the truncated seminorm*

$$\left(\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dy\, dx\right)^{\frac{1}{p}}$$

are comparable, that is there exists a constant $C = C(\theta, d, s, p, \alpha, \beta, \Omega) > 0$ such that

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \frac{dy}{d_\Omega(y)^\beta} \frac{dx}{d_\Omega(x)^\alpha} \le C \int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \frac{dy}{d_\Omega(y)^\beta} \frac{dx}{d_\Omega(x)^\alpha}\, dy\, dx,$$

for all $f \in L^1_{loc}(\Omega)$.

It is clear that the reverse inequality is trivial with constant equal to one, hence we indeed obtain the comparability between the full and the truncated weighted Gagliardo seminorms. Moreover, when $p = 1$, the comparability can be formulated in a more general setting, for all A_1 class Muckenhoupt weights, see Theorem 23.

Section 5 contains proofs of our main results, Theorems 3 and 4. Theorem 3 is a generalization of [9, Theorem 2] and Theorem 4 is a generalization of [9, Theorem 3], provided that Ω is a uniform domain.

Theorem 3. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty, bounded, open set, let $0 < s < 1$, $1 \leq p < \infty$ and $\alpha, \beta \geq 0$.*
(I) If $sp + \alpha + \beta < d - \overline{\dim}_M(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) = W^{s,p;\,\alpha,\beta}(\Omega)$.
(II) If Ω is $(d - sp - \alpha - \beta)$-homogeneous, $p > 1$ and $sp + \alpha + \beta = \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) = W^{s,p;\,\alpha,\beta}(\Omega)$.
(III) If Ω is κ-plump and $sp + \alpha + \beta > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, then $W_0^{s,p;\,\alpha,\beta}(\Omega) \neq W^{s,p;\,\alpha,\beta}(\Omega)$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty, bounded, uniform and open set, let $0 < s < 1$, $1 \leq p < \infty$ and $0 \leq \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. If $sp + \alpha + \beta > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, then*

$$W_0^{s,p;\,\alpha,\beta}(\Omega) = \left\{ f \in W^{s,p;\,\alpha,\beta}(\Omega) : \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\, dx < \infty \right\}.$$

Theorem 4 reveals the property known partially also for classical (unweighted) Sobolev spaces $W^{1,p}(\Omega)$, see [18, Example 9.12] or [17].

Remark 5. In the proof of the case II in the Theorem 3 we use a reflexivity property of the space $W^{s,p;\,\alpha\,\beta}(\Omega)$ (see Proposition 25). This explains why $p = 1$ is excluded from the assumptions. It is not clear if the density property holds in this case and we leave it as an open problem.

To prove the case (III) of the Theorem 3, we use a (weak) weighted fractional Hardy inequality, which can can be easily derived from the (weak) fractional (s, p, a)-Hardy inequality, given in [11, Corollary 3] and also in [9, Theorem 5] in the case (T') of the result below. It suffices to take the function $\phi(x) = x^{sp+\alpha+\beta}$ and notice that $\operatorname{dist}(y, \partial\Omega) \lesssim \operatorname{dist}(x, \partial\Omega)$ on the ball $B(x, R \operatorname{dist}(x, \partial\Omega))$. We present this version below.

Theorem 6. *([11, Corollary 3], [9, Theorem 5]) Let $0 < p < \infty$, $0 < s < 1$ and $\alpha, \beta \geq 0$. Suppose that $\Omega \neq \emptyset$ is an open, κ-plump set so that either condition (T), or condition (T'), or condition (F) holds*

(T) *$sp + \alpha + \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, Ω is unbounded and $\xi = 0$,*
(T') *$sp + \alpha + \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, Ω is bounded and $\xi = 1$,*
(F) *$sp + \alpha + \beta > \overline{\operatorname{co\,dim}}_A(\partial\Omega)$, Ω is bounded or $\partial\Omega$ is unbounded and $\xi = 0$.*

Then there exist constants c and R such that the following inequality

$$\int_\Omega \frac{|u(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\, dx \leq c \int_\Omega \int_{\Omega \cap B(x, R\, d_\Omega(x))} \frac{|u(x) - u(y)|^p}{|x - y|^{d+sp}} \frac{dy}{d_\Omega(y)^\beta} \frac{dx}{d_\Omega(x)^\alpha} + c\xi \|u\|^p_{L^p(\Omega)}, \tag{1}$$

holds for all measurable functions u for which the left-hand side is finite.

As an easy corollary in the case (T'), deriving directly from Theorem 3 and Theorem 6, we obtain the embedding $W^{s,p;\,\alpha,\beta}(\Omega) \subset L^p(\Omega, \operatorname{dist}(\cdot, \partial\Omega)^{-sp-\alpha-\beta})$.

Theorem 7. *Let $1 \leq p < \infty$ and $0 < s < 1$. Suppose that $\Omega \neq \emptyset$ is an open, uniform, bounded set such that $0 \leq \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ and $sp + \alpha + \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. Then there* *exists a constant c such that*

$$\int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\,dx \le c\|f\|^p_{W^{s,p;\,\alpha,\beta}(\Omega)} < \infty,$$

for all $f \in W^{s,p;\,\alpha,\beta}(\Omega)$.

Theorem 7 is a generalization of the unweighted case from [9, Theorem 4], provided that Ω is a uniform domain.

Notation. Having two nonnegative functions A and B we use a symbol '$\lesssim$' if there exists a constant $c > 0$ such that $A \le cB$. The constant c usually depends on some parameters, like α, β, d, s, p, Ω, but not on the arguments of the functions A, B and the set of these parameters arises from context. Moreover, we write $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

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2. Definitions

We will use the same definitions as in [9, Section 2]; for Reader's convenience we repeat them below.

2.1. Assouad and Minkowski dimensions.

Recall that we denote the distance from $x \in \mathbb{R}^d$ to a set $E \subset \mathbb{R}^d$ by $\operatorname{dist}(x, E) = \inf_{y\in E} |x - y|$.

Definition 8. Let $r > 0$. For open sets $\Omega \subset \mathbb{R}^d$ we define the *inner tubular neighbourhood* of Ω as

$$\Omega_r = \{x \in \Omega : d_\Omega(x) \le r\},$$

and for arbitrary sets $E \subset \mathbb{R}^d$ we define the *tubular neighbourhood* of E as

$$\widetilde{E}_r = \left\{x \in \mathbb{R}^d : \operatorname{dist}(x, E) \le r\right\}.$$

Definition 9. [15, Section 3] Let $E \subset \mathbb{R}^d$. The *lower Assouad codimension* $\underline{\operatorname{co\,dim}}_A(E)$ is defined as the supremum of all $q \ge 0$, for which there exists a constant $C = C(q) \ge 1$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$ it holds

$$\left|\widetilde{E}_r \cap B(x,R)\right| \le C\,|B(x,R)|\left(\frac{r}{R}\right)^q.$$

Conversely, the *upper Assouad codimension* $\overline{\operatorname{co\,dim}}_A(E)$ is defined as the infimum of all $s \ge 0$, for which there exists a constant $c = c(s) > 0$ such that for all $x \in E$ and $0 < r < R < \operatorname{diam} E$ it holds

$$\left|\widetilde{E}_r \cap B(x,R)\right| \ge c\,|B(x,R)|\left(\frac{r}{R}\right)^s.$$

We remark that having strict inequality $R < \operatorname{diam} E$ above makes the definitions applicable also for unbounded sets E; for bounded sets E we could have $R \le \operatorname{diam} E$.

In Euclidean space $\mathbb{R}^d$ (more general - in Ahlfors d-regular measure metric spaces) it holds

$$\underline{\dim}_A(E) + \overline{\operatorname{co\,dim}}_A(E) = \overline{\dim}_A(E) + \underline{\operatorname{co\,dim}}_A(E) = d,$$

where $\underline{\dim}_A(E)$ and $\overline{\dim}_A(E)$ denote respectively the well known lower and upper Assouad dimension – see for example [15, Section 2]. Moreover, if $\underline{\operatorname{co\,dim}}_A(E) = \overline{\operatorname{co\,dim}}_A(E)$, we simply denote both of these values by $\operatorname{co\,dim}_A(E)$.

We recall a notion of σ-homogenity, coming from [20, Theorem A.12].

Definition 10. Let $E \subset \mathbb{R}^d$ and let $V(E, x, \lambda, r) = \{y \in \mathbb{R}^d : \operatorname{dist}(y, E) \le r, |x-y| \le \lambda r\}$. We say that E is *σ-homogeneous*, if there exists a constant L such that

$$|V(E, x, \lambda, r)| \le L r^d \lambda^\sigma$$

for all $x \in E$, $\lambda \ge 1$ and $r > 0$.

If $0 < r < R < \operatorname{diam}(E)$, then taking $\lambda = R/r$ in the definition gives

$$\left|\widetilde{E}_r \cap B(x, R)\right| = \left|V\Big(E, x, \frac{R}{r}, r\Big)\right| \le C\, |B(x, R)| \left(\frac{r}{R}\right)^{d-\sigma},$$

where $C = C(d, E)$ is a constant. This means that if $\underline{\operatorname{co\,dim}}_A(E) = s$, then $(d-s)$-homogeneous sets are precisely these sets E, for which the supremum in the definition of the lower Assouad codimension is attained. For the definition of the concept of homogenity from a different point of view the Reader may also see [20, Definition 3.2].

Definition 11. The *upper Minkowski dimension* of a set $E \subset \mathbb{R}^d$ is defined as

$$\overline{\dim}_M(E) = \inf\{s \ge 0 : \limsup_{r \to 0} \left|\widetilde{E}_r\right| r^{d-s} = 0\},$$

see for example [14, Section 2].

It is not hard to see that $\underline{\operatorname{co\,dim}}_A(E) \le d - \overline{\dim}_M(E)$ and the equality holds if E is $(d - \underline{\operatorname{co\,dim}}_A(E))$ - homogeneous. Moreover (considering again open, bounded sets Ω), the *distance zeta function*

$$\zeta_\Omega(q) := \int_\Omega \frac{dx}{d_\Omega(x)^q}$$

is finite if $q < d - \overline{\dim}_M(\partial\Omega)$ and infinite if $q > d - \overline{\dim}_M(\partial\Omega)$ (see [14, Lemma 3.3 and Lemma 3.5]).

We recall a geometric notion from [27], appearing among other assumptions in Theorem 6.

Definition 12. A set $E \subset \mathbb{R}^d$ is *κ-plump* with $\kappa \in (0, 1)$ if, for each $0 < r < \operatorname{diam}(E)$ and each $x \in \overline{E}$, there is $z \in \overline{B}(x, r)$ such that $B(z, \kappa r) \subset E$.

2.2. **Whitney decomposition and operator P^η.** Let Ω be an open, nonempty, proper subset of $\mathbb{R}^d$. Let Q be any closed cube in $\mathbb{R}^d$. We denote by $l(Q)$ the length of the side of Q and by x_Q the center of Q. Following [23], there exists a family of dyadic cubes $\mathcal{W} = \{Q_n\}_{n \in \mathbb{N}}$, called the *Whitney decomposition*, satisfying for all $Q, S \in \mathcal{W}$ the conditions:

- $\Omega = \bigcup_n Q_n$;
- if $Q \ne S$, then $\operatorname{Int} Q \cap \operatorname{Int} S = \emptyset$;
- there exists a constant $C = C(\mathcal{W})$ such that $C \operatorname{diam} Q \le \operatorname{dist}(Q, \partial\Omega) \le 4C \operatorname{diam} Q$;
- if $Q \cap S \ne \emptyset$, then $l(Q) \le 2l(S)$;
- if $Q \subset 5S$ then $l(S) \le 2l(Q)$.

The dilation of the cube Q, cQ for $c > 0$, is always taken with respect to its center, that is cQ is a cube with the same center as Q, but the length of the side $cl(Q)$.

