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Perpetual American options with asset-dependent discounting

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Jonas Al-Hadad

Nieskończone opcje amerykańskie z dyskontowaniem zależnym od aktywa bazowego

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Streszczenie

Niniejsza rozprawa przedstawia analizę problemu optymalnego zatrzymania postaci

$$V_{\mathrm{A}}^{\omega}(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\int_0^\tau \omega(S_w)dw} g(S_\tau) \right],$$

gdzie S_t jest procesem dyfuzyjnym ze skokami, $\mathcal{T}$ jest rodziną czasów zatrzymania, natomiast g i ω są odpowiednio funkcją wypłaty i funkcją dyskontującą. Zakładamy ponadto, że powyższa wartość oczekiwana jest liczona względem miary martyngałowej. Wówczas, zgodnie z ogólną teorią wyceny opcji finansowych, wzór ten interpretujemy jako funkcja wartości nieskończonej opcji amerykańskiej z dyskontowaniem zależnym od aktywa bazowego. Rozpatrywany przez nas problem stanowi uogólnienie klasycznego przypadku wyceny opcji amerykańskiej ze stałym dyskontowaniem, tzn. gdy $\omega(s) = r$, gdzie r jest stopą wolną od ryzyka. W kontekście zastosowań finansowych najczęściej przyjmuje się, że funkcja wypłaty jest postaci $g(s) = (K - s)^+$ lub $g(s) = (s - K)^+$, co odpowiada kolejno opcji sprzedaży i opcji kupna. W rozprawie analizujemy dokładnie pierwszy z tych przypadków.

Motywacją do analizy tak zdefiniowanego problemu jest rozwój instrumentów finansowych, w szczególności instrumentów pochodnych, które pojawiają się coraz częściej w literaturze naukowej. Jest to pewnego rodzaju odpowiedź na zapotrzebowanie rynków finansowych i ich dynamiczną ekspansję rozpoczętą w drugiej połowie XX w. Rynek pozagiełdowy, na którym duże instytucje finansowe, takie jak na przykład banki inwestycyjne czy fundusze hedgingowe, zawierają ze sobą transakcje, jest stałym polem wyzwań dla naukowców prowadzących badania w obszarze matematyki finansowej. Jednym z dominujących zagadnień współczesnej matematyki finansowej jest wycena instrumentów pochodnych, a rynek pozagiełdowy umożliwia jego uczestnikom stworzenie własnych, unikalnych produktów finansowych, które byłyby zgodne z prognozami i celami danej firmy. Zaliczyć do nich możemy między innymi zabezpieczenie przed ryzykiem w sytuacji dużej zmienności na giełdach czy spekulację mającą na celu przyniesienie nadmiarowych zysków. Proces wyceny instrumentów pochodnych odbywa się w ścisły, zmatematyzowany sposób, dlatego też wykorzystywany aparat matematyczny jest stale rozwijany. W przypadku opcji amerykańskich, które charakteryzują się tym, że nabywca może zdecydować się na ich wykonanie w dowolnym momencie czasu trwania kontraktu, proces wyceny sprowadza się do rozwiązania pewnego problemu optymalnego zatrzymania. W ogólności, problemy optymalnego zatrzymania pojawiają się w różnych dziedzinach matematyki jak teoria ruiny, teoria sterowania czy teoria kolejek, ale również w innych naukach, na przykład w fizyce. To sprawia, że badany przez nas problem ma charakter interdyscyplinarny i nie jest ukierunkowany jedynie na zastosowania w obszarze matematyki finansowej.

Do głównych wyników pracy zaliczamy udowodnienie wypukłości analizowanej funkcji wartości, określenie postaci optymalnego czasu zatrzymania w przypadku opcji sprzedaży i przede wszystkim uzyskanie jawnego wzoru funkcji wartości, gdy aktywo bazowe modelowane jest spektralnie ujemnym wykładniczym procesem Lévy'ego. Formułujemy również szereg pomocniczych

twierdzeń i lematów, w tym te dotyczące równania Hamiltona-Jacobiego-Bellmana czy parytetu opcji kupna/sprzedaży. Praca zawiera również część numeryczną, w której przedstawiamy przykłady wzorów analitycznych funkcji wartości wraz z wykresami dla różnych funkcji dyskontujących. Opisujemy również zastosowaną metodologię numeryczną, która pozwala nam wyznaczyć funkcję wartości, gdy nie jesteśmy w stanie wyrazić jej wzorem analitycznym.

W rozdziale pierwszym przytaczamy podstawowe informacje i pojęcia stosowane w pracy, takie jak wstęp dotyczący rynków finansowych, podstawy teorii wyceny opcji, procesów Lévy'ego i funkcji skalujących. Ponadto opisujemy główny problem badań wraz z przeglądem literatury i motywacją, którą kierowaliśmy się w analizie tego rodzaju zagadnienia. Pod koniec rozdziału prezentujemy notację stosowaną w pracy.

Rozdział drugi zawiera główne wyniki niniejszej rozprawy. W początkowej części przedstawiamy ogólne założenia, na których operujemy. Dotyczą one głównie rozpatrywanego procesu dyfuzji ze skokami, który modeluje zachowanie aktywa bazowego. Następnie formułujemy twierdzenie o wypukłości funkcji wartości. Jest ono kluczowe przy określeniu postaci optymalnego czasu zatrzymania. W dalszej części pracy koncentrujemy się na szczególnym przypadku analizowanej przez nas opcji, tzn. opcji sprzedaży, a następnie prezentujemy twierdzenia dotyczące tego instrumentu. W pierwszej kolejności wnioskujemy o postaci optymalnego czasu zatrzymania, tzn. dowodzimy, że jest on pierwszym momentem, w którym cena aktywa bazowego wpada w dany odcinek. Wynik ten pozwala nam sformułować główne twierdzenie pracy, tj. Twierdzenie 3, w którym przedstawiona jest jawna postać funkcji wartości w przypadku, gdy cena aktywa bazowego modelowana jest przez spektralnie ujemny wykładniczy proces Lévy'ego. Następnie prezentujemy szczególne przypadki głównego twierdzenia, gdy aktywo bazowe modelowane jest geometrycznym ruchem Browna oraz wykładniczym procesem Lévy'ego z ujemnymi skokami wykładniczymi. Dla drugiego z wymienionych przypadków pokazujemy, że funkcja wartości składa się z tzw. uogólnionych funkcji skalujących, które są rozwiązaniami pewnych równań różniczkowych zwyczajnych. W dalszej części rozdziału udowadniamy, że rozpatrywany przez nas problem spełnia równanie Hamiltona-Jacobiego-Bellmana i wskazujemy warunki wystarczające, aby warunek gładkości był spełniony. Dowodzimy także tzw. parytet opcji kupna/sprzedaży, czyli zależność, jaka zachodzi między funkcją wartości dla opcji kupna i sprzedaży.

W rozdziale trzecim przedstawiamy numeryczną część pracy, tzn. przykłady, które demonstrują analityczne wzory funkcji wartości wraz z odpowiednimi wykresami dla różnych funkcji dyskontujących. Prezentujemy również metodologię numeryczną do wyznaczenia funkcji wartości bazującą na rozwiązaniu równań różniczkowych zwyczajnych metodą rozwinięcia funkcji w szereg Taylora. Procedura ta wykonana jest za pomocą języka programowania *Python* i biblioteki *mpmath* używanej do arytmetyki zmiennoprzecinkowej na liczbach rzeczywistych i zespolonych o dowolnie zdefiniowanej precyzji. Pod koniec rozdziału przedstawiamy wykresy uzyskanych funkcji wartości.

Czwarty rozdział zawiera dowody głównych twierdzeń, jak również twierdzeń pomocniczych i lematów.

Treść rozprawy powstała na podstawie dwóch artykułów napisanych wspólnie z promotorem: *Perpetual American options with asset-dependent discounting* (złożony do publikacji i dostępny pod adresem `https://arxiv.org/pdf/2007.09419.pdf`) oraz *Pricing perpetual American put options with asset-dependent discounting* opublikowany w czasopiśmie Journal of Risk and Financial Management.

Contents

Introduction

This thesis provides an analysis of a perpetual American option with asset-dependent discounting[1]. In a bit of a nutshell, we can say that the problem we consider extends the classical theory of American option pricing, where a deterministic discount rate is considered.

Before we present the main problem of our deliberation, let us present the basic assumptions and notation on which we rely throughout this dissertation.

We assume that the uncertainty associated with the stock[2] price process S_t is described by a jump-diffusion process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with natural filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions and $\mathbb{P}$ being a risk-neutral measure under which the discounted (with respect to a risk-free interest rate) asset price process S_t is a local martingale. We point out that, as noted in [42, Table 1.1, p. 29], introducing jumps into the model implies a loss of completeness of the market, which results in the lack of uniqueness of an equivalent martingale measure. However, this class of stochastic processes reflects stock price movements quite accurately. Empirical observations show that the logarithmic prices of stocks have a heavier left tail than the normal distribution on which the seminal Black-Scholes model is founded, see e.g. [41]. The introduction of jumps in the financial market dates back to Merton's paper [113], who added a compound Poisson process to the standard Brownian motion to describe the dynamic of the logarithm of stocks more precisely. Since then, there have been many papers and books working in this set-up, e.g. [42, 133] and references therein. In particular, [42, Table 1.1, p. 29] gives many other reasons to consider this type of market.

With the general set-up already presented, we can move on to the main topic of the discussion, which is the analysis of the optimal stopping problem given by

$$V_{\mathrm{A}}^{\omega}(s) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\int_0^\tau \omega(S_w)dw} g(S_\tau) \right], \tag{1}$$

where $\mathcal{T}$ is a family of $\mathbb{F}$-stopping times (τ is a stopping time if $\tau : \Omega \to [0, \infty]$ and $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$), g is a payoff function and ω is a discount function. Above $\mathbb{E}_s$ represents the expectation with respect to $\mathbb{P}_s$, while $\mathbb{P}_s$ denotes the measure $\mathbb{P}$ when $S_0 = s$. We assume that the function g is convex and allow ω to take negative values. In financial terms, this function can be interpreted as the value function of a perpetual American option[3] with asset-dependent discounting and the payoff function g. Typically, the payoff function takes the form $g(s) = (K - s)^+$ or $g(s) = (s - K)^+$, which corresponds to a put and call option, respectively.

[1]Throughout the thesis, we use the terms *asset-dependent discounting* and *functional discounting* interchangeably.

[2]Stocks are financial assets, however in this thesis we use interchangeably these terms when we refer to the process S_t.

[3]To be more precise, one should include additional factor $e^{-r\tau}$ in (1) and treat the term $e^{-\int_0^\tau \omega(S_w)dw} g(S_\tau)$ as a payoff function in order to describe (1) as the value function of a perpetual American option. Of course, this corresponds to replacing the discount function ω in (1) with its shifted version $\omega - r$.

As we already mentioned, the value function given in (1) is a generalised case for the typical American option with the deterministic discount rate, that is $\omega(s) = r$. In this case, we obtain

$$V_{\mathrm{A}}(s) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-r\tau} g(S_\tau) \right],$$

which represents the perpetual American option's value function with constant discount rate r.

To emphasise the motivation for the conducted research, let us note that the discount rate changing in time or a random discount rate is widely used in pricing derivatives in financial markets. It has proven to be a valuable and flexible tool for determining the value of various options. Usually, either a discount rate is independent of the asset price or this dependence is introduced via a correlation between the Gaussian components of these two processes. Our object of study is completely different. We want to understand an extreme case where we have a robust and functional dependence between the discount rate and the asset price. One of the advantages of this type of functional discounting is that an option buyer can customise an option by selecting an appropriate functional rate according to his risk aversion and the degree of confidence in how the asset price will look during the whole option's life. In particular, we look closely at the American put option with the discount function ω having the opposite monotonicity to the payoff function g. At first glance, such a case seems counterintuitive, since in the case of the put option, if the asset price is in a higher region, one can expect the discount rate to be lower, while the opposite effect can be expected for a lower range of asset prices. This dependence somehow balances the discount function with the payoff function. However, we can think of an investor who has strong confidence in the movement of the asset price and wishes to make an extra profit when he/she is right and suffers a more significant loss when he/she is wrong. This concept resembles an idea that stands behind barrier options. If the investor believes that it is unlikely that the asset price will hit a given level, he/she can add a knock-out provision with the barrier set at the support level to reduce the price of the option. By including the barrier provision, he/she can eliminate paying for these scenarios he/she feels are unlikely. In our approach, we work in two ways by reducing the premium thanks to improbably incidents from the investor's perspective and increasing it for scenarios that are more likely for him/her. Such a description of the analysed option adequately describes a financial instrument suitable for a risky investor and due to its complexity can be traded on the over-the-counter market.

Our research focuses only on financial applications, but one can look at optimisation problem (1) from a broader perspective. In the case of the general theory of stochastic processes, multiplying by the discount factor $e^{-\int_0^\tau \omega(S_w)dw}$ corresponds to killing a generator of S_t by the potential ω. The killing by potential ω has been known widely in physics and other applied sciences. Therefore, formula (1) can be seen as a specific functional that describes the gain or energy, and the goal is to optimise it by choosing the optimal stopping time.

The first main goal of this dissertation is to find a closed-form expression of (1) for $g(s) = (K - s)^+$, which corresponds to the put option and S_t being a spectrally negative exponential Lévy process, that is $S_t = e^{X_t}$, where X_t is a Lévy process without positive jumps. The methodology we use combines the theory of partial differential equations with the fluctuation theory of Lévy processes. To do this, we start by proving in Theorem 1 an inheritance of convexity property from the payoff function to the value function (we recall that we assume that g is convex). The proof of this result requires a few key steps. First, we prove in Theorem 10 (available in Chapter 4) the convexity of the value function for a European option, i.e.

$$V_{\mathrm{E}}^{\omega}(s,t) := \mathbb{E}_{s,t} \left[e^{-\int_t^T \omega(S_w)dw} g(S_T) \right] \tag{2}$$

for fixed time horizon T, where $\mathbb{E}_{s,t}$ is the expectation with respect to $\mathbb{P}_{s,t}$, which denotes the measure $\mathbb{P}$ when $S_t = s$. In the proof, we follow the idea given by Ekström and Tysk in [68], that is the value function $V_{\mathrm{E}}^{\omega}(s,t)$ given in (2) can be presented as a unique viscosity solution to a certain Cauchy problem for a second-order operator related to the generator of the process S_t. In fact, applying similar arguments like in [122, Proposition 5.3, p. 23] and [68, Lemma 3.1, p. 386] one can show that, under some additional assumptions, this solution can be treated as the classical one. Then, we can formulate sufficient locally convexity preserving conditions for the infinitesimal preservation of convexity at some point. This characterisation is given in terms of a differential inequality on the coefficients of the considered operator. Eventually, it allows us to prove the convexity of $V_{\mathrm{E}}^{\omega}(s,t)$. In the next step, we apply the dynamic programming principle (see [66]) in order to generalise the convexity property of $V_{\mathrm{E}}^{\omega}(s,t)$ to the Bermudan option's value function. This fact is stated in Lemma 6. Ultimately, we conclude about the convexity of $V_{\mathrm{A}}^{\omega}(s)$.

In the remaining part of the thesis, we focus on the perpetual American put option with the value function

$$V_{\mathrm{A^{Put}}}^{\omega}(s) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\int_0^{\tau} \omega(S_w)dw} (K - S_\tau)^+ \right]$$

for some strike price $K > 0$ and S_t being a spectrally negative exponential Lévy process.

Using the classical optimal stopping theory presented in [120], we identify the optimal stopping region for this problem as an interval and we consider the function

$$v_{\mathrm{A^{Put}}}^{\omega}(s,l,u) := \mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw} (K - S_{\tau_{l,u}})^+ \right], \tag{3}$$

where

$$\tau_{l,u} := \inf\{t \geq 0 : S_t \in [l,u]\} \tag{4}$$

for $0 \leq l \leq u \leq K$. To determine the closed-form of $V_{\mathrm{A^{Put}}}^{\omega}(s)$ we need to take the maximum over levels l and u in formula (3). This fact is stated in Theorem 2. Finally, these results lead us to the crucial theorem, that is Theorem 3 with the closed-form of $v_{\mathrm{A^{Put}}}^{\omega}(s,l,u)$.

We recall that the spectrally negative Lévy processes do not have positive jumps. Hence, our analysis could be applied to the Black-Scholes model, as well as to the spectrally negative exponential Lévy process with downward exponential jumps. In Theorem 4 and Theorem 5 we present the closed-form of $v_{\mathrm{A^{Put}}}^{\omega}(s,l,u)$ in both these scenarios. In addition, in the latter case, we assume a non-negative discount function ω, which implies $l = 0$, and therefore we can express $v_{\mathrm{A^{Put}}}^{\omega}(s,0,u)$ in terms of the generalised scale functions introduced in Chapter 1. It is a consequence of the use of first passage time laws and the fluctuation theory considered in [105]. In this analysis, the change of measure technique developed in [119] is also a crucial step. In Theorem 6, we show that the generalised scale functions satisfy certain ordinary differential equations, which in some cases can be solved analytically.

For optimal stopping problem (1), we give sufficient conditions under which we can formalise the classical approach. In particular, in Theorem 7 we prove that if the value function $V_{\mathrm{A}}^{\omega}(s)$ is smooth enough, it is a unique solution to a certain Hamilton-Jacobi-Bellman (HJB) system. Moreover, considering an exponential Lévy process of the asset price S_t, we prove that the regularity of 1 for $(0,1)$ and $(1,\infty)$ gives the smooth fit property at the ends of the stopping region. We want to underline here that proving the regularity of the value function for jump-diffusion processes (which allows one to formulate the HJB equation) in general is a challenging problem (see [52] for some deep results related to it). Nevertheless, it is possible in our case due to Theorem 7 and Remark 9. We rely on the classical approach of [102] and [120]. Further, even solving the HJB equation does not provide the form of the stopping region (besides the fact that it is the set where the value function equals the payoff function). This is why we do not follow this path, but stick with our methodology.

We also show that in this general setting of functional discounting, one can express the price of a call option in terms of the price of a put option. It is called put-call symmetry (or parity) and is provided in Theorem 8. The proof is based on the exponential change of measure introduced in [119]. This result supplements [64, 74], where the authors extended to the Lévy market the findings obtained in [32].

The last part of our dissertation contains examples in which we analytically or numerically determine the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ for different discount functions ω. As the underlying process S_t, we consider the Black-Scholes model and the exponential Lévy process with downward exponential jumps. In the first scenario, we take the negative ω function and show that a double continuation region appears in this case. In other words, the optimal stopping region is an interval $[l^*, u^*]$, where $l^* > 0$ which is a rare event in the study of option pricing. For the selected discount function, we obtain the analytical form of the value function, which consists of Gaussian hypergeometric functions. For the latter scenario, we take a linear and power discount function ω. For these functions, we again derive analytical forms of the value function. This time, they include special functions like the Kummer confluent hypergeometric function and Bessel functions of the first and second kind. Lastly, we present how we can numerically determine the value function for different discount functions for which we are unable to obtain the analytical solution.

The dissertation is structured as follows. Chapter 1 presents the preliminaries, which cover the basics of financial markets, some background on option pricing theory and basic information about Lévy processes and scale functions. Moreover, we state the main problem of the thesis with the motivation and purpose justification for the conducted study. A comprehensive literature review on the subject is also provided. Lastly, we present the notation used throughout this dissertation.

Chapter 2 contains the main results of this thesis. First, we present the general set-up with which we work and state assumptions used in theorems and lemmas in this chapter. We then formulate Theorem 1 on the convexity of the value function. Next, we focus only on the put option and define the optimal stopping time as the first moment when the asset price enters a given interval. This observation is stated in Theorem 2. In Section 2.4, we formulate the main theorem, that is Theorem 3, in which we present the closed-form of the value function for the case when the asset price process follows the spectrally negative exponential Lévy process. Next, in Theorem 4 and Theorem 5 we present specific instances where the asset price process follows the geometric Brownian motion and the exponential Lévy process with downward exponential jumps, respectively. In the latter case, the value function consists of the so-called ξ-scale functions that satisfy certain ordinary differential equations, as stated in Theorem 6. Later, we show that our set-up makes the classical approach via the HJB system possible. In other words, in Theorem 7 we prove that the value function $V^{\omega}_{\mathrm{A}}(s)$ satisfies the HJB system. Our last primary result is put-call parity, which allows us to calculate the price of the perpetual American call option having the price of the put option. It is given in Theorem 8.

Chapter 3 presents examples of closed-form value functions for different discount functions ω with their figures. First, we introduce the pricing methodologies we use, i.e. the analytical and the numerical approach. Then we present an example of the Black-Scholes model and the negative discount function ω. This case generates a double continuation region. In addition, we indicate some examples for the case of a spectrally negative Lévy process with downward exponential jumps. We can obtain analytical solutions for some cases and compare them with numerical ones. Finally, we show how we can proceed only numerically to obtain the value function when we cannot find an analytical solution. This chapter contains many figures for the various cases considered.

Proofs of the essential theorems, together with auxiliary lemmas and theorems, are included in Chapter 4, as they could unnecessarily blur the main picture of the dissertation due to their length and complexity.

Chapter 1

Preliminaries

In this chapter, we present some preliminary facts that form the basis for this dissertation. We start with the basics of financial markets. We briefly discuss option pricing theory with its history and risk-neutral pricing methodology, a crucial concept in financial mathematics[1]. In addition, the fluctuation theory of Lévy processes is quoted, together with the scale functions, which are the tools corresponding to different boundary-crossing problems related to the Lévy processes. We state the main problem considered in the thesis with the motivation that has driven us towards this scientific research. An extensive literature overview is also provided with the notation used throughout the thesis. Although the facts mentioned here could be commonly known, we decided to recall them briefly to provide completeness to this thesis and unify the notation.

1.1 Basics of financial markets

Financial markets refer to any marketplace where financial products, such as stocks, bonds, derivatives and others, are traded between two sides. Simply put, companies and individuals can go to financial markets to meet various financial objectives, e.g. raising money by issuing bonds or stocks to grow their businesses. In contrast, in the case of financial surpluses, they can also lend money to other companies. For individual investors (whether large institutions such as banks or hedge funds), financial markets offer the opportunity to invest money in exchange for a return called a dividend and the prospect of added value if their assets appreciate. There are undoubtedly many more possibilities for allocating and investing capital in financial markets. As more complex instruments were developed starting in the seventies of the twentieth century, it turned out that the mathematical apparatus became an integral part of financial markets. In general, finance is unique among the application areas of mathematics both in the level of mathematics involved and the short gap between pure mathematical research and its application in a commercial environment. In fact, the multitude of financial instruments, the complexity of hedging strategies and risk management techniques have made the application of mathematics to financial markets seem irreversible.

According to the European Central Bank (ECB), see [73], financial markets can be divided into a money market, a debt market and an equity market. The money market consists of the unsecured and secured cash and derivatives segments. The debt market is the market where debt instruments are traded, whereas the equity market is a market in which stocks of companies are

[1] In this dissertation, we use the terms *financial mathematics* and *mathematical finance* interchangeably.

issued and traded, either through exchanges or over-the-counter markets. We will pay special attention to the first category as it contains derivative instruments (derivatives) that constitute the most mathematical part of all financial instruments.

International Accounting Standard IAS 32 defines a financial instrument as *any contract that gives rise to a financial asset of one entity and a financial liability or equity instrument of another entity*, see [75]. There are several ways to categorise financial instruments. They may be divided according to an asset class which depends on whether they are equity-based (reflecting ownership of the issuing entity) or debt-based (reflecting a loan the investor has made to the issuing entity). However, a group of financial instruments, such as foreign exchange instruments, is neither debt-based nor equity-based and belongs to its own category. Another way to look at them is through the lens of cash versus derivative. Cash instruments include products such as deposits, loans and easily transferable securities. The market determines this type of instrument so that any market fluctuations will be directly reflected in its value. On the contrary, derivative instruments derive their value from the value of one or more underlying assets, such as stocks, indices or interest rates. They do not require any principal investment in those assets. In simple terms, derivatives are designed to create exposure to market prices to changes in an underlying asset. Some of the more common derivatives include forwards, futures, options, swaps and variations such as collateralised debt obligations or credit default swaps, which played a significant role in the financial crisis of 2007–2008. In recent years, the traditional scope of derivative contracts has been extended and more often they involve non-traditional underlying assets such as energy, real estate and even insurance loss indices or weather, see [77] and [95] for surveys of these areas. The power of derivatives is based on reducing the market risks associated with oscillations of stock prices, interest rates or exchange rates. In other words, financial derivatives trading is based on leverage techniques, i.e. it allows one to make enormous profits with a small amount of initial capital. In general, derivatives are broadly categorised. One of the classifications includes lock or option products. Lock products obligate the contractual parties to the terms over the duration of the contract's life (swaps, futures and forwards belong to this group). In turn, the second group provides the buyer with the right, but not the obligation, to exercise the contract under the specified terms. Another division concerns the way they are traded in the market: over-the-counter derivatives (abbreviated as OTC) and exchange-traded derivatives (abbreviated as ET). The first group contains contracts that are privately negotiated and traded directly between two parties, without going through an exchange or other intermediary. The OTC derivative market is the largest derivative market. It is predominantly unregulated with respect to the disclosure of information between parties, since the OTC market is made up of banks and other highly sophisticated parties, such as hedge funds. Reporting OTC transactions is complicated because trades can occur privately without the activity being visible on any exchange. In contrast, ETD derivatives are traded via specialised derivative exchanges or other exchanges, where individuals trade standardised contracts that the exchange has defined. A derivative exchange acts as an intermediary and takes an initial margin from both sides of the trade as a guarantee, making this type of transaction safer for both parties.

Large financial corporations mainly use derivatives for various investment purposes, such as risk management (e.g. to hedge by providing offsetting compensation in case of an undesired event) or for speculation (making a financial bet, often based on the sentiment of market participants). Recently, many funds have begun to use financial derivatives as an alternative to a long-term buy-and-hold strategy. For example, some portfolio managers may hold a portfolio of index futures instead of the underlying stocks that make up the indices. In addition, the ability to create instruments based on any asset is conducive to the constant growth of this market.

To give an idea of the size of the derivative market, *The Economist* has reported that in June 2011, the over-the-counter derivative market amounted to approximately \$700 trillion and the size of the market traded on exchanges totalled an additional \$83 trillion. Other sources, such as [134], report that the derivative market is estimated to be worth more than \$1.2 quadrillion. Some analysts estimate that the derivative market is worth more than ten times the world's gross domestic product.

The explosive growth in derivative contracts occurred after 1999, when the Glass-Steagall Act was repealed, which allowed banks to operate as brokerage houses. Glass-Steagall, adopted in 1933, separated brokerage houses and banks to ensure banks would no longer be involved in risky transactions, which was the root cause of the crash that led to the Great Depression in 1929. Today, there is a degree of consensus that derivatives positively impact the financial system as a whole. For a comprehensive review in this area, we refer to [134].

Many annual surveys of derivative exchange volumes highlight strong growth in futures and options trading in recent years. According to [2], at the global level, the total number of futures and options traded on exchanges around the world increased to 24.78 billion contracts in 2016. In particular, the expansion of options is visible, providing much space for investment manoeuvres for institutional and individual investors. In the markets, there are exchange-traded options and OTC options. The former are standardised call and put contracts on, for example, the major stock indices, typically with a range of strike levels and maturity times less than one year. On the other hand, OTC options are negotiated on a case-by-case basis between banks and may involve longer maturities and more exotic features. Their prices are not publicly quoted.

A critical moment in option development in the modern study of options was 1973. As originally presented in [28], Fischer Black and Myron Scholes came up with the celebrated option pricing formula. This formula provides a closed-form solution for the price of a European call option on a non-dividend-paying stock. Robert Merton shortly after published a paper, see [112], expanding the mathematical understanding of the option pricing model and coined the term *Black-Scholes option pricing model*[2]. This formula immediately became very influential in finance and led to a boom in option trading on real markets. The same year, on 26 April 1973, the options were first publicly traded on the Chicago Board Options Exchange (CBOE). The first created standardised, listed options were the call options on 16 stocks, whereas the put options were not even introduced until 1977.

The Black-Scholes formula and related concepts of hedging and replication of derivative securities had an enormous impact on the paradigms of financial markets. In particular, stochastic models became ubiquitous in the financial industry. These factors have made the range of options that can be traded a function of investor demand. This new wave of option trading seems unlikely to recede. Furthermore, technology has made access to financial markets easier for small and more prominent investors, so trading options and other derivatives will be a significant part of financial markets over time.

