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Report on the PhD thesis
"Fractional Sobolev spaces and Hardy inequalities", by M. Kijaczko

The thesis studies fine properties of fractional Sobolev spaces; an important class of functional spaces that has been widely studied during the 20th and 21st century. In addition to their intrinsic interest, these spaces have important connections to integro-differential operators (most notably, the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$), as well as to stable Lévy processes in probability theory.

Fractional Sobolev spaces extend (and interpolate) the notion of classical Sobolev spaces

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \forall |\alpha| \le k\}.$$

A classical result in this context is the Meyers-Serrin Theorem, which states that C^∞ functions are dense in $W^{k,p}(\Omega)$ for *any* domain $\Omega \subset \mathbb{R}^n$. Such density property is very useful in applications, since often computations are easier for smooth functions.

The first part of the PhD thesis of M. Kijaczko studies the same question in case of fractional Sobolev spaces

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dx\, dy < \infty \right\},$$

with $1 \le p < \infty$ and $s \in (0,1)$, or their weighted version

$$\tilde{W}^{s,p}(\Omega, w) := \left\{ u \in L^p_{\text{loc}}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, w(x) w(y)\, dx\, dy < \infty \right\}.$$

The thesis includes several new and interesting results in this direction, most notably the following:

- If w is locally comparable to a constant and $\int_\Omega \frac{w(x)}{1+|x|^{n+ps}}\, dx < \infty$, then C^∞ functions are dense in $\tilde{W}^{s,p}(\Omega, w)$, for *any* domain $\Omega \subset \mathbb{R}^n$.
- Under some geometric conditions on $\partial\Omega$, a new characterization of $W^{s,p}_0(\Omega)$ is established.
- The same question is investigated for

$$W^{s,p,\alpha,\beta}(\Omega) := \left\{ u \in L^p_{\text{loc}}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, d(x)^{-\alpha} d(y)^{-\beta}\, dx\, dy < \infty \right\},$$

  where $d(x) := \text{dist}(x, \partial\Omega)$.

To establish these results, he uses a variety of tools, and shows that has a very good general knowledge of several areas of mathematical analysis. More precisely, he uses tools like Whitney decomposition from harmonic analysis, smoothing operators and other tools from functional analysis, and various notions of dimension of rough sets from geometric measure theory.

The second part of the PhD thesis of M. Kijaczko studies fractional Hardy inequalities. In case of the classical Sobolev space $W^{1,p}(\Omega)$, the Hardy inequality reads as

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p}\, dx \leq C \int_{\mathbb{R}^n} |\nabla u|^p dx.$$

Moreover, for any bounded convex domain $\Omega \subset \mathbb{R}^n$, one has

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{d(x)^p}\, dx \leq C \int_{\mathbb{R}^n} |\nabla u|^p dx,$$

where d is the distance to the boundary.

This part of the thesis includes some new results for fractional versions of these inequalities, more precisely:

- A fractional Hardy-type inequality for Sobolev-Bregman forms in a half-space.
- A version of it for general domains $\Omega \subset \mathbb{R}^n$.

These results are interesting and establishing them also requires mastering several tools from mathematical analysis.

Summarizing, the thesis includes several new and interesting results, mostly studying two types of questions: the density of C^∞ functions in these spaces, or the finding of new Hardy-type inequalities associated with these spaces.

Moreover, the candidate had to develop new ideas and go beyond the current state of the art in the topics that he studied. This clearly shows his ability to conduct independent scientific work, as well as the original solution of several open problems in the context of weighted fractional Sobolev spaces.

Overall, I believe this is a very good PhD thesis. Please do not hesitate to contact me if you have further questions.

Sincerely,

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