Inspired by [23] we define a *shadow* of a cube $Q \in \mathcal{W}$ as

$$\mathbf{Sh}_\theta(Q) = \{S \in \mathcal{W} : S \subset B\left(x_Q, \theta l(Q)\right)\}.$$

The „realization" of $\mathbf{Sh}_\theta$ is $\mathbf{SH}_\theta(Q) = \bigcup \mathbf{Sh}_\theta(Q)$. When θ is fixed, we abbreviate the notation as $\mathbf{Sh}_\theta(Q) =: \mathbf{Sh}(Q)$ and $\mathbf{SH}_\theta(Q) =: \mathbf{SH}(Q)$.

For all $Q, S \in \mathcal{W}$ we define their *long distance* D as

$$D(Q, S) = l(Q) + \operatorname{dist}(Q, S) + l(S).$$

We say that a sequence of cubes $(Q, R_1, R_2, \dots, R_n, S)$ is a *chain*, if all two adjacent cubes have nonempty intersection. We denote $(Q, R_1, R_2, \dots, R_n, S) = [Q, S]$ and $[Q, S) = [Q, S] \setminus S$.

The Whitney decomposition is *admissible*, if there exists $a > 0$ such that for all $Q, S \in \mathcal{W}$ there exists a chain $[Q, S] = (Q_1, Q_2, \dots, Q_n)$ satisfying

- $\displaystyle\sum_{i=1}^{n} l(Q_i) \le \frac{1}{a} D(Q, S)$;
- there exists $1 \le i_0 \le n$ such that $l(Q_i) \ge aD(Q, Q_i)$ for all $1 \le i \le i_0$ and $l(Q_i) \ge aD(Q_i, S)$ for all $i_0 \le i \le n$. We denote $Q_{i_0} =: Q_S$. This is the so-called *central* cube in the chain $[Q, S]$.

As stated in [23], for a γ-admissible Whitney decomposition we can always take sufficiently large $\rho = \rho_\gamma > 1$ such that for every γ-admissible chain of cubes $[Q, S]$ we have $Q \in \mathbf{Sh}_{\rho_\gamma}(P)$ for $P \in [Q, Q_S]$ and $5Q \subset \mathbf{SH}_{\rho_\gamma}(Q)$ for every Whitney cube $Q \in \mathcal{W}$.

Next, we recall the definition and basic properties of the operator P^η, defined in [10]. From now on we fix a Whitney decomposition $\mathcal{W}$ such that $C(\mathcal{W}) = 1$ (see [26]) and $0 < \varepsilon < \sqrt{5/4} - 1 < \frac{1}{4}$. If Q is a cube, we denote by Q^* the cube Q „blown up" $(1 + \varepsilon)$ times, that is the cube with the same center $x_{Q^*} = x_Q$, but the length of the side $l(Q^*) = (1 + \varepsilon) l(Q)$. The cube Q_n^{**} is defined in a similar way, that is $Q_n^{**} = (Q_n^*)^*$. Notice that our choice of ε guarantees that $(1 + \varepsilon)^2 < \frac{5}{4}$ and in consequence $Q_n^{**} \subset \frac{5}{4} Q_n$. Moreover, each point $x \in \Omega$ belongs to at most 12^d cubes Q_n^{**}.

Let $\{\psi_n\}_{n \in \mathbb{N}}$ be a partition of unity adjusted to the Whitney decomposition $\mathcal{W} = \{Q_n\}_{n \in \mathbb{N}}$ of Ω, that is a family of functions satisfying $0 \le \psi_n \le 1$, $\psi_n = 1$ on Q_n, $\operatorname{supp} \psi_n \subset Q_n^*$, $\psi_n \in C_c^\infty(\Omega)$, $\sum_n \psi_n = 1$ and $|\psi_n(x) - \psi_n(y)| \le C|x - y|/l(Q_n)$ for some positive constant C independent of Q_n. Let us also fix a nonnegative function $h \colon \mathbb{R}^d \to \mathbb{R}$ with the following properties: $\operatorname{supp} h = B(0, 1)$, $\int_{R^d} h(x)\,dx = 1$, $h \in C^\infty(\mathbb{R}^d)$. For $\delta > 0$ we define its dilation as $h_\delta(x) = \delta^{-d} h(x/\delta)$. Moreover, let $\eta \colon \mathcal{W} \to (0, \infty)$ be any function satisfying $\eta(Q) < \frac{\varepsilon}{2} l(Q)$ for all $Q \in \mathcal{W}$ (a typical example is $\eta(Q) = \delta\, l(Q)$ for any $\delta < \varepsilon/2$). For $f \in L^1_{loc}(\Omega)$, extended by 0 on $\mathbb{R}^d \setminus \Omega$, we define the operator P^η as

$$P^\eta f = \sum_{n=1}^{\infty} (f \psi_n) * h_{\eta(Q_n)}.$$

Here $f * g(x) = \int_{\mathbb{R}^d} f(y) g(x - y)\,dy$ is the standard convolution operation. It was proved in [10] that P^η is well defined, $P^\eta f \in C^\infty(\Omega)$ and P^η maps the space of all compactly supported, locally integrable functions into $C_c^\infty(\Omega)$ (see [10], Propositions 1 and 2).

2.3. **Uniform domains.** There are two equivalent ways to define the notion of uniform domain. The first one comes from [27], and the second one uses the Whitney decomposition and chains of cubes and can be found for example in [23]. We present both definitions here.

Definition 13. A domain (i.e. connected, open set) $\Omega \subset \mathbb{R}^d$ is *uniform*, if there exists a constant $C \ge 1$ such that for all points $x, y \in \Omega$ there is a curve $\gamma \colon [0, l] \to \Omega$ joining them, parameterized by arc length and satisfying $l \le C|x - y|$ and $\operatorname{dist}(z, \partial\Omega) \ge \frac{1}{C} \min\{|z - x|, |z - y|\}$ for all $z \in \gamma$. Equivalently, a domain $\Omega \subset \mathbb{R}^d$ is uniform, if there exists an admissible Whitney decomposition of Ω.

Uniform domains and various reformulations of the definitions above appear also in [13], [21] and [22]. To give a concrete, nontrivial example, we remark here that the Koch snowflake is known to be uniform, despite the highly irregular behaviour of its boundary. It is also σ-homogeneous with $\sigma = \log_3 4$, according to [19, Theorem 1.1].

2.4. **Muckenhoupt class A_1 and Hardy-Littlewood maximal operator.**

Definition 14. For $f \in L^1_{loc}(\mathbb{R}^d)$ the (non-centered) maximal Hardy-Littlewood operator is defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y)\, dy,$$

where supremum is taken over all cubes Q containing x. Equivalently, M can be defined using balls containing x instead of cubes (up to a multiplicative constant). It is well known that this operator is bounded on $L^p(\mathbb{R}^d)$, whenever $1 < p \leq \infty$.

Definition 15. We say that a positive weight w belongs to the *Muckenhoupt class* A_1, if there exists a constant $A > 0$ such that for all cubes $Q \subset \mathbb{R}^d$ it holds

$$\frac{1}{|Q|} \int_Q w(x)\, dx \leq A \inf_{y \in Q} w(y). \tag{2}$$

Notice that by (2) we can easily see that if $w \in A_1$, then the maximal Hardy-Littlewood operator acting on the function w satisfies

$$Mw(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q w(y)\, dy \leq Aw(x), \tag{3}$$

where A depends on w. This property will be important for us later in the proof of Theorem 2. Moreover, it was proved in [8, Theorem 1.1 (B)] that the weight $d_\Omega^{-\alpha}$ belongs to the Muckenhoupt class A_1 if and only if $0 \leq \alpha < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. Hence, by (3), $Md_\Omega^{-\alpha}$ satisfies

$$Md_\Omega^{-\alpha}(x) \leq A\, d_\Omega(x)^{-\alpha}, \tag{4}$$

where the constant A depends on Ω and $\alpha \in [0, \underline{\operatorname{co\,dim}}_A(\partial\Omega))$.

3. Lemmas

We start with showing that under some assumptions $C_c^\infty(\Omega)$ is a subset of $W^{s,p;\,\alpha,\beta}(\Omega)$ and in consequence the latter is not trivial. This is an analogue of [6, Lemma 2.1], where the same fact was established for $\Omega = \mathbb{R}^d \setminus \{0\}$. Although we consider bounded domains, it agrees with the cited result in some aspects, as we have $\operatorname{co\,dim}_A(\{0\}) = d$. Noteworthy, if one of the exponents α, β is nonpositive, then the corresponding weight is bounded and this case is trivial.

Lemma 16. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, uniform domain. Suppose that $0 < s < 1$, $1 \leq p < \infty$, $0 \leq \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ and $\alpha + \beta < d - \overline{\dim}_M(\partial\Omega) + p(1-s)$. Then $C_c^\infty(\Omega) \subset W^{s,p;\,\alpha,\beta}(\Omega)$.*

Proof. Let $\varphi \in C_c^\infty(\Omega)$. Then φ is Lipschitz and locally integrable, so, by Theorem 2 with $\theta = \frac{1}{2}$ we have

$$\begin{aligned}
[\varphi]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} &\lesssim \int_\Omega \int_{B\left(x, \frac{1}{2} d_\Omega(x)\right)} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx \\
&\lesssim \int_\Omega d_\Omega(x)^{-\alpha-\beta}\, dx \int_{B\left(x, \frac{1}{2} d_\Omega(x)\right)} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{d+sp}}\, dy \\
&\lesssim \int_\Omega d_\Omega(x)^{-\alpha-\beta}\, dx \int_{B\left(x, \frac{1}{2} d_\Omega(x)\right)} \frac{dy}{|x-y|^{d+sp-p}} \\
&\lesssim \int_\Omega d_\Omega(x)^{-\alpha-\beta+p(1-s)}\, dx < \infty,
\end{aligned}$$

where the last inequality follows from the properties of the distance zeta function. ☐

Remark 17. We make an easy observation that for any Borel subset $A \subset \Omega$ and $\alpha, \beta \geq 0$ it holds
(5)
$$\int_A \int_A \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx \leq 2 \int_A \int_A \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha-\beta}\, dy\, dx.$$
Indeed, to prove (5) it suffices to split the inner integral into integrals over $A \cap \{d_\Omega(x) \geq d_\Omega(y)\}$ and $A \cap \{d_\Omega(x) < d_\Omega(y)\}$ and use the symmetry between variables x and y. According to above, if we abandon the assumption about the uniformity of Ω in Lemma 16, then, using (5), if Ω is bounded, we can analogously show that $C_c^\infty(\Omega) \subset W^{s,p;\,\alpha,\beta}(\Omega)$ for $\alpha + \beta < d - \overline{\dim}_M(\partial\Omega)$. Interestingly, this is a different range of parameters than in the Lemma 16.

Moreover, if $C_c^\infty(\Omega) \subset W^{s,p;\,\alpha,\beta}(\Omega)$ and Ω is bounded, then we cannot have $\alpha, \beta > d - \overline{\dim}_M(\partial\Omega)$. Indeed, if $\varphi \in C_c^\infty(\Omega)$, then simple calculation shows that
$$\begin{aligned}[\varphi]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} &\geq \operatorname{diam}(\Omega)^{-d-sp} \int_\Omega \int_\Omega |\varphi(x)-\varphi(y)|^p d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dy\, dx \\ &\geq \operatorname{diam}(\Omega)^{-d-sp} \int_{\operatorname{supp}\varphi} \int_{\Omega\setminus\operatorname{supp}\varphi} |\varphi(x)|^p d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dy\, dx.\end{aligned}$$
The inner integral $\int_{\Omega\setminus\operatorname{supp}\varphi} d_\Omega(y)^{-\beta}\, dy$ is infinite if $\beta > d - \overline{\dim}_M(\partial\Omega)$. The case when $\alpha > d - \overline{\dim}_M(\partial\Omega$ can be obtained similarly.

Definition 18. A weight $w\colon \Omega \to \mathbb{R}^d$ is *locally comparable to a constant* if for every compact subset $K \subset \Omega$ there exists $C_K > 0$ such that $\frac{1}{C_K} \leq w(x) \leq C_K$ for almost all $x \in K$.

The following Theorem is a generalization of [10, Theorem 12], where the same fact was proved for $w = v$.