1.2 Option pricing theory

To better understand the concept of how financial options work, let us consider the following example. We assume that a trader buys the option to buy wheat at £100 per bushel in six months from now. After this time, the trader would profit if the market price of wheat per bushel exceeded £100 per bushel because he/she would be able to pay less for the product than

[2] In the modern literature it is very common to encounter the term *Black-Scholes formula*.

its market price. On the other hand, if the market price drops below £100 per bushel, the trader would not exercise this option and would limit the loss of this transaction to the cost of buying the option. In the case of a forward contract, the situation is very similar, but the buyer must buy wheat at £100 per bushel regardless of its market price. In this situation, determining the price of this contract is simple because we know precisely the cash flow that will occur on the expiration date. Therefore, the fair price of a forward contract is a discounted cash flow from the expiry day. Note that the final cash flow is unknown for options, as it depends on the future market price, making option pricing a more mathematically demanding task. This example illustrates a commodity call option; while there are many other examples of options and new types of options are constantly emerging.

The cost of the option is often called an option premium or an option value. It is often the leading focus of research in option pricing theory, as the valuation of financial options is carried out in a formalised mathematical manner. In some simplification, we can say that this premium is calculated by taking the conditional expectation of a discounted cash flow under the risk-neutral measure. It is a key element of the whole theory and this topic will be expanded in one of the subsections below.

As we already know what a financial option is and how it works, we try to give a flavour of mathematics to indicate how option pricing theory can be complementary to the practical side of option trading. In short, option pricing theory is a probabilistic approach to assigning a value to an option contract. It is simultaneously the primary goal of this theory. However, it also consists of side tasks, like deriving various risk measures (known as the option Greeks). Since market conditions are constantly changing, the Greeks provide traders with a means to determine how sensitive the value of a derivative contract is to factors such as price fluctuations, volatility or time to expiry. The most common of the Greeks are simply the first- and second-order derivatives of the value function. Option traders and portfolio managers consider these measures essential as they can benefit from them to hedge risk and understand how the P&L (Profit and Loss Statement) will behave as other factors fluctuate.

As we mentioned in the previous section, the 1970s turned out to be a breakthrough in the option pricing theory. Published in 1973, the Black-Scholes option pricing model brings a new quantitative approach to pricing options, helping fuel the growth of derivative investing. It was the first widely used mathematical method to calculate the theoretical value of an option contract using current stock prices, dividends, option strike price, interest rates, time to expiration and volatility.

1.2.1 Historical background

We now reveal some crucial facts in the development and application of options contracts that have happened over the centuries. We pay special attention to the events of the twentieth century, which made it possible to formalise the valuation of options from a mathematical point of view and thus introduce these instruments into everyday use in financial markets.

The history of option contracts dates back to ancient times, while the development of exchange trading for option contracts took place from the 16th to 18th centuries.

In *Politics* [5, Book I, Chapter 11, Sections 5-10], Aristotle relates a history of how the Greek philosopher Thales of Miletus profited from an option-type agreement around the 6th century BC. According to the story, one year ahead, he predicted that the next olive harvest would be exceptionally good and used what he had to place a deposit on the local olive presses. Consequently, Thales secured the rights to the presses at a relatively low rate. When the harvest proved to be bountiful, demand for the presses was high, so Thales charged a high price for their use and reaped a considerable profit. Paraphrasing it in modern trading terms, Thales bought a call option on olive presses and paid a small premium for this option. Another often quoted ancient reference to an option-feature transaction can be found in Genesis 29 of the Bible, where Laban offers Jacob an option to marry his youngest daughter, Rachel, in exchange for seven years of labour. This story illustrates an important issue associated with option trading, that is the possibility of delivery failure. Luckily, that did not happen in this case. Although Aristotlean and Biblical anecdotes provide notable evidence of option contracting in ancient times, following the evolution of options through time is complicated by the similarity of option contracts to other types of contracts, such as gambles. For more information on option contracts in ancient times, see [123].

Moving on to more modern times and the expansion of trade, the rise of urban centres caused that forward and option contracts became essential for urban merchants, as they could contract with agricultural producers for crops before harvest or fishermen for catches before arrival at port. The evolution of options contracts revolved around two critical elements: enhanced securitisation of transactions and the emergence of speculative trading. Both these developments are closely connected with the concentration of commercial activity, initially at the sizeable medieval market fairs and, later, on the bourses. Over time, medieval market fairs were surpassed by trade in urban centres such as Bruges, Antwerp and Lyon. Due to the rapid expansion of seaborne trade during the period, speculative transactions in grain were still particularly active at sea. The trade in whale oil, herring and salt was also important, see [11], [72] and [76]. For more details on the emergence of futures and options contracts trading on the Antwerp Exchange, see [142]. The collapse of Antwerp in 1585 and the resulting diaspora of merchants contributed significantly to the rise of the financial and commodity exchanges in Amsterdam and London.

During the 17th and 18th centuries, trading forward and option contracts on the Amsterdam exchange exhibited many essential features of exchange trading in modern derivative markets. By the middle of the 17th century, trading on the Amsterdam bourse of options on the Dutch East Indies Company and the Dutch West Indies Company had progressed to where the put and call options with regular expiration dates were traded, see [76] and [144]. By the 18th century, the trade involved Dutch joint stock shares and British funds. This trading on the Amsterdam bourse is the first historical instance of exchange trading in financial derivative contracts.

Over time, more speculators began appearing in the commodity markets. The reasons for this were the lack of significant price variability, the practise of using forward contracts with terms in years or a few days, and the inability of speculators not connected to the trade to handle physical delivery. One of the more famous examples is the tulipmania of 1634–1637 when contract prices for some bulbs of the recently introduced and fashionable tulip reached extraordinarily high levels. It triggered actions restricting speculative participation in commodity markets. Since late 1636, the Dutch parliament had considered a decree that changed how tulip contracts functioned. Legal changes were eventually introduced in 1637 and forward contracts were transformed into option contracts to limit the speculative bubble. Following the Glorious Revolution of 1688, many of the speculative practises used in Amsterdam were adopted in England, where stock trading had a highly developed spot market by the mid-1690s.

The modern perception of option contracts as a sophisticated risk management tool is inconsistent with the long history of attempts to impose legal restrictions on option trading. The basis for such restrictions is the close correspondence between option contracts and gambles. Since these contracts were often used for gambling purposes, the parties to the contract could not expect the protection of the courts if the transaction did not go as planned. Brokers and other agents with public recognition or registration were not allowed to facilitate such contracts.

As a consequence, option trading was generally restricted to private transactions between individuals in which professional or social reputation was used to control the risk of contract default. During the emergence of trade in free-standing option contracts, the conventional legal view was that such contracts could be entered into by private parties willing to conduct such business without the guarantee that the courts could be used to enforce such contracts. However, in periods of speculative excess, the abuse of option contracts produced a subsequent demand for regulation.

There is limited information about the methods used for pricing options contracts at that time. De la Vega ([54]) and de Pinto ([56]) indicate that the options were used primarily to speculate and not to manage the risk by participants in the cash market. Therefore, it is possible that the forces of supply and demand mainly determined prices. On the other hand, Wilson ([144]) points out that there was also some understanding and application of the concept of cash-and-carry arbitrage, especially for time bargains. He provides, among other things, quotes for options on East India Company and South Sea Company shares in 1719 that reflect some pricing inefficiencies. However, there is evidence that option writers understood a put-call parity and, consequently, could have created fully hedged written option arbitrage profits. Both de la Vega and de Pinto contain statements indicating that the put-call parity was understood, as it was applied in specific circumstances of the late 17th and 18th century on the Amsterdam option market.

The history of option pricing theory is sparse. Relatively little was written until the appearance of Bachelier ([9]) and Bronzin ([31]), although Lefèvre ([87]) introduced the valuation using expiration date profit diagrams. Before this time, there was evidence that market participants had a subtle understanding of option pricing. However, market convention rather than competitive pricing was more important to determine the actual premiums of the options, for more information see [44]. For various reasons, including a history of speculative abuses, option trading was held in low esteem by the majority of stock and commodity market participants, especially in the United States. Consequently, the trade was generally conducted by a specialised group of traders catering to a relatively small clientele. This changed in the 19th century, when the popularity of options began to increase rapidly. It was related to the dramatic expansion of stock issues associated with railway, canal and industrial growth. At some point, this trade expanded to include retail investors. Although important merchant manuals from the first half of the century, such as [139], do not contain a discussion of options, similar manuals at the time [31], such as [61], include a detailed discussion indicating active trading of options on stocks in Paris and, to a lesser extent, in London and Berlin.

1.2.1.1 Modern history

Modern mathematical finance is a child of the twentieth century. As written in [45]: *The date March 29, 1900, should be considered as the birthdate of mathematical finance.* On this day, Louis Bachelier defended his doctoral dissertation *Théorie de la Spéculation* [9], at the Sorbonne University in France. Bachelier's extraordinary thesis was years, and in some respects decades, ahead of its time. His pioneering analysis of the stock and option markets contains several ideas of enormous value in finance and probability. In particular, the theory of Brownian motion, one of the most important mathematical discoveries of the twentieth century, was initiated and used to develop a rational theory of option pricing. He also explicitly discovered the fundamental relation between Brownian motion and the heat equation. This fact was rediscovered five years later by Einstein [65]. It resulted in a goldmine of mathematical investigation through the work of Kolmogorov, Kakutani, Feynman, Kac and many others up to recent research. It is worth noting that Henri Poincaré, in his report on Bachelier's thesis, expressed regret that Bachelier did not study in detail the discovered relationship of stochastic processes with equations in partial derivatives. He was probably intrigued by deeper perspectives in this area. A more detailed description of Louis Bachelier's life and scientific work can be found in [45].

After Bachelier's pioneering work, it remained silent around the theme of option pricing for almost 70 years. The time was not ripe for sophisticated financial instruments, technology could not have handled them, and there was little matter of two world wars and the Great Depression. The Bretton Woods agreements [89] on fixed exchange rates and barriers to capital movements from 1944 provided little scope for financial intermediation. Meanwhile, however, the mathematicians were far from idle. In 1905, Einstein, in his substantial paper [65], derived the Brownian transition function of the form

$$q(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

by analysing the diffusion of particles in a perfect gas. This result put the Brownian motion as a mathematical model firmly on the map. As mentioned above, five years earlier, Louis Bachelier showed that $q(x,t)$ is the solution of the Chapman-Kolmogorov equation and solves the heat equation. In 1923, Norbert Wiener [143] provided a rigorous treatment, showing that it is possible to define a probability measure on the space of continuous functions, which corresponds to the Brownian transition function. In 1933, Kolmogorov published his book [92], which laid the modern axiomatic foundations of probability theory. In the late 1930s, Joseph Leo Doob formally introduced the concept of martingale in [62]. Then, in 1944 Kiyoshi Itô, attempting to elucidate the connection between partial differential operators and Markov processes, introduced stochastic differential equations and the famous *Itô stochastic calculus*, see [83]. In later years, major contributions to the development of stochastic calculus were made by McKean in [111] and Meyer, who formulated the supermartingale decomposition theorem, see [114, 115]. It opened the way to defining stochastic calculus for general classes of semimartingales, not just Brownian motion. Stroock and Varadhan in [135, 136] definitively demonstrated the connection between martingales and Markov processes. The net effect of these developments was to turn stochastic calculus from a niche topic of interest to a few initiates into a substantial body of techniques accessible to a wide range of applied scientists. More details on the history of probability theory and stochastic calculus can be found in the excellent textbook by Rogers and Williams [126].

An intense period of progress in financial mathematics was 1965–1980. An American economist Paul Samuelson rediscovered Bachelier's thesis in the library of Harvard University in 1965, following a request of the statistician J. Savage. He was immediately fascinated by Bachelier's work and started a line of research on option pricing and related topics, which at this time had much more repercussions than Bachelier's thesis. In his pioneering paper [130], Samuelson proposed a multiplicative version of Bachelier's model by introducing a geometric Brownian motion to model the stock price behaviour. Compared to Bachelier's model, the geometric Brownian motion takes positive values with probability one and the logarithmic stock price returns are normally distributed. These characteristics reflect real stock price movements much more accurately. Samuelson's study provided a viable model for stock prices that led, eight years later, to the central result of modern finance, the Black-Scholes option pricing formula [28]. Bernstein, in [22], recounts in detail the background to the discovery of the glorious formula by Fischer Black, Myron Scholes and their collaborator Robert Merton.

The Black-Scholes formula was published in 1973. The same year that option trading began on the Chicago Board Options Exchange (CBOE). Once the formula was digested and researchers recognised the power of stochastic calculus for analysing business and theoretical problems, the range and depth of applications expanded rapidly. Among the most critical early applications, beyond option pricing, were dynamic models of the term structure of interest rates, beginning with those of Vasicek [141] and Cox, Ingersoll and Ross [48]. In addition, in 1973, the Bretton Woods system finally collapsed, leading to an immediate requirement for managing exchange rate volatility. By 1980, the arbitrage pricing theory had become well understood; the close link with martingale theory was established by Harrison, Kreps and Pliska, see [78, 79]. It is coincidental but relevant that 1979 was the date of the first IBM PC, ushering in the era of massive computational capacity without which the industry could not exist.

Another model that played a decisive role in the development of option pricing was the binomial tree model, introduced by Cox, Ross and Rubinstein [47] in 1979. Its simple structure and easy implementation have given analysts the ability to price a wide range of financial derivatives almost systematically. The key results regarding this model are as follows: there is a unique martingale measure, the price of an option is obtained by computing the discounted expectation with respect to this measure and it can be characterised as the unique measure such that the discounted underlying price process is a martingale. The question of to what extent these properties generalise to other market models turned out to be surprisingly delicate and definitive answers were not given until the 1990s.

Today, most traded stock and futures options are American style, but most index options are European. The former can be exercised at any time up to and including the expiration date. In turn, the European options can only be exercised on the expiration date. In general, the price of an American option is equal to that of a European option, plus an additional non-negative early exercise premium.

1.2.2 Risk-neutral pricing

A common issue that arises frequently in different financial problems is the valuation of future cash flows, which are risky because the payment is not deterministic. A classical way to proceed is to estimate future cash flows and discount them to the present date. Nevertheless, of course, there is some uncertainty involved in estimating these future cash flows. The usual way to compensate for this uncertainty is to apply an interest rate higher than the riskless rate of return corresponding to the rate of return of government bonds. The spread between the risk-free rate of return and the interest rate used to discount future cash flows can be quite substantial to compensate for the riskiness. In mathematical terms, the above procedure may be described as follows: first, one determines the expected value of the future cash flows and then discounts by using an elevated discount factor. However, there is no systematic way to assess the degree of uncertainty in determining the expected value that can be quantified and how this should be considered to determine the spread between the interest rates.

The foundation of option pricing theory is based on a different approach, which is based on the concept of a risk-neutral probability measure rather than a real-world probability measure[3]. The mathematical model of a financial market under the risk-neutral measure refers to a virtual world, not a real one. As under the risk-neutral measure, the asset price process discounted by the risk-free interest rate is a martingale; it is common to call this measure a martingale measure. This approach was applied in the seminal paper [28] of Fischer Black and Myron Scholes. It simply consists of calculating the expected value of future cash flows under the risk-neutral probability

[3]Typically in the literature, the risk-neutral probability measure is denoted by $\mathbb{Q}$, while the real-world probability measure is denoted by $\mathbb{P}$. In our work, we focus exclusively on option pricing and denote by $\mathbb{P}$ the risk-neutral probability measure, while we do not use the notation $\mathbb{Q}$ at all. We also use the symbols $\mathbb{P}_s$ and $\mathbb{P}_{(x)}$ to indicate that $S_0 = s$ and $X_0 = x$, respectively.

measure and discounting them with the risk-free rate. The natural question that arises is about the existence and uniqueness of the risk-neutral probability measure. Unfortunately, there is no simple answer and it depends on the model under consideration.

In the case of the Black-Scholes model, there is a unique martingale measure, which can be derived via the Girsanov theorem. More on this topic will be presented in the section on the Black-Scholes model. In many applications, it is not necessary to even consider the original real-world probability measure. It is common in the literature that authors work under the risk-neutral measure and all assumptions are made under this set-up. We also proceed in this way in the central part of the thesis.

The technique of a risk-neutral measure was not the novel feature of the Black and Scholes work. It was used decades earlier by Bachelier in his dissertation thesis [9]. In the first pages of his thesis, Bachelier lists two kinds of probability, i.e. the probability which might be called mathematical, which can be determined a priori and which is studied in games of chance and the probability which depends on future events and consequently is impossible to predict in a mathematical manner. The latter is the probability that the speculator tries to predict. In retrospect, one can interpret the first statement as the risk-neutral probability measure and the second as the real-world probability measure.

1.2.3 Fundamental Theorem of Asset Pricing

The Fundamental Theorem of Asset Pricing is one of the pillars supporting mathematical finance. In vague terms, it states that the no-arbitrage possibility in the market is equivalent to the existence of a probability measure being equivalent to the real-world measure and under which the asset price process is a martingale.

The story of this theorem started with the work of Black, Scholes [28] and Merton [112], where these authors considered geometric Brownian motion as a model that describes the behaviour of asset prices. They used a technique to price options, where one changes the underlying measure to an equivalent measure under which the discounted stock price process is a martingale. Subsequently, the option value was obtained by taking the expectation with respect to this measure, which is called the risk-neutral measure. This technique was not the novel feature of [28] and [112]. It was used by Bachelier [9], who considered the Brownian motion as a model of a stock price process. The prices obtained by Bachelier were, at least for the empirical data he considered, very close to those derived from the celebrated Black-Scholes formula, see [132].

The decisive novel feature of the Black-Scholes model was the argument linking the option pricing technique with the notion of arbitrage. In other words, the payoff function of an option can be precisely replicated by hedging, that is by dynamically trading in the underlying asset. This idea is credited in [112, footnote no 3] to Merton who opened up a new perspective on how to deal with options. The technique of replicating the option is absent in Bachelier's work, whereas the idea of spanning a market by forming linear combinations of primitive assets first appears in the classic paper [6] by Arrow. The mathematically delightful situation that the market is complete, which means that all derivatives can be replicated, occurs in the Black-Scholes model and in Bachelier's original model. Another example of a continuous-time model that shares this property is the compensated Poisson process, as observed by Cox and Ross [46]. Roughly speaking, these are the only models in continuous time sharing this beautiful *martingale representation property*, see [80] and [104] for a precise statement on the uniqueness of martingale measure for these families of models. As attractive as it might be, the consideration of complete markets is somewhat dangerous from an economic point of view. The precise replicability of options, a sound mathematical theorem, may lead to the illusion that this is also true in economic reality. However, of course, these models are far from matching reality one-to-one. Instead, they only highlight essential aspects of reality and should not be considered universally appropriate.

When the merits and limitations of the Black-Scholes model unfolded in the late 1970s, the investigations on the Fundamental Theorem of Asset Pricing started. As Harrison and Pliska formulate it in the introduction to their classic paper [79]: *it was a desire to better understand their formula which originally motivated our study, ...* The challenge was to obtain a deeper insight into the relation of the following two aspects: on the one hand, the pricing methodology by taking expectations with respect to a properly chosen risk-neutral measure and, on the other hand, the pricing methodology without arbitrage considerations. It was unclear why these two seemingly unrelated approaches yield identical results in the Black-Scholes model. Perhaps even more relevant was the question: How far can this phenomenon be extended to more involved models? The first to discuss these questions in a systematic way was Ross [129], see also [46], [127] and [128]. He formulated the first precise version of the Fundamental Theorem of Asset Pricing in [129] with the proof based on the Hahn-Banach theorem. After this early work by Ross, a major advance was achieved between 1979 and 1981 by three seminal papers [78], [79], [96] by Harrison, Kreps and Pliska. They also formulated a version of the Fundamental Theorem of Asset Pricing for finite, filtered probability space, see [79, Theorem 2.7, p. 228]. The proof again relies on the Hahn-Banach theorem (a finite-dimensional version) plus an extra argument, making sure to find a measure which is equivalent to the real-world measure. The restriction to a finite probability space is very severe in applications. The concept of continuous time is the theory's flavour, building on the Black-Scholes model. Nevertheless, this involves infinite probability spaces. Many interesting concerns were formulated in the papers [78] and [79], hinting at the difficulties of proving a version of the Fundamental Theorem of Asset Pricing beyond the setting of finite probability spaces. Kreps, in his paper [96], achieved a breakthrough in this direction. He introduced the concept of *no free lunch*. The economic interpretation of the no free lunch condition is a sharpening of the no-arbitrage condition. This remarkable work by Kreps set new standards and, for the first time, a mathematically precise statement of the Fundamental Theorem of Asset Pricing was achieved for a general class of models in continuous time. The heroic period of development of the Fundamental Theorem of Asset Pricing marked by Ross [129], Harrison-Kreps [78], Harrison-Pliska [79] and Kreps [96] put the issue on safe mathematical grounds and brought some spectacular results. However, there are still some limitations and many questions remain open. Some of them were answered in subsequent years, while others opened new perspectives. For a thorough overview of this topic, we refer to the extensive monograph [55].

1.2.4 Black-Scholes model

In 1973, in the Journal of Political Economy, Black and Scholes published their seminal paper [28], which influenced the dynamic growth of option pricing theory in the second half of the twentieth century and played a profound role in the economics of everyday life. These authors presented their model for pricing options. Shortly after that, Merton in [112] expanded the mathematical understanding of this model. Departing from the no-arbitrage principle and using the concept of dynamic trading, these authors derived the so-called Black-Scholes formula for the price of a European call option. This formula provided, for the first time, a theoretical method of fairly pricing a risk-hedging security. It can be presented as follows

$$V_0 = s\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

with

$$d_1 := \frac{\log(\frac{s}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 := d_1 - \sigma\sqrt{T}.$$

The parameters s, K, r, σ and T are constants and describe the specific characteristics of a stock and an option, while $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

The fundamental insight in their work is the idea of perfect replication. This technique has no parallel in previous studies in the work of Bachelier. It turns out that in a market model where prices follow geometric Brownian motion, perfect replication is possible, giving a unique option price.

The groundwork assumption of the Black-Scholes model is that the market consists of at least one risky asset, usually called the stock with the price denoted by S_t and one riskless asset, usually called the money market, cash or bonds with the price indicated by F_t. The following equations model these prices

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t, \\ dF_t = rF_t dt, \end{cases}$$

where μ, σ and r are constants in the model representing drif, volatility and riskless rate, respectively, while B_t is a Brownian motion under the real-world measure[4].

From the Girsanov theorem, we know that there exists a measure, called a risk-neutral measure, under which the discounted stock price process $e^{-rt}S_t$ is a martingale. Moreover, under this measure, the process S_t follows a geometric Brownian motion as in the initial settings, but with another drift parameter, that is

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where B_t is a Brownian motion under the risk-neutral probability measure. The point is that the drift term in the above SDE is equal to r, the risk-free interest rate. So in such a case, we say that the market is risk-neutral. In other words, when we price an option, we use the measure with respect to which the drift of the underlying asset is equal to the risk-free interest rate r, so it is independent of the individual preferences of the two parties to the transaction. In the Black-Scholes model, there is only one risk-neutral measure. As we mentioned earlier, this feature characterises complete financial markets, meaning that all derivatives can be replicated. For more details, refer to [42].

In general, the formula that provides a theoretical price for a European option with the payoff function g can be written as

$$V_0 = e^{-rT}\mathbb{E}_s\left[g(S_T)\right],$$

which can be described in words as the discounted expectation from the payoff function taken under the risk-neutral measure.

In the modern financial industry, the Black-Scholes model is widely used. However, with some adjustments, the methodology often extends to pricing a whole range of complex option products, such as barrier options, basket options, look-back options, American options and many others. Moreover, several of the assumptions of the original model have been removed in subsequent extensions of the model, e.g. no dividends [112], continuous stock returns [113], continuous

[4]Later in this dissertation, in Chapter 2, we denote by B_t the risk-neutral measure as we do not focus on the real-world measure at all.

evolution of the stock price [47], constant variance of the underlying returns [81] or constant interest rates [10].

The landmark work on option pricing theory was highlighted in particular by the Royal Swedish Academy of Sciences in 1997 when the Nobel Prize was awarded to Robert Merton and Myron Scholes [5]. In the commission's official press release, we can read: *Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black, developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.* In the later part of this document, we can also read: *Black, Merton and Scholes made a vital contribution by showing that it is in fact not necessary to use any risk premium when valuing an option. This does not mean that the risk premium disappears; instead, it is already included in the stock price.*

1.3 Exponential Lévy processes

In this section, we look at exponential Lévy processes[6] which form a generalisation of the Black-Scholes model by allowing stock prices to jump. In general, the exponential Lévy process S_t is defined by

$$S_t = se^{L_t},$$

where L_t is a Lévy process and $S_0 = s$.

However, later in the thesis, we will use the notation

$$S_t = e^{X_t}, \tag{1.1}$$

where X_t inherits the same properties as L_t, but is shifted by $\log s$.

It turns out that the use of such models to describe the behaviour of financial assets has become very common in recent years. Extensive empirical studies have shown that the Gaussian model is not capable of capturing certain features such as skewness, asymmetry and heavy tails, which are commonly encountered in financial data, see [41]. To overcome these problems, we can replace the Brownian motion as a model for logarithmic prices with a general Lévy process X_t. Then X_t as a Lévy process satisfies the property of independence and stationary increments. These conditions go hand in hand with real market stock price movements and justify the utility of exponential Lévy processes in financial modelling.

For a comprehensive survey on exponential Lévy models, we recommend textbooks such as [42, 133] for a more financial perspective and [4, 99] for a more mathematical perspective. It is worth mentioning that Lévy processes appear in a wide range of applications, not only in the financial industry but also in physics, biology and other sciences.

In general, the exponential Lévy models fall into two categories. The first category, called *jump-diffusion models*, assumes that the evolution of prices is given by a diffusion process punctuated by jumps at random moments. In this situation, jumps represent rare events such as crashes, large drawdowns, or rapid growths. A Lévy process can represent such an evolution with a non-zero Gaussian component and a jump part with finitely many jumps, i.e.

$$X_t = \xi t + \sigma B_t + \sum_{i=1}^{N_t} Y_i,$$

[5] Fischer Black was not awarded the Nobel Prize due to his death in 1995, but he was cited as a key contributor.

[6] In this thesis, we use the terms *exponential Lévy process* and *exponential Lévy model* interchangeably.

where $\xi \in \mathbb{R}$ and $\sigma \geq 0$ are constants, N_t is a Poisson process independent of Brownian motion B_t and $\{Y_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables independent of N_t and B_t. The first model considered in the literature of this type is the Merton model [113] from 1976. It establishes that Y_i has a normal distribution. Another model is the Kou model [94], where jump sizes are distributed according to an asymmetric double exponential distribution.

The second category consists of models with an infinite number of jumps in every interval, called *infinite activity* or *infinite intensity* models. In these models, one does not need to introduce a Brownian component since the dynamic of jumps is already rich enough to generate nontrivial small-time behaviour, see [34]. An example of the process in this group is the variance gamma process [110], which is a three-parameter generalisation of the Brownian motion and is obtained by evaluating a Brownian motion with constant drift and volatility at a random time given by a gamma process. Contrary to previous models, the variance gamma process does not have a continuous martingale component. Instead, it is a pure jump process with infinite activity, see [43]. The density of the Lévy measure of the variance gamma process is given by

$$v(x) = \frac{\mu_p^2}{\nu_p x} e^{-\frac{\mu_p}{\nu_p} x} \mathbb{1}_{\{x>0\}} + \frac{\mu_n^2}{\nu_n |x|} e^{-\frac{\mu_n}{\nu_n}|x|} \mathbb{1}_{\{x<0\}},$$

where $\mu_p = \frac{1}{2}\sqrt{\vartheta^2 + \frac{2\sigma^2}{\nu}} + \frac{\vartheta}{2}$, $\nu_p = \mu_p^2 \nu$, $\mu_n = \frac{1}{2}\sqrt{\vartheta^2 + \frac{2\sigma^2}{\nu}} - \frac{\vartheta}{2}$, $\nu_n = \mu_n^2 \nu$, while ϑ, ν and σ are the parameters of this model. Another example is the the CGMY model [34] as it has four parameters: C, G, M, Y. It can be specified directly by the Lévy measure density of the form

$$v(x) = \frac{C}{x^{1+Y}} e^{-Mx} \mathbb{1}_{\{x>0\}} + \frac{C}{|x|^{1+Y}} e^{-G|x|} \mathbb{1}_{\{x<0\}},$$

where $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. It is easy to see that by choosing $Y = 0$, $C = \frac{1}{\nu} = \frac{\mu_p^2}{\nu_p} = \frac{\mu_n^2}{\nu_n}$, $G = \frac{\mu_n}{\nu_n}$ and $M = \frac{\mu_p}{\nu_p}$ we obtain the Lévy density corresponding to the measure presented above for the variance gamma process. Hence the variance gamma process can be seen as a particular case of the CGMY process.

In the next part of the thesis, our considerations centre around the exponential Lévy processes with negative jumps. This class of stochastic processes appears frequently in scientific research, including risk theory [146], option pricing [8] or insurance risk models [103].

1.3.1 Spectrally negative Lévy processes

Spectrally negative Lévy processes form a subclass of Lévy processes and are commonly used in various financial applications. The fundamental feature of this class of stochastic processes is the fact that they can only move upward in a continuous way. An arbitrary Lévy process can be written as the difference of two independent spectrally negative Lévy processes, which gives the possibility of establishing general results by studying this subclass of processes. Moreover, by adding independent copies of any spectrally negative Lévy processes together, the resulting process remains within the class of spectrally negative Lévy processes.

Later in this dissertation, we restrict ourselves to the model where X_t, from (1.1), is given by the spectrally negative Lévy process (possibly starting at some positive value). This restriction is mainly motivated by analytical tractability and the availability of many results regarding this class of Lévy processes that we provide later in this thesis. It is also worth mentioning that our choice goes hand in hand with market practise, that is in [35] the authors have offered empirical evidence to support the case of a model in which the spectrally negative Lévy process models the risky asset.

Let us begin with a brief overview of what is meant by a spectrally negative Lévy process. We suppose that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions. A stochastic process X_t is said to be a Lévy process on this space if it is a strong Markov, $\mathbb{F}$-adapted process with càdlàg paths, stationary and independent increments and $\mathbb{P}(X_0 = 0) = 1$. From these properties, it can be shown that X_t is continuous in probability and at any fixed time the probability of having a jump is zero.

The distribution of the Lévy process X_t is characterised by its characteristic function $\varphi : \mathbb{R} \to \mathbb{C}$ of the form

$$\varphi(\theta) := \mathbb{E}_{(0)}\left[e^{i\theta X_t}\right],$$

where the subscript with brackets in $\mathbb{E}_{(0)}$ denotes the initial value of X_0. In this case, we have $X_0 = 0$. It can be shown that there exists a unique continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ such that

$$\varphi(\theta) = e^{t\Psi(\theta)}.$$

Throughout the thesis, we call Ψ a characteristic exponent of X_t.

The Lévy-Khintchine formula provides us with the general form of Ψ, that is

$$\Psi(\theta) = i\zeta\theta - \frac{\sigma^2}{2}\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(dx),$$

where $\zeta \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. As the characteristic function uniquely determines the underlying probability distribution, each Lévy process is uniquely determined by the Lévy-Khintchine triplet (ζ, σ, Π).

We say that X_t is a spectrally negative Lévy process if the measure Π is carried by $(-\infty, 0)$, i.e. $\Pi(0, \infty) = 0$. Notationally, we say that X_t is a spectrally positive Lévy process when $-X_t$ is spectrally negative.

We can also represent a spectrally negative Lévy process X_t as

$$X_t = \zeta t + \sigma B_t + J_t^{(-)},$$

where $\zeta \in \mathbb{R}$ is a drift parameter, $\sigma \geq 0$ is a volatility parameter, B_t is a Brownian motion and $J_t^{(-)}$ is a spectrally negative Lévy process without a Gaussian component that is independent of B_t. Here, we exclude the case where X_t has monotonic paths. The jumps of $J_t^{(-)}$ are all non-positive, so the moment generating function of X_t exists for all $\theta \geq 0$. It allows us to talk about the Laplace exponent that is defined by

$$\psi(\theta) := \frac{1}{t} \log \mathbb{E}_{(0)}\left[e^{\theta X_t}\right], \tag{1.2}$$

which is well-defined at least for $\theta \geq 0$.

Taking into account an analytical extension of the characteristic exponent Ψ, we have $\psi(\theta) = \Psi(-i\theta)$, which is equal to

$$\psi(\theta) = \zeta\theta + \frac{\sigma^2}{2}\theta^2 + \int_{-\infty}^{0} \left(e^{\theta x} - 1 - \theta x \mathbb{1}_{\{|x|<1\}}\right) \Pi(dx).$$

Using Hölder's inequality, or alternatively differentiating, it is easy to check that ψ is strictly convex and tends to infinity as θ tends to infinity. Therefore, it allows us to define the right-inverse of ψ given by

$$\Phi(q) := \sup\{\theta \geq 0 : \psi(\theta) = q\},$$

where $q \in \mathbb{R}$. It denotes the largest root of equation $\psi(\theta) = q$ when it exists. We can observe that there exist at most two roots for a given q (there is always a root at zero since $\psi(0) = 0$) and precisely one root when $q > 0$. By differentiating (1.2), we can see that $\psi'(0^+) = \mathbb{E}_{(0)}[X_1] \in [-\infty, \infty)$ which determines the long-term behaviour of X_t, see [98, Lemma 7, p. 23].

Let us now define the family of martingales given by

$$\mathcal{E}_t(\alpha) := e^{\alpha X_t - \psi(\alpha) t}$$

for any $\alpha \geq 0$ and the corresponding family of probability measures $\{\mathbb{P}_{(x)} : x \in \mathbb{R}\}$ referring to the conditional version of $\mathbb{P}$ where $X_0 = x$ is given. Applying the Girsanov theorem, we can define a new probability measure $\mathbb{P}^{(\alpha)}_{(x)}$ via

$$\left.\frac{d\mathbb{P}^{(\alpha)}_{(x)}}{d\mathbb{P}_{(x)}}\right|_{\mathcal{F}_t} = \frac{\mathcal{E}_t(\alpha)}{\mathcal{E}_0(\alpha)}. \tag{1.3}$$

Under this change of measure, X_t remains within the class of spectrally negative Lévy processes (see [99, Corollary 3.10, p. 80]) with the Laplace exponent, under $\mathbb{P}^{(\alpha)}_{(0)}$, given by

$$\psi^{(\alpha)}(\theta) = \psi(\theta + \alpha) - \psi(\alpha) \tag{1.4}$$

for $\theta \geq -\alpha$.

In the next part of the thesis, we focus our attention on the specific case of X_t, that is

$$X_t = \zeta t + \sigma B_t - \sum_{i=1}^{N_t} Y_i \tag{1.5}$$

where $\{Y_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables which are exponentially distributed with mean $\frac{1}{\rho} > 0$ and N_t is a Poisson process independent of the Brownian motion B_t. Its Laplace exponent takes the form

$$\psi(\theta) = \zeta\theta + \frac{\sigma^2}{2}\theta^2 - \frac{\lambda\theta}{\rho + \theta}. \tag{1.6}$$

Taking into account the behaviour of $\psi(\theta)$ as $\theta \to \pm\infty$ and $\theta \to \rho^{\pm}$ we can easily verify that for every $q > 0$ equation $\psi(\theta) = q$ has exactly three real solutions $\{\gamma_1, \gamma_2, \Phi(q)\}$, which satisfy $\gamma_2 < -\rho < \gamma_1 < 0 < \Phi(q)$.

Using (1.4), we can derive the Laplace exponent $\psi^{(\alpha)}(\theta)$ of X_t under $\mathbb{P}^{(\alpha)}_{(0)}$, that is

$$\psi^{(\alpha)}(\theta) = \zeta^{(\alpha)}\theta + \frac{{\sigma^{(\alpha)}}^2}{2}\theta^2 - \frac{\lambda^{(\alpha)}\theta}{\rho^{(\alpha)} + \theta},$$

where

$$\zeta^{(\alpha)} = \zeta + \sigma^2\alpha, \quad \sigma^{(\alpha)} = \sigma, \quad \lambda^{(\alpha)} = \frac{\lambda\rho}{\rho + \alpha} \quad \text{and} \quad \rho^{(\alpha)} = \rho + \alpha.$$

Further details about the class of spectrally negative Lévy processes and how they embed within the general class of Lévy processes can be found in the monographs of Applebaum [4], Bertoin [24], Kyprianou [99] and Sato [131].

1.4 Scale functions

The main aim of this section is to provide some general definitions and facts involving *q-scale functions* as well as specific generalisations of these functions that play a critical role in our thesis.

Common factors that bind together the scale functions and spectrally negative Lévy processes are the so-called one- and two-sided exit problems for spectrally negative Lévy processes. The exit problems essentially consist of characterising the Laplace transforms of σ_a^+, σ_0^- and $\sigma_a^+ \wedge \sigma_0^-$, where

$$\sigma_a^- := \inf\{t \geq 0 : X_t \leq a\} \quad \text{and} \quad \sigma_a^+ := \inf\{t \geq 0 : X_t \geq a\}$$

for $a \in \mathbb{R}$. Note that X_t as a spectrally negative Lévy process starting at some point between 0 and a can hit the point a when crossing upward, as it can only continuously move upward. On the other hand, it can hit 0 continuously or jump below zero. It has turned out that one- and two-sided exit problems of spectrally negative Lévy processes can be characterised by the exponential function together with two families, $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ and $\{Z^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ known as the *q-scale functions*, see [24], [25], [26], [27], [71], [125], [138], [147].

Definition 1. *For a given spectrally negative Lévy process X_t with Laplace exponent $\psi(\theta)$, we define a family of functions indexed by $q \geq 0$, $W^{(q)} : \mathbb{R} \to [0, \infty)$, as follows. For each given $q \geq 0$, we have $W^{(q)}(x) = 0$ when $x < 0$ and otherwise on $[0, \infty)$, $W^{(q)}(x)$ is the unique right continuous function whose Laplace transform is*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x) dx := \frac{1}{\psi(\theta) - q} \tag{1.7}$$

for $\theta > \Phi(q)$.

Adding the subscript α to the q-scale function $W^{(q)}(x)$ means that we work under the $\mathbb{P}_{(0)}^{(\alpha)}$ measure defined in (1.3). We can establish the following relationship for $W_\alpha^{(q)}(x)$ with different values of q and α.

Lemma 1 ([99, Lemma 8.4, p. 222])**.** *For any $q \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ such that $\psi(\alpha) < \infty$ we have*

$$W^{(q)}(x) = e^{\alpha x} W_\alpha^{(q - \psi(\alpha))}(x) \tag{1.8}$$

for all $x \in \mathbb{R}$ and $q \geq \psi(\alpha)$.

Another q-scale function considered in this thesis is the function $Z^{(q)}(x)$, which is defined as follows.

Definition 2. *For $q \geq 0$, we define $Z^{(q)} : \mathbb{R} \to [1, \infty)$ by*

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy \tag{1.9}$$

for $x > 0$ and $Z^{(q)}(x) = 1$ for $x \leq 0$.

Like the function $W^{(q)}(x)$, the function $Z^{(q)}(x)$ may be characterised by its Laplace transform and continuity on $(0, \infty)$. Indeed, we can check that

$$\int_0^\infty e^{-\theta x} Z^{(q)}(x) dx = \frac{\psi(\theta)}{\theta} \left(\psi(\theta) - q\right)$$

for $\theta > \Phi(q)$.

For convenience, we denote $W(x) := W^{(0)}(x)$ and $Z(x) := Z^{(0)}(x)$. For clarity of notation, we refer to $W(x)$ and $Z(x)$ as the scale functions, while to $W^{(q)}(x)$ and $Z^{(q)}(x)$ as the q-scale functions. Furthermore, we use the notations $W_\alpha^{(q)}(x)$ and $Z_\alpha^{(q)}(x)$ to indicate the q-scale functions for X_t under the $\mathbb{P}_{(0)}^{(\alpha)}$ probability measure.

For the review of one- and two-sided exit problems that contain the q-scale functions $W^{(q)}(x)$ and $Z^{(q)}(x)$, we refer to [98, Section 3, p. 18]. To give only one immediate example of the so-called two-sided exit problem, we provide an identity with a long history, see [25], [26], [125], [138] and [147].

Remark 1 ([98, Formula (3), p. 19]). *For any $x \leq a$ and $q \geq 0$,*

$$\mathbb{E}_{(x)}\left[e^{-q\sigma_a^+}\mathbb{1}_{\{\sigma_a^+<\sigma_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}.$$

In fact, it is through this identity that the *scale function* gets its name. As noted in [40], possibly the first reference to this terminology can be found in [23].

Now we state the result for the limit of $\frac{Z^{(q)}(x)}{W^{(q)}(x)}$ as x tends to infinity. For the formulation of this result and the later part of the thesis, we shall understand $\frac{0}{\Phi(0)}$ as $\lim_{\theta\to 0}\frac{\theta}{\psi(\theta)} \vee \psi'(0)$.

Lemma 2 ([99, Exercise 8.5, p. 234]). *For $q \geq 0$,*

$$\lim_{x\to\infty}\frac{Z^{(q)}(x)}{W^{(q)}(x)} = \frac{q}{\Phi(q)}.$$

Considering X_t given in (1.5), we can obtain a convenient expression for the q-scale functions. If we take the partial fraction decomposition of the rational function $\frac{1}{\psi(\theta)-q}$ with $\psi(\theta)$ given in (1.6) and invert the Laplace transform in (1.7), we conclude that

$$W^{(q)}(x) = \frac{e^{\gamma_1 x}}{\psi'(\gamma_1)} + \frac{e^{\gamma_2 x}}{\psi'(\gamma_2)} + \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))},$$

where $\{\gamma_1, \gamma_2, \Phi(q)\}$ is the set of real solutions to $\psi(\theta) = q$. From (1.9) we calculate

$$Z^{(q)}(x) = 1 + q\left(\frac{e^{\gamma_1 x}-1}{\gamma_1\psi'(\gamma_1)} + \frac{e^{\gamma_2 x}-1}{\gamma_2\psi'(\gamma_2)} + \frac{e^{\Phi(q)x}-1}{\Phi(q)\psi'(\Phi(q))}\right).$$

If we take $\sigma = 0$ or $\lambda = 0$ in (1.6), then $W^{(q)}(x)$ and $Z^{(q)}(x)$ take simplified forms, that is

$$W^{(q)}(x) = \frac{e^{\gamma_1 x}}{\psi'(\gamma_1)} + \frac{e^{\gamma_2 x}}{\psi'(\gamma_2)}$$

and

$$Z^{(q)}(x) = 1 + q\left(\frac{e^{\gamma_1 x}-1}{\gamma_1\psi'(\gamma_1)} + \frac{e^{\gamma_2 x}-1}{\gamma_2\psi'(\gamma_2)}\right)$$

for γ_1 and γ_2 again being the real solutions to $\psi(\theta) = q$.

We also state here one more remark about the joint Laplace transform of the time to overshoot and overshoot itself, which will be used later in our thesis.

Lemma 3 ([99, Exercise 8.7, p. 235]). *For $x > 0$, $\alpha \geq 0$ and $u \geq 0$,*

$$\mathbb{E}_{(x)}\left[e^{-u\sigma_0^- + \alpha X_{\sigma_0^-}} \mathbb{1}_{\{\sigma_0^- < \infty\}}\right] = e^{\alpha x}\left(Z_\alpha^{(q)}(x) - \frac{q}{\Phi(q)} W_\alpha^{(q)}(x)\right),$$

where $q = u - \psi(\alpha)$.

Now, we turn our attention to the specific generalisations of the q-scale functions.

Definition 3. *For any measurable function ξ, we define the ξ-scale functions $\{\mathcal{W}^{(\xi)}(x), x \in \mathbb{R}\}$, $\{\mathcal{Z}^{(\xi)}(x), x \in \mathbb{R}\}$ and $\{\mathcal{H}^{(\xi)}(x), x \in \mathbb{R}\}$ as unique solutions to the following renewal-type equations*

$$\mathcal{W}^{(\xi)}(x) := W(x) + \int_0^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y)dy, \tag{1.10}$$

$$\mathcal{Z}^{(\xi)}(x) := 1 + \int_0^x W(x-y)\xi(y)\mathcal{Z}^{(\xi)}(y)dy, \tag{1.11}$$

$$\mathcal{H}^{(\xi)}(x) := e^{\Phi(c)x} + \int_0^x W^{(c)}(x-z)(\xi(z)-c)\mathcal{H}^{(\xi)}(z)dz, \tag{1.12}$$

where $W(x) = W^{(0)}(x)$ is a classical zero scale function and in equation (1.12) *it is additionally assumed that $\xi(x) = c$ for all $x \leq 0$ and some constant $c \in \mathbb{R}$.*

We also define a two-variable equivalent to $\mathcal{W}^{(\xi)}(x)$.

Definition 4. *For any measurable function ξ, we define the ξ-scale function $\{\mathcal{W}^{(\xi)}(x,z), (x,z) \in \mathbb{R}^2\}$ by*

$$\mathcal{W}^{(\xi)}(x,z) := W(x-z) + \int_z^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y,z)dy. \tag{1.13}$$

We introduce the following S_t counterparts of the scale functions (1.10), (1.11), (1.12) and (1.13)

$$\mathscr{W}^{(\xi)}(s) := \mathcal{W}^{(\xi\circ\exp)}(\log s), \tag{1.14}$$

$$\mathscr{Z}^{(\xi)}(s) := \mathcal{Z}^{(\xi\circ\exp)}(\log s), \tag{1.15}$$

$$\mathscr{H}^{(\xi)}(s) := \mathcal{H}^{(\xi\circ\exp)}(\log s), \tag{1.16}$$

$$\mathscr{W}^{(\xi)}(s,z) := \mathcal{W}^{(\xi\circ\exp)}(\log s, z), \tag{1.17}$$

where $\xi \circ \exp(x) := \xi(e^x)$.

Similarly as before, we can add the subscript α to the functions (1.14)–(1.17), which means that we work under the $\mathbb{P}_{(0)}^{(\alpha)}$ measure. Therefore, we have $\mathscr{W}_\alpha^{(\xi)}(s)$, $\mathscr{Z}_\alpha^{(\xi)}(s)$, $\mathscr{H}_\alpha^{(\xi)}(s)$ and $\mathscr{W}_\alpha^{(\xi)}(s,z)$.

We also define the following functions

$$\eta(x) := \omega(e^x) = \omega(s), \quad \eta_u(x) := \eta(x + \log u) \tag{1.18}$$

and

$$\omega_u(s) := \omega(su), \quad \omega_u^\alpha(s) := \omega_u(s) - \psi(\alpha). \tag{1.19}$$

Lastly, we present the resolvent density at the point z of X_t starting at $\log s - \log u$ killed by the potential ω_u and when exiting the positive half-line. It is given by

$$r(s,u,z) := \mathscr{W}^{(\omega_u)}(\log s - \log u)c_{\mathscr{W}^{(\omega)}/\mathscr{W}^{(\omega)}}(z) - \mathscr{W}^{(\omega_u)}(\log s - \log u, z), \tag{1.20}$$

where

$$c_{\mathscr{W}^{(\omega)}/\mathscr{W}^{(\omega)}}(z) := \lim_{y\to\infty} \frac{\mathscr{W}^{(\omega)}(\log y, z)}{\mathscr{W}^{(\omega)}(\log y)}.$$

1.5 Main problem

As we mentioned in Introduction, the main objective of this dissertation is to price a perpetual American put option with asset-dependent discounting. In other words, we want to obtain a closed-form solution to the following problem

$$V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\int_0^\tau \omega(S_w)dw} (K - S_\tau)^+ \right]. \tag{1.21}$$

In financial terms, we call this function the value function of the particular option. Apart from the financial nomenclature that we use in the dissertation, we can consider problem (1.21) as a certain optimal stopping problem.

Asset-dependent discounting is reflected in the ω function, which is a crucial concept considered in this thesis. We underline that the discount function ω for various economic reasons can be different from the risk-free interest rate $r > 0$; more on this topic will be discussed later.

The way we choose to discount is distinctive, i.e. we assume a strong dependence between the discount factor and the asset price. Such a procedure aims to understand various economic phenomena that might appear in this extreme case. Our approach differs from typical studies considered in the literature, where the interest rate is independent of the asset price, or there is a weak dependence between these two factors. Therefore, we believe that the research we have conducted is noteworthy in the context of American option pricing and other areas where optimisation problems are studied.

In the following, we present one of the main theorems of our thesis, which also appears in Section 2.4 as Theorem 3. It shows us a closed-form of the function $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u)$ under certain general assumptions. This function is related to the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ by the equality

$$V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s) = v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l^*, u^*),$$

where

$$v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l^*, u^*) = \sup_{0 \le l \le u \le K} v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u).$$

Furthermore, the optimal stopping time in our problem takes the following form $\tau_{l,u} = \inf\{t \ge 0 : S_t \in [l, u]\}$. More detailed explanations on this topic are provided in Chapter 2. What is essential here is that having the information that the optimal stopping region is the interval $[l, u]$ is enough to maximise function (1.23) with respect to both l and u to obtain the final form of the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$. Moreover, by choosing a specific process S_t, we are able to obtain a more simplified form of $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u)$ which is presented in Subsection 2.4.1 and Subsection 2.4.2. Furthermore, we derive analytical expressions of (1.21) for the specific discount functions ω, these results are presented in Chapter 3.

Theorem. *Assume that the stock price process S_t is described by (2.2) with X_t being the spectrally negative Lévy process and ω is a measurable, bounded from below, concave and non-decreasing function such that*

$$\omega(s) = c \text{ for all } s \in (0, 1] \text{ and some constant } c \in \mathbb{R}. \tag{1.22}$$

Then

$$
\begin{aligned}
v^{\omega}_{\mathrm{A^{Put}}}(s,l,u) = {} & \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(l)}(K-l)\mathbb{1}_{\{s<l\}} + (K-s)\mathbb{1}_{\{s\in[l,u]\}} \\
& + \Bigg\{ \int_0^\infty \int_0^\infty \frac{\mathscr{H}^{(\omega_u)}((\frac{u}{e^y}) \wedge l)}{\mathscr{H}^{(\omega_u)}(l)}(K - e^{\log l \vee (\log u - y)}) r(s,u,z)\Pi(-z-dy)dz \\
& + (K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(\omega_u^\alpha)}_{\alpha}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha}\mathscr{W}^{(\omega_u^\alpha)}_{\alpha}\left(\frac{s}{u}\right)\right)\right)\Bigg\}\mathbb{1}_{\{s>u\}},
\end{aligned}
\tag{1.23}
$$

where

$$
c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha} = \lim_{z\to\infty} \frac{\mathscr{Z}^{(\omega^\alpha)}_\alpha(z)}{\mathscr{W}^{(\omega^\alpha)}_\alpha(z)}
$$

and $r(s,u,z)$ *is given in* (1.20).
If $l = 0$ *then condition* (1.22) *is superfluous and*

$$
\begin{aligned}
v^{\omega}_{\mathrm{A^{Put}}}(s,0,u) = {} & (K-s)\mathbb{1}_{\{s\in[0,u]\}} \\
& + \Bigg\{ \int_0^\infty \int_0^\infty (K - e^{\log u - y}) r(s,u,z)\Pi(-z-dy)dz \\
& + (K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(\omega_u^\alpha)}_{\alpha}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha}\mathscr{W}^{(\omega_u^\alpha)}_{\alpha}\left(\frac{s}{u}\right)\right)\right)\Bigg\}\mathbb{1}_{\{s>u\}}.
\end{aligned}
\tag{1.24}
$$

From a practical perspective, formula (1.24) is much easier to handle than (1.23). In other words, under certain assumptions, the lower bound of the optimal stopping region $l = 0$ and then (1.24) reduces to (1.23), which is a much simpler form and more convenient to generate numerical examples.

1.6 Literature Overview

Let us recall that the main goal of this thesis is to find a closed-form expression of (1.21) for different stock price processes S_t and discount functions ω.

As we mentioned in Introduction, our primary approach to this problem is the assumption of a robust and functional dependence between the discount function and the asset price process. In particular, we take a closer look at the discount function, which has the opposite monotonicity to the payoff function. At first sight, such a case seems counter-intuitive because, for the put option, if the asset price is in a higher region, one can expect that the interest rate will be lower and the opposite effect one expects for a smaller range of asset prices. This dependence somehow balances the discount function with the payoff function. On the other hand, we can think of an investor who has strong confidence in the movement of the asset price and wishes to make an extra profit when he/she is right and suffers a more significant loss when he/she is wrong. This concept resembles an idea that stands behind barrier options. If the investor believes that it is unlikely that the asset price will hit a given level, he/she can add a knock-out provision with the barrier set at the support level to reduce the price of an option. Including the barrier provision can eliminate paying for these scenarios that he/she feels are improbable. In our approach, we work in two ways by reducing the premium thanks to improbably incidents from the investor's perspective and increasing it for scenarios that are more likely to happen for him/her. Such a description of the analysed option adequately describes a financial instrument tailored to the risky investor.

Let us list here, for example, an up-and-out step put option analysed by [106, Formula (2.6b), p. 60]. It is a particular case of our option, except that it is of a European type, not an American one, but its mechanism of action is very similar to the one we consider. Another example in which asset-dependent discounting is considered is a so-called gold loan (see [39] for the survey related to this financial instrument). This contract is characterised as follows: a borrower receives at time 0 (the date of the contract inception) a loan amount $K > 0$ using one mass unit (one troy ounce) of gold as a collateral, which must be physically delivered to a lender. This amount grows at the functional borrowing rate given in the contract that can depend on the gold spot price $\bar{S}_t$. When repaying the loan, the borrower can redeem the gold at any time and the contract is terminated. Of course, the dynamic of $\bar{S}_t$ under the risk-neutral measure is such that the discounted price $e^{-rt}\bar{S}_t$ is a martingale, that is $\mathbb{E}_s\left[\bar{S}_t\right] = e^{rt}\mathbb{E}_s\left[\bar{S}_0\right]$. Assuming that the storage costs are equal to the borrowing rate plus some fixed cost $c > 0$ per unit of time and that the borrowing rate is a function $\bar{\omega}$ of the gold spot price increased by this fixed cost $\bar{S}_t e^{ct}$, the value of this contract, with an infinite maturity date, at time 0 equals

$$\begin{aligned}
&\sup_{\tau\in\mathcal{T}}\mathbb{E}_s\left[e^{-r\tau}\left(\bar{S}_\tau e^{\int_0^\tau \bar{\omega}(\bar{S}_w e^{cw})dw + c\tau} - Ke^{\int_0^\tau \bar{\omega}(\bar{S}_w e^{cw})dw}\right)^+\right] \\
&\qquad = \sup_{\tau\in\mathcal{T}}\mathbb{E}_s\left[e^{-\int_0^\tau \omega(S_w)dw}\left(S_\tau - K\right)^+\right],
\end{aligned} \tag{1.25}$$

where $S_t = \bar{S}_t e^{ct}$ and $\omega(S_t) = r - \bar{\omega}(S_t)$. As we can see from formula (1.25), such an instrument is equivalent to a call option for the problem we analyse in this thesis.

Our dissertation seems to be the first to analyse the optimal problem of the form (1) in this generality for jump-diffusion processes. For classical diffusion processes, Lamberton in [101] proved that the value function given in (1) is continuous and can be characterised as the unique solution to a variational inequality in the sense of distributions. Another crucial paper for our considerations is [18] which introduced discounting via a positive continuous additive functional of the process S_t and used the approach of Itô and McKean [84, Section 4.6, p. 128] to characterise the value function. We can see that $t \to \int_0^t \omega(S_w)dw$ is indeed an additive functional. A similar problem was also considered in [108], where the authors developed an average problem approach to prove the optimality of threshold type strategies for optimal stopping of Lévy models with continuous additive functional discounting. If we take $\omega(s) = (\log s - \log K)^+$ for a strike K, then $\int_0^t \omega(S_w)dw = \int_0^t (X_w - \log K)^+ dw$, which is equal to the area under the trajectory of $(X_t - \log K)^+$. Therefore, in this special case, we can talk about the so-called area options, see [51] for details.

Another interesting paper by Rodosthenous and Zhang [124] concerns the optimal stopping of an American call option in a random time horizon under an exponential spectrally negative Lévy model. An omega default clock models the random time horizon. In their case, the first time the occupation time of the asset price below a fixed level y exceeds an independent exponential random variable with mean $\frac{1}{\varrho}$. This corresponds to the special case of our discounting with $\omega(s) = r + \varrho\mathbb{1}_{\{s\leq y\}}$, where r is a risk-free interest rate. Similar discounting was analysed in [60] where American step options were considered. In this case $\omega(s) = \varrho\mathbb{1}_{\{s\in A^\pm(H)\}}$, where $A^\pm(H) = \{s > 0 : \pm(s - H) \geq 0\}$ with H being a constant barrier. Furthermore, the payoff function of a step option is the same as the payoff of a vanilla option, except that a factor

$e^{-\int_0^t \omega(S_w)dw}$ deflates it when the knock-out rate ϱ is positive or inflates it by the same factor when the knock-in rate ϱ is negative. In both cases, this factor depends exponentially on the cumulative excursion time above or below a given barrier for the entire lifetime of the option. This exponential functional can be interpreted as a knock-out (knock-in) discount factor. Step options, as they are modifications of barrier options and belong to the group of non-standardised financial products, are mainly traded over-the-counter.