Theorem 19. *Let $\Omega \subset \mathbb{R}^d$ be an nonempty open set, $0 < s < 1$, $p \in [1, \infty)$. Denote*
$$\widetilde{W}^{s,p;\,w,v}(\Omega) = \left\{ f\colon \Omega \to \mathbb{R}^d \text{ measurable} : \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\, w(x)\, v(y)\, dx\, dy < \infty \right\}.$$
We understand $\widetilde{W}^{s,p;\,w,v}(\Omega)$ as a semi-normed space. If w and v are locally bounded and satisfy the integral condition
$$\int_\Omega \frac{w(x)}{(1+|x|)^{d+sp}}\, dx < \infty, \int_\Omega \frac{v(x)}{(1+|x|)^{d+sp}}\, dx < \infty, \tag{6}$$
then $C^\infty(\Omega) \cap \widetilde{W}^{s,p;\,w,v}(\Omega)$ is dense in $\widetilde{W}^{s,p;\,w,v}(\Omega)$. Moreover, we have
$$W_0^{s,p;\,w,v}(\Omega) = W_c^{s,p;\,w,v}(\Omega).$$

Proof. The proof follows the proof of [10, Theorem 12]. First, we fix a Whitney decomposition $\mathcal{W} = \{Q_n\}_{n\in\mathbb{N}}$ of Ω with a constant $C(\mathcal{W}) = 1$. We extend w and v by 0 outside Ω. If w or v take the value zero on Ω, then we can artificially augment them by adding a positive, locally comparable to a constant weights w', v', which in addition satisfy (6). New weights $w + w'$ and $v + v'$ are also locally comparable to a constant, positive and satisfy (6). In this case w and v should be replaced by $w + w'$ and $v + v'$ in all the computations below.

Denote by τ_y the translation operator, that is $\tau_y f(x) = f(x-y)$, $x, y \in \mathbb{R}^d$, and let $M = 12^{d(p-1)}$. Moreover, let $f \in \widetilde{W}^{s,p;\,w,v}(\Omega)$ and
$$g_n(x,y) = \frac{f(x)\psi_n(x) - f(y)\psi_n(y)}{|x-y|^{\frac{d}{p}+s}}\, \mathbb{1}_{\Omega\times\Omega}\,(x,y).$$

We have

$$[P^{\eta_k} f - f]^p_{W^{s,p;\,w,v}(\Omega)} \leq M \sum_{n=1}^{\infty} \int_{\mathbb{R}^d} \|\tau_{\eta_k(Q_n)u} g_n - g_n\|^p_{L^p(\mathbb{R}^{2d}, w\times v)} \, h(u)\, du$$

and, for $t < \eta_k(Q_n)$,

$$\begin{aligned}
&\|\tau_t g_n - g_n\|^p_{L^p(\mathbb{R}^{2d}, w\times v)} \\
&\leq \int_{Q_n^*}\int_{Q_n^{**}} \frac{|f(x-t)\psi_n(x-t) - f(y-t)\psi_n(y-t) - f(x)\psi_n(x) + f(y)\psi_n(y)|^p}{|x-y|^{d+sp}} w(x)\, v(y)\, dx\, dy \\
&+ \int_{Q_n^*}\int_{\Omega\setminus Q_n^{**}} \frac{|f(x)\psi_n(x) - f(x-t)\psi_n(x-t)|^p}{|x-y|^{d+sp}} w(x)\, v(y)\, dx\, dy \\
&+ \int_{Q_n^*}\int_{\Omega\setminus Q_n^{**}} \frac{|f(x)\psi_n(x) - f(x-t)\psi_n(x-t)|^p}{|x-y|^{d+sp}} w(y)\, v(x)\, dx\, dy \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

The estimates of the integrals I_1, I_2 and I_3 and completely analogous to these from [9, Proof of Theorem 12]. Notice that the properly modified version of [10, Proposition 9] also holds. The equality between $W_0^{s,p;\,w,v}(\Omega)$ and $W_c^{s,p;\,w,v}(\Omega)$ is a consequence of [10, Proposition 2] and the fact that the approximating functions are of the form $P^{\eta_k} f$. □

Remark 20. Suppose that Ω is bounded. Then we trivially have

$$1 \leq (1+|x|)^{d+sp} \leq M := \sup_{x\in\Omega} (1+|x|)^{d+sp} < \infty,$$

hence, the condition (6) is equivalent to $w, v \in L^1(\Omega)$. Moreover, if $w(x) = d_\Omega(x)^{-\alpha}$, $v(x) = d_\Omega(x)^{-\beta}$, then (6) is satisfied when $0 \leq \alpha, \beta < d - \overline{\dim}_M(\partial\Omega)$ (we refer again to [14]). Of course, the function $d_\Omega(x)^{-a}$ is locally comparable to a constant on Ω for every $a \in \mathbb{R}$.

Lemma 21. *Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open set such that $|\Omega| < \infty$. Then we have*

$$W_0^{s,p;\,w,v}(\Omega) = W^{s,p;\,w,v}(\Omega) \iff \mathbb{1}_\Omega \in W_0^{s,p;\,w,v}(\Omega).$$

Proof. Using the result of Theorem 19 about the equality between $W_0^{s,p;\,w,v}(\Omega)$ and $W_c^{s,p;\,w,v}(\Omega)$, the proof is a copy of [9, Lemma 13]. □

Lemma 22. *Let Ω be an open, uniform, bounded domain and let*

$$v_n(x) = \max\left\{\min\left\{2 - n d_\Omega(x), 1\right\}, 0\right\} = \begin{cases} 1 & \text{when } d_\Omega(x) \leq 1/n, \\ 2 - n d_\Omega(x) & \text{when } 1/n < d_\Omega(x) \leq 2/n, \\ 0 & \text{when } d_\Omega(x) > 2/n. \end{cases}$$

There exists a constant $C = C(d,s,p,\alpha,\beta,\Omega) > 0$ such that the following inequality holds for all functions $f \in W^{s,p;\,\alpha,\beta}(\Omega)$ and $0 \leq \alpha, \beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$,

(7)

$$[f v_n]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} \leq C n^{sp} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p}{d_\Omega(x)^{\alpha+\beta}}\, dx + C \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx.$$

Moreover, without assuming the uniformity of Ω, the following weaker inequality is satisfied for all $\alpha, \beta \geq 0$, $\alpha + \beta < d - \overline{\dim}_M(\partial\Omega)$ and $f \in L^\infty(\Omega)$,

(8)

$$[f v_n]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} \leq C\|f\|^p_\infty n^{sp} \int_{\Omega_{\frac{3}{n}}} \frac{dx}{d_\Omega(x)^{\alpha+\beta}} + C \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx.$$

Proof. The following proof is a modification of [9, Lemma 10]. By Theorem 2, taking $\theta = \frac{1}{2}$ we have

$$\begin{aligned}
[fv_n]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} &\lesssim \int_\Omega \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)} \frac{|f(x)v_n(x)-f(y)v_n(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&= \int_{\Omega_{\frac{3}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\Omega_{\frac{3}{n}}} \frac{|f(x)v_n(x)-f(y)v_n(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&+ \int_{\Omega_{\frac{3}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\left(\Omega\setminus\Omega_{\frac{3}{n}}\right)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&+ \int_{\Omega\setminus\Omega_{\frac{3}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\Omega_{\frac{3}{n}}} \frac{|f(y)v_n(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

Starting with estimating the integral J_1, we obtain

$$\begin{aligned}
J_1 &\lesssim \int_{\Omega_{\frac{3}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\Omega_{\frac{3}{n}}} \frac{|v_n(y)|^p|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&+ \int_{\Omega_{\frac{3}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\Omega_{\frac{3}{n}}} \frac{|f(x)|^p|v_n(x)-v_n(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&=: K_1 + K_2.
\end{aligned}$$

The integral K_1 can be trivially bounded from above by the remainder of the weighted Gagliardo seminorm, that is

$$K_1 \le \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx.$$

Moreover, using the bound $|v_n(x)-v_n(y)| \le \min\{1, |x-y|\}$ and the fact that $d_\Omega(x) \approx d_\Omega(y)$ on the ball $B\left(x, \frac{1}{2}d_\Omega(x)\right)$ we can estimate K_2 as follows,

$$K_2 \lesssim \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p\left(\min\{1,|x-y|\}\right)^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha-\beta}\,dy\,dx.$$

Splitting the inner integral over dy into $|x-y| > 1/n$ and $|x-y| \le 1/n$ gives the first term in (7).

Going back to the integral J_2 and remembering that $v_n = 0$ on $\Omega_{\frac{3}{n}} \setminus \Omega_{\frac{2}{n}}$ we have

$$\begin{aligned}
J_2 &= \int_{\Omega_{\frac{2}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\left(\Omega\setminus\Omega_{\frac{3}{n}}\right)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&\lesssim \int_{\Omega_{\frac{2}{n}}} \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)\cap\left(\Omega\setminus\Omega_{\frac{3}{n}}\right)} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha-\beta}\,dy\,dx \\
&\le \int_{\Omega_{\frac{2}{n}}} \int_{\Omega\setminus\Omega_{\frac{3}{n}}} \frac{|f(x)v_n(x)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha-\beta}\,dy\,dx \\
&\le \int_{\Omega_{\frac{2}{n}}} |f(x)|^p d_\Omega(x)^{-\alpha-\beta}\,dx \int_{B(x,1/n)^c} \frac{dy}{|x-y|^{d+sp}} \\
&\lesssim n^{sp} \int_{\Omega_{\frac{2}{n}}} |f(x)|^p d_\Omega(x)^{-\alpha-\beta}\,dx.
\end{aligned}$$

The integral J_3 can be estimated in the similar way as J_2. That ends the proof of (7). We note that the proof of (8) is analogous to the previous part. K_1 estimates by (5) and in the integrals J_2 and J_3 we use the fact that $d_\Omega(y) \geq d_\Omega(x)$ for $y \notin \Omega_{\frac{3}{n}}$ and $x \in \Omega_{\frac{3}{n}}$, hence, the comparability is not necessary here. The only thing that essentially changes is the estimation of K_2. In this case we bound $|f(x)|$ from above by its L^∞-norm, use (5) and then proceed similarly as before to obtain the desired result. That proves (8). □

4. Proof of the comparability

Proof of Theorem 2. In the proof of this Theorem we use techniques coming from [23]. We start with fixing sufficiently fragmented Whitney decomposition $\mathcal{W} = \mathcal{W}(\theta)$, so that for $(x, y) \in Q \times 5Q$ it holds $y \in B(x, \theta\, d_\Omega(x))$. Suppose first that $p > 1$. Let $q = \frac{p}{p-1}$ be the Hölder conjugate exponent to p. Using the duality between spaces $L^p(\Omega \times \Omega)$ and $L^q(\Omega \times \Omega)$ we can write the weighted Gagliardo seminorm wherewithal dual norm, that is

$$\begin{aligned}
&\left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\, dx\right)^{\frac{1}{p}} \\
&= \sup \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{\frac{d}{p}+s}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\, dy\, dx,
\end{aligned}$$

where the supremum is taken over all nonnegative $g \in L^q(\Omega \times \Omega)$ satisfying $\|g\|_{L^q(\Omega\times\Omega)} \leq 1$. For now on we fix such a function g. Now, we split the integration range as follows,

$$\int_\Omega \int_\Omega = \sum_Q \int_Q \int_{2Q} + \sum_{Q,S} \int_Q \int_{S\setminus 2Q} =: S_1 + S_2.$$