The option we consider in this thesis also has this feature and can be used in direct transactions between two parties without the supervision of an exchange. The idea of step options can be further generalised in that the discount function ω can be treated in real options as a more general knock-out (knock-in) factor. One of the advantages of this functional discounting is that an option buyer can customise the option by selecting an appropriate functional rate according to his risk aversion and the degree of confidence in what the asset price will look like during the whole option life. From a risk management perspective, we can still hedge this option by trading the underlying asset. We can also identify the value of the contract. Additionally, since different market participants can select different discount rates, short-term manipulation by traders is substantially more difficult. Therefore, considering such options may help reduce market volatility, as noted in [106].

The pricing technique developed in this thesis can be applied to a wide range of financial contracts where the discounting in the above vein is affected by the underlying asset price process. Apart from the options mentioned earlier, one can consider Executive Stock Options (ESOs), in which the executive may exercise the ESO prematurely and leave the firm if an interesting opportunity arises or for diversification or liquidity reasons. Therefore, this policy can be determined by publicly available information, such as stock prices. As Carr and Linetsky [33] noted, this option corresponds to $\omega(s) = \lambda_f + \lambda_e \mathbb{1}_{\{s>K\}}$ or $\omega(s) = \lambda_f + \lambda_e \mathbb{1}_{\{\log s > \log K\}}$, where λ_f is a constant intensity of early exercise or forfeiture due to exogenous voluntary or involuntary employment termination and λ_e is the constant intensity of early exercise due to the executive's exogenous desire for liquidity or diversification. Another relevant application concerns R&D projects. Here, the probability of success before a competitor can depend on the ability of the firm to invest resources in the discovery process. If performance is poor, for example, due to mismanagement, the company does not invest resources in the discovery process. In the opposite scenario, more resources are devoted to research activities. Therefore, the price of this type of project depends on the path-dependent discounting as well, see [140] for a survey.

The convexity of the value function and the convexity preserving property, which is a crucial component of our analysis, have been studied quite extensively, see e.g. [20, 21, 36, 67, 70, 82, 85, 90] for diffusion models and [69, 86] for one-dimensional jump-diffusion models.

We model asset price dynamics in a financial market by the jump-diffusion process. Based on the empirical observations, we conclude that this class of stochastic processes modelling asset prices is more reasonable than the one used in the seminal Black-Scholes model. The logarithmic prices of stocks are more skew, asymmetric and have a heavier left tail than the normal distribution, on which the seminal Black-Scholes model is founded. The introduction of jumps in the financial market dates back to [113], who added a compound Poisson process to the standard Brownian motion to accurately describe the dynamic of the logarithm of stocks. Since then, there have been many papers and books working in this set-up, see [42, 133] and references therein. In particular, [42, Table 1.1, p. 29] gives many other ample reasons to consider this type of market. In addition to the classical Black-Scholes market, one can consider the normal inverse Gaussian model of [12], the hyperbolic model of [63], the variance gamma model of [109], the CGMY model of [34] and the tempered stable process analysed in [29, 93]. Many papers have also studied American options in the jump-diffusion markets, see [1, 3, 7, 16, 29, 37, 91, 116].

Identifying the solution of the optimal stopping problem by solving the corresponding HJB equation (as is done in this thesis as well) has been widely used in the literature, see [97, 120] for details. In the context of American options with the constant discount function, both methods of variational inequalities and viscosity solutions to boundary value problems in the spirit of Bensoussan and Lions [19] are also well known, see [102, 121, 122].

The smooth fit condition is usually applied to determine the unknown boundaries of the stopping region, see e.g. [100, 102] for the exponential Lévy process of asset prices. As Lamberton and Mikou [102] and Kyprianou and Surya [100] showed, the continuous fit condition is always satisfied, but not necessarily the smooth fit property. Therefore, we prove that the appropriate regularities of the process S_t at the critical points mentioned above give smooth paste conditions. It represents a generalisation of the classical results derived by [100, 102]. We want to emphasise here that using our approach (proving the convexity of the value function and maximising it over the ends l and u of the stopping interval $[l, u]$) allows us to avoid identifying critical points via smooth paste conditions.

Apart from this, the interval form of the stopping region (hence producing a double-sided continuation region) is much rarer. It might come, for example, from the fact that when at time $t = 0$ the discount rate is negative, it is worth waiting, since discounting might increase the profit of such an option. This phenomenon has already been observed for fixed negative discounting, see [13, 14, 15, 53, 145], or in the case of American capped options with a positive interest rate, see [30, 59]. Therefore, in this case, one can obtain a *double continuation region*.

In this thesis, we also prove that in this general setting of asset-dependent discounting, the price of the call option can be expressed in terms of the price of the put option. It is called a put-call symmetry. Our finding supplements [64, 74] which extends to the Lévy market the findings of [32]. Moreover, an analogous negative discount rate case result was obtained in [13, 14, 15, 53]. A comprehensive review of the put-call duality for American options is given in [57]. We also refer to [58, Section 7, p. 480] and other references therein for a general survey on the American options in the jump-diffusion models.

1.7 Notation

Let us introduce a set $E \subset \mathbb{R} \times [0, T]$. We use the following notation

- $C_\alpha(E)$ is the set of locally Hölder(α) functions with $\alpha \in (0, 1)$,
- $C_{\text{pol}}(E)$ is the set of functions of at most polynomial growth,
- $C^{p,q}(E)$ is the set of functions for which all derivatives $\frac{\partial^k}{\partial s^k}\left(\frac{\partial^l f(s,t)}{\partial t^l}\right)$ with $|k| + 2l \le p$ and $0 \le l \le q$ exist in the interior of E and have continuous extensions to E,
- $C^{p,q}_\alpha(E)$ and $C^{p,q}_{\text{pol}}(E)$ are the sets of functions $f \in C^{p,q}(E)$ for which all derivatives $\frac{\partial^k}{\partial s^k}\left(\frac{\partial^l f(s,t)}{\partial t^l}\right)$ with $|k| + 2l \le p$ and $0 \le l \le q$ belong to $C_{\text{pol}}(E)$ and $C_\alpha(E)$, respectively.

Furthermore, in many places in the thesis, we use the processes S_t and X_t interchangeably, making use of the fact that $S_t = e^{X_t}$. Then, if the process S_t occurs at the expected value, we mark it with the subscript, that is $\mathbb{E}_s[\ldots]$ to indicate that $S_0 = s$, while for the process X_t, we write $\mathbb{E}_{(x)}[\ldots]$ to specify that $X_0 = x$. Both of these formulas are equivalent. The first of the listed expectations corresponds to the measure $\mathbb{P}_s$, while the second corresponds to $\mathbb{P}_{(x)}$. When we use the symbol $\mathbb{P}$ we understand it as $\mathbb{P}_1$ (or equivalently $\mathbb{P}_{(0)}$), the anagolic designation we use for the symbol $\mathbb{E}$.

Chapter 2

American options with asset-dependent discounting

This chapter contains a precise formulation of our problem and presents in detail our pricing approach of American put options with asset-dependent discounting. First, we introduce a class of jump-diffusion stochastic processes together with all assumptions used throughout the thesis. For this general set-up, we prove several facts leading to the main theorem, that is Theorem 3. Firstly, we prove a significant Theorem 1, concerning the convexity of the value function analysed. As we mentioned in Introduction, the convexity property of the value function underlies our approach to option pricing, as it allows us to define the form of the optimal stopping region. In the next step, we prove Theorem 2 about the optimal stopping rule and then deduce the form of the stopping region. Then, we formulate a principal Theorem 3 that presents a closed-form expression of $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u)$. Maximising it over l and u leads to the final form of the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$. In Theorem 4 and Theorem 5 we give particular expressions of $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u)$ for the Black-Scholes model and the exponential Lévy process with downward exponential jumps. In particular, we note that in the latter case, the value function consists of the ξ-scale functions introduced in the previous chapter. In Theorem 6, we show that these functions are solutions of specific second- or third-order differential equations. The classic approach via the HJB system works as well in our set-up. Moreover, the appropriate regularities of the asset price process imply smooth paste conditions for the value function. These facts are stated in Theorem 7. Finally, we present the put-call parity, which shows the relation between the price of a call option and a put option, as indicated in Theorem 8. Proofs of all theorems stated in this chapter are available in Chapter 4.

2.1 Jump-diffusion process

In this thesis, we operate on a class of jump-diffusion processes. Therefore, we introduce here some definitions and assumptions that are used throughout the dissertation. We assume a jump-diffusion financial market, which is defined as follows. On a complete filtered risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with natural filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions, we define $\mathbb{F}$-adapted couple (B_t, v), where B_t is a standard Brownian motion and $v = v(dt, dz)$ is a homogeneous Poisson random measure on $\mathbb{R}^+_0 \times \mathbb{R}$ for $\mathbb{R}^+_0 = [0, \infty)$, which is independent of B_t. Then the stock price process S_t solves the following stochastic differential equation

$$dS_t = \mu(S_{t-}, t)dt + \sigma(S_{t-}, t)dB_t + \int_{\mathbb{R}} \gamma(S_{t-}, t, z)\tilde{v}(dt, dz), \tag{2.1}$$

where

- $\tilde{v}(dt, dz) = (v - q)(dt, dz)$ is a compensated jump martingale random measure of v,
- v is a homogenous Poisson random measure defined on $\mathbb{R}_0^+ \times \mathbb{R}$ with intensity measure

$$q(dt, dz) = dt\, \Pi(dz).$$

We additionally assume that the jump-diffusion process has a finite activity of jumps, that is

$$\lambda := \int_{\mathbb{R}} \Pi(dz) < \infty,$$

where Π is a Lévy measure. Then $N_t = v([0, t] \times \mathbb{R})$ is a Poisson process and Π can be represented as

$$\Pi(dz) = \lambda \mathbb{P}\left(e^{Y_i} - 1 \in dz\right),$$

where $\{Y_i\}_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables, independent of N_t, with distribution μ_Y. Note that B_t and N_t are also independent of each other.

We note that if $\mu(s, t) = \mu s$, $\sigma(s, t) = \sigma s$ and $\gamma(s, t, z) = sz$, then the asset price process S_t is an exponential Lévy process, that is

$$S_t = e^{X_t}, \tag{2.2}$$

where X_t is a Lévy process that starts at $x = \log s$ with a triple (ζ, σ, Π) for

$$\zeta = \mu - \frac{\sigma^2}{2} \quad \text{and} \quad \Pi(dz) = \lambda \mu_Y(dz). \tag{2.3}$$

This observation follows directly from the Itô's lemma.

2.1.1 Assumptions

Assumptions about the model parameters used throughout this dissertation are as follows.

Assumption (A).

(A1) The drift parameter μ: $\mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ and the diffusion parameter σ: $\mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ are continuous functions, while the jump size γ: $\mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is measurable and for each fixed $z \in \mathbb{R}$, the function $(s, t) \to \gamma(s, t, z)$ is continuous.

(A2) There exists a constant $C > 0$ such that

$$\mu^2(s, t) + \sigma^2(s, t) + \gamma^2(s, t, z) \le Cs^2$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A3) There exists a constant $C > 0$ such that

$$|\mu(s_2, t) - \mu(s_1, t)| + |\sigma(s_2, t) - \sigma(s_1, t)| + |\gamma(s_2, t, z) - \gamma(s_1, t, z)| \le C|s_2 - s_1|$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A4) There exists a constant $C > -1$ such that

$$\gamma(s, t, z) > Cs$$

for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

(A5) $g(s) \in C_{pol}(\mathbb{R}^+)$, *where* $C_{pol}(\mathbb{R}^+)$ *denotes the set of functions of at most polynomial growth.*

(A6) $V_{\mathrm{A}}^{\omega}(s) < \infty$ *for all* $s \in \mathbb{R}^+$.

The Assumptions (A1), (A2), (A3) guarantee that there exists a unique solution to (2.1). Moreover, (A2) and (A4) imply that

$$\mathbb{P}\left(S_t \leq 0 \text{ for some } t \in \mathbb{R}_0^+\right) = 0$$

which is a natural assumption, since the process S_t describes the stock price dynamic, so its value must be positive. Additionally, to make the pricing problem well-defined, we assume (A6). To recall, the form of $V_{\mathrm{A}}^{\omega}(s)$ is defined in (1). We do not provide here the necessary conditions for ω and g that guarantee (A6), we focus only on the finiteness of $V_{\mathrm{A}}^{\omega}(s)$. However, it can be easily shown that $\omega \geq 0$ and the boundedness of g are sufficient conditions for (A6) to hold.

Remark 2. Note that Assumptions (A1)–(A4) are all satisfied for the exponential Lévy process given in (2.2).

If we talk about convexity and concavity in this thesis, we mean it in a weak sense, allowing these functions to be constants within some regions.

2.2 Convexity

Our first crucial result concerns the convexity of the value function $V_{\mathrm{A}}^{\omega}(s)$.

Theorem 1. *Let Assumptions (A) hold. Assume that the payoff function g is convex, ω is concave, the stock price process S_t follows* (2.1)*, and the following inequalities are satisfied*

$$\frac{\partial^2 \gamma(s,t,z)}{\partial s^2}\gamma(s,t,z) \geq 0, \tag{2.4}$$

$$\left(\frac{\partial^2 \mu(s,t)}{\partial s^2} - 2\frac{d\omega(s)}{ds}\right)\frac{\partial V_{\mathrm{E}}^{\omega}(s,t)}{\partial s} - \frac{d^2\omega(s)}{ds^2}V_{\mathrm{E}}^{\omega}(s,t) \geq 0, \tag{2.5}$$

where $V_{\mathrm{E}}^{\omega}(s,t)$ is defined in (2)*. Then the value function $V_{\mathrm{A}}^{\omega}(s)$ is convex as a function of s.*

Remark 3. We now give sufficient conditions in terms of the model parameters for (2.5) to be satisfied. If S_t is the exponential Lévy process (hence $\mu(s,t) = \mu s$, $\sigma(s,t) = \sigma s$ and $\gamma(s,t,z) = sz$) then (2.4) is satisfied. Also, let $g(s) = (K - s)^+$ and, therefore, consider the value function $V_{\mathrm{A}^{\mathrm{Put}}}^{\omega}(s)$ defined in (1.21). If ω is a non-decreasing function, then the function $s \to V_{\mathrm{E}}^{\omega}(s,t)$ is non-increasing. Moreover, the concavity of ω implies that the second term in (2.5) is non-negative. Combining all these conditions, we can conclude that (2.5) is satisfied.

Remark 4. We stated the above assumptions to look at the put option. We can simply note that an analogous approach can be applied to the call option.

2.2.1 Perpetual American put option

Assume now the particular case of (1) with the payoff function of the form

$$g(s) = (K - s)^+.$$

Then, instead of $V^{\omega}_{\mathrm{A}}(s)$ we use the expression $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ and interpret it as the value function of a perpetual American put option with asset-dependent discounting. It is given by

$$V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_s\left[e^{-\int_0^\tau \omega(S_w)dw}(K - S_\tau)\right]. \tag{2.6}$$

Note that above we used the fact that the option will not be realised when it equals zero. Therefore, the plus sign in the payoff function could be skipped.

From now on, we will focus only on the asset price modelled by a spectrally negative exponential Lévy process, that is

$$S_t = e^{X_t}, \tag{2.7}$$

where X_t is a spectrally negative Lévy process starting at $x = \log s$. This means that X_t does not have positive jumps, which corresponds to the inclusion of the support of the Lévy measure Π on the negative half-line. It is a prevalent assumption used in financial mathematics that faithfully reflects the behaviour of stock prices, see [3, 8]. One can easily observe that the dual case of the spectrally positive Lévy process X_t can be treated similarly. We decided to skip this analysis and focus only on a more predominant, from a practical perspective, spectrally negative scenario.

2.3 Optimal stopping time

To solve the optimal stopping problem given in (2.6), we need to determine the optimal stopping rule. From the general theory of optimal stopping, see [120, Chapter III, p. 122], we can conclude that the optimal stopping rule is the first time when the value function is equal to the payoff function[1], that is

$$\tau^* = \inf\{t \geq 0 : V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(S_t) = K - S_t\}.$$

From Theorem 1 we know that $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ is convex. Moreover, from the definition of the value function it follows that $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s) \geq K - s$. Taking these facts into account, together with the linearity of the payoff function, it follows that $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ and g can cross each other in at most two points. This observation leads straight to the conclusion about the form of the stopping region, which is stated in Theorem 2. We recall that in (4) and (3) we introduced the entry time into the interval $[l, u]$, that is

$$\tau_{l,u} = \inf\{t \geq 0 : S_t \in [l, u]\}$$

and the corresponding value function

$$v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u) = \mathbb{E}_s\left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(K - S_{\tau_{l,u}})\right],$$

where $0 \leq l \leq u \leq K$.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then the value function given in* (2.6) *is equal to*

$$V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s) = v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l^*, u^*),$$

where

$$v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l^*, u^*) = \sup_{0\leq l\leq u\leq K} v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u).$$

The optimal stopping rule is τ_{l^*,u^*}, where l^*, u^* realise the supremum above.

[1]More precisely, one can observe that [120, Formula (6.3.1), p. 127] is equivalent to [120, Formula (6.0.1), p. 124] that can be understood as [120, Formula (2.2.3), p. 36] as noted in [120, p. 125].

Remark 5. Another characterisation of critical points l^* and u^* via a smooth fit property is given in Theorem 7.

Theorem 2 indicates that the optimal stopping rule in our problem is the first time the process S_t enters the interval $[l^*, u^*]$ for some $l^* \leq u^*$. When $l^* = u^*$, the interval becomes a point. In some cases, the observation above allows us to identify the value function in a much more transparent way. Finally, note that if the discount function ω is non-negative, then it is never optimal to wait to exercise the option for sufficiently small asset prices. In other words, it means that $l^* = 0$ and that the stopping region is one-sided.

2.4 Main result

The main result of this thesis is the closed-form expression of the function $v^{\omega}_{\mathrm{A^{Put}}}(s, l, u)$, as presented in Theorem 3.

Theorem 3. *Assume that the stock price process S_t is described by* (2.7) *and ω is a measurable, bounded from below, concave and non-decreasing function such that*

$$\omega(s) = c \text{ for all } s \in (0, 1] \text{ and some constant } c \in \mathbb{R}. \tag{2.8}$$

Then

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s, l, u) &= \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(l)}(K - l)\mathbb{1}_{\{s<l\}} + (K - s)\mathbb{1}_{\{s\in[l,u]\}} \\ &+ \Bigg\{ \int_0^\infty \int_0^\infty \frac{\mathscr{H}^{(\omega_u)}((\frac{u}{e^y}) \wedge l)}{\mathscr{H}^{(\omega_u)}(l)}(K - e^{\log l \vee (\log u - y)})r(s, u, z)\Pi(-z - dy)dz \\ &+ (K - u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(\omega_u^\alpha)}_\alpha\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha}\mathscr{W}^{(\omega_u^\alpha)}_\alpha\left(\frac{s}{u}\right)\right)\right)\Bigg\}\mathbb{1}_{\{s>u\}}, \end{aligned}$$

where

$$c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha} = \lim_{z\to\infty}\frac{\mathscr{Z}^{(\omega^\alpha)}_\alpha(z)}{\mathscr{W}^{(\omega^\alpha)}_\alpha(z)}$$

and $r(s, u, z)$ is given in (1.20).
If $l = 0$ then condition (2.8) *is superfluous and*

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s, 0, u) &= (K - s)\mathbb{1}_{\{s\in[0,u]\}} \\ &+ \Bigg\{ \int_0^\infty \int_0^\infty (K - e^{\log u - y})r(s, u, z)\Pi(-z - dy)dz \\ &+ (K - u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(\omega_u^\alpha)}_\alpha\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega^\alpha)}_\alpha/\mathscr{W}^{(\omega^\alpha)}_\alpha}\mathscr{W}^{(\omega_u^\alpha)}_\alpha\left(\frac{s}{u}\right)\right)\right)\Bigg\}\mathbb{1}_{\{s>u\}}. \end{aligned}$$

Remark 6. For the general case where $l > 0$, condition (2.8) is a technical one and is a consequence of the assumption made in [105, Theorem 2.5, p. 3279], which is used in the proof. However, this assumption is probably superfluous.

2.4.1 Black-Scholes model

We can give a more detailed analysis in the case of the Black-Scholes model. The stock price process is of the form $S_t = e^{X_t}$, where

$$X_t = \log s + \zeta t + \sigma B_t \tag{2.9}$$

with $\zeta = \mu - \frac{\sigma^2}{2}$, while $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are the model parameters. Under the martingale measure, the drift parameter $\mu = r$, where r is a risk-free interest rate.

Theorem 4. *Assume that ω is a bounded from below, concave and non-decreasing function. For the Black-Scholes model with X_t given in* (2.9)*, the function $v^{\omega}_{\text{A}^{\text{Put}}}(s,l,u)$ defined in* (3) *is given by*

$$\begin{aligned} v^{\omega}_{\text{A}^{\text{Put}}}(s,l,u) = &\frac{h(s)}{h(l)}(K-l)\mathbb{1}_{\{s<l\}} + (K-s)\mathbb{1}_{\{s\in[l,u]\}} \\ &+ \frac{h(s)}{h(u)}(K-u)\mathbb{1}_{\{s>u\}}, \end{aligned}$$

where $h(s)$ is a solution to

$$\frac{\sigma^2 s^2}{2}h''(s) + rsh'(s) - \omega(s)h(s) = 0, \tag{2.10}$$

which satisfies

$$\begin{cases} h(s) = K - s, & s \in [l^*, u^*], \\ \lim_{s\to\infty} h(s) = \text{const}. \end{cases} \tag{2.11}$$

Remark 7. The optimal limits l^* and u^* can be found from the smooth fit property given in Theorem 7.

2.4.2 Lévy exponential jumps

We can construct a more explicit form of the function $v^{\omega}_{\text{A}^{\text{Put}}}(s,l,u)$ for the exponential Lévy process with downward exponential jumps. In this case, the stock price process is given by $S_t = e^{X_t}$ for

$$X_t = \log s + \zeta t + \sigma B_t - \sum_{i=1}^{N_t} Y_i, \tag{2.12}$$

where $\zeta = \mu - \frac{\sigma^2}{2}$, while $\mu \in \mathbb{R}$ and $\sigma \geq 0$. Furthermore, N_t is the Poisson process with intensity $\lambda > 0$, independent of Brownian motion B_t, and $\{Y_i\}_{i\in\mathbb{N}}$ is a sequence of i.i.d. exponentially distributed random variables with mean $\frac{1}{\rho} > 0$, independent of B_t and N_t. Furthermore, under the martingale measure, the drift parameter $\mu = r + \frac{\lambda}{\rho+1}$ with r being a risk-free interest rate. We recall that the Laplace exponent of X_t starting at 0 is as follows

$$\psi(\theta) = \zeta\theta + \frac{\sigma^2}{2}\theta^2 - \frac{\lambda\theta}{\rho+\theta}. \tag{2.13}$$

Theorem 5. *Assume that ω is a non-negative, concave and non-decreasing function. For the exponential Lévy model with X_t given in* (2.12)*, we have $l = 0$. Furthermore,*
(i) if $\sigma = 0$ and $\lambda > 0$ then

$$v^{\omega}_{\mathrm{APut}}(s, 0, u) = \left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right), \tag{2.14}$$

(ii) if $\sigma > 0$ and $\lambda = 0$ then

$$v^{\omega}_{\mathrm{APut}}(s, 0, u) = (K - u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}}\mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right), \tag{2.15}$$

(iii) if $\sigma > 0$ and $\lambda > 0$ then

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s, 0, u) = &\left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right) \\ &+ (K - u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}}\mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right), \end{aligned} \tag{2.16}$$

where

$$c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}} = \lim_{z\to\infty}\frac{\mathscr{Z}^{(\omega)}(z)}{\mathscr{W}^{(\omega)}(z)} \quad \text{and} \quad c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}} = \lim_{z\to\infty}\frac{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}(z)}{\mathscr{W}_{\alpha}^{(\omega^{\alpha})}(z)}. \tag{2.17}$$

The optimal boundary u^* in (2.15) *and* (2.16) *can be determined by the smooth fit condition*

$$(v^{\omega}_{\mathrm{APut}})'(u^*, 0, u^*) = -1,$$

while the optimal boundary u^* in (2.14) *can be determined by the continuous fit condition*

$$v^{\omega}_{\mathrm{APut}}(u^*, 0, u^*) = K - u^*.$$

From (1.4) (see also [119, Proposition 5.6, p. 782]) we simply note that the Laplace exponent of X_t taken under $\mathbb{P}^{(\alpha)}_{(0)}$ is of the same form as (2.13), that is

$$\psi^{(\alpha)}(\theta) = \zeta^{(\alpha)}\theta + \frac{{\sigma^{(\alpha)}}^2}{2}\theta^2 - \frac{\lambda^{(\alpha)}\theta}{\rho^{(\alpha)} + \theta}, \tag{2.18}$$

where $\zeta^{(\alpha)} = \zeta + \sigma^2\alpha$, $\sigma^{(\alpha)} = \sigma$, $\lambda^{(\alpha)} = \frac{\lambda\rho}{\rho+\alpha}$ and $\rho^{(\alpha)} = \rho + \alpha$. Therefore, finding the scale functions under $\mathbb{P}_{(0)}$ and $\mathbb{P}^{(\alpha)}_{(0)}$ works in the same way. To do so, we recall that in (1.14) and (1.15) we introduced them via regular ξ-scale functions, that is $\mathscr{W}^{(\xi)}(s) = \mathcal{W}^{(\xi\circ\exp)}(x)$ and $\mathscr{Z}^{(\xi)}(s) = \mathcal{Z}^{(\xi\circ\exp)}(x)$ for $x = \log s$. Therefore, to identify closed-forms of (2.14), (2.15) and (2.16) it suffices to find ξ-scale functions $\mathcal{W}^{(\xi)}(x)$ and $\mathcal{Z}^{(\xi)}(x)$ for a given generic function ξ. We recall that both ξ-scale functions are given as solutions of the renewal equations (1.10) and (1.11) formulated in terms of the classical scale function $W(x)$.

From the definition of the first scale function given in (1.7) with $q = 0$ and from (2.13) with $\sigma > 0$, we derive the following

$$W(x) = \sum_{i=1}^{3}\Upsilon_i e^{\gamma_i x},$$

where γ_i solves

$$\psi(\gamma_i) = 0 \tag{2.19}$$

and

$$\Upsilon_i = \frac{1}{\psi'(\gamma_i)}.$$

Note that one of the solutions to (2.19) equals 0, so we can set $\gamma_1 = 0$. In turn, if $\sigma = 0$ in (2.13), then

$$W(x) = \sum_{i=1}^{2} \Upsilon_i e^{\gamma_i x}$$

with $\gamma_1 = 0$, $\gamma_2 = \frac{\lambda - \rho\zeta}{\zeta}$, $\Upsilon_1 = -\frac{\rho}{\lambda - \rho\zeta}$ and $\Upsilon_2 = \frac{\lambda}{\zeta(\lambda - \rho\zeta)}$. Theorem 6 shows that the ξ-scale functions $\mathcal{W}^{(\xi)}(x)$ and $\mathcal{Z}^{(\xi)}(x)$ satisfy specific ordinary differential equations.

Theorem 6. *We assume that the function ξ is continuously differentiable. For the exponential Lévy model with X_t given in* (2.12) *we have*
(i) If $\sigma = 0$ and $\lambda > 0$ or $\lambda = 0$ and $\sigma > 0$, then $\mathcal{W}^{(\xi)}(x)$ solves

$$\mathcal{W}^{(\xi)''}(x) = ((\Upsilon_1 + \Upsilon_2)\xi(x) + \gamma_2)\,\mathcal{W}^{(\xi)'}(x) + \big((\Upsilon_1 + \Upsilon_2)\xi'(x) - \gamma_2\Upsilon_1\xi(x)\big)\,\mathcal{W}^{(\xi)}(x) \tag{2.20}$$

with

$$\begin{cases} \mathcal{W}^{(\xi)}(0) = \Upsilon_1 + \Upsilon_2, \\ \mathcal{W}^{(\xi)'}(0) = (\Upsilon_1 + \Upsilon_2)^2\xi(0) + \Upsilon_2\gamma_2. \end{cases} \tag{2.21}$$

Moreover, the function $\mathcal{Z}^{(\xi)}(x)$ solves the same equation (2.20) *with*

$$\begin{cases} \mathcal{Z}^{(\xi)}(0) = 1, \\ \mathcal{Z}^{(\xi)'}(0) = (\Upsilon_1 + \Upsilon_2)\xi(0). \end{cases} \tag{2.22}$$

(ii) If $\sigma > 0$ and $\lambda > 0$, then the function $\mathcal{W}^{(\xi)}(x)$ solves

$$\begin{aligned} \mathcal{W}^{(\xi)'''}(x) = {} & (\gamma_2 + \gamma_3)\,\mathcal{W}^{(\xi)''}(x) \\ & + (\Upsilon_2(\gamma_2 - \gamma_3)\xi(x) - \gamma_2\gamma_3 - \gamma_3\Upsilon_1\xi(x))\,\mathcal{W}^{(\xi)'}(x) \\ & + \big(\Upsilon_2(\gamma_2 - \gamma_3)\xi'(x) + \gamma_2\gamma_3\Upsilon_1\xi(x) - \gamma_3\Upsilon_1\xi'(x)\big)\,\mathcal{W}^{(\xi)}(x) \end{aligned} \tag{2.23}$$

with

$$\begin{cases} \mathcal{W}^{(\xi)}(0) = 0, \\ \mathcal{W}^{(\xi)'}(0) = \Upsilon_2\gamma_2 + \Upsilon_3\gamma_3, \\ \mathcal{W}^{(\xi)''}(0) = \Upsilon_2{\gamma_2}^2 + \Upsilon_3{\gamma_3}^2. \end{cases} \tag{2.24}$$

Moreover, the function $\mathcal{Z}^{(\xi)}(x)$ solves the same equation (2.23) *with*

$$\begin{cases} \mathcal{Z}^{(\xi)}(0) = 1, \\ \mathcal{Z}^{(\xi)'}(0) = 0, \\ \mathcal{Z}^{(\xi)''}(0) = \Upsilon_2(\gamma_2 - \gamma_3)\xi(0) - \gamma_3\Upsilon_1\xi(0). \end{cases} \tag{2.25}$$

Remark 8. Note that in our case $l^* = 0$, so from Theorem 6 it follows that assumption (2.8) is not required.

2.5 Hamilton-Jacobi-Bellman equation

A classical approach via the Hamilton-Jacobi-Bellman equation is also possible in our set-up. More precisely, as before in (2.2), we consider $S_t = e^{X_t}$ for the Lévy process X_t that starts at $x = \log s$ with the triple (ζ, σ, Π).

We note that using [131, Theorem 31.5, p. 208] and Itô's lemma, one can conclude that the process S_t is a Markov process with an infinitesimal generator

$$\mathcal{A}f(s) = A^C f(s) + A^J f(s),$$

where A^C is a second-order linear differential operator of the form

$$A^C f(s) = \frac{\sigma^2 s^2}{2} f''(s) + \left(\zeta + \frac{\sigma^2}{2}\right) s f'(s)$$

and A^J is an integral operator given by

$$A^J f(s) = \int_{-\infty}^{0} \left(f(se^z) - f(s) - s|z|f'(s)\mathbb{1}_{\{|z|\leq 1\}}\right) \Pi(dz).$$

The domain $D(\mathcal{A})$ of this generator consists of the functions belonging to $C^2(\mathbb{R}^+)$ if $\sigma > 0$ and $C^1(\mathbb{R}^+)$ if $\sigma = 0$. In this dissertation, we prove that $V_{\mathrm{A}}^{\omega}(s)$ satisfies the HJB equation given in Theorem 7 with appropriate smooth fit conditions. We recall that 1 is regular for $(0, 1)$ and for the process S_t if $\mathbb{P}_1\left(\tau_{(0,1)} = 0\right) = 1$ for $\tau_{(0,1)} = \inf\{t \geq 0 : S_t \in (0, 1)\}$. Similarly, we can define the regularity for $(1, \infty)$. Lastly, we observe that the regularity of S_t at 1 corresponds to that of X_t at 0 for the negative or positive half-line.

Theorem 7. *Assume that the asset price is a spectrally negative exponential Lévy process* (2.7). *Let ω be a concave function bounded from below with the opposite monotonicity to the payoff function g. Assume that $V_{\mathrm{A}}^{\omega}(s) \in D(\mathcal{A})$ and $g(s) \in C^1(\mathbb{R}^+)$. Then $V_{\mathrm{A}}^{\omega}(s)$ uniquely solves the following HJB system*

$$\begin{cases} \mathcal{A}V_{\mathrm{A}}^{\omega}(s) - \omega(s)V_{\mathrm{A}}^{\omega}(s) = 0, & s \notin [l^*, u^*], \\ V_{\mathrm{A}}^{\omega}(s) = g(s), & s \in [l^*, u^*]. \end{cases} \tag{2.26}$$

Moreover, if 1 *is regular for* $(0, 1)$ *and for the process S_t, then there is a smooth fit at the right end of the stopping region*

$$(V_{\mathrm{A}}^{\omega})'(u^*) = g'(u^*).$$

Similarly, if 1 *is regular for* $(1, \infty)$ *and for the process S_t then there is a smooth fit at the left end of the stopping region*

$$(V_{\mathrm{A}}^{\omega})'(l^*) = g'(l^*).$$

Remark 9. Let us consider the put option. Then from Theorem 2 and Theorem 3, we can conclude that the smoothness of the value function $V_{\mathrm{A}^{\mathrm{Put}}}^{\omega}(s)$ corresponds to the smoothness of the ξ-scale functions for ω, ω_u and ω_u^{α} (defined in (1.19)). From the definitions of these functions given in (1.14), (1.15) and (1.16), it follows that the smoothness of the latter function is equivalent to the smoothness of the first scale function observed under measures $\mathbb{P}_{(0)}$ and $\mathbb{P}_{(0)}^{(\alpha)}$. By [99, Lemma 8.4, p. 222] the smoothness of the first scale function does not change under the exponential change of measure (1.3). Thus, from [40, Lemma 2.4 (p. 117), Theorem 3.10 (p. 136) and Theorem 3.11 (p. 140)], it follows that

- if $\sigma > 0$ then $V^{\omega}_{\mathrm{A_{Put}}}(s) \in C^2(\mathbb{R}^+)$,
- if $\sigma = 0$ and the jump measure Π is absolutely continuous or $\int_{-1}^{0} |z|\Pi(dz) = \infty$, then $V^{\omega}_{\mathrm{A_{Put}}}(s) \in C^1(\mathbb{R}^+)$.

Furthermore, by [3, Proposition 7, p. 11], 1 is regular for both $(0,1)$ and $(1,\infty)$ if $\sigma > 0$. Hence, HJB system (2.26) with the smooth fit property could be used without additional assumptions as long as $\sigma > 0$. If one has a single continuation region $[u^*, \infty)$ and $\sigma = 0$, then by [3, Proposition 7, p. 11] to get the smooth fit condition at u^*, it suffices to assume that the drift ζ of the process X_t is strictly negative.

2.6 Put-call parity

The put-call parity allows one to calculate the price of the American call option having the put option price. We formulate this relation again for S_t being a general exponential Lévy process defined in (2.2) with the Lévy triple (ζ, σ, Π) and $S_0 = s$. Apart from the following function

$$v^{\omega}_{\mathrm{A_{Put}}}(s, K, \zeta, \sigma, \Pi, l, u) := \mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw} (K - S_{\tau_{l,u}})^+ \right]$$

defined in (3), we denote

$$v^{\omega}_{\mathrm{A_{Call}}}(s, K, \zeta, \sigma, \Pi, l, u) := \mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw} (S_{\tau_{l,u}} - K)^+ \right].$$

Theorem 8. *Assume that $\psi(1) < \infty$. Let $0 \le l \le u \le K$. Then we have the following*

$$v^{\omega}_{\mathrm{A_{Call}}}(s, K, \zeta, \sigma, \Pi, l, u) = v^{\vartheta^{(1)}}_{\mathrm{A_{Put}}} \left(K, s, -\zeta, \sigma, \hat{\Pi}, \frac{lK}{s}, \frac{uK}{s} \right), \tag{2.27}$$

where

$$\hat{\Pi}(dx) = e^{-x}\Pi(-dx), \tag{2.28}$$
$$\vartheta^{(1)}(\cdot) = \omega \left(\frac{1}{\cdot} \frac{s}{K} \right) - \psi(1).$$

Moreover, if the assumptions of Theorem 1 hold for the function $\vartheta^{(1)}$ then the American call option admits a double continuation region with optimal stopping boundaries l^*_c and u^*_c such that

$$\frac{l^*}{l^*_c} = \frac{u^*}{u^*_c} = \frac{K}{s}, \tag{2.29}$$

where l^* and u^* are the stopping limits for the put option.

Remark 10. Note that the value function of the American call option is expressed in terms of the American put option calculated for the Lévy process $\hat{X}_t$ being dual to the process X_t observed under the measure $\mathbb{P}^{(1)}_{(\log K)}$. In particular, the jumps of $\hat{X}_t$ have a direction opposite to those of X_t, for which the put option is priced. In general, determining the conditions for ω such that $\vartheta^{(1)}$ satisfies all the assumptions of Theorem 1 seems severe, and then we can only work on a case-by-case basis.

Chapter 3

Examples

This chapter shows examples of a closed-form of $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ along with figures for different discount functions ω and asset price processes S_t. In some cases, these functions were obtained analytically, while for others the analytical formula could not be determined explicitly, so we proceeded numerically to generate the figures of the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$.

3.1 Pricing methodology

In Theorem 3 we state the exact form of $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u)$ for S_t given in (2.7), while in Theorem 4 and Theorem 5 we provide this formula for more specific cases of S_t. In this chapter, we use these theorems to represent the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ in the closed-form. For this purpose, we choose a specific form of the process S_t and the discount function ω for both the classical Black-Scholes model and the exponential Lévy process with downward exponential jumps. Then, in the case of Theorem 5, we still need to identify the generalised scale functions to determine the form of $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u)$. As we know from Theorem 6, they are the solutions of some ordinary differential equations. We present examples in which we can explicitly solve these equations and obtain analytical solutions of the generalised scale functions, as well as the function $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u)$. Then, maximising it with respect to the parameters l and u, we can derive the closed-form of $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$. We call this procedure an analytical approach.

On the other hand, in other situations, we solve the differential equations mentioned above numerically and generate figures of the generalised scale functions. It allows us to create the final figure of the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ for the optimal values of l and u. This procedure is called a numerical approach.

It is also worth highlighting that we used a numerical method for all analytical cases and compared the results obtained between these two approaches. The numerical part of our work is done in the *Python* programming language. We use the package *mpmath* for numerical calculations, which is a designated library for real and complex floating-point arithmetic with arbitrary precision. It allows us to derive very accurate numerical results. In addition, these results are indeed close to their analytical counterparts. A more detailed description of the numerical computation is provided in Subsection 3.1.2.

3.1.1 Analytical approach

As indicated above, in the analytical approach, we explicitly solve ordinary differential equations from Theorem 4 and Thereom 6. We can do this by selecting the appropriate function ω in

(2.10) and ξ in (2.20) and (2.23). In the first of these cases, the solution of (2.10) with boundary conditions (2.11) allows us to immediately obtain the form of $v^{\omega}_{\mathrm{A^{Put}}}(s,l,u)$. In some scenarios, it is possible that a double continuation region appears, e.g. when ω is negative.

In turn, in the second case, solving equations (2.20) and (2.23) with boundary conditions (2.21), (2.22), (2.24) and (2.25) allows us to obtain only forms of the generalised scale functions. Later, we still need to calculate constants (2.17) to derive $v^{\omega}_{\mathrm{A^{Put}}}(s,0,u)$. We note that in Theorem 5 it is assumed that ω is non-negative and therefore $l=0$.

3.1.2 Numerical approach

Analogously to the previous section, the numerical approach involves solving the same differential equations as above, but this time we use a numerical algorithm.

In general, solving a high-order ordinary differential equation consists of transforming it into a first-order vector form and then applying an appropriate algorithm that returns the numerical solution of the $n+1-$dimensional system of first-order ordinary differential equations.

For practical purposes, such as financial engineering problems, numerical approximations to the solutions of ordinary differential equations are often sufficient. In our case, we focus on the Higher-Order Taylor Method. This method uses the Taylor polynomial for the solution of the equation. Using the differential equation, it approximates the $0-$th-order term using the value of the previous step (the initial condition for the first step) and the subsequent terms of the Taylor expansion.

We perform all numerical calculations in the *Python* programming language using the *mpmath* library for arbitrary-precision floating-point arithmetic, which enables us to obtain results with arbitrarily high accuracy. The consequence of this is that the differences between the analytical and numerical results are negligible. We can manipulate them, which results in more or less computation time of our algorithm.

3.2 Black-Scholes model

Let us take the discount function ω of the form

$$\omega(s) = -\frac{C}{s+1} - D,$$

where C and D are positive constants. According to Theorem 4, we have

$$v^{\omega}_{\mathrm{A^{Put}}}(s,l,u) = \frac{h(s)}{h(l)}(K-l)\mathbb{1}_{\{s\in(0,l)\}} + (K-s)\mathbb{1}_{\{s\in[l,u]\}} + \frac{h(s)}{h(u)}(K-u)\mathbb{1}_{\{s\in(u,\infty)\}},$$

where h is a solution to

$$\frac{\sigma^2 s^2}{2}h''(s) + rsh'(s) + \left(\frac{C}{s+1} + D\right)h(s) = 0 \tag{3.1}$$

which satisfies

$$\begin{cases} h(s) = K - s, \quad s \in [l^*, u^*], \\ \lim\limits_{s\to\infty} h(s) = \text{const.} \end{cases} \tag{3.2}$$

First, we solve the above equation and identify the form of $h(s)$. Then, we look for boundaries l^* and u^* such that

$$v^{\omega}_{\mathrm{A^{Put}}}(s,l^*,u^*) = \sup_{0\le l\le u\le K} v^{\omega}_{\mathrm{A^{Put}}}(s,l,u).$$

We can find them by applying the smooth and continuous fit conditions. The general solution to (3.1) is given by

$$h(s) = K_1 s^{d_1} {}_2F_1(a_1, b_1, c_1; -s) + K_2 s^{d_2} {}_2F_1(a_2, b_2, c_2; -s), \tag{3.3}$$

where ${}_2F_1(\cdot, \cdot, \cdot; \cdot)$ is the Gaussian hypergeometric function. Moreover, $a_i = (-1)^{i+1}(M - G)$, $b_i = (-1)^i(M + G)$, $c_i = 1 + 2(-1)^i G$, $d_i = (-1)^i G + L$ for $i = 1, 2$ and K_1, K_2 are some constants, where $L = \frac{1}{2} - \frac{r}{\sigma^2}$, $M = \sqrt{L^2 - \frac{2D}{\sigma^2}}$, $G = \sqrt{L^2 - \frac{2(C+D)}{\sigma^2}}$.

Using formula (3.3) and the boundary conditions given in (3.2) we can identify the form of value function (2.6). Since we consider the negative function ω, we obtain a double continuation region. We take one of the summands from (3.3) for $s \in (0, l^*)$ and the second for $s \in (u^*, \infty)$. This choice is made in such a way that, on the given interval, we impose a greater function of these two. Hence, we derive

$$V^{\omega}_{\mathrm{A^{Put}}}(s) = \begin{cases} K_2 s^{d_2} {}_2F_1(a_2, b_2, c_2; -s), & s \in (0, l^*), \\ K - s, & s \in [l^*, u^*], \\ K_1 s^{d_1} {}_2F_1(a_1, b_1, c_1; -s), & s \in (u^*, \infty). \end{cases} \tag{3.4}$$

Using the smooth and continuous fit properties, we can find K_1 and K_2 and show that l^* and u^* solve the following equation

$$1 + {}_2F_1(a_i, b_i, c_i; -s) K_i D_i + s^{d_i} P_i = 0, \tag{3.5}$$

where

$$\begin{aligned} K_i &= (K - s) \frac{s^{-d_i}}{{}_2F_1(a_i, b_i, c_i; -s)}, \\ D_i &= d_i s^{d_i - 1}, \\ P_i &= -\frac{a_i b_i {}_2F_1(a_i + 1, b_i + 1, c_i + 1; -s)}{c_i} \end{aligned}$$

for $i = 1, 2$. We numerically calculate the roots of (3.5) for $i = 1, 2$ and assign the smaller result to l^* and the larger one to u^*.

Let us assume the given set of parameters $C = 0.001$, $D = 0.01$, $K = 20$, $r = 5\%$ and $\sigma = 20\%$. The numerical procedure above produces $l^* \approx 7.23$ and $u^* \approx 8.34$. Figure 3.1 presents the value function that arises in this case.

Remark 11. Let us note that $\lim_{s \to 0^+} V^{\omega}_{\mathrm{A^{Put}}}(s) = \infty$ which means that the price of the option is unlimited even for an arbitrarily low stock price. This is a consequence of the fact that the discount function is strictly negative for $s \to 0^+$.

3.3 Lévy exponential jumps

3.3.1 Constant discount function

The case when ω is constant, that is $\omega(s) = q$, is the standard example that appears extensively in the literature. However, this case is quite special, as it turns out that the second term in the sum in (2.16) simplifies and we do not need to deal with the measure $\mathbb{P}^{(\alpha)}_{(0)}$ (and thus calculate the limit for $\alpha \to \infty$) to find the form of $v^{\omega}_{\mathrm{A^{Put}}}(s, 0, u)$. This fact is stated in the following theorem.

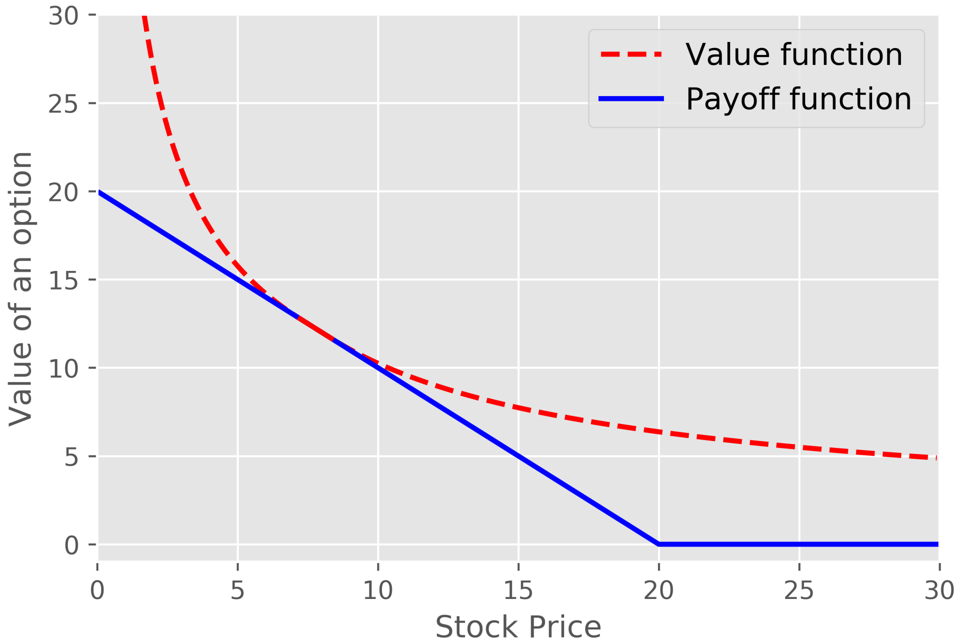


Figure 3.1: The payoff function and the value function $V^{\omega}_{\mathrm{APut}}(s)$ given in (3.4) for the following set of parameters: $C = 0.001$, $D = 0.01$, $K = 20$, $r = 5\%$ and $\sigma = 20\%$.

Theorem 9. *Assume that* $\omega(s) = q$. *Then*

$$\begin{aligned}\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}&\left(\mathscr{Z}^{(q-\psi(\alpha))}_{\alpha}\left(\frac{s}{u}\right)-c_{\mathscr{Z}^{(q-\psi(\alpha))}_{\alpha}/\mathscr{W}^{(q-\psi(\alpha))}_{\alpha}}\mathscr{W}^{(q-\psi(\alpha))}_{\alpha}\left(\frac{s}{u}\right)\right)\\&=\frac{\sigma^2}{2}\left(\mathscr{W}^{(q)\prime}\left(\frac{s}{u}\right)-\Phi(q)\mathscr{W}^{(q)}\left(\frac{s}{u}\right)\right).\end{aligned}\tag{3.6}$$

Proof. Note that

$$\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(q-\psi(\alpha))}_{\alpha}\left(\frac{s}{u}\right)-c_{\mathscr{Z}^{(q-\psi(\alpha))}_{\alpha}/\mathscr{W}^{(q-\psi(\alpha))}_{\alpha}}\mathscr{W}^{(q-\psi(\alpha))}_{\alpha}\left(\frac{s}{u}\right)\right)\tag{3.7}$$

corresponds to the continuous transition of the process S_t to the interval $(0, u]$, or, in other words, to the continuous exit of the half-line (u, ∞). We define

$$\sigma_0^- = \inf\{t \geq 0 : X_t \leq 0\} \quad \text{and} \quad \sigma_a^+ = \inf\{t \geq 0 : X_t \geq a\}.$$

It turns out that formula (3.7) is equivalent to

$$\mathbb{E}_{\frac{s}{u}}\left[e^{-q\sigma_0^-};\sigma_0^-<\sigma_a^+;X_{\sigma_0^-}=0\right].$$

More details can be found in the proof of Theorem 3. Then using [107, Formula (13), p. 1417] for $x = \log\left(\frac{s}{u}\right)$, $a = 0$ and $v^{(q)}(x) = W^{(q)\prime}(x)$ together with the fact that $W^{(q)\prime}(0) = \frac{2}{\sigma^2}$ (see [99, Exercise 8.5, p. 235]), we obtain

$$\mathbb{E}_{\frac{s}{u}}\left[e^{-q\sigma_0^-};\sigma_0^-<\sigma_a^+;X_{\sigma_0^-}=0\right]=\frac{\sigma^2}{2}\left(W^{(q)\prime}(x-\log u)-\frac{W^{(q)\prime}(a)}{W^{(q)}(a)}W^{(q)}(x-\log u)\right).$$

Lastly, we complete the proof by taking the limit $a \to \infty$ and applying L'Hospital rule. □

Ultimately, (2.16) for the constant discount function $\omega(s) = q$ can be written as

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s,0,u) = &\left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(q)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(q)}/\mathscr{W}^{(q)}}\mathscr{W}^{(q)}\left(\frac{s}{u}\right)\right) \\ &+ (K-u)\frac{\sigma^2}{2}\left(\mathscr{W}^{(q)\prime}\left(\frac{s}{u}\right) - \Phi(q)\mathscr{W}^{(q)}\left(\frac{s}{u}\right)\right). \end{aligned} \tag{3.8}$$

Remark 12. For the case of $\lambda = 0$, it can be shown that (2.15) simplifies to the well-known formula in the Black-Scholes model, that is

$$v^{\omega}_{\mathrm{A^{Put}}}(s,0,u) = (K-u)\left(\frac{s}{u}\right)^{-\frac{2r}{\sigma^2}},$$

where we used substitutions $q = r$ and $\zeta = r - \frac{\sigma^2}{2}$. Therefore, we are not forced to apply the smooth fit condition to find the optimal value of u. Instead, we can do this analytically by finding the maximum of $v^{\omega}_{\mathrm{A^{Put}}}(s,0,u)$ with respect to u and derive the form of the value function $V^{\omega}_{\mathrm{A^{Put}}}(s)$.

Figure 3.2 presents the value function $V^{\omega}_{\mathrm{A^{Put}}}(s)$ for three different values of q, that is $q \in \{0.3, 0.6, 0.9\}$.

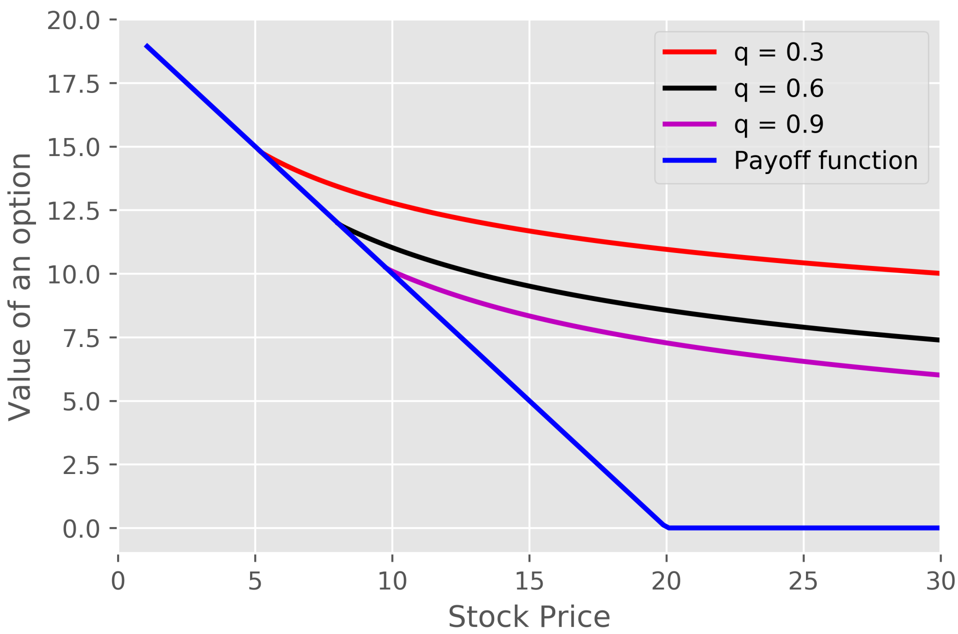


Figure 3.2: The payoff function and the value function $V^{\omega}_{\mathrm{A^{Put}}}(s)$ corresponding to (3.8) for the following set of parameters: $K = 20$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 6$, $\rho = 2$ and $q \in \{0.3, 0.6, 0.9\}$.

Based on Figure 3.2 we can simply note that a higher value of ω results in a smaller value of $V^{\omega}_{\mathrm{A^{Put}}}(s)$ which is in line with (2.6) and financial intuition.

In turn, Figure 3.3 shows a comparison of the value function $V^{\omega}_{\mathrm{A^{Put}}}(s)$ corresponding to three cases (2.14), (2.15), (2.16) and for the same value of $q = 0.5$.

The resulting relation between these functions is again consistent with economic expectations.

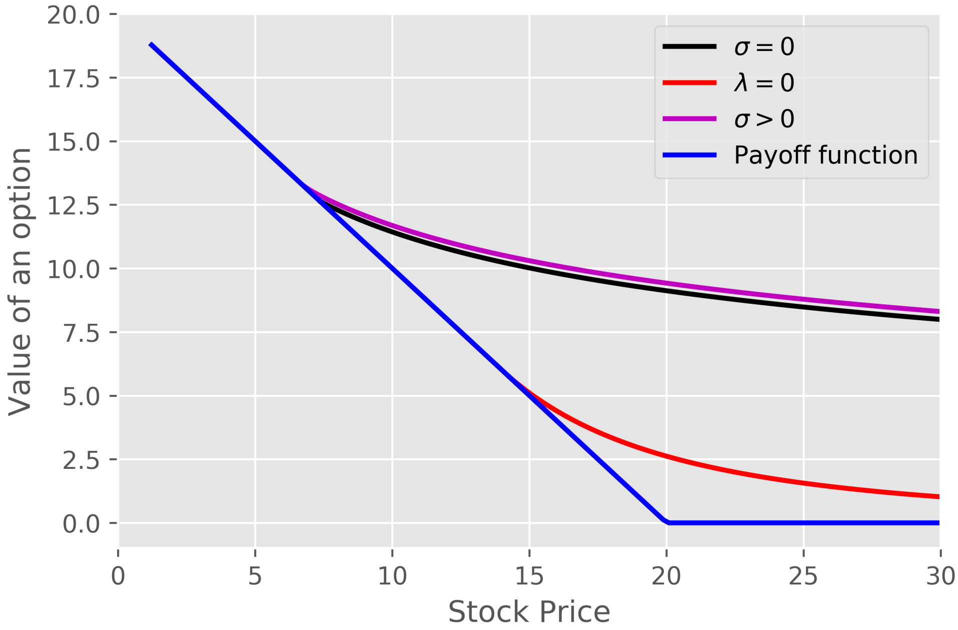


Figure 3.3: The payoff function and the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ corresponding to (3.8) for three cases $\sigma = 0$, $\lambda = 0$, $\sigma > 0$ and for the following set of parameters: $K = 20$, $r = 0.05$, $\sigma = 0.4$, $\lambda = 6$, $\rho = 2$ and $q = 0.5$.