Thanks to our assumption about the Whitney decomposition, the first sum can be immediately estimated by the truncated seminorm with making use of the Hölder inequality,

$$\begin{aligned}
S_1 &\leq \left(\sum_Q \int_Q \int_{2Q} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx\right)^{\frac{1}{p}} \|g\|_{L^q(\Omega\times\Omega)} \\
&\leq \left(\sum_Q \int_Q \int_{2Q} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x) - f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, we only need to estimate the second part, S_2. We denote by f_Q the average value of f on the cube Q, that is $f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx$ (the latter is finite by assumption). Using similar arguments as in [23, Section 4] we observe that for $x \in Q$ and $y \in S \setminus 2Q$

it holds $|x-y| \approx D(Q,S)$, hence, triangle inequality yields

$$\begin{aligned}
S_2 &\lesssim \sum_Q \sum_S \int_Q \int_S \frac{|f(x)-f(y)|}{D(Q,S)^{\frac{d}{p}+s}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\,dy\,dx \\
&\leq \sum_Q \sum_S \int_Q \int_S \frac{|f(x)-f_Q|}{D(Q,S)^{d+sp}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\,dy\,dx \\
&+ \sum_Q \sum_S \int_Q \int_S \frac{|f_Q - f_{Q_S}|}{D(Q,S)^{d+sp}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\,dy\,dx \\
&+ \sum_Q \sum_S \int_Q \int_S \frac{|f_{Q_S} - f_S|}{D(Q,S)^{d+sp}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\,dy\,dx \\
&+ \sum_Q \sum_S \int_Q \int_S \frac{|f_S - f(y)|}{D(Q,S)^{d+sp}} d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}} g(x,y)\,dy\,dx \\
&=: \boldsymbol{(A)} + \boldsymbol{(B)} + \boldsymbol{(C)} + \boldsymbol{(D)}.
\end{aligned}$$

Let us estimate $\boldsymbol{(A)}$ first. By Hölder inequality and Fubini-Tonelli theorem we get

$$\begin{aligned}
\boldsymbol{(A)} &\leq \sum_Q \int_Q |f(x)-f_Q| d_\Omega(x)^{-\frac{\alpha}{p}} \left(\sum_S \int_S g(x,y)^q\,dy \right)^{\frac{1}{q}} \left(\sum_S \int_S \frac{d_\Omega(y)^{-\beta}}{D(Q,S)^{d+sp}} \right)^{\frac{1}{p}} dx \\
&\leq \left(\sum_Q \int_Q |f(x)-f_Q|^p d_\Omega(x)^{-\alpha} \sum_S \int_S \frac{d_\Omega(y)^{-\beta}}{D(Q,S)^{d+sp}}\,dx \right)^{\frac{1}{p}}.
\end{aligned}$$

By [23, Lemma 2.7] with $r = l(Q)$ and the Muckenhoupt condition (4) we have

$$\begin{aligned}
\sum_S \int_S \frac{d_\Omega(y)^{-\beta}}{D(Q,S)^{d+sp}}\,dy &\lesssim l(Q)^{-sp} \inf_{y\in Q} M d_\Omega^{-\beta}(y) \\
&\lesssim l(Q)^{-sp} \inf_{y\in Q} d_\Omega(y)^{-\beta} \\
&\lesssim l(Q)^{-sp} d_\Omega(y)^{-\beta}
\end{aligned}$$

for any $y \in Q$, where M is the Hardy-Littlewood maximal function. Hence, by Jensen inequality and Whitney decomposition properties, $\boldsymbol{(A)}$ can be bounded from above as follows,

$$\begin{aligned}
\boldsymbol{(A)}^p &\lesssim \sum_Q \int_Q \frac{1}{|Q|} \int_Q |f(x)-f(y)|^p d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta} l(Q)^{-sp}\,dy\,dx \\
&\lesssim \sum_Q \int_Q \int_Q |f(x)-f(y)|^p d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta} l(Q)^{-sp-d}\,dy\,dx \\
&\lesssim \sum_Q \int_Q \int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx \\
&\leq \int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\,dy\,dx,
\end{aligned}$$

as it holds $|x-y| \lesssim l(Q)$ for $x, y \in Q$.

Now, we face the estimation of the component $\boldsymbol{(B)}$. We denote by $\mathcal{N}(P)$ the successor of the cube P in the chain $[Q,S]$. It holds $\mathcal{N}(P) \subset 5P$ and $Q \in \mathbf{Sh}(P)$ for $P \in [Q, Q_S]$.

Also, $D(Q,S) \approx D(P,S)$ for such P. Hence, analogously to [23], by triangle inequality and Jensen inequality we can estimate $\boldsymbol{(B)}$ as follows,

$$\begin{aligned}
\boldsymbol{(B)} &\le \sum_{Q,S} \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(Q,S)^{\frac{d}{q}+s}} g(x,y) \sum_{P\in[Q,Q_S)} |f_P - f_{\mathcal{N}(P)}| \, dy \, dx \\
&\le \sum_{Q,S} \int_Q \int_S \sum_{P\in[Q,Q_S)} \frac{1}{|P|} \frac{1}{|\mathcal{N}(P)|} \int_P \int_{\mathcal{N}(P)} |f(\xi) - f(\zeta)| \, d\xi \, d\zeta \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(Q,S)^{\frac{d}{p}+s}} g(x,y) \, dy \, dx \\
&\lesssim \sum_P \frac{1}{|P||5P|} \int_P \int_{5P} |f(\xi) - f(\zeta)| \, d\xi \, d\zeta \sum_{Q\in \mathbf{Sh}(P)} \sum_S \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(P,S)^{\frac{d}{q}+s}} g(x,y) \, dy \, dx.
\end{aligned}$$

Define

$$G(x) = \left(\int_\Omega g(x,y)^q \, dy \right)^{\frac{1}{q}}, \; x \in \Omega.$$

Using again Hölder inequality, Muckenhoupt condition (4) and Whitney covering properties we have

$$\begin{aligned}
&\sum_{Q\in \mathbf{Sh}(P)} \sum_S \int_Q \int_S \frac{d_\Omega(x)^{-\frac{\alpha}{p}} d_\Omega(y)^{-\frac{\beta}{p}}}{D(P,S)^{\frac{d}{p}+s}} g(x,y) \, dy \, dx \\
&\le \sum_{Q\in \mathbf{Sh}(P)} \int_Q d_\Omega(x)^{-\frac{\alpha}{p}} \left(\sum_S \int_S \frac{d_\Omega(y)^{-\beta}}{D(P,S)^{d+sp}} \right)^{\frac{1}{p}} G(x) \, dx \\
&\lesssim l(P)^{-\frac{\beta}{p}-s} \int_{\mathbf{SH}(P)} G(x) d_\Omega(x)^{-\frac{\alpha}{p}} \, dx.
\end{aligned}$$

Let us take small $\varepsilon > 0$, to be established in a moment. We apply Hölder inequality with exponents $q-\varepsilon$ and $\frac{q-\varepsilon}{q-\varepsilon-1}$ to the integral above to obtain

$$\int_{\mathbf{SH}(P)} G(x) d_\Omega(x)^{-\frac{\alpha}{p}} \, dx \le \left(\int_{\mathbf{SH}(P)} G(x)^{q-\varepsilon}(x) \, dx \right)^{\frac{1}{q-\varepsilon}} \left(\int_{\mathbf{SH}(P)} d_\Omega(x)^{-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}} \, dx \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}}.$$

Notice that

$$\lim_{\varepsilon\to 0^+} \frac{q-\varepsilon}{p(q-\varepsilon-1)} = \frac{q}{p(q-1)} = 1,$$

hence, remembering that by assumption $0 \le \alpha < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$, for sufficiently small ε we still have $0 \le \frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)} < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$ (this condition defines ε, as well as $q-\varepsilon > 1$). According to this, by [23, Lemma 2.7] and (4), we have

$$\begin{aligned}
&\left(\int_{\mathbf{SH}(P)} G(x)^{q-\varepsilon}(x) \, dx \right)^{\frac{1}{q-\varepsilon}} \left(\int_{\mathbf{SH}(P)} d_\Omega(x)^{-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}} \, dx \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\lesssim \left(l(P)^d \inf_{x\in P} MG^{q-\varepsilon}(x) \right)^{\frac{1}{q-\varepsilon}} \left(l(P)^d \inf_{x\in P} M d_\Omega^{-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}}(x) \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\lesssim \left(l(P)^d \inf_{x\in P} MG^{q-\varepsilon}(x) \right)^{\frac{1}{q-\varepsilon}} \left(l(P)^{d-\frac{\alpha(q-\varepsilon)}{p(q-\varepsilon-1)}} \right)^{\frac{q-\varepsilon-1}{q-\varepsilon}} \\
&\le l(P)^{d-\frac{\alpha}{p}} \left(MG^{q-\varepsilon}(\zeta) \right)^{\frac{1}{q-\varepsilon}}
\end{aligned}$$

for any $\zeta \in P$. Finally, summing up all the considerations above, by Jensen inequality, Hölder inequality and boundendess of the Hardy-Littlewood maximal function on $L^{\frac{q}{q-\varepsilon}}(\mathbb{R}^d)$ we get the following result,

$$
\begin{aligned}
\boldsymbol{(B)} &\lesssim \sum_P \frac{l(P)^{d-\frac{\alpha}{p}-\frac{\beta}{p}-s}}{|P||5P|} \int_P \int_{5P} |f(\xi)-f(\zeta)| \left(MG^{q-\varepsilon}(\zeta)\right)^{\frac{1}{q-\varepsilon}}\, d\xi\, d\zeta \\
&= \sum_P \frac{l(P)^{-\frac{\alpha}{p}-\frac{\beta}{p}-s}}{|5P|} \int_P \int_{5P} |f(\xi)-f(\zeta)| \left(MG^{q-\varepsilon}(\zeta)\right)^{\frac{1}{q-\varepsilon}}\, d\xi\, d\zeta \\
&\leq \sum_P l(P)^{-\frac{\alpha}{p}-\frac{\beta}{p}-s} \left(\int_P \left(\frac{1}{|5P|}\int_{5P} |f(\xi)-f(\zeta)|\, d\xi\right)^p d\zeta\right)^{\frac{1}{p}} \left(\int_P \left(MG^{q-\varepsilon}(\zeta)\right)^{\frac{q}{q-\varepsilon}}\, d\zeta\right)^{\frac{1}{q}} \\
&\leq \sum_P l(P)^{-\frac{\alpha}{p}-\frac{\beta}{p}-s} \left(\int_P \frac{1}{|5P|}\int_{5P} |f(\xi)-f(\zeta)|^p\, d\xi\, d\zeta\right)^{\frac{1}{p}} \left(\int_P \left(MG^{q-\varepsilon}(\zeta)\right)^{\frac{q}{q-\varepsilon}}\, d\zeta\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_P \int_P \int_{5P} \frac{|f(\xi)-f(\zeta)|^p}{|\xi-\zeta|^{d+sp}} d_\Omega(\zeta)^{-\alpha} d_\Omega(\xi)^{-\beta}\right)^{\frac{1}{p}} \left(\int_\Omega \left(MG^{q-\varepsilon}(\zeta)\right)^{\frac{q}{q-\varepsilon}}\, d\zeta\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_P \int_P \int_{5P} \frac{|f(\xi)-f(\zeta)|^p}{|\xi-\zeta|^{d+sp}} d_\Omega(\zeta)^{-\alpha} d_\Omega(\xi)^{-\beta}\right)^{\frac{1}{p}} \left(\int_\Omega G^q(\zeta)\, d\zeta\right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_P \int_P \int_{5P} \frac{|f(\xi)-f(\zeta)|^p}{|\xi-\zeta|^{d+sp}} d_\Omega(\zeta)^{-\alpha} d_\Omega(\xi)^{-\beta}\, d\xi\, d\zeta\right)^{\frac{1}{p}} \\
&\lesssim \left(\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(\xi)-f(\zeta)|^p}{|\xi-\zeta|^{d+sp}} d_\Omega(\zeta)^{-\alpha} d_\Omega(\xi)^{-\beta}\, d\xi\, d\zeta\right)^{\frac{1}{p}}.
\end{aligned}
$$

That ends $\boldsymbol{(B)}$. We observe that the case $\boldsymbol{(C)}$ is symmetric to $\boldsymbol{(B)}$ (as we may have $Q_S = S_Q$). We will obtain the same estimate as in $\boldsymbol{(B)}$, but with α and β changed, however it holds $d_\Omega(x) \approx d_\Omega(y)$ for $x, y \in 5P$, hence, we will obtain exactly the same bound. The case $\boldsymbol{(D)}$ is symmetric to $\boldsymbol{(A)}$. That ends the proof in the case $p > 1$. When $p = 1$, we proceed similarly and actually this case is simpler and does not require the usage of dual norms. ☐

When $p = 1$, we can formulate even a more general result.

Theorem 23. *Let Ω be a nonempty, bounded, uniform domain and $0 < s < 1$, $0 < \theta \leq 1$. If the weights w, v belong to the Muckenhoupt class A_1, then the full seminorm $[f]_{W^{s,1;w\,v}(\Omega)}$ and the truncated seminorm*

$$\int_\Omega \int_{B(x,\theta d_\Omega(x))} \frac{|f(x)-f(y)|}{|x-y|^{d+s}} \left(w(x)v(y) + w(y)v(x)\right)\, dy\, dx$$

are comparable for all $f \in L^1_{loc}(\Omega)$. The comparability constant depends on Ω, s, d, w, v and θ.

Proof. The proof is similar to the proof of Theorem 2. The additional term in the truncated seminorm above follows from the fact, that components $\boldsymbol{(B)}$ and $\boldsymbol{(C)}$ are symmetric with respect to w and v, but we cannot use the comparability of $w(x)$ and $v(y)$ on the cube $5P$, as for the distance weights. ☐

Remark 24. Interestingly, the result of Theorem 2 allow to deduce in some cases another comparability property, between weighted Gagliardo seminorms $[f]_{W^{s,p;\,\alpha\,\beta}(\Omega)}$ and

$[f]_{W^{s,p;\,\alpha+\beta,0}(\Omega)}$. Suppose that Ω is a nonempty, bounded, uniform domain and the parameters α, β satisfy $0 \le \alpha, \beta, \alpha+\beta < \underline{\operatorname{co\,dim}}_A(\partial\Omega)$. Take $f \in L^1_{loc}(\Omega)$. By (5) we have

$$[f]_{W^{s,p;\,\alpha\,\beta}(\Omega)} \le 2^{\frac{1}{p}} [f]_{W^{s,p;\,\alpha+\beta,0}(\Omega)}.$$

To obtain a converse inequality, we use Theorem 2 with $\theta = \frac{1}{2}$ and get

$$\begin{aligned}[f]^p_{W^{s,p;\,\alpha\,\beta}(\Omega)} &\ge \int_\Omega \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\beta} d_\Omega(x)^{-\alpha}\,dy\,dx \\ &\approx \int_\Omega \int_{B\left(x,\frac{1}{2}d_\Omega(x)\right)} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(y)^{-\alpha-\beta}\,dy\,dx \\ &\gtrsim [f]^p_{W^{s,p;\,\alpha+\beta,0}(\Omega)}.\end{aligned}$$

Overall, we indeed get that

$$[f]_{W^{s,p;\,\alpha\,\beta}(\Omega)} \approx [f]_{W^{s,p;\,\alpha+\beta,0}(\Omega)}.$$

5. Proofs of main results

Before we proceed to prove our main results, we need the following Proposition.

Proposition 25. *Let Ω be a nonempty open set. Then the space $W^{s,p;\,w,\,v}(\Omega)$ is reflexive for $0 < s < 1$, $1 < p < \infty$ and all weights w and v.*

Proof. The proof is a modification of the proof of the reflexivity of the classical Sobolev space $W^{1,p}(\Omega)$ from [3, Proposition 8.1]. We define the isometry $T\colon W^{s,p;\,w,\,v}(\Omega) \to L^p(\Omega) \times L^p(\Omega\times\Omega, w\times v)$ (the latter endowed with the natural product norm) by

$$T(u) = \left(u, \frac{u(x)-u(y)}{|x-y|^{\frac{d}{p}+s}}\right).$$

The reflexitiy of $W^{s,p;\,w,\,v}(\Omega)$ is a consequence of reflexivity of $L^p(\Omega)\times L^p(\Omega\times\Omega, w\times v)$. ☐

Proof of Theorem 3, case I. By Lemma 21 and Theorem 19 to prove the density of $C_c^\infty(\Omega)$ in $W^{s,p;\,\alpha,\beta}(\Omega)$ it suffices to approximate the function $f = \mathbb{1}_\Omega$ by functions with compact support. By (8) (keeping the same notation), we have

$$[fv_n]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} \le Cn^{sp} \int_{\Omega_{\frac{3}{n}}} \frac{dx}{d_\Omega(x)^{\alpha+\beta}}.$$

We have

$$n^{sp} \int_{\Omega_{\frac{3}{n}}} d_\Omega(x)^{-\alpha-\beta}\,dx \lesssim \int_{\Omega_{\frac{3}{n}}} d_\Omega(x)^{-\alpha-\beta-sp}\,dx \longrightarrow 0,$$

when $n \longrightarrow \infty$, because $\int_\Omega d_\Omega(x)^{-\alpha-\beta-sp}\,dx = \zeta_\Omega(\alpha+\beta+sp) < \infty$. ☐

Proof of Theorem 3, case II. Recall that in this case we assume that Ω is $(d-sp-\alpha-\beta)$-homogeneous. Define the layers $\Omega_{i,n} = \left\{x \in \Omega : \frac{3}{2^{i+1}n} < d_\Omega(x) \le \frac{3}{2^i n}\right\}$. We observe that

$$n^{sp} \int_{\Omega_{\frac{3}{n}}} d_\Omega(x)^{-\alpha-\beta}\, dx = n^{sp} \sum_{i=0}^{\infty} \int_{\Omega_{i,n}} d_\Omega(x)^{-\alpha-\beta}\, dx$$
$$\approx n^{sp+\alpha+\beta} \sum_{i=0}^{\infty} 2^{-i(\alpha+\beta)}\, |\Omega_{i,n}|$$
$$\lesssim n^{sp+\alpha+\beta-\underline{d}} \sum_{i=0}^{\infty} 2^{-i(\alpha+\beta+\underline{d})} = C,$$

where C is a constant independent of n. That means that the sequence $\{fv_n\}_{n\in\mathbb{N}}$ is bounded in $W^{s,p;\,\alpha,\beta}(\Omega)$. Now, the proof follows [9, Proof of Theorem 2, case II]: we use Banach–Alaoglu and Eberlein–Šmulian theorems to conclude that there exists a subsequence $\{fv_{n_k}\}_{k\in\mathbb{N}}$ convergent to $\mathbb{1}_\Omega$ in $W^{s,p;\,\alpha,\beta}(\Omega)$. The reflexivity of $W^{s,p;\,\alpha,\beta}(\Omega)$ is essential here. □

Proof of Theorem 3, case III. We proceed analogously as in the unweighted case in [9, Proof of Theorem 2, case III]. In this case we just need to use the fractional weighted Hardy inequality (1) in the case (F) and Fatou's lemma to prove that the function $f = \mathbb{1}_\Omega$ cannot be approximated by $C_c^\infty(\Omega)$ functions in $W^{s,p;\,\alpha,\beta}(\Omega)$. □

Remark 26. Notice that in the proof of the case III we use the fact that if $u_n \longrightarrow \mathbb{1}_\Omega$ in $L^p(\Omega)$, then there exists a subsequence u_{n_k} convergent to $\mathbb{1}_\Omega$ almost everywhere; the same fact holds if we replace $L^p(\Omega)$ by the weighted space $L^p(\Omega, W)$ for almost everywhere positive $W \in L^1(\Omega)$.

Proof of Theorem 4. If $\int_\Omega |f(x)|^p d_\Omega(x)^{-sp-\alpha-\beta}\, dx < \infty$, then $f \in W_0^{s,p;\,\alpha,\beta}(\Omega)$, because by Lemma 22 we have

$$[fv_n]^p_{W^{s,p;\,\alpha,\beta}(\Omega)} \lesssim n^{sp} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p}{d_\Omega(x)^{\alpha+\beta}}\, dx + \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx$$
$$\lesssim \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)|^p}{d_\Omega(x)^{sp+\alpha+\beta}}\, dx + \int_{\Omega_{\frac{3}{n}}} \int_{\Omega_{\frac{3}{n}}} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} d_\Omega(x)^{-\alpha} d_\Omega(y)^{-\beta}\, dy\, dx \to 0,$$

when $n \longrightarrow \infty$. On the other side, if $f \in W_0^{s,p;\,\alpha,\beta}(\Omega)$, then by the fractional Hardy inequality (1) and Fatou's lemma we obtain that $\int_\Omega |f(x)|^p d_\Omega(x)^{-sp-\alpha-\beta}\, dx < \infty$. That proves the desired characterization of $W_0^{s,p;\,\alpha,\beta}(\Omega)$. □

Proof of Theorem 7. This is a straightforward consequence of the fractional Hardy inequality (1) in the case (T'), case I of the Theorem 3 and Fatou's lemma. We can easily see that uniform domains are κ-plump, so (6) is applicable. □

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SHARP HARDY INEQUALITIES FOR SOBOLEV-BREGMAN FORMS

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Abstract. We obtain sharp fractional Hardy inequalities for the half-space and for convex domains. We extend the results of Bogdan and Dyda and of Loss and Sloane to the setting of Sobolev-Bregman forms.

1. Introduction and main results

Let $0 < \alpha < 2$ and $d = 1, 2, \ldots$. Bogdan and Dyda [2] proved the following Hardy inequality in the half-space $D = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$. For every $u \in C_c(D)$,

$$\frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,\alpha}\int_D \frac{u(x)^2}{x_d^\alpha}dx\,, \tag{1}$$

where

$$\kappa_{d,\alpha} = \frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\frac{\mathrm{B}\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right)-2^\alpha}{\alpha 2^\alpha}\,, \tag{2}$$

and (1) fails to hold for some $u \in C_c(D)$ if $\kappa_{d,\alpha}$ is replaced by a bigger constant. Here, Γ is the Euler gamma function, B is the Euler beta function, and $C_c(D)$ denotes the class of all the continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ with compact support in D.

The main purpose of this note is to prove a generalization of this inequality, where the left-hand side of (1) is replaced with the following form:

$$\mathcal{E}_p[u] := \frac{1}{2}\int_D\int_D (u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})|x-y|^{-d-\alpha}\,dy\,dx, \tag{3}$$

defined for $p \in (1, \infty)$ and $u : \mathbb{R}^d \to \mathbb{R}$, where

$$a^{\langle k\rangle} := |a|^k \operatorname{sgn} a, \quad a, k \in \mathbb{R}.$$

We call such integral forms the *Sobolev-Bregman forms.*

For $\alpha \neq 1$ let

$$\kappa_{d,p,\alpha} = -\frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\left(\mathrm{B}\left(\tfrac{\alpha-1}{p}+1, -\alpha\right)+\mathrm{B}\left(\alpha-\tfrac{\alpha-1}{p}, -\alpha\right)+\tfrac{1}{\alpha}\right) \geq 0. \tag{4}$$

Recall that $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and $1/\Gamma$ can be extended analytically to the whole of $\mathbb{R}$, hence $\mathrm{B}(x,y)$ is well defined for all $x, y \neq 0, -1, -2, \ldots$. Noteworthy, $\kappa_{d,p,1} = 0$ (understood as the limit of $\kappa_{d,p,\alpha}$ as $\alpha \to 1$). Furthermore, observe that $\kappa_{d,p,\alpha} = \kappa_{d,p',\alpha}$, where $p' = p/(p-1)$.

Our first main result reads as follows.

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Theorem 1. *Let $0 < \alpha < 2$, $d = 1, 2, \dots$ and $1 < p < \infty$. For every $u \in C_c(D)$,*

$$\frac{1}{2}\int_D\int_D \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,p,\alpha}\int_D \frac{|u(x)|^p}{x_d^\alpha}dx\,, \tag{5}$$

and the constant in (5) *is the best possible, i.e. it cannot be replaced by a bigger one.*

Our work is motivated by a recent paper of Bogdan, Jakubowski, Lenczewska and Pietruska-Pałuba [4], who obtained a similar inequality for the whole space $\mathbb{R}^d$ instead of D (see [4, Theorems 1 and 2]), namely they proved that if $0 < \alpha < d \wedge 2$, then for $u \in L^p(\mathbb{R}^d)$,

$$\frac{\mathcal{A}_{d,-\alpha}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{\frac{d-\alpha}{p}}\int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha}dx\,,$$

where $\mathcal{A}_{d,-\alpha} = \frac{2^\alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}$ and the constant $\kappa_{\frac{d-\alpha}{p}}$ is explicit and optimal. They used this inequality to characterize the L^p contractivity property of the Feynman-Kac semigroup generated by $\Delta^{\alpha/2} + \kappa|x|^{-\alpha}$.