3.3.2 Linear discount function

In this subsection, we consider a linear discount function of the form $\omega(s) = Cs$ for some positive constant C.

3.3.2.1 $\sigma = 0$

Let us consider the case of $\sigma = 0$. Then the asset price process S_t can only jump from (u, ∞) to the stopping region $(0, u]$. From Theorem 5 we know that

$$v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, 0, u) = \left(K - \frac{u\rho}{\rho + 1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right), \tag{3.9}$$

where $\omega_u\left(\frac{s}{u}\right) = \omega(s) = Cs$. Equivalently, (3.9) can be rewritten as

$$v^{\eta}_{\mathrm{A}^{\mathrm{Put}}}(x, 0, u) = \left(K - \frac{u\rho}{\rho + 1}\right)\left(\mathcal{Z}^{(\eta_u)}(x - \log u) - c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}\mathcal{W}^{(\eta_u)}(x - \log u)\right), \tag{3.10}$$

where $x = \log s$ and $\eta_u(x - \log u) = \eta(x) = Ce^x$.

To find the closed-form of (3.10) we need to identify $\mathcal{W}^{(\eta_u)}(x - \log u)$ and $\mathcal{Z}^{(\eta_u)}(x - \log u)$. From Theorem 6 it follows that both $\mathcal{W}^{(\eta)}(x)$ and $\mathcal{Z}^{(\eta)}(x)$ solve the following ordinary differential equation

$$f''(x) = (Ae^x + B)f'(x) + De^x f(x) \tag{3.11}$$

with $A = \frac{C}{\zeta}$, $B = \frac{\lambda - \rho\zeta}{\zeta}$ and $D = C\frac{1+\rho}{\zeta}$, while the initial conditions are as follows

$$\begin{cases} \mathcal{W}^{(\eta)}(0) = \frac{1}{\zeta}, \\ \mathcal{W}^{(\eta)\prime}(0) = \frac{C+\lambda}{\zeta^2} \end{cases} \tag{3.12}$$

and

$$\begin{cases} \mathcal{Z}^{(\eta)}(0) = 1, \\ \mathcal{Z}^{(\eta)\prime}(0) = \frac{C}{\zeta}. \end{cases} \tag{3.13}$$

Substituting $t = Ae^x$ and $F(t) = f(x)$ into (3.11) we obtain the Kummer's equation of the form

$$tF''(t) + (b - t)F'(t) - aF(t) = 0, \tag{3.14}$$

where $b = 1 - B$ and $a = \frac{D}{A}$. If b is not an integer, then the general solution to (3.14) has the form

$$F(t) = K_{1\,1}F_1(a_1, b_1; t) + K_2 t^{1-b}{}_1F_1(a_2, b_2; t), \tag{3.15}$$

where K_1 and K_2 are the constants that can be found based on the initial conditions, $a_1 = a$, $b_1 = b$, $a_2 = a - b + 1$, $b_2 = 2 - b$, while ${}_1F_1(\cdot, \cdot; \cdot)$ is the Kummer confluent hypergeometric function.

We denote by K_1^W, K_2^W and K_1^Z, K_2^Z the constants corresponding to $\mathcal{W}^{(\eta)}(x)$ and $\mathcal{Z}^{(\eta)}(x)$, respectively. Using initial conditions (3.12) and (3.13), we can simply calculate these constants for both $\mathcal{W}^{(\eta)}(x)$ and $\mathcal{Z}^{(\eta)}(x)$. By shifting these functions by $\log u$, we produce $\mathcal{W}^{(\eta_u)}(x - \log u)$ and $\mathcal{Z}^{(\eta_u)}(x - \log u)$.

The asymptotic behaviour of ${}_1F_1\,(a, b; t)$ for $t \to \infty$ is as follows

$${}_1F_1(a, b; t) = \frac{\Gamma(b)}{\Gamma(a)} e^t t^{a-b} \left[1 + O\left(\frac{1}{t}\right)\right]. \tag{3.16}$$

Based on (3.16) we calculate the constant $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}$ (or equivalently $c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}$) that occurs in (3.10). It has the following form

$$c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}} = \frac{K_1^Z \frac{\Gamma(b_1)}{\Gamma(a_1)} A^{a_1 - b_1} + K_2^Z \frac{\Gamma(b_2)}{\Gamma(a_2)} A^{a_2 - 1}}{K_1^W \frac{\Gamma(b_1)}{\Gamma(a_1)} A^{a_1 - b_1} + K_2^W \frac{\Gamma(b_2)}{\Gamma(a_2)} A^{a_2 - 1}}. \tag{3.17}$$

Combining all the results obtained and substituting them into (3.10), we can present the analytical form of $v^{\omega}_{\text{APut}}(s, 0, u)$. Then we maximise it with respect to u and derive the graphical form of the value function $V^{\omega}_{\text{APut}}(s)$. Figure 3.4 presents a comparison of the value function $V^{\omega}_{\text{APut}}(s)$ when the generalised scale functions were calculated analytically and numerically by solving equation (3.11).

Furthermore, Figure 3.5 illustrates the constant $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}$ obtained in (3.17) together with the quotient of the functions $\mathcal{Z}^{(\eta)}(x)$ and $\mathcal{W}^{(\eta)}(x)$.

3.3.2.2 $\lambda = 0$

Consider the case of $\lambda = 0$. In this case, the asset price process S_t leaves (u, ∞) and enters the stopping region $(0, u]$ only continuously. Therefore, from Theorem 5 we have

$$v^{\omega}_{\text{APut}}(s, 0, u) = (K - u) \left(\lim_{\alpha \to \infty} \left(\frac{s}{u}\right)^{\alpha} \left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}} \mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right) \tag{3.18}$$

which is equivalent to

$$v^{\eta}_{\text{APut}}(x, 0, u) = (K - u) \left(\lim_{\alpha \to \infty} e^{\alpha(x - \log u)} \left(\mathcal{Z}_{\alpha}^{(\eta_u^{\alpha})}(x - \log u) - c_{\mathcal{Z}_{\alpha}^{(\eta^{\alpha})}/\mathcal{W}_{\alpha}^{(\eta^{\alpha})}} \mathcal{W}_{\alpha}^{(\eta_u^{\alpha})}(x - \log u)\right)\right). \tag{3.19}$$

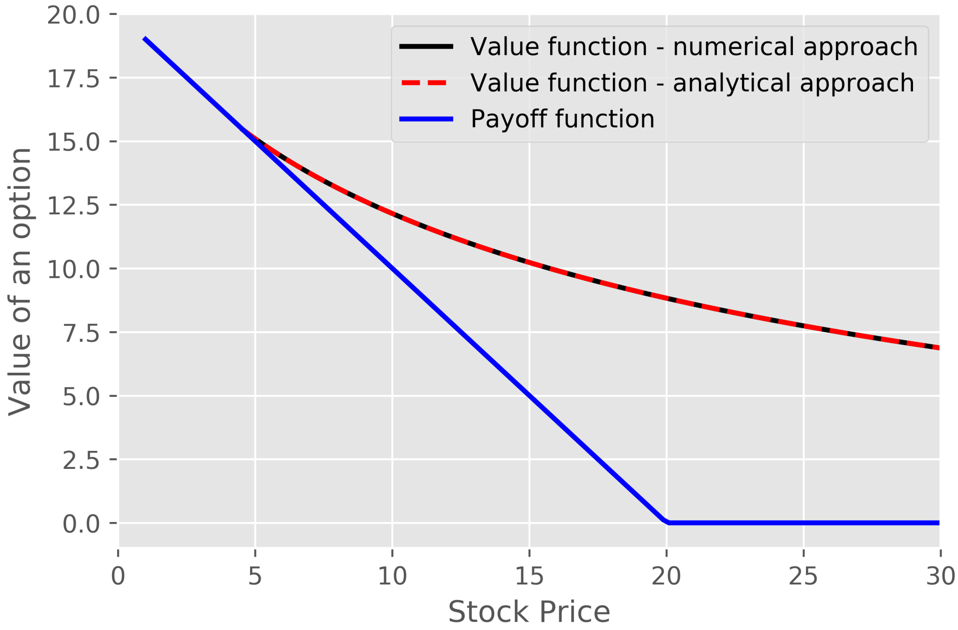


Figure 3.4: The payoff function and the value function $V^{\omega}_{A^{\text{Put}}}(s)$ corresponding to (3.9) for both methods of determining the generalised scale functions: analytical and numerical one, and for the following set of parameters: $K = 20$, $C = 0.1$, $r = 0.05$, $\lambda = 6$, $\rho = 2$.

It suffices to find $\mathcal{W}_{\alpha}^{(\eta_u^{\alpha})}(x - \log u)$ and $\mathcal{Z}_{\alpha}^{(\eta_u^{\alpha})}(x - \log u)$. From Theorem 6 it follows that both $\mathcal{W}_{\alpha}^{(\eta^{\alpha})}(x)$ and $\mathcal{Z}_{\alpha}^{(\eta^{\alpha})}(x)$ solve

$$f''(x) = B_{\alpha} f'(x) + (D_{\alpha} e^x + E_{\alpha}) f(x) \tag{3.20}$$

with $B_{\alpha} = -\frac{2}{\sigma^2}(\zeta + \sigma^2 \alpha)$, $D_{\alpha} = \frac{2C}{\sigma^2}$ and $E_{\alpha} = -\frac{2}{\sigma^2}\left(\zeta\alpha + \frac{\sigma^2}{2}\alpha^2\right)$. The initial conditions have the following form

$$\begin{cases} \mathcal{W}_{\alpha}^{(\eta^{\alpha})}(0) = 0, \\ \mathcal{W}_{\alpha}^{(\eta^{\alpha})\prime}(0) = \frac{2}{\sigma^2} \end{cases} \tag{3.21}$$

and

$$\begin{cases} \mathcal{Z}_{\alpha}^{(\eta^{\alpha})}(0) = 1, \\ \mathcal{Z}_{\alpha}^{(\eta^{\alpha})\prime}(0) = 0. \end{cases} \tag{3.22}$$

Substituting $t = 2\sqrt{-D_{\alpha} e^x}$ and $F(t) = e^{-\frac{B_{\alpha} x}{2}} f(x)$ into (3.20) we obtain the Bessel differential equation of the form

$$t^2 F''(t) + tF'(t) + (t^2 - v^2) F(t) = 0, \tag{3.23}$$

where $v = \sqrt{{B_{\alpha}}^2 + 4E_{\alpha}} = \frac{2\zeta}{\sigma^2}$. The general solution to (3.23) is equal to

$$F(t) = K_1 J_v(t) + K_2 Y_v(t),$$

where $J_v(\cdot)$ and $Y_v(\cdot)$ are the Bessel functions of the first and second kinds, while K_1 and K_2 are constants. In the following, we use the symbols K_1^W, K_2^W and K_1^Z, K_2^Z which correspond to

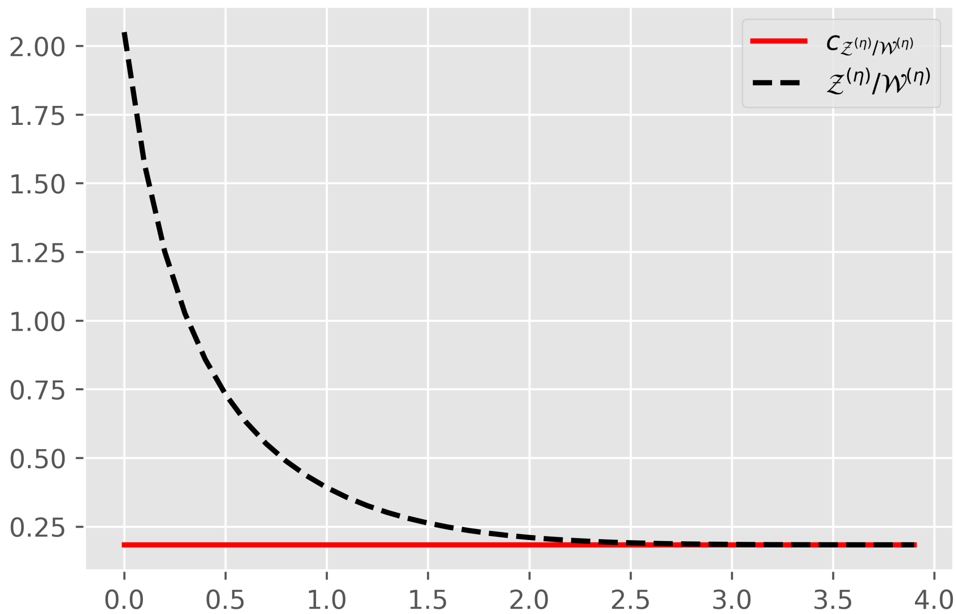


Figure 3.5: Comparison of the constant $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}$ given in (3.17) and the ratio of $\mathcal{Z}^{(\eta)}(x)$ and $\mathcal{W}^{(\eta)}(x)$ for a linear discount function $\omega(s) = Cs$ ($\eta(x) = Ce^x$) and for the following set of parameters: $K = 20$, $C = 0.1$, $r = 0.05$, $\lambda = 6$, $\rho = 2$.

$\mathcal{W}_\alpha^{(\eta^\alpha)}(x)$ and $\mathcal{Z}_\alpha^{(\eta^\alpha)}(x)$, respectively. Therefore, we derive

$$f(x) = e^{\frac{B_\alpha x}{2}}\Big(K_1 J_v(2\sqrt{-D_\alpha e^x}) + K_2 Y_v(2\sqrt{-D_\alpha e^x})\Big). \tag{3.24}$$

Based on the form of (3.24) and the fact that D_α does not depend on α, we can simply note that (3.19) is also independent of α. Therefore, we can take an arbitrary value of α in (3.20). This key observation allows us to rewrite (3.24) in a simplified form. Indeed, for $\alpha = 0$ equation (3.20) is equal to

$$f''(x) = B_0 f'(x) + D_0 e^x f(x), \tag{3.25}$$

where $B_0 = \frac{-2\zeta}{\sigma^2}$ and $D_0 = \frac{2C}{\sigma^2}$. Hence, the general solution to (3.25) takes the following form

$$f(x) = e^{\frac{B_0 x}{2}}\Big(K_1 J_v(2\sqrt{-D_0 e^x}) + K_2 Y_v(2\sqrt{-D_0 e^x})\Big). \tag{3.26}$$

For $B_0 = \frac{1}{2} - n$, where $n \in \mathbb{N}_0$ and $D_0 t > 0$, equation (3.26) reduces to

$$f(x) = K_1\Big(\cosh(4\sqrt{D_0 e^x})\Big)^n + K_2\Big(\sinh(4\sqrt{D_0 e^x})\Big)^n.$$

If we take the following sample parameters $r = 0.05$ and $\sigma = 0.2$, we obtain $n = 2$ and therefore

$$\begin{aligned} f(x) = K_1\bigg(&\frac{3\sinh(2\sqrt{e^x})}{4e^{\frac{5}{2}x}} + \frac{\sinh(2\sqrt{e^x})}{e^{\frac{3}{2}x}} - \frac{3\cosh(2\sqrt{e^x})}{2e^{2x}}\bigg) \\ + K_2\bigg(&\frac{3\cosh(2\sqrt{e^x})}{4e^{\frac{5}{2}x}} + \frac{\cosh(2\sqrt{e^x})}{e^{\frac{3}{2}x}} - \frac{3\sinh(2\sqrt{e^x})}{2e^{2x}}\bigg). \end{aligned} \tag{3.27}$$

Applying initial conditions (3.21) and (3.22) we can simply obtain K_1^W, K_2^W and K_1^Z, K_2^Z. Using equality (3.27) that holds for both $\mathcal{W}_\alpha^{(\eta^\alpha)}(x)$ and $\mathcal{Z}_\alpha^{(\eta^\alpha)}(x)$, we can calculate the following

$$c_{\mathcal{Z}_\alpha^{(\eta^\alpha)}/\mathcal{W}_\alpha^{(\eta^\alpha)}} = c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}} = \frac{K_1^Z + K_2^Z}{K_1^W + K_2^W}. \tag{3.28}$$

Taking into account all the results, we can obtain the analytical form of (3.19) and then maximise it with respect to u to obtain the value function $V^\omega_{\mathrm{A}^{\mathrm{Put}}}(s)$ for the sample data.

Figure 3.6 presents a comparison of the value function $V^\omega_{\mathrm{A}^{\mathrm{Put}}}(s)$ corresponding to (3.18) for the generalised scale functions obtained analytically and numerically.

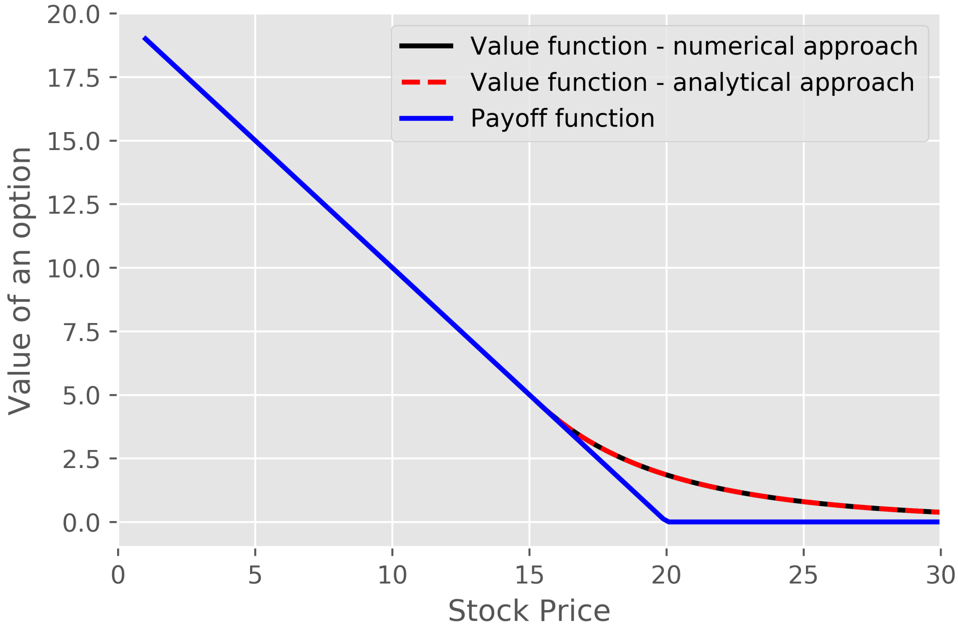


Figure 3.6: The payoff function and the value function $V^\omega_{\mathrm{A}^{\mathrm{Put}}}(s)$ corresponding to (3.18) for both methods of determining the generalised scale functions: analytical and numerical one, and for the following set of parameters: $K = 20$, $C = 0.1$, $r = 0.05$, $\sigma = 0.2$.

In turn, Figure 3.7 shows the constant $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}$ given in (3.28) with the ratio of $\mathcal{Z}^{(\eta)}(x)$ and $\mathcal{W}^{(\eta)}(x)$.

3.3.3 Power discount function

This time, we take into account a power function of the form $\omega(s) = Cs^n$ for $n \in (0, 1]$ and C being some positive constant. This case is a generalisation of a linear discount function scenario.

3.3.3.1 $\sigma = 0$

Similarly to the case of a linear discount function, the functions $\mathcal{W}^{(\eta)}(x)$ and $\mathcal{Z}^{(\eta)}(x)$ solve

$$f''(x) = (Ae^{nx} + B)f'(x) + De^{nx}f(x) \tag{3.29}$$

with $A = \frac{C}{\zeta}$, $B = \frac{\lambda - \rho\zeta}{\zeta}$ and $D = C\frac{n+\rho}{\zeta}$, while the initial conditions are the same as those provided in (3.12) and (3.13). Applying a substitution $t = \frac{A}{n}e^{nx}$ and $F(t) = f(x)$, we transform

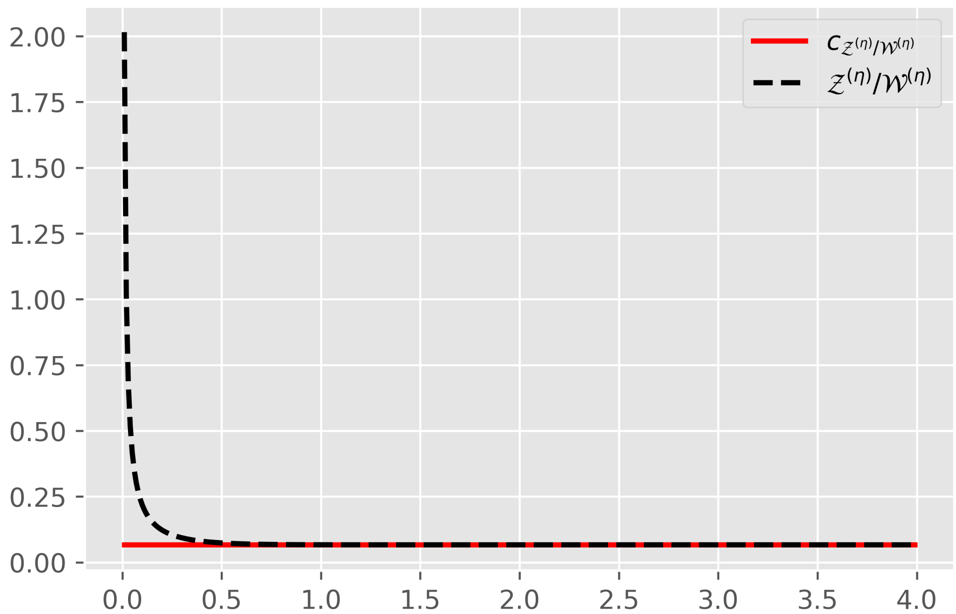


Figure 3.7: Comparison of the constant $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}$ and the ratio of $\mathcal{Z}^{(\eta)}(x)$ and $\mathcal{W}^{(\eta)}(x)$ for a linear discount function $\omega(s) = Cs$ ($\eta(x) = Ce^x$) and for the following set of parameters: $K = 20$, $C = 0.1$, $r = 0.05$, $\sigma = 0.2$.

(3.29) into

$$tF''(t) + (b - t)F'(t) - aF(t) = 0, \tag{3.30}$$

where $b = 1 - \frac{B}{n}$ and $a = \frac{D}{An}$. The general solution to (3.30) has the same form as provided in (3.15). Therefore, for both the linear and the power discount function ω, the form of the value function $V^{\omega}_{\text{APut}}(s)$ is identical.

3.3.3.2 $\lambda = 0$

As in the above case, the idea of finding the closed-form of the value function can be borrowed from the linear case. This time, the functions $\mathcal{W}^{(\eta^\alpha)}_\alpha(x)$ and $\mathcal{Z}^{(\eta^\alpha)}_\alpha(x)$ satisfy the equation

$$f''(x) = B_\alpha f'(x) + (D_\alpha e^{nx} + E_\alpha)f(x)$$

with $B_\alpha = -\frac{2}{\sigma^2}(\zeta + \sigma^2\alpha)$, $D_\alpha = \frac{2C}{\sigma^2}$ and $E_\alpha = -\frac{2}{\sigma^2}\left(\zeta\alpha + \frac{\sigma^2}{2}\alpha^2\right)$, while the initial conditions are of the form (3.21) and (3.22). If we substitute $t = \frac{2}{n}\sqrt{-D_\alpha e^{nx}}$ and $F(t) = e^{-\frac{B_\alpha x}{2}} f(x)$, we receive the Bessel differential equation for $F(t)$ with the solution

$$F(t) = K_1 J_v(t) + K_2 Y_v(t).$$

Therefore, we have

$$f(x) = e^{\frac{B_\alpha x}{2}} \left(K_1 J_v\left(\frac{2}{n}\sqrt{-D_\alpha e^{nx}}\right) + K_2 Y_v\left(\frac{2}{n}\sqrt{-D_\alpha e^{nx}}\right)\right), \tag{3.31}$$

where $v = \frac{\sqrt{B_\alpha{}^2+4E_\alpha}}{n} = \frac{2\zeta}{n\sigma^2}$. As in the previous section, we can show that the value function that arises in this scenario does not depend on α. Thus, for $\alpha = 0$, (3.31) takes the form

$$f(x) = e^{\frac{B_0 x}{2}}\left(K_1 J_v\left(\frac{2}{n}\sqrt{-D_0 e^{nx}}\right) + K_2 Y_v\left(\frac{2}{n}\sqrt{-D_0 e^{nx}}\right)\right).$$

Again, having exact formulas for generalised scale functions, we can easily represent the form of $v^\omega_{\mathrm{APut}}(s, 0, u)$ and maximise it with respect to u to derive the value function $V^\omega_{\mathrm{APut}}(s)$.

3.3.4 Other discount functions

For some discount functions ω we cannot find the analytical forms of $\mathcal{W}^{(\eta)}(x)$, $\mathcal{Z}^{(\eta)}(x)$, $\mathcal{W}_\alpha^{(\eta^\alpha)}(x)$ and $\mathcal{Z}_\alpha^{(\eta^\alpha)}(x)$. These functions are solutions to the ordinary differential equations that occur in Theorem 6. Thus, we cannot also explicitly identify the value function $V^\omega_{\mathrm{APut}}(s)$. In this situation, we can proceed with a numerical analysis of these equations. We apply the approach explained in Subsection 3.1.2.

3.3.4.1 $\sigma = 0$

As we mentioned at the beginning of this section, we cannot always get the analytical solutions to the generalised scale functions $\mathcal{W}^{(\eta)}(x)$, $\mathcal{Z}^{(\eta)}(x)$, $\mathcal{W}_\alpha^{(\eta^\alpha)}(x)$ and $\mathcal{Z}_\alpha^{(\eta^\alpha)}(x)$. This is the case, for example, when $\sigma = 0$ and the discount function is of the form $\omega(s) = C\arctan(s)$ for some positive C. Then, we can only generate these functions numerically. Figure 3.8 presents the value function $V^\omega_{\mathrm{APut}}(s)$ for both $\omega(s) = Cs$ and $\omega(s) = C\arctan(s)$, respectively. Since for all positive s we have $s > \arctan(s)$, we expect that the value function corresponding to $\omega(s) = C\arctan(s)$ takes higher values than those for $\omega(s) = Cs$. We can also note that the difference between these functions increases with higher values of s, which is in line with economic intuition, since the difference between $\omega(s) = s$ and $\omega(s) = \arctan(s)$ also increases as s increases.

3.3.4.2 $\lambda = 0$

The case of $\lambda = 0$ corresponds to the situation when the process X_t from (2.12) does not have any jumps. Then, the function $v^\omega_{\mathrm{APut}}(s, 0, u)$ takes the form (2.15). From a numerical point of view, the problem is to choose a large enough value of α in (2.15) to obtain the final form of the value function $V^\omega_{\mathrm{APut}}(s)$. In this section, we avoid this problem by selecting discount functions for which the value function is independent of the parameter α. In Figure 3.9 we can observe the value functions for both $\omega(s) = C\sqrt{s}$ and $\omega(s) = C\sqrt{s} + Z$ for some positive Z, that is we compare two discount functions that differ in a shift. This time, we can see that the value functions obtained in this way also differ only in a shift, which is in line with financial intuition.