For the sake of completeness, let us mention that the sharp fractional Hardy inequality

$$\int_D\int_D \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,dy\,dx \geq \mathcal{D}_{d,s,p}\int_D \frac{|u(x)|^p}{x_d^{sp}}\,dx, \tag{6}$$

where $u \in C_0^\infty(\overline{D})$ if $ps < 1$ and $u \in C_0^\infty(D)$ if $ps > 1$, was obtained by Frank and Seiringer in [10]. The constant $\mathcal{D}_{d,s,p}$ is optimal and has an explicit form (see [10, (1.4)]). However, this result is not directly comparable to ours, as the integral forms in (3) and on the left-hand side of (6) are different for $p \neq 2$. The reader interested in fractional Hardy inequalities may also see Frank and Seiringer [9] for an analogous result on $\mathbb{R}^d$, Frank, Lieb and Seiringer [8] for other optimal inequalities and [5–7] for more general Hardy inequalities, but with unknown sharp constants.

Loss and Sloane [13] proved that if $\alpha \in (1, 2)$, then a fractional Hardy inequality similar to (6) holds for all convex, proper subsets of $\mathbb{R}^d$, with the same optimal constant (see [13, Theorem 1.2]). We obtain an analogous formula for Sobolev-Bregman forms and this is our second main result.

Theorem 2. *Let Ω be an open, proper subset of $\mathbb{R}^d$ and let $1 < \alpha < 2$. Then, for $u \in C_c(\Omega)$,*

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,p,\alpha}\int_\Omega \frac{|u(x)|^p}{m_\alpha(x)^\alpha}\,dx, \tag{7}$$

where

$$m_\alpha(x)^\alpha = \frac{\int_{\mathbb{S}^{d-1}}|\omega_d|^\alpha\,d\omega}{\int_{\mathbb{S}^{d-1}}d_{\omega,\Omega}(x)^{-\alpha}\,d\omega}, \quad d_{\omega,\Omega}(x) = \min\{|t| : x + t\omega \notin \Omega\}.$$

In particular, if Ω is convex, then

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \geq \kappa_{d,p,\alpha}\int_\Omega \frac{|u(x)|^p}{\mathrm{dist}(x,\partial\Omega)^\alpha}\,dx. \tag{8}$$

The constant in (8) *is optimal.*

Here, $\mathrm{dist}(x, \partial\Omega)$ denotes the distance from the point $x \in \Omega$ to $\partial\Omega$, i.e. $\mathrm{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega}|x-y|$. We note that (8) is an easy consequence of (7) since $m_\alpha(x) \leq \mathrm{dist}(x, \partial\Omega)$ if Ω is convex and $x \in \Omega$ (see [13]). If $\alpha \leq 1$ and Ω is a bounded convex domain, then the inequality (8) cannot hold with any positive constant, see Remark 1.

Fractional Hardy inequalities are of interest not only from the analytical point of view, but also due to their connection with stochastic processes by means of Dirichlet forms. In particular, our result is related to the censored α-stable process in D, which, informally speaking, is a stable process „forced" to stay inside D, see Bogdan, Burdzy and Chen [1]. If we denote its Dirichlet form by $\mathcal{C}(u,v)$, then similarly as in [2], (5) is for $u \in C_c^\infty(D)$ equivalent to

$$\mathcal{C}(u, u^{\langle p-1\rangle}) \geq \mathcal{A}_{d,-\alpha}\kappa_{d,p,\alpha} \int_D \frac{|u(x)|^p}{x_d^\alpha} dx.$$

Due to the relation between $\mathcal{C}$ and the Dirichlet form of the α-stable process killed upon leaving D, which we denote by $\mathcal{K}(u,v)$ (we again refer to [1]), we get

$$\mathcal{K}(u, u^{\langle p-1\rangle}) \geq \mathcal{A}_{d,-\alpha}\left(\kappa_{d,p,\alpha} + \frac{1}{\alpha}\frac{\pi^{(d-1)/2}\Gamma\left((1+\alpha)/2\right)}{\Gamma\left((\alpha+d)/2\right)}\right) \int_D \frac{|u(x)|^p}{x_d^\alpha} dx,$$

for all $u \in C_c^\infty(D)$.

The generator of the censored α-stable process in D is the *regional fractional Laplacian*, defined for $u \in C_c^2(D)$ by the formula

$$\Delta_D^{\alpha/2} u(x) = \mathcal{A}_{d,-\alpha} \lim_{\varepsilon \to 0^+} \int_{D \cap \{|y-x|>\varepsilon\}} \frac{u(y)-u(x)}{|x-y|^{d+\alpha}}\, dy\,,$$

see [11, 12]. We will use the notation $\mathcal{L} = \mathcal{A}_{d,-\alpha}^{-1}\Delta_D^{\alpha/2}$.

For $a, b \in \mathbb{R}$ and $p > 1$, we define the *Bregman divergence*

$$F_p(a,b) := |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a).$$

The function F_p is nonnegative as the second-order Taylor remainder of the convex function $x \mapsto |x|^p$. Moreover, we have the identity $F_p(a,b) + F_p(b,a) = p(b-a)(b^{\langle p-1\rangle} - a^{\langle p-1\rangle})$, and the latter expression appears on the left-hand side of (5). We refer the reader to [4] for more references on Sobolev-Bregman forms.

We denote by $|x| = (x_1^2 + \cdots + x_d^2)^{1/2}$ the Euclidean norm of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, and $B(x,r)$ stands for the Euclidean ball of radius $r > 0$ centered at x. For $d \geq 2$ we write $x = (x', x_d)$, where $x' = (x_1, \ldots, x_{d-1})$, and we let $\|x'\| = \max_{k=1,\ldots,d-1} |x_k|$.

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2. Proof of Theorem 1

In order to prove our first result, we will need an analogue of [4, Theorem 1].

Let $w = w_\beta = x_d^\beta$ for $\beta \in (-1, \alpha)$. By [1, (5.4) and (5.5)],

$$\mathcal{L}w_\beta(x) = \gamma(\alpha,\beta)\,\frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\, x_d^{-\alpha}\, w_\beta(x)\,, \tag{9}$$

where

$$\gamma(a,b) = \int_0^1 \frac{(t^b-1)(1-t^{a-b-1})}{(1-t)^{1+a}}\, dt, \quad a \in (0,2),\ b \in (-1,a), \tag{10}$$

is absolutely convergent. We note that $\gamma(\alpha,\beta) \leq 0$ if and only if $\beta(\alpha-\beta-1) \geq 0$.

Lemma 1. *Let $w(x) = w_\beta(x) = x_d^\beta$, $\beta \in (-1, \alpha)$ for $p \in (1,2)$ and $\beta \in \left(-\frac{1}{p-1}, \frac{\alpha}{p-1}\right)$ for $p \in [2,\infty)$, and $u \in C_c(D)$. Then we have*

$$\mathcal{E}_p[u] = \frac{(p-1)\gamma(\alpha,\beta) + \gamma(\alpha,(p-1)\beta)}{p} \frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)} \int_D \frac{|u(x)|^p}{x_d^\alpha} dx + \frac{1}{p}\int_D\int_D F_p\left(\frac{u(x)}{w(x)}, \frac{u(y)}{w(y)}\right) w(x)^{p-1}\, w(y) \frac{dy\, dx}{|x-y|^{d+\alpha}}.$$

In particular, for $\beta = \frac{\alpha-1}{p}$,

$$\mathcal{E}_p[u] = \kappa_{d,p,\alpha} \int_D \frac{|u(x)|^p}{x_d^\alpha} dx + \frac{1}{p}\int_D\int_D F_p\left(\frac{u(x)}{w(x)}, \frac{u(y)}{w(y)}\right) w(x)^{p-1}\, w(y) \frac{dy\, dx}{|x-y|^{d+\alpha}}.$$

Proof. Let $w = w_\beta$, $u \in C_c(D)$, $x, y \in D$. We have

$$\begin{aligned}
(11)\quad & pu(x)^{\langle p-1\rangle}(u(x)-u(y)) + |u(y)|^p \frac{w(x)^{p-1} - w(y)^{p-1}}{w(y)^{p-1}} + (p-1)|u(x)|^p \frac{w(y)-w(x)}{w(x)} \\
&= |u(y)|^p \frac{w(x)^{p-1} - w(y)^{p-1}}{w(y)^{p-1}} + p|u(x)|^p \frac{w(y)-w(x)}{w(x)} - |u(x)|^p \frac{w(y)-w(x)}{w(x)} \\
&\quad - pu(x)^{\langle p-1\rangle}(u(y)-u(x)) \\
&= |u(y)|^p \frac{w(x)^{p-1} - w(y)^{p-1}}{w(y)^{p-1}} - |u(x)|^p \frac{w(y)-w(x)}{w(x)} \\
&\quad - p\left(u(x)^{\langle p-1\rangle}u(y) - |u(x)|^p - |u(x)|^p \frac{w(y)-w(x)}{w(x)}\right).
\end{aligned}$$

We integrate both sides with respect to the measure $\mu_\varepsilon(dx\,dy) := \mathbb{1}_{\{|x-y|>\varepsilon\}}|x-y|^{-d-\alpha}\,dx\,dy$ and use the symmetry of μ_ε. We obtain

$$\begin{aligned}
\text{LHS} &= \frac{p}{2}\int_D\int_D (u(x)-u(y))(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle})\,\mu_\varepsilon(dx\,dy) \\
&\quad + \int_D\int_D |u(x)|^p \frac{w(y)^{p-1} - w(x)^{p-1}}{w(x)^{p-1}}\,\mu_\varepsilon(dx\,dy) \\
&\quad + (p-1)\int_D\int_D |u(x)|^p \frac{w(y)-w(x)}{w(x)}\,\mu_\varepsilon(dx\,dy)
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= \int_D\int_D \left(|u(y)|^p \frac{w(x)^{p-1}}{w(y)^{p-1}} - |u(y)|^p - |u(x)|^p \frac{w(y)}{w(x)} + |u(x)|^p\right)\mu_\varepsilon(dx\,dy) \\
&\quad - p\int_D\int_D \left(u(x)^{\langle p-1\rangle}u(y) - |u(x)|^p - |u(x)|^p \frac{w(y)-w(x)}{w(x)}\right)\mu_\varepsilon(dx\,dy) \\
&= \int_D\int_D \left(|u(y)|^p \frac{w(x)^{p-1}}{w(y)^{p-1}} - |u(x)|^p \frac{w(y)}{w(x)} - pu(x)^{\langle p-1\rangle}u(y) + p|u(x)|^p \frac{w(y)}{w(x)}\right)\mu_\varepsilon(dx\,dy) \\
&= \int_D\int_D \left(\frac{|u(y)|^p}{w(y)^p} - \frac{|u(x)|^p}{w(x)^p} - p\frac{u(x)^{\langle p-1\rangle}}{w(x)^{p-1}}\left(\frac{u(y)}{w(y)} - \frac{u(x)}{w(x)}\right)\right) w(x)^{p-1}w(y)\,\mu_\varepsilon(dx\,dy) \\
&= \int_D\int_D F_p\left(\frac{u(y)}{w(y)}, \frac{u(x)}{w(x)}\right) w(x)^{p-1}w(y)\,\mu_\varepsilon(dx\,dy).
\end{aligned}$$

We note that for $b \in \{1, p-1\}$,

$$\int_{\{y\in D:|x-y|>\varepsilon\}} \frac{w(x)^b - w(y)^b}{|x-y|^{d+\alpha}} dy \to -\mathcal{L}w(x)^b \qquad \text{as} \quad \varepsilon \to 0,$$

uniformly in $x \in \operatorname{supp} u$, thus letting $\varepsilon \to 0$ yields

$$\begin{aligned}
&\frac{p}{2}\int_D\int_D (u(x)-u(y))(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle})|x-y|^{-d-\alpha}\,dx\,dy \\
&\quad= \int_D |u(x)|^p \lim_{\varepsilon\to 0}\int_{\{y\in D:|x-y|>\varepsilon\}} \left(w(x)^{p-1} - w(y)^{p-1}\right)|x-y|^{-d-\alpha}\,dy\,\frac{dx}{w(x)^{p-1}} \\
&\qquad+ (p-1)\int_D |u(x)|^p \lim_{\varepsilon\to 0}\int_{\{y\in D:|x-y|>\varepsilon\}} (w(x)-w(y))\,|x-y|^{-d-\alpha}\,dy\,\frac{dx}{w(x)} \\
&\qquad+ \int_D\int_D F_p\left(\frac{u(y)}{w(y)}, \frac{u(x)}{w(x)}\right) w(x)^{p-1}w(y)|x-y|^{-d-\alpha}\,dx\,dy \\
&\quad= \int_D |u(x)|^p \left(\frac{-\mathcal{L}w(x)^{p-1}}{w(x)^{p-1}} + (p-1)\frac{-\mathcal{L}w(x)}{w(x)}\right)dx \\
&\qquad+ \int_D\int_D F_p\left(\frac{u(y)}{w(y)}, \frac{u(x)}{w(x)}\right) w(x)^{p-1}w(y)|x-y|^{-d-\alpha}\,dx\,dy.
\end{aligned}$$

Hence, the first assertion of the Lemma follows. Further,

$$\begin{aligned}
&(p-1)\frac{\mathcal{L}w(x)}{w(x)} + \frac{\mathcal{L}w(x)^{p-1}}{w(x)^{p-1}} = ((p-1)\gamma(\alpha,\beta) + \gamma(\alpha,(p-1)\beta))\,\frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\,x_d^{-\alpha} \\
&\quad= \frac{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{\alpha+d}{2}\right)}\int_0^1 \frac{(p-1)(t^\beta-1)(1-t^{\alpha-\beta-1}) + (t^{(p-1)\beta}-1)(1-t^{\alpha-(p-1)\beta-1})}{(1-t)^{1+\alpha}}\,dt\,x_d^{-\alpha}.
\end{aligned}$$

It is easy to check that, for $t \in (0,1)$, the minimum of the function

$$\beta \mapsto (p-1)(t^\beta-1)(1-t^{\alpha-\beta-1}) + (t^{(p-1)\beta}-1)(1-t^{\alpha-(p-1)\beta-1})$$

is $p(t^\beta - 1)(t^{\beta'}-1) < 0$, where $\beta = \frac{\alpha-1}{p}$ and $\beta' = \frac{(p-1)(\alpha-1)}{p}$. Further, since $\Gamma(x+1) = x\Gamma(x)$, by [2, (2.2)] we get

$$\gamma(\alpha,\beta) = \mathrm{B}(\beta+1,-\alpha) + \mathrm{B}(\alpha-\beta,-\alpha) + \frac{1}{\alpha}, \tag{12}$$

for $\alpha \in (0,2)\setminus\{1\}$ and $\beta \in (-1,\alpha)$. Hence, the second assertion of the Lemma follows. □

Proof of Theorem 1. By Lemma 1, (5) holds. To complete the proof, we will verify the optimality of $\kappa_{d,p,\alpha}$. Our proof is a modification of [2, Proof of Theorem 1.1]. In what follows let $\beta = \frac{\alpha-1}{p}$. If $\alpha \geq 1$, there are real functions v_n, $n = 1, 2, \ldots$, such that

- (i) $v_n = 1$ on $[-n^2, n^2]^{d-1} \times \left[\frac{1}{n}, 1\right]$,
- (ii) $\operatorname{supp} v_n \subset [-n^2-1, n^2+1]^{d-1} \times \left[\frac{1}{2n}, 2\right]$,
- (iii) $0 \leq v_n \leq 1$, $|\nabla v_n(x)| \leq c x_d^{-1}$ and $|\nabla^2 v_n(x)| \leq c x_d^{-2}$ for $x \in D$.

If $\alpha < 1$, then instead we take v_n satisfying

- (i') $v_n = 1$ on $[-n^2, n^2]^{d-1} \times [1, n]$,
- (ii') $\operatorname{supp} v_n \subset [-n^2-n, n^2+n]^{d-1} \times \left[\frac{1}{2}, 2n\right]$,
- (iii) $0 \leq v_n \leq 1$, $|\nabla v_n(x)| \leq c x_d^{-1}$ and $|\nabla^2 v_n(x)| \leq c x_d^{-2}$ for $x \in D$.

Now, for any $\alpha \in (0,2)$, we define

$$u_n(x) = v_n(x)^{\frac{2}{p}} x_d^{\beta}\,. \tag{13}$$

By Lemma 1,

$$\mathcal{E}_p[u_n] = \kappa_{d,p,\alpha} \int_D \frac{|u_n(x)|^p}{x_d^\alpha}\,dx + \frac{1}{p}\int_D\int_D \frac{F_p\left(v_n(x)^{\frac{2}{p}}, v_n(y)^{\frac{2}{p}}\right)}{|x-y|^{d+\alpha}} w(x)^{p-1} w(y)\,dx\,dy.$$

We have, for $\alpha \geq 1$,

$$\int_D \frac{|u_n(x)|^p}{x_d^\alpha}\,dx \geq \int\limits_{\{x\colon \|x'\|\leq n^2,\ \frac{1}{n}<x_d<1\}} \frac{x_d^{\alpha-1}}{x_d^\alpha}\,dx = (2n^2)^{d-1}\log n, \tag{14}$$

and, for $\alpha < 1$,

$$\int_D \frac{|u_n(x)|^p}{x_d^\alpha} dx \geq \int\limits_{\{x\colon \|x'\|\leq n^2,\, 1<x_d<n\}} \frac{x_d^{\alpha-1}}{x_d^\alpha}\,dx = (2n^2)^{d-1}\log n. \tag{15}$$

Now, it suffices to show that there exists a constant c independent of n such that

$$\int\limits_D\int\limits_D \frac{F_p\left(v_n(x)^{\frac{2}{p}}, v_n(y)^{\frac{2}{p}}\right)}{|x-y|^{d+\alpha}} w(x)^{p-1} w(y)\,dx\,dy \leq cn^{2(d-1)}.$$

To this end, we adapt [2, Proof of Lemma 2.3]. Recall that by [3, Lemma 2.3], for $p > 1$ and $a, b \in \mathbb{R}$, there exist $c_p, C_p > 0$, such that

$$c_p\left(b^{\langle p/2\rangle} - a^{\langle p/2\rangle}\right)^2 \leq F_p(a,b) \leq C_p\left(b^{\langle p/2\rangle} - a^{\langle p/2\rangle}\right)^2 \tag{16}$$

and hence

$$\begin{aligned}\int_D\int_D \frac{F_p\left(v_n(x)^{\frac{2}{p}}, v_n(y)^{\frac{2}{p}}\right)}{|x-y|^{d+\alpha}} w(x)^{p-1} w(y)\,dx\,dy &\leq C_p \int_D\int_D \frac{(v_n(x)-v_n(y))^2}{|x-y|^{d+\alpha}} w(x)^{p-1} w(y)\,dx\,dy \\ &=: C_p I.\end{aligned}$$

Let $B(x,s,t) = B(x,t)\setminus B(x,s)$. We can bound the latter integral by $cn^{2(d-1)}$, as in [2]. Considering first $\alpha \geq 1$, we obtain

$$\begin{aligned} I &\leq \int_D\int_{B(x,\frac{1}{4n})} + \int_{\{x\colon x_d\geq\frac{1}{2}\}}\int_{B(x,\frac{1}{4})} + \int_D\int_{D\setminus B(x,\frac{1}{4})} + \int_{P_n}\int_{D\cap B(x,\frac{1}{4n},\frac{1}{4})} \\ &\quad + \int_{R_n}\int_{D\cap B(x,\frac{1}{4n},\frac{1}{4})} + \int_{L_n}\int_{D\cap B(x,\frac{1}{4n},\frac{1}{4})} \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

where, recalling that $x = (x', x_d)$,

$$\begin{aligned} P_n &= \{x \in \mathbb{R}^d : \|x'\| \geq n^2 - 1\,,\ 0 < x_d < \tfrac{1}{2}\}\,, \\ R_n &= \{x \in \mathbb{R}^d : \|x'\| < n^2 - 1\,,\ 0 < x_d < \tfrac{2}{n}\}\,, \\ L_n &= \{x \in \mathbb{R}^d : \|x'\| < n^2 - 1\,,\ \tfrac{2}{n} \leq x_d < \tfrac{1}{2}\}\,, \end{aligned}$$

for $d \geq 2$, and $P_n = \emptyset$, $R_n = \{x \in \mathbb{R} : 0 < x < \frac{2}{n}\}$, $L_n = (\frac{2}{n}, \frac{1}{2})$ for $d = 1$. For simplicity of notation, let $K_n = \operatorname{supp} v_n$. Now, estimates for the integrals I_k are analogous to those in [2], although we have the non-symmetric factor $w(x)^{p-1}w(y)$ here. For example, if

$x \in K_n$ and $y \in B(x, \frac{1}{4n})$, then $|v(x) - v(y)| \le c|x-y|x_d^{-1}$, as follows from (ii) and (iii). Hence

$$\begin{aligned} I_1 &= \int_D \int_{B(x,\frac{1}{4n})} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{d+\alpha}}\, w(x)^{p-1} w(y)\, dy\, dx \\ &\le \int_{K_n} \int_{B(x,\frac{1}{4n})} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{d+\alpha}}\, w(x)^{p-1} w(y)\, dy\, dx \\ &\quad + \int_{K_n} \int_{B(x,\frac{1}{4n})} \frac{(v_n(x)-v_n(y))^2}{|x-y|^{d+\alpha}}\, w(x) w(y)^{p-1}\, dy\, dx \\ &\le c \int_{K_n} \int_{B(x,\frac{1}{4n})} \frac{x_d^{(p-1)\beta-2} y_d^{\beta}}{|x-y|^{d+\alpha-2}}\, dy\, dx \\ &\quad + c \int_{K_n} \int_{B(x,\frac{1}{4n})} \frac{x_d^{\beta-2} y_d^{(p-1)\beta}}{|x-y|^{d+\alpha-2}}\, dy\, dx \\ &\le c' \int_{K_n} \int_{B(x,\frac{1}{4n})} \frac{x_d^{\alpha-3}}{|x-y|^{d+\alpha-2}}\, dy\, dx \\ &\le c'' n^{2(d-1)}, \end{aligned}$$

where in the last line we used the inequality $y_d \le \frac{3}{2} x_d$, which follows from (ii) and the triangle inequality. We now turn to the case $\alpha < 1$. We have

$$\begin{aligned} I &= \int_D \int_D \frac{(v_n(x)-v_n(y))^2}{|x-y|^{d+\alpha}}\, w(x)^{p-1} w(y)\, dx\, dy \\ &\le \int_D \int_{B(x,\frac{1}{4})} + \int_{\{x:\, x_d \ge \frac{n}{2}\}} \int_{B(x,\frac{n}{4})} + \int_D \int_{D\setminus B(x,\frac{n}{4})} + \int_{P_n} \int_{D\cap B(x,\frac{1}{4},\frac{n}{4})} \\ &\quad + \int_{\{x:0<x_d<2\}} \int_{D\cap B(x,\frac{1}{4},\frac{n}{4})} + \int_{L_n} \int_{D\cap B(x,\frac{1}{4},\frac{n}{4})} \\ &= I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

where

$$\begin{aligned} P_n &= \{x \in \mathbb{R}^d :\ \|x'\| \ge n^2 - n\,,\ 0 < x_d < \tfrac{n}{2}\}\,, \\ L_n &= \{x \in \mathbb{R}^d : \|x'\| < n^2 - n\,,\ 2 \le x_d < \tfrac{n}{2}\}\,, \end{aligned}$$

for $d \ge 2$, and $P_n = \emptyset$, $L_n = (2, \frac{n}{2})$ for $d = 1$.