3.3.4.3 $\sigma > 0$ and $\lambda > 0$

The most general case is when $\sigma > 0$ and $\lambda > 0$. Then, the value function $V^\omega_{\mathrm{APut}}(s)$ corresponds to formula (2.16). For the linear discount function $\omega(s) = Cs$, the functions $\mathcal{W}^{(\eta)}(x)$ and $\mathcal{Z}^{(\eta)}(x)$ are the solutions to the following ordinary differential equation

$$f'''(x) = Af''(x) + (Be^x + D)f'(x) + Ee^x f(x) \tag{3.32}$$

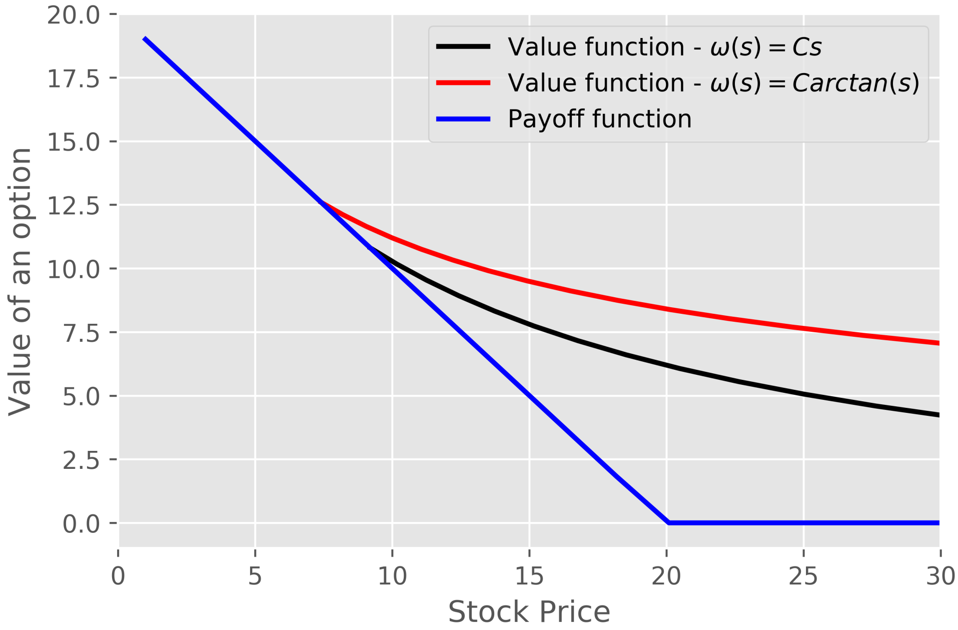


Figure 3.8: The payoff function and the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ corresponding to (3.9) for both $\omega(s) = Cs$ and $\omega = C\arctan(s)$ and for the following set of parameters: $K = 20$, $C = 0.5$, $r = 0.05$, $\lambda = 6$, $\rho = 2$.

with parameters of the form

$$\begin{cases} A = \gamma_2 + \gamma_3, \\ B = C\left[\Upsilon_2(\gamma_2 - \gamma_3) - \Upsilon_1\gamma_3\right], \\ D = -\gamma_2\gamma_3, \\ E = C\left[\Upsilon_2(\gamma_2 - \gamma_3) + \Upsilon_1\gamma_2\gamma_3 - \Upsilon_1\gamma_3\right]. \end{cases}$$

The initial conditions are as follows

$$\begin{cases} \mathcal{W}^{(\eta)}(0) = 0, \\ \mathcal{W}^{(\eta)\prime}(0) = \Upsilon_2\gamma_2 + \Upsilon_3\gamma_3, \\ \mathcal{W}^{(\eta)\prime\prime}(0) = \Upsilon_2{\gamma_2}^2 + \Upsilon_3{\gamma_3}^2 \end{cases}$$

and

$$\begin{cases} \mathcal{Z}^{(\eta)}(0) = 1, \\ \mathcal{Z}^{(\eta)\prime}(0) = 0, \\ \mathcal{Z}^{(\eta)\prime\prime}(0) = C\left[\Upsilon_2(\gamma_2 - \gamma_3) - \Upsilon_1\gamma_3\right]. \end{cases}$$

In turn, $\mathcal{W}^{(\eta^\alpha)}_\alpha(x)$ and $\mathcal{Z}^{(\eta^\alpha)}_\alpha(x)$ solve

$$f'''(x) = A_\alpha f''(x) + (B_\alpha e^x + D_\alpha)f'(x) + (E_\alpha e^x + F_\alpha)f(x) \tag{3.33}$$

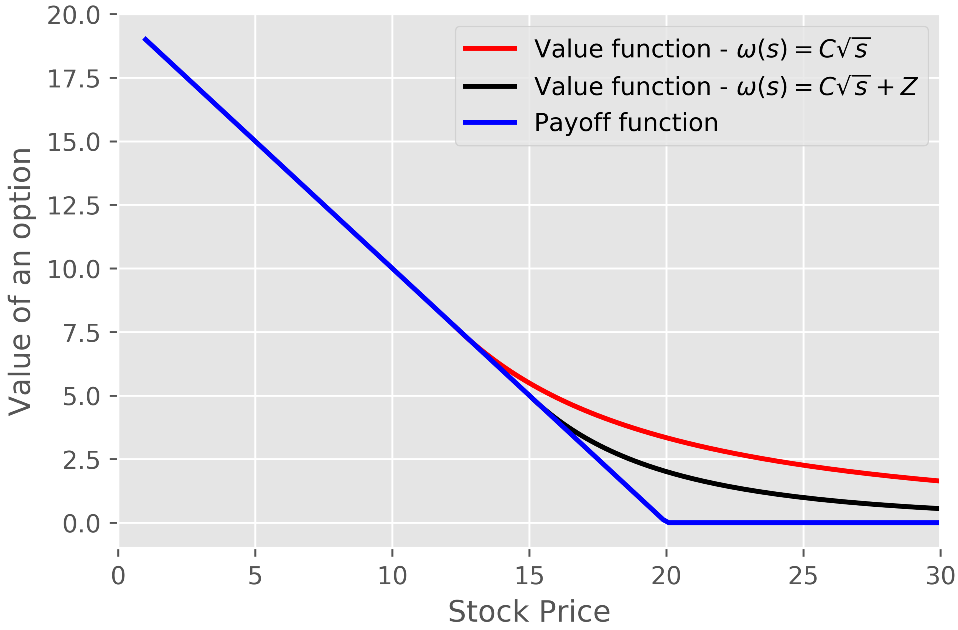


Figure 3.9: The payoff function and the value function $V^{\omega}_{\mathrm{APut}}(s)$ corresponding to (3.18) for both $\omega(s) = C\sqrt{s}$ and $\omega = C\sqrt{s} + Z$ and for the following set of parameters: $K = 20$, $C = 0.005$, $Z = 0.1$, $r = 0.05$, $\sigma = 0.2$.

with parameters of the form

$$\begin{cases} A_\alpha = \gamma_{\alpha_2} + \gamma_{\alpha_3}, \\ B_\alpha = C\left[\Upsilon_{\alpha_2}(\gamma_{\alpha_2} - \gamma_{\alpha_3}) - \Upsilon_{\alpha_1}\gamma_{\alpha_3}\right], \\ D_\alpha = -\Upsilon_{\alpha_2}(\gamma_{\alpha_2} - \gamma_{\alpha_3})\psi(\alpha) - \gamma_2\gamma_3 + \Upsilon_{\alpha_1}\gamma_{\alpha_3}\psi(\alpha), \\ E_\alpha = C\left[\Upsilon_{\alpha_2}(\gamma_{\alpha_2} - \gamma_{\alpha_3}) + \Upsilon_{\alpha_1}\gamma_{\alpha_2}\gamma_{\alpha_3} - \Upsilon_{\alpha_1}\gamma_{\alpha_3}\right], \\ F_\alpha = -\Upsilon_{\alpha_1}\gamma_{\alpha_2}\gamma_{\alpha_3}\psi(\alpha). \end{cases}$$

The initial conditions are as follows

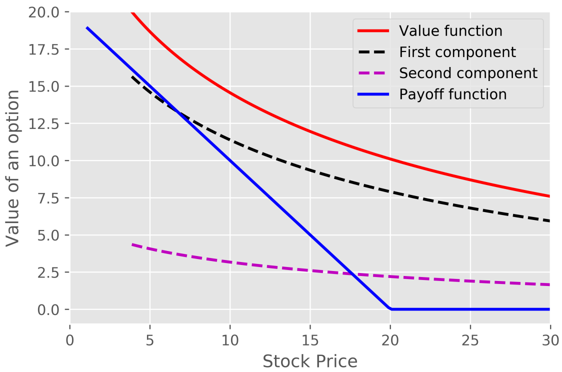
$$\begin{cases} \mathcal{W}_\alpha^{(\eta^\alpha)}(0) = 0, \\ \mathcal{W}_\alpha^{(\eta^\alpha)\prime}(0) = \Upsilon_{\alpha_2}\gamma_{\alpha_2} + \Upsilon_{\alpha_3}\gamma_{\alpha_3}, \\ \mathcal{W}_\alpha^{(\eta^\alpha)\prime\prime}(0) = \Upsilon_{\alpha_2}{\gamma_{\alpha_2}}^2 + \Upsilon_{\alpha_3}{\gamma_{\alpha_3}}^2 \end{cases}$$

and

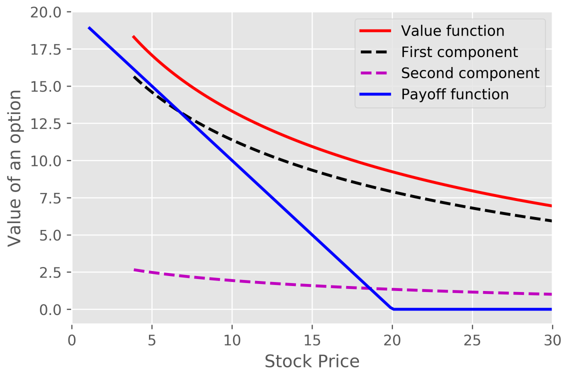
$$\begin{cases} \mathcal{Z}_\alpha^{(\eta^\alpha)}(0) = 1, \\ \mathcal{Z}_\alpha^{(\eta^\alpha)\prime}(0) = 0, \\ \mathcal{Z}_\alpha^{(\eta^\alpha)\prime\prime}(0) = C\left[\Upsilon_{\alpha_2}(\gamma_{\alpha_2} - \gamma_{\alpha_3}) - \Upsilon_{\alpha_1}\gamma_{\alpha_3}\right]. \end{cases}$$

In this case, we cannot identify explicit solutions to third-order ordinary differential equations (3.32) and (3.33), so we are forced to use a numerical algorithm to generate the generalised scale functions and hence the value function $V^{\omega}_{\mathrm{APut}}(s)$.

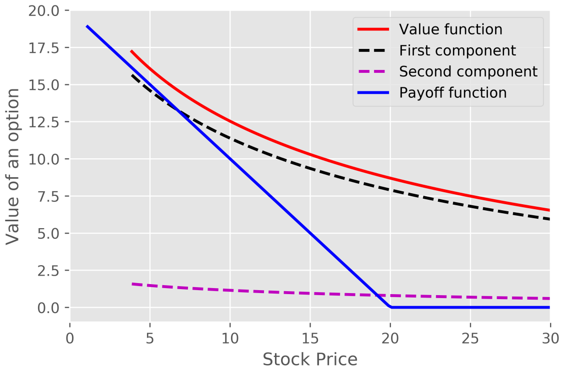
Figure 3.10 shows several graphs of the value function $V^{\omega}_{\mathrm{APut}}(s)$ for different values of the parameter α together with the first and second components that occur in (2.16). We can observe that a higher value of the α parameter allows us to obtain the value function we are looking for.



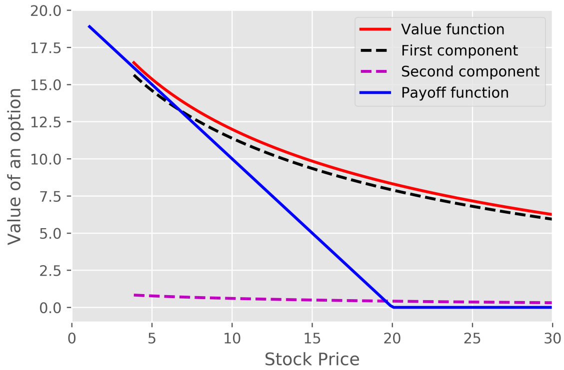
(a) $\alpha = 5$



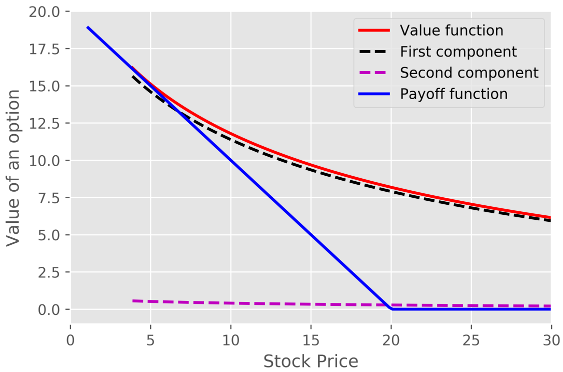
(b) $\alpha = 10$



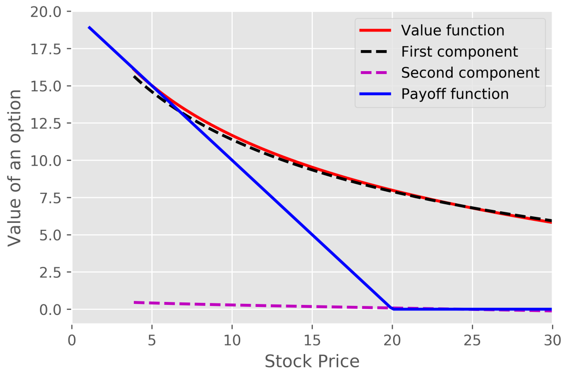
(c) $\alpha = 20$



(d) $\alpha = 50$



(e) $\alpha = 100$



(f) $\alpha = 150$

Figure 3.10: The payoff function and the value function $V^{\omega}_{\mathrm{APut}}(s)$ corresponding to (2.16) for the particular choice of α and $\omega(s) = Cs$, and for the following set of parameters: $K = 20$, $C = 0.1$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 6$, $\rho = 2$.

Chapter 4

Proofs

This chapter presents the proofs of the theorems and lemmas contained in Chapter 2 along with their content, as well as auxiliary theorems and lemmas. We decided to leave the content of the theorems from Chapter 2 unchanged, so the formulas from these theorems are characterised by the numbering (2.x), and the rest of the formulas in this chapter by (4.x).

Before we prove the first important theorem in our thesis, that is Theorem 1 on the convexity of $V_{\mathrm{A}}^{\omega}(t)$, we show the convexity of a European option price $V_{\mathrm{E}}^{\omega}(s,t)$ defined in (2) with additional Assumptions (B) and (C) presented below. This fact is stated in Theorem 10. Furthermore, in this theorem, we formulate Lemma 4 and Lemma 5 on specific properties of the function $V_{\mathrm{E}}^{\omega}(s,t)$. Then, we relax unnecessary conditions and formulate Theorem 11. Ultimately, we prove Theorem 1, which is based on showing an inheritance of the convexity of the value function from the European option to the Bermudan option (Lemma 6), and then the American option. Then, we present the proof of the most relevant result in our work, i.e. Theorem 3, followed by specific versions of it, namely Theorem 4 and Theorem 5. In the next proof of Theorem 6 we show the derivations of ordinary differential equations that are satisfied by the generalised scale functions defined in (1.10) and (1.11). Lastly, we prove Theorem 7 about the HJB equation and Theorem 8 related to the put-call parity.

We state the following assumptions.

Assumptions (B)

There exist constants $C > 0$ and $\alpha \in (0,1)$ such that

(B1) $\mu(s,t) \in C_{\alpha}^{2,1}(\mathbb{R}^{+} \times [0,T])$;
(B2) $\sigma^2(s,t) \geq Cs^2$ *for all* $(s,t) \in \mathbb{R}^{+} \times [0,T]$;
(B3) $\sigma(s,t) \in C_{\alpha}^{2,1}(\mathbb{R}^{+} \times [0,T])$;
(B4) $\gamma(s,t,z) \in C_{\alpha}^{2,1}(\mathbb{R}^{+} \times [0,T])$ *with the Hölder continuity being uniform in* z;
(B5) $|\omega(s)| \leq C$ *for all* $s \in \mathbb{R}^{+}$;
(B6) $\omega(s) \in C_{\alpha}^{2}(\mathbb{R}^{+})$;
(B7) $g(s)$ *is Lipschitz continuous;*
(B8) $g(s) \in C_{\alpha}^{4}(\mathbb{R}^{+})$.

Assumptions (C)

There exists a constant $C > 0$ such that

(C1) $|\frac{\partial \mu(s,t)}{\partial t}| \leq Cs$, $|\frac{\partial^2 \mu(s,t)}{\partial s^2}| \leq \frac{C}{s}$ *for all* $(s,t) \in \mathbb{R}^{+} \times [0,T]$;
(C2) $|\frac{\partial \sigma(s,t)}{\partial t}| \leq Cs$, $|\frac{\partial^2 \sigma(s,t)}{\partial s^2}| \leq \frac{C}{s}$ *for all* $(s,t) \in \mathbb{R}^{+} \times [0,T]$;

(C3) $|\frac{\partial\gamma(s,t,z)}{\partial t}| \le Cs$, $|\frac{\partial^2\gamma(s,t,z)}{\partial s^2}| \le \frac{C}{s}$ *for all* $(s,t,z) \in \mathbb{R}^+ \times [0,T] \times \mathbb{R}$*;*
(C4) $|\frac{d\omega(s)}{ds}| \le \frac{C}{s}$, $|\frac{d^2\omega(s)}{ds^2}| \le \frac{C}{s^2}$ *for all* $s \in \mathbb{R}^+$*;*
(C5) $g(s) \in C^3_{pol}(\mathbb{R}^+)$.

Theorem 10. *Let all the assumptions of Theorem 1 be satisfied. We also assume that Assumptions (B) and (C) hold. Then* $V^\omega_{\mathrm{E}}(s,t)$ *is convex with respect to* s *at all times* $t \in [0,T]$.

Proof. The first part of the proof proceeds similarly to the proof of [68, Proposition 4.1, p. 389].

Let
$$\mathcal{L}V^\omega_{\mathrm{E}}(s,t) = -\frac{\partial V^\omega_{\mathrm{E}}(s,t)}{\partial t} - A^C_t V^\omega_{\mathrm{E}}(s,t) - A^J_t V^\omega_{\mathrm{E}}(s,t) + \omega(s)V^\omega_{\mathrm{E}}(s,t),$$
where A^C_t is a second-order linear differential operator of the form
$$A^C_t V^\omega_{\mathrm{E}}(s,t) = \beta(s,t)\frac{\partial^2 V^\omega_{\mathrm{E}}(s,t)}{\partial s^2} + \mu(s,t)\frac{\partial V^\omega_{\mathrm{E}}(s,t)}{\partial s}$$
with $\beta(s,t) = \frac{\sigma^2(s,t)}{2}$ and A^J_t is an integro-differential operator given by
$$A^J_t V^\omega_{\mathrm{E}}(s,t) = \int_{\mathbb{R}} \left(V^\omega_{\mathrm{E}}(s+\gamma(s,t,z),t) - V^\omega_{\mathrm{E}}(s,t) - \gamma(s,t,z)\frac{\partial V^\omega_{\mathrm{E}}(s,t)}{\partial s} \right) \Pi(dz).$$
Before proceeding further, we formulate two auxiliary lemmas.

Lemma 4. *Let Assumptions (A) and (B) hold and assume that the stock price process* S_t *follows* (2.1)*. Then* $V^\omega_{\mathrm{E}}(s,t) \in C^{4,1}_\alpha(\mathbb{R}^+ \times [0,T]) \cap C_{pol}(\mathbb{R}^+ \times [0,T])$ *and it is the solution to the Cauchy problem given by*
$$\begin{cases} \mathcal{L}V^\omega_{\mathrm{E}}(s,t) = 0, & (s,t) \in \mathbb{R}^+ \times [0,T), \\ V^\omega_{\mathrm{E}}(s,T) = g(s), & s \in \mathbb{R}^+. \end{cases} \tag{4.1}$$

Proof of Lemma 4. First, we define the function $f : \mathbb{R}^+ \to \mathbb{R}$ of the form
$$f(s) = \begin{cases} -\frac{1}{s}, & s \in (0,1], \\ s, & s \in [2,\infty) \end{cases}$$
such that $f(s) \in C^2(\mathbb{R}^+)$ and $f'(s) > 0$ for all $s \in \mathbb{R}^+$.

Taking $Y_t = f(S_t)$ and applying Itô's lemma on (2.1), we obtain
$$dY_t = \tilde{\mu}(Y_{t-},t)dt + \tilde{\sigma}(Y_{t-},t)dB_t + \int_{\mathbb{R}} \tilde{\gamma}(Y_{t-},t,z)\tilde{v}(dt,dz),$$
where
$$\begin{aligned} \tilde{\mu}(y,t) &= \mu(f^{-1}(y),t)f'(f^{-1}(y)) + \frac{\sigma^2(f^{-1}(y),t)}{2}f''(f^{-1}(y)) \\ &\quad + \int_{\mathbb{R}} \left(\tilde{\gamma}(y,t,z) - f'(f^{-1}(y))\gamma(f^{-1}(y),t,z)\right)\Pi(dz), \\ \tilde{\sigma}(y,t) &= f'(f^{-1}(y))\sigma(f^{-1}(y),t), \\ \tilde{\gamma}(y,t,z) &= f(f^{-1}(y) + \gamma(f^{-1}(y),t,z)) - y. \end{aligned}$$
We also define the function
$$\tilde{\omega}(y) := \omega(f^{-1}(y))$$

and

$$\tilde{g}(y) := g(f^{-1}(y)).$$

We can now verify that the functions $\tilde{\mu}(y,t)$, $\tilde{\sigma}(y,t)$, $\tilde{\gamma}(y,t,z)$ and $\tilde{g}(y)$ satisfy conditions (2.2) – (2.5) from [122, Section 2, p. 4]. Let

$$v(y,t) := V_{\mathrm{E}}^{\omega}(f^{-1}(y),t).$$

From [122, Theorem 3.1, p. 11] it follows that $v(y,t)$ is a viscosity solution to

$$\begin{cases} \tilde{\mathcal{L}}v(y,t) = \tilde{f}(y,t), & (y,t) \in \mathbb{R} \times [0,T), \\ v(y,T) = \tilde{g}(y), & y \in \mathbb{R}, \end{cases} \tag{4.2}$$

where

$$\tilde{\mathcal{L}}v(y,t) = -\frac{\partial v(y,t)}{\partial t} - \frac{\tilde{\sigma}^2(y,t)}{2}\frac{\partial^2 v(y,t)}{\partial y^2} - \hat{\mu}(y,t)\frac{\partial v(y,t)}{\partial y} + \tilde{\omega}(y)v(y,t)$$

with

$$\hat{\mu}(y,t) = \tilde{\mu}(y,t) - \int_{\mathbb{R}} \tilde{\gamma}(y,t,z)\Pi(dz)$$

and

$$\tilde{f}(y,t) = -\int_{\mathbb{R}} \left(v(y+\tilde{\gamma}(y,t,z),t) - v(y,t)\right)\Pi(dz).$$

Furthermore, using [122, Proposition 3.3, p. 10] yields that $v(y,t) \in C(\mathbb{R} \times [0,T])$ and satisfies

$$|v(y_2,t_2) - v(y_1,t_1)| \le C((1+|y_2|)|t_2-t_1|^{\frac{1}{2}} + |y_2-y_1|) \tag{4.3}$$

for some $C > 0$ and for all $t_1, t_2 \in [0,T]$ and $y_1, y_2 \in \mathbb{R}$. Based on (4.3) and the assumptions made on γ, we can conclude that $\tilde{f}(y,t) \in C_\alpha(\mathbb{R} \times [0,T]) \cap C_{\mathrm{pol}}(\mathbb{R} \times [0,T])$. Then applying [86, Theorem A.14, p. 222] give us the existence of a unique classical solution $w(y,t)$ to (4.2) such that $w(y,t) \in C^{2,1}(\mathbb{R} \times [0,T)) \cap C_{\mathrm{pol}}(\mathbb{R} \times [0,T])$. In view of the fact that $w(y,t)$ is continuous, we can observe that $\tilde{f}(y,t)$ is Lipschitz continuous in y, uniformly in t. Therefore, from [122, Lemma 3.1, p. 9] we know that $w(y,t)$ is also Lipschitz continuous in y, uniformly in t. From the uniqueness result given in [122, Theorem 4.1, p. 14] we can deduce that $v(y,t) = w(y,t)$. Applying [86, Theorem A.18, p. 224] we find $v(y,t) \in C_\alpha^{4,1}(\mathbb{R} \times [0,T])$. Returning to the original coordinates, it follows that $V_{\mathrm{E}}^{\omega}(s,t) \in C_\alpha^{4,1}(\mathbb{R}^+ \times [0,T]) \cap C_{\mathrm{pol}}(\mathbb{R}^+ \times [0,T])$ and satisfies (4.1). □

Lemma 5. *Let Assumptions (A), (B) and (C) hold and assume that the stock price process S_t follows* (2.1). *Then there exist constants $n > 0$ and $K > 0$ such that the value function $V_{\mathrm{E}}^{\omega}(s,t)$ satisfies*

$$\left|\frac{\partial^2 V_{\mathrm{E}}^{\omega}(s,t)}{\partial s^2}\right| \le K(s^{-n} + s^n)$$

for all $(s,t) \in \mathbb{R}^+ \times [0,T]$.

Proof of Lemma 5. The proof follows in the same way as the proof of Lemma 4. However, this time we apply [86, Theorem A.20, p. 225] which guarantees the existence of a unique classical solution $w(y,t)$ to (4.2) that satisfies $w(y,t) \in C_{\mathrm{pol}}^{2,1}(\mathbb{R} \times [0,T])$. Therefore, going back to the original coordinates, we conclude that $V_{\mathrm{E}}^{\omega}(s,t) \in C_{\mathrm{pol}}^{2,1}(\mathbb{R}^+ \times [0,T])$. Therefore, there exist constants $n > 0$ and $K > 0$ such that

$$\left|\frac{\partial^2 V_{\mathrm{E}}^{\omega}(s,t)}{\partial s^2}\right| \le K(s^{-n} + s^n)$$

for all $(s,t) \in \mathbb{R}^+ \times [0,T]$. This completes the proof. □

We introduce the function $u^\omega : \mathbb{R}^+ \times [0,T] \to \mathbb{R}^+$ of the form

$$u^\omega(s,t) := V_{\mathrm{E}}^\omega(s, T-t)$$

and we prove the convexity of $u^\omega(s,t)$ with respect to s. Note that it is equivalent to the convexity of the value function $V_{\mathrm{E}}^\omega(s,t)$ in s. Furthermore, based on Lemma 4, the function $u^\omega(s,t)$ solves the Cauchy problem of the form

$$\begin{cases} \frac{\partial u^\omega(s,t)}{\partial t} = \hat{\mathcal{L}} u^\omega(s,t), & (s,t) \in \mathbb{R}^+ \times (0,T], \\ u^\omega(s,0) = g(s), & s \in \mathbb{R}^+, \end{cases}$$

where

$$\begin{aligned} \hat{\mathcal{L}} u^\omega(s,t) = {} & \beta(s,t) \frac{\partial^2 u^\omega(s,t)}{\partial s^2} + \mu(s,t) \frac{\partial u^\omega(s,t)}{\partial s} - \omega(s) u^\omega(s,t) \\ & + \int_{\mathbb{R}} \left(u^\omega(s+\gamma(s,t,z),t) - u^\omega(s,t) - \gamma(s,t,z) \frac{\partial u^\omega(s,t)}{\partial s} \right) \Pi(dz) \end{aligned}$$

with $\beta(s,t) = \frac{\sigma^2(s,t)}{2}$. Observe that by Lemma 5 there exist constants $n > 0$ and $K > 0$ such that

$$\left| \frac{\partial^2 u^\omega(s,t)}{\partial s^2} \right| \le K(s^{-n} + s^n) \tag{4.4}$$

for all $(s,t) \in \mathbb{R}^+ \times [0,T]$.

Let us now define a convex function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ of the form

$$\kappa(s) := s^{n+3} + s^{-n+1}$$

with

$$\frac{d^2\kappa(s)}{ds^2} = (n+3)(n+2)s^{n+1} + n(n-1)s^{-n-1}$$

and

$$\begin{aligned} \frac{d^2(\hat{\mathcal{L}}\kappa(s))}{ds^2} = {} & \frac{\partial^2 \beta(s,t)}{\partial s^2} \frac{d^2\kappa(s)}{ds^2} + 2 \frac{\partial \beta(s,t)}{\partial s} \frac{d^3\kappa(s)}{ds^3} + \beta(s,t) \frac{d^4\kappa(s)}{ds^4} \\ & + \frac{\partial^2 \mu(s,t)}{\partial s^2} \frac{d\kappa(s)}{ds} + 2 \frac{\partial \mu(s,t)}{\partial s} \frac{d^2\kappa(s)}{ds^2} + \mu(s,t) \frac{d^3\kappa(s)}{ds^3} \\ & - \frac{d^2\omega(s)}{ds^2} \kappa(s) - 2 \frac{d\omega(s)}{ds} \frac{d\kappa(s)}{ds} - \omega(s) \frac{d^2\kappa(s)}{ds^2} \\ & + \int_{\mathbb{R}} \left(\frac{d^2\kappa(s+\gamma(s,t,z))}{ds^2} \left(1 + \frac{\partial \gamma(s,t,z)}{\partial s} \right)^2 \right. \\ & + \frac{d\kappa(s+\gamma(s,t,z))}{ds} \frac{\partial^2 \gamma(s,t,z)}{\partial s^2} - \gamma(s,t,z) \frac{d^3\kappa(s)}{ds^3} \\ & \left. - \left(1 + 2 \frac{\partial \gamma(s,t,z)}{\partial s} \right) \frac{d^2\kappa(s)}{ds^2} - \frac{\partial^2 \gamma(s,t,z)}{\partial s^2} \frac{d\kappa(s)}{ds} \right) \Pi(dz). \end{aligned}$$

The assumptions we make on the coefficients μ, σ, γ and the function ω and their derivatives imply that each component of the above expression grows at most as s^{n+1} for large s and as s^{-n-1} for small s. The same behaviour characterises $\frac{d^2\kappa(s)}{ds^2}$.