The integrals I_k can be estimated similarly as in the case $\alpha \ge 1$.

□

3. Proof of Theorem 2

Frank and Seiringer [9] proved an abstract form of the fractional Hardy inequality in a more general setting. Loosely speaking, they showed that under some assumptions the inequality

$$\int_\Omega \int_\Omega |u(x) - u(y)|^p\, k(x,y)\, dy\, dx \ge \int_\Omega |u(x)|^p\, V(x)\, dx \tag{17}$$

holds for symmetric kernels $k(x, y)$ and related functions V (see [9, Proposition 2.2]). In order to prove Theorem 2, we first need to formulate an analogue of (17) for Sobolev-Bregman forms.

Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open set. Suppose that $\{k_\varepsilon(x,y)\}_{\varepsilon>0}$ is a family of measurable, symmetric kernels satisfying

$$0 \le k_\varepsilon(x,y) \le k(x,y), \quad \lim_{\varepsilon\to 0^+} k_\varepsilon(x,y) = k(x,y),$$

for almost all x, y and some measurable function $k(x,y)$. Moreover, let w be a positive, measurable function. Define

$$V_\varepsilon(x) := \int_\Omega \left(\frac{1}{p} \frac{w(x)^{p-1} - w(y)^{p-1}}{w(x)^{p-1}} + \frac{p-1}{p} \frac{w(x)-w(y)}{w(x)} \right) k_\varepsilon(x,y)\, dy \tag{18}$$

and suppose that there exists a function V such that $V_\varepsilon \to V$ weakly in $L^1_{loc}(\Omega)$, that is, for any bounded g with compact support in Ω, $\int_\Omega V_\varepsilon(x) g(x)\, dx \to \int_\Omega V(x) g(x)\, dx$ as $\varepsilon \to 0$.

In what follows, we will use the notation

$$\mathcal{E}_p[u] = \mathcal{E}_p^{\Omega,k}[u] := \frac{1}{2} \int_\Omega \int_\Omega (u(x)-u(y)) \left(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle} \right) k(x,y)\, dy\, dx.$$

Lemma 2. *For $u \in C_c(\Omega)$,*

$$\mathcal{E}_p[u] \ge \int_\Omega |u(x)|^p\, V(x)\, dx. \tag{19}$$

Proof. Proceeding as in the proof of Lemma 1, we arrive at the equality

$$\begin{aligned} &\frac{1}{2} \int_\Omega \int_\Omega (u(x)-u(y)) \left(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle} \right) k_\varepsilon(x,y)\, dy\, dx \\ &\qquad = \int_\Omega |u(x)|^p\, V_\varepsilon(x)\, dx + \frac{1}{p} \int_\Omega \int_\Omega F_p \left(\frac{u(x)}{w(x)}, \frac{u(y)}{w(y)} \right) w(x)^{p-1} w(y) k_\varepsilon(x,y)\, dy\, dx \\ &\qquad \ge \int_\Omega |u(x)|^p\, V_\varepsilon(x)\, dx, \end{aligned}$$

since $F_p(a,b) \ge 0$. Now we let $\varepsilon \to 0$ and use Lebesgue's Dominated Convergence Theorem on the left-hand side (provided that $\mathcal{E}_p[u] < \infty$) and weak convergence on the right-hand side to obtain the desired result. ☐

Lemma 3. *Let $1 < \alpha < 2$ and $J \subset \mathbb{R}$ be an open set. Then for $u \in C_c(J)$,*

$$\frac{1}{2} \int_J \int_J \frac{(u(x)-u(y)) \left(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle} \right)}{|x-y|^{1+\alpha}}\, dy\, dx \ge \kappa_{1,p,\alpha} \int_J \frac{|u(x)|^p}{\operatorname{dist}(x,\partial J)^\alpha}\, dx. \tag{20}$$

Proof. Our proof relies on an appropriate modification of [13, Proof of Theorem 2.5]. We will first consider the case $J = (0,1)$. Set $w = w_{(\alpha-1)/p}$ and, for $x \in (0,1)$,

$$V(x) = \text{P.V.} \int_0^1 \left(\frac{1}{p} \frac{w(x)^{p-1} - w(y)^{p-1}}{w(x)^{p-1}} + \frac{p-1}{p} \frac{w(x)-w(y)}{w(x)} \right) |x-y|^{-1-\alpha}\, dy.$$

By (9), we have

$$\begin{aligned} V(x) &= \text{P.V.} \left(\int_0^\infty - \int_1^\infty \right) \left(\frac{1}{p} \frac{w(x)^{p-1} - w(y)^{p-1}}{w(x)^{p-1}} + \frac{p-1}{p} \frac{w(x)-w(y)}{w(x)} \right) |x-y|^{-1-\alpha}\, dy \\ &\ge \text{P.V.} \int_0^\infty \left(\frac{1}{p} \frac{w(x)^{p-1} - w(y)^{p-1}}{w(x)^{p-1}} + \frac{p-1}{p} \frac{w(x)-w(y)}{w(x)} \right) |x-y|^{-1-\alpha}\, dy \\ &= \kappa_{1,p,\alpha} x^{-\alpha}, \end{aligned}$$

since the integrand is nonpositive for $y \in [1, \infty)$. In addition, the latter principal value integral is uniformly convergent on every compact set $K \subset (0, \infty)$. In consequence,

$$V_\varepsilon(x) = \left(\int_0^{x-\varepsilon} + \int_{x+\varepsilon}^1\right)\left(\frac{1}{p}\frac{w(x)^{p-1} - w(y)^{p-1}}{w(x)^{p-1}} + \frac{p-1}{p}\frac{w(x) - w(y)}{w(x)}\right)|x-y|^{-1-\alpha}\,dy > 0$$

for small $\varepsilon > 0$ and all $x \in K$. Hence, by the proof of Lemma 2 combined with Fatou's lemma,

$$\frac{1}{2}\int_0^1\int_0^1 \frac{(v(x)-v(y))\left(v(x)^{\langle p-1\rangle} - v(y)^{\langle p-1\rangle}\right)}{|x-y|^{1+\alpha}}\,dy\,dx \geq \kappa_{1,p,\alpha}\int_0^1 \frac{|v(x)|^p}{x^\alpha}\,dx, \tag{21}$$

for any function v such that $\operatorname{supp} v \subset (0, 1]$. Moreover, for $u \in C_c((0, 1))$, observe that using (21) gives

$$\begin{aligned}
&\int_0^1 \frac{|u(x)|^p}{\min\{x, 1-x\}^\alpha}\,dx \\
&= \int_0^{\frac12} \frac{|u(x)|^p}{x^\alpha}\,dx + \int_{\frac12}^1 \frac{|u(x)|^p}{(1-x)^\alpha}\,dx \\
&= 2^{\alpha-1}\left(\int_0^1 \frac{\left|u\left(\frac{x}{2}\right)\right|^p}{x^\alpha}\,dx + \int_0^1 \frac{\left|u\left(1-\frac{x}{2}\right)\right|^p}{x^\alpha}\,dx\right) \\
&\leq 2^{\alpha-1}\kappa_{1,p,\alpha}^{-1}\left(\int_0^1\int_0^1 \frac{\left(u\left(\frac{x}{2}\right) - u\left(\frac{y}{2}\right)\right)\left(u\left(\frac{x}{2}\right)^{\langle p-1\rangle} - u\left(\frac{y}{2}\right)^{\langle p-1\rangle}\right)}{|x-y|^{1+\alpha}}\,dy\,dx\right. \\
&\quad\left. + \int_0^1\int_0^1 \frac{\left(u\left(1-\frac{x}{2}\right) - u\left(1-\frac{y}{2}\right)\right)\left(u\left(1-\frac{x}{2}\right)^{\langle p-1\rangle} - u\left(1-\frac{y}{2}\right)^{\langle p-1\rangle}\right)}{|x-y|^{1+\alpha}}\,dy\,dx\right) \\
&= \kappa_{1,p,\alpha}^{-1}\left(\int_0^{\frac12}\int_0^{\frac12} + \int_{\frac12}^1\int_{\frac12}^1\right)\frac{(u(x)-u(y))\left(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle}\right)}{|x-y|^{1+\alpha}}\,dy\,dx \\
&\leq \kappa_{1,p,\alpha}^{-1}\int_0^1\int_0^1 \frac{(u(x)-u(y))\left(u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle}\right)}{|x-y|^{1+\alpha}}\,dy\,dx.
\end{aligned}$$

By translation and scaling, an analogous formula holds for any interval (a, b) and since every open subset of $\mathbb{R}$ is a countable union of disjoint intervals, (20) is an easy consequence of the above computations. □

Proof of Theorem 2. We will use arguments similar to those presented by Loss and Sloane in [13]. We denote by $\mathcal{L}_\omega$ the $(d-1)$ dimensional Lebesgue measure on the plane $x \cdot \omega = 0$. Calculations analogous to [13, Proof of Lemma 2.4] and Lemma 3 give

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(u(x)^{\langle p-1\rangle}-u(y)^{\langle p-1\rangle})}{|x-y|^{d+\alpha}}\,dx\,dy \\
&\quad = \frac{1}{4}\int_{\mathbb{S}^{d-1}} d\omega \int_{\{x:x\cdot\omega=0\}} d\mathcal{L}_\omega(x) \int_{\{x+s\omega\in\Omega\}} ds \int_{\{x+t\omega\in\Omega\}} \\
&\qquad \times \frac{(u(x+s\omega)-u(x+t\omega))\left(u(x+s\omega)^{\langle p-1\rangle}-u(x+t\omega)^{\langle p-1\rangle}\right)}{|s-t|^{1+\alpha}}\,dt \\
&\quad \geq \frac{1}{2}\kappa_{1,p,\alpha}\int_{\mathbb{S}^{d-1}}\int_{\{x:x\cdot\omega=0\}}\int_{\{x+s\omega\in\Omega\}} \frac{|u(x+s\omega)|^p}{d_{\omega,\Omega}(x+s\omega)^\alpha}\,ds\,d\mathcal{L}_\omega(x)\,d\omega \\
&\quad = \frac{1}{2}\kappa_{1,p,\alpha}\int_{\mathbb{S}^{d-1}}\int_\Omega \frac{|u(x)|^p}{d_{\omega,\Omega}(x)^\alpha}\,dx\,d\omega \\
&\quad = \kappa_{d,p,\alpha}\int_\Omega \frac{|u(x)|^p}{m_\alpha(x)^\alpha}\,dx,
\end{aligned}$$

where the last equality follows from

$$\int_{\mathbb{S}^{d-1}} |\omega_d|^\alpha\,d\omega = \frac{2\pi^{\frac{d-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{d+\alpha}{2}\right)}.$$

This proves (7) and, in consequence, (8). Since Ω is convex, there exists a hyperplane Π tangent to Ω at a point $P \in \partial\Omega$. Now, calculations analogous to those in [14, Proof of Theorem 5] yield the optimality of the constant in (8). □

Remark 1. Noteworthy, if $\alpha \leq 1$ and Ω is a bounded convex domain, then the best constant in the inequality (7) is zero. Indeed, first notice that every convex set is a Lipschitz domain. Dyda constructed in [5] a sequence of functions $u_n \in C_c(\Omega)$ such that $0 \leq u_n \leq 1$, $u_n \to 1$ pointwise and $\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{d+\alpha}}\,dy\,dx \to 0$ for $\alpha < 1$, as $n \to \infty$, $\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{d+\alpha}}\,dy\,dx \leq C$ for $\alpha = 1$. Taking $v_n = u_n^{2/p}$ and using (16), we see that the inequality (7) cannot hold with a positive constant, as the right-hand side of (7) tends to a positive value, when $\alpha < 1$ and to infinity, when $\alpha = 1$.

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