In addition, we define the function $\vartheta : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ given by

$$\vartheta(s,t) := \left(\frac{\partial^2 \mu(s,t)}{\partial s^2} - 2\frac{d\omega(s)}{ds} \right) \frac{d\kappa(s)}{ds} - \frac{d^2\omega(s)}{ds^2}\kappa(s)$$

which also behaves as $\frac{d^2(\hat{\mathcal{L}}\kappa(s))}{ds^2}$ at 0 and ∞.

Hence, we claim that there exists a positive constant C such that

$$C\frac{d^2\kappa(s)}{ds^2} - \frac{d^2(\hat{\mathcal{L}}\kappa(s))}{ds^2} > -\vartheta(s,t). \tag{4.5}$$

In the second part of the proof, we define the auxiliary function

$$u^\omega_\varepsilon(s,t) := u^\omega(s,t) + \varepsilon e^{Ct}\kappa(s) \tag{4.6}$$

for some $\varepsilon > 0$.

We carry out a proof by contradiction. Then assume that $u^\omega_\varepsilon(s,t)$ is not convex. For this purpose, we denote by Λ the set of points for which $u^\omega_\varepsilon(s,t)$ is not convex, that is

$$\Lambda := \{(s,t) \in \mathbb{R}^+ \times [0,T] : \frac{\partial^2 u^\omega_\varepsilon(s,t)}{\partial s^2} < 0\}$$

and we assume that the set Λ is not empty.

From Lemma 5 we know that $u^\omega(s,t)$ satisfies (4.4). Due to this fact and using (4.6), we claim that there exists a positive constant R such that $\Lambda \subseteq [R^{-1}, R] \times [0,T]$. This is a direct consequence of the choice of $u^\omega_\varepsilon(s,t)$ in (4.6) that $\frac{d^2\kappa(s)}{ds^2}$ grows faster than $\frac{\partial^2 u^\omega(s,t)}{\partial s^2}$ for large and small values of s.

Consequently, the set Λ is a bounded set. Since the closure of a bounded set is also bounded, we conclude that the closure of Λ, i.e. $\mathrm{cl}(\Lambda)$, is compact.

Since a compact set always contains its infimum, we can define

$$t_0 := \inf\{t \geq 0 : (s,t) \in \mathrm{cl}(\Lambda) \text{ for some } s \in \mathbb{R}^+\}.$$

From the initial condition, that is $u^\omega(s,0) = g(s)$ and the convexity of g, we have the following

$$\frac{d^2 u^\omega_\varepsilon(s,0)}{ds^2} = \frac{d^2(g(s) + \varepsilon\kappa(s))}{ds^2} \geq \varepsilon\frac{d^2\kappa(s)}{ds^2} > 0$$

for all $s \in \mathbb{R}^+$. Therefore, we can conclude that $t_0 > 0$.

Furthermore, at the point where the infimum is reached, that is (s_0, t_0) for some $s_0 \in \mathbb{R}^+$

$$\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2} = 0.$$

This is a consequence of the continuity of the function $\frac{\partial^2 u^\omega_\varepsilon(s,t)}{\partial s^2}$ in s. In addition, for $t \in [0, t_0)$ we have $\frac{\partial^2 u^\omega_\varepsilon(s_0,t)}{\partial s^2} > 0$ and applying the symmetry of the second derivatives at $t = t_0$, we derive

$$\frac{\partial^2}{\partial s^2}\left(\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2}\right) \leq 0. \tag{4.7}$$

Furthermore, at (s_0,t_0) we also have

$$\begin{aligned}\frac{\partial^2(\hat{\mathcal{L}}u^\omega_\varepsilon(s_0,t_0))}{\partial s^2} &= \frac{\partial^2\beta(s_0,t_0)}{\partial s^2}\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2} + 2\frac{\partial\beta(s_0,t_0)}{\partial s}\frac{\partial^3 u^\omega_\varepsilon(s_0,t_0)}{\partial s^3} \\ &+ \beta(s_0,t_0)\frac{\partial^4 u^\omega_\varepsilon(s_0,t_0)}{\partial s^4} + \frac{\partial^2\mu(s_0,t_0)}{\partial s^2}\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s} \\ &+ 2\frac{\partial\mu(s_0,t_0)}{\partial s}\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2} + \mu(s_0,t_0)\frac{\partial^3 u^\omega_\varepsilon(s_0,t_0)}{\partial s^3} \\ &- \frac{d^2\omega(s_0)}{ds^2}u^\omega_\varepsilon(s_0,t_0) - 2\frac{d\omega(s_0)}{ds}\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s} - \omega(s_0)\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2} \\ &+ \int_{\mathbb{R}}\left(\frac{\partial^2 u^\omega_\varepsilon(s_0+\gamma(s_0,t_0,z),t_0)}{\partial s^2}\left(1+\frac{\partial\gamma(s_0,t_0,z)}{\partial s}\right)^2\right. \\ &+ \frac{\partial u^\omega_\varepsilon(s_0+\gamma(s_0,t_0,z),t_0)}{\partial s}\frac{\partial^2\gamma(s_0,t_0,z)}{\partial s^2} - \gamma(s_0,t_0,z)\frac{\partial^3 u^\omega_\varepsilon(s_0,t_0)}{\partial s^3} \\ &\left. - \left(1+2\frac{\partial\gamma(s_0,t_0,z)}{\partial s}\right)\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2} - \frac{\partial^2\gamma(s_0,t_0,z)}{\partial s^2}\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s}\right)\Pi(dz).\end{aligned}$$

Since $\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2}=0$ and $\frac{\partial^2 u^\omega_\varepsilon(s,t_0)}{\partial s^2}$ has a local minimum at $s=s_0$, we have $\frac{\partial^3 u^\omega_\varepsilon(s_0,t_0)}{\partial s^3}=0$ and $\frac{\partial^4 u^\omega_\varepsilon(s_0,t_0)}{\partial s^4}\geq 0$. Thus,

$$\begin{aligned}\frac{\partial^2(\hat{\mathcal{L}}u^\omega_\varepsilon(s_0,t_0))}{\partial s^2} &\geq \frac{\partial^2\mu(s_0,t_0)}{\partial s^2}\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s} - \frac{d^2\omega(s_0)}{ds^2}u^\omega_\varepsilon(s_0,t_0) \\ &- 2\frac{d\omega(s_0)}{ds}\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s} \\ &+ \int_{\mathbb{R}}\left(\frac{\partial u^\omega_\varepsilon(s_0+\gamma(s_0,t_0,z),t_0)}{\partial s}\frac{\partial^2\gamma(s_0,t_0,z)}{\partial s^2}\right. \\ &\left. - \frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial s}\frac{\partial^2\gamma(s_0,t_0,z)}{\partial s^2}\right)\Pi(dz).\end{aligned}$$

Since $u^\omega_\varepsilon(s,t_0)$ is convex in s and $\frac{\partial^2 u^\omega_\varepsilon(s_0,t_0)}{\partial s^2}=0$, applying (2.4), we can conclude that the integral part of the above expression is non-negative. Moreover, (2.5) implies that

$$\frac{\partial^2(\hat{\mathcal{L}}u^\omega_\varepsilon(s_0,t_0))}{\partial s^2} \geq \varepsilon e^{Ct_0}\left(\left(\frac{\partial^2\mu(s_0,t_0)}{\partial s^2} - 2\frac{d\omega(s_0)}{ds}\right)\frac{d\kappa(s_0)}{ds} - \frac{d^2\omega(s_0)}{ds^2}\kappa(s_0)\right) = \varepsilon e^{Ct_0}\vartheta(s_0,t_0). \tag{4.8}$$

Combining (4.5) with (4.7) and (4.8) at (s_0,t_0), we derive the following result

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\left(\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial t} - \hat{\mathcal{L}}u^\omega_\varepsilon(s_0,t_0)\right) &= \varepsilon e^{Ct_0}\frac{d^2}{ds^2}(C\kappa(s_0) - \hat{\mathcal{L}}\kappa(s_0)) \\ &> -\varepsilon e^{Ct_0}\vartheta(s_0,t_0) \geq \frac{\partial^2}{\partial s^2}\left(\frac{\partial u^\omega_\varepsilon(s_0,t_0)}{\partial t} - \hat{\mathcal{L}}u^\omega_\varepsilon(s_0,t_0)\right)\end{aligned}$$

which is a contradiction. It confirms that the set Λ is empty and thus $u^\omega_\varepsilon(s,t)$ is a convex function. Finally, letting $\varepsilon\to 0$ we conclude that $u^\omega(s,t)$ is convex in s for all $t\in[0,T]$. $\square$

Using the same arguments as in the proof of [68, Theorem 4.1, p. 389], we can resign from Assumptions (B) and (C) in Theorem 10, that is the following theorem holds.

Theorem 11. *Let the assumptions of Theorem 1 hold. Then $V^{\omega}_{\mathrm{E}}(s,t)$ is convex with respect to s at all times $t \in [0,T]$.*

We are ready to give the proof of our first main result, that is Theorem 1. We also recall the content of this theorem.

Theorem 1. *Let Assumptions (A) hold. Assume that the payoff function g is convex, ω is concave, the stock price process S_t follows (2.1), and the following inequalities are satisfied*

$$\frac{\partial^2 \gamma(s,t,z)}{\partial s^2}\gamma(s,t,z) \geq 0, \tag{2.4}$$

$$\left(\frac{\partial^2 \mu(s,t)}{\partial s^2} - 2\frac{d\omega(s)}{ds}\right)\frac{\partial V^{\omega}_{\mathrm{E}}(s,t)}{\partial s} - \frac{d^2\omega(s)}{ds^2}V^{\omega}_{\mathrm{E}}(s,t) \geq 0, \tag{2.5}$$

where $V^{\omega}_{\mathrm{E}}(s,t)$ is defined in (2). Then the value function $V^{\omega}_{\mathrm{A}}(s)$ is convex as a function of s.

Proof of Theorem 1. As noted in [68, Section 7, p. 395], under Assumptions (A1)–(A4), for each $p \geq 1$ there exists a constant C such that the stock price process given in (2.1) satisfies

$$\mathbb{E}_s\left[\sup_{0\leq t\leq T}|S_t|^p\right] \leq C(1+s^p).$$

Together with (A5) and (A6) it implies that the value function given by

$$V^{\omega}_{\mathrm{A}_T}(s,t) := \sup_{\tau\in\mathcal{T}_t^T}\mathbb{E}_{s,t}\left[e^{-\int_t^\tau \omega(S_w)dw}g(S_\tau)\right]$$

is well-defined, where $\mathcal{T}_t^T$ is the family of $\mathbb{F}$-stopping times with values in $[t,T]$ for fixed maturity $T>0$. Moreover, we denote

$$V^{\omega}_{\mathrm{A}_T}(s) := V^{\omega}_{\mathrm{A}_T}(s,0).$$

Let us now define a Bermudan option with the value function of the form

$$V^{\omega}_{\mathrm{B}_\Xi}(s,t) := \sup_{\tau\in\mathcal{T}_\Xi}\mathbb{E}_{s,t}\left[e^{-\int_t^\tau \omega(S_w)dw}g(S_\tau)\right],$$

where $\mathcal{T}_\Xi$ is the set of stopping times with values in

$$\mathrm{B}_\Xi = \left\{\frac{n}{2^\Xi}(T-t)+t : n = 0,1,\ldots,2^\Xi\right\},$$

where Ξ is some positive integer number. To simplify the notation, we denote

$$V^{\omega}_{\mathrm{B}_\Xi}(s) := V^{\omega}_{\mathrm{B}_\Xi}(s,0).$$

In contrast to the American options, the Bermudan options are the options that can be exercised in one of finitely many times.

Now, we show that $V^{\omega}_{\mathrm{B}_\Xi}(s,t)$ inherits the property of convexity from its European equivalent $V^{\omega}_{\mathrm{E}}(s,t)$. Next, we generalise this result to the American case $V^{\omega}_{\mathrm{A}}(s)$.

Lemma 6. *Let the assumptions of Theorem 1 hold. Then $V^{\omega}_{B_\Xi}(s,t)$ is convex with respect to s at all times $t \in [0,T]$.*

As the possible exercise times of the Bermudan option become more dense, the value function $V^{\omega}_{\mathrm{B}_\Xi}(s,t)$ converges to $V^{\omega}_{\mathrm{A}_T}(s,t)$. To formalise this result, we proceed as follows. For a given stopping time τ_0^T that takes values in $[0,T]$, we define

$$\tau_\Xi := \inf\{t \in \mathrm{B}_\Xi : t \geq \tau_0^T\}.$$

Then $\tau_\Xi \in \mathrm{B}_\Xi$ is a stopping time and $\tau_\Xi \to \tau_0^T$ almost surely as $\Xi \to \infty$. Moreover, by the dominated convergence theorem, we obtain the following

$$\begin{aligned}&\left|\mathbb{E}_s\left[e^{-\int_0^{\tau_\Xi}\omega(S_w)dw}g(S_{\tau_\Xi})\right] - \mathbb{E}_s\left[e^{-\int_0^{\tau_0^T}\omega(S_w)dw}g(S_{\tau_0^T})\right]\right|\\
&\quad\leq \mathbb{E}_s\left|e^{-\int_0^{\tau_\Xi}\omega(S_w)dw}g(S_{\tau_\Xi}) - e^{-\int_0^{\tau_0^T}\omega(S_w)dw}g(S_{\tau_0^T})\right| \to 0\end{aligned}$$

as $\Xi \to \infty$. Therefore, it follows that

$$\liminf_{\Xi\to\infty} V^{\omega}_{\mathrm{B}_\Xi}(s) \geq V^{\omega}_{\mathrm{A}_T}(s)$$

It is obvious that

$$V^{\omega}_{\mathrm{B}_\Xi}(s) \leq V^{\omega}_{\mathrm{A}_T}(s),$$

so we finally derive

$$V^{\omega}_{\mathrm{B}_\Xi}(s) \to V^{\omega}_{\mathrm{A}_T}(s)$$

as $\Xi \to \infty$. We take maturity T tending to infinity to receive our claim. □

Now we present the proof of the most significant result in our work, that is Theorem 3 with its content.

Theorem 3. *Assume that the stock price process S_t is described by* (2.7) *and ω is a measurable, bounded from below, concave and non-decreasing function such that*

$$\omega(s) = c \textit{ for all } s \in (0,1] \textit{ and some constant } c \in \mathbb{R}. \tag{2.8}$$

Then

$$\begin{aligned}v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s,l,u) &= \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(l)}(K-l)\mathbb{1}_{\{s<l\}} + (K-s)\mathbb{1}_{\{s\in[l,u]\}}\\
&+\left\{\int_0^\infty\int_0^\infty \frac{\mathscr{H}^{(\omega_u)}((\frac{u}{e^y})\wedge l)}{\mathscr{H}^{(\omega_u)}(l)}(K-e^{\log l\vee(\log u-y)})r(s,u,z)\Pi(-z-dy)dz\right.\\
&\left.+(K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^\alpha\left(\mathscr{Z}_\alpha^{(\omega_u^\alpha)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_\alpha^{(\omega^\alpha)}/\mathscr{W}_\alpha^{(\omega^\alpha)}}\mathscr{W}_\alpha^{(\omega_u^\alpha)}\left(\frac{s}{u}\right)\right)\right)\right\}\mathbb{1}_{\{s>u\}},\end{aligned}$$

where

$$c_{\mathscr{Z}_\alpha^{(\omega^\alpha)}/\mathscr{W}_\alpha^{(\omega^\alpha)}} = \lim_{z\to\infty}\frac{\mathscr{Z}_\alpha^{(\omega^\alpha)}(z)}{\mathscr{W}_\alpha^{(\omega^\alpha)}(z)}$$

and $r(s,u,z)$ *is given in* (1.20).
If $l=0$ *then condition* (2.8) *is superfluous and*

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s,0,u) &= (K-s)\mathbb{1}_{\{s\in[0,u]\}} \\ &+ \Bigg\{ \int_0^\infty \int_0^\infty (K-e^{\log u - y}) r(s,u,z)\Pi(-z-dy)dz \\ &+ (K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}}\mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right)\Bigg\}\mathbb{1}_{\{s>u\}}. \end{aligned}$$

Proof of Theorem 3. We recall the following exit identities

$$\sigma_a^- = \inf\{t\geq 0 : X_t \leq a\} \quad \text{and} \quad \sigma_a^+ = \inf\{t\geq 0 : X_t \geq a\}$$

for $a\in\mathbb{R}$. We also define the following

$$\sigma_{a,b} := \inf\{t\geq 0 : X_t \in [a,b]\}$$

for $a\leq b$ and $a,b\in\mathbb{R}$.

From [105, Theorem 2.5, p. 3279] and [105, Corollary 2.1, p. 3276] we have

$$\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_a^+}\eta(X_w)\,dw}; \sigma_a^+ < \infty\right] = \frac{\mathcal{H}^{(\eta)}(x)}{\mathcal{H}^{(\eta)}(a)}, \tag{4.9}$$

$$\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_0^-}\eta(X_w)\,dw}; \sigma_0^- < \infty\right] = \mathcal{Z}^{(\eta)}(x) - c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}\mathcal{W}^{(\eta)}(x), \tag{4.10}$$

where $c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}} = \lim_{z\to\infty}\frac{\mathcal{Z}^{(\eta)}(z)}{\mathcal{W}^{(\eta)}(z)}$ and η is defined in (1.18). In (4.9) we additionally assume that $\eta(x)=c$ for all $x\leq 0$ and some constant $c\in\mathbb{R}$.

Denoting

$$\tau_a^- := \inf\{t\geq 0 : S_t \leq a\} \quad \text{and} \quad \tau_a^+ := \inf\{t\geq 0 : S_t \geq a\},$$

where $S_t = e^{X_t}$, we can conclude, from (4.9) and (4.10), that

$$\begin{aligned} \mathbb{E}_s\left[e^{-\int_0^{\tau_a^+}\omega(S_w)\,dw}; \tau_a^+ < \infty\right] &= \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(a)}, \\ \mathbb{E}_s\left[e^{-\int_0^{\tau_1^-}\omega(S_w)\,dw}; \tau_1^- < \infty\right] &= \mathscr{Z}^{(\omega)}(s) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega)}(s), \end{aligned} \tag{4.11}$$

where $\omega(s)=\omega(e^x)=\eta(x)$ and the functions $\mathscr{Z}^{(\omega)}(s)$, $\mathscr{W}^{(\omega)}(s)$, $\mathscr{H}^{(\omega)}(s)$ are defined in (1.14), (1.15) and (1.16).

We consider three possible cases of the initial state $S_0 = s$:

1. $s<l$: As the process S_t is spectrally negative and starts below the interval $[l,u]$, it can enter this interval only in a continuous way, and hence $\tau_{l,u}=\tau_l^+$ and $S_{\tau_{l,u}}=l$. Thus, from (4.11)

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s,l,u) &= \mathbb{E}_s\left[e^{-\int_0^{\tau_l^+}\omega(S_w)dw}; S_{\tau_l^+}=l\right](K-l) \\ &= \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(l)}(K-l). \end{aligned}$$

2. $s \in [l,u]$: If the process S_t starts within the interval $[l,u]$, which is an optimal stopping region, we decide to exercise our option immediately, that is $\tau_{l,u} = 0$. Therefore, we have

$$v^{\omega}_{\mathrm{A^{Put}}}(s,l,u) = K - s.$$

3. $s > u$: There are three possible cases of entering the interval $[l,u]$ by the process S_t when it starts above u: S_t enters $[l,u]$ continuously going down, or jumps from (u,∞) to (l,u), or S_t jumps from the interval (u,∞) to the interval $(0,l)$ and then enters $[l,u]$ continuously.

We can distinguish these cases in the following way

$$\begin{aligned} v^{\omega}_{\mathrm{A^{Put}}}(s,l,u) = \mathbb{E}_s &\left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(K - S_{\tau_{l,u}}); \tau_u^- < \tau_l^- \right] \\ &+ \mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(K - S_{\tau_{l,u}}); \tau_u^- = \tau_l^- \right]. \end{aligned} \tag{4.12}$$

To analyse the first component in (4.12), note that

$$\begin{aligned} &\mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(K - S_{\tau_{l,u}}); \tau_u^- < \tau_l^- \right] = \mathbb{E}_s \left[e^{-\int_0^{\tau_u^-} \omega(S_w)dw}(K - S_{\tau_u^-}); S_{\tau_u^-} \in [l,u] \right] \\ &= \int_l^u (K-z)\mathbb{E}_s \left[e^{-\int_0^{\tau_u^-} \omega(S_w)dw}; S_{\tau_u^-} \in dz \right] + (K-u)\mathbb{E}_s \left[e^{-\int_0^{\tau_u^-} \omega(S_w)dw}; S_{\tau_u^-} = u \right]. \end{aligned}$$

Now we express the above formulas in the $X_t = \log S_t$ process. We recall that in (1.18) we also introduced the function $\eta_u(x) = \eta(x + \log u)$. Then

$$\begin{aligned} &\mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(K - S_{\tau_{l,u}}); \tau_u^- < \tau_l^- \right] \\ &= \int_{\log l}^{\log u} (K - e^z)\mathbb{E}_{(x)} \left[e^{-\int_0^{\sigma^-_{\log u}} \eta(X_w)dw}; X_{\sigma^-_{\log u}} \in dz \right] \\ &\quad + (K-u)\mathbb{E}_{(x)} \left[e^{-\int_0^{\sigma^-_{\log u}} \eta(X_w)dw}; X_{\sigma^-_{\log u}} = \log u \right] \\ &= \int_0^{\log u - \log l} (K - e^{\log u - y})\mathbb{E}_{(x-\log u)} \left[e^{-\int_0^{\sigma_0^-} \eta_u(X_w)dw}; -X_{\sigma_0^-} \in dy \right] \\ &\quad + (K-u)\mathbb{E}_{(x-\log u)} \left[e^{-\int_0^{\sigma_0^-} \eta_u(X_w)dw}; X_{\sigma_0^-} = 0 \right]. \end{aligned} \tag{4.13}$$

From the compensation formula for Lévy processes given in [99, Theorem 4.4, p. 95] we have

$$\mathbb{E}_{(x-\log u)} \left[e^{-\int_0^{\sigma_0^-} \eta_u(X_w)dw}; -X_{\sigma_0^-} \in dy \right] = \int_0^{\infty} r^{(\eta_u)}(x - \log u, z)\Pi(-z - dy)dz, \tag{4.14}$$

where $r^{(\eta_u)}(x - \log u, z)$ is the resolvent density of X_t killed by the potential η_u and when exiting the positive half-line, which is, by [105, Theorem 2.2, p. 3278], given by

$$r^{(\eta_u)}(x - \log u, z) = \mathcal{W}^{(\eta_u)}(x - \log u) \lim_{y\to\infty} \frac{\mathcal{W}^{(\eta_u)}(y,z)}{\mathcal{W}^{(\eta_u)}(y)} - \mathcal{W}^{(\eta_u)}(x - \log u, z).$$

Note that $r^{(\eta_u)}(\log s - \log u, z) = r(s,u,z)$ for $r(s,u,z)$ given in (1.20).

To find $\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; X_{\sigma_0^-}=0\right]$, we consider

$$\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw+\alpha X_{\sigma_0^-}}; \sigma_0^-<\infty\right] \tag{4.15}$$

for some $\alpha>0$. Note that using the change of measure given in (1.3), it is equal to

$$e^{\alpha(x-\log u)}\mathbb{E}^{(\alpha)}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u^\alpha(X_w)dw}; \sigma_0^-<\infty\right], \tag{4.16}$$

where $\mathbb{E}^{(\alpha)}_{(x-\log u)}$ is the expectation with respect to $\mathbb{P}^{(\alpha)}_{(x-\log u)}$ and $\eta_u^\alpha(x):=\eta_u(x)-\psi(\alpha)$. From (4.10) we know that

$$\mathbb{E}^{(\alpha)}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u^\alpha(X_w)dw}; \sigma_0^-<\infty\right]=\mathcal{Z}_\alpha^{(\eta_u^\alpha)}(x-\log u)-c_{\mathcal{Z}_\alpha^{(\eta^\alpha)}/\mathcal{W}_\alpha^{(\eta^\alpha)}}\mathcal{W}_\alpha^{(\eta_u^\alpha)}(x-\log u).$$

Moreover, observe that (4.15) can be written as

$$\begin{aligned}\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw+\alpha X_{\sigma_0^-}}; \sigma_0^-<\infty\right]&=\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; X_{\sigma_0^-}=0\right]\\&\quad+\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw+\alpha X_{\sigma_0^-}}; X_{\sigma_0^-}<0\right].\end{aligned}$$

Taking the limit $\alpha\to\infty$ and using (4.16), we derive

$$\lim_{\alpha\to\infty}e^{\alpha(x-\log u)}\mathbb{E}^{(\alpha)}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u^\alpha(X_w)dw}; \sigma_0^-<\infty\right]=\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; X_{\sigma_0^-}=0\right] \tag{4.17}$$

and, therefore, we have

$$\begin{aligned}&\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; X_{\sigma_0^-}=0\right]\\&=\lim_{\alpha\to\infty}e^{\alpha(x-\log u)}\left(\mathcal{Z}_\alpha^{(\eta_u^\alpha)}(x-\log u)-c_{\mathcal{Z}_\alpha^{(\eta^\alpha)}/\mathcal{W}_\alpha^{(\eta^\alpha)}}\mathcal{W}_\alpha^{(\eta_u^\alpha)}(x-\log u)\right).\end{aligned} \tag{4.18}$$

Furthermore, the second component of (4.12) is equal to

$$\begin{aligned}&\mathbb{E}_s\left[e^{-\int_0^{\tau_{l,u}}\omega(S_w)dw}(K-S_{\tau_{l,u}}); \tau_u^-=\tau_l^-\right]\\&=\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log l,\log u}}\eta(X_w)dw}(K-e^{X_{\sigma_{\log l,\log u}}}); \sigma_{\log u}^-=\sigma_{\log l}^-\right]\\&=\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log l,\log u}}\eta(X_w)dw}(K-e^{X_{\sigma_{\log l,\log u}}}); X_{\sigma_{\log u}^-}<\log l\right]\\&=\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log u}^-}\eta(X_w)dw}\mathbb{E}\left[e^{-\int_0^{\sigma_{\log l,\log u}}\eta(X_w)dw}(K-e^{X_{\sigma_{\log l,\log u}}}); X_{\sigma_{\log u}^-}<\log l\right]\right]\\&=\int_{\log u-\log l}^\infty\mathbb{E}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}\mathbb{E}_{(\log u-y)}\left[e^{-\int_0^{\sigma_{\log l}^+}\eta_u(X_w)dw}(K-e^{X_{\sigma_{\log l}^+}})\right]; -X_{\sigma_0^-}\in dy\right]\\&=\int_{\log u-\log l}^\infty\frac{\mathcal{H}^{(\eta_u)}(\log u-y)}{\mathcal{H}^{(\eta_u)}(\log l)}(K-l)\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; -X_{\sigma_0^-}\in dy\right].\end{aligned} \tag{4.19}$$

Now we have to express all the generalised scale functions in terms of S_t as defined in (1.14)–(1.17) with $x = \log s$ and using (1.18). Finally, using (4.12) together with (4.13), (4.14), (4.17) and (4.19) completes the proof of the first part of the theorem. If $l^* = 0$, then we can proceed as before, except that we do not need identity (4.11), and hence condition (2.8) is indeed superfluous. $\square$

The special cases of Theorem 3, that is Theorem 4 and Theorem 5, along with the proofs, are provided below.

Theorem 4. *Assume that ω is a bounded from below, concave and non-decreasing function. For the Black-Scholes model with X_t given in* (2.9)*, the function $v^{\omega}_{\mathrm{APut}}(s,l,u)$ defined in* (3) *is given by*

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s,l,u) = &\frac{h(s)}{h(l)}(K-l)\mathbb{1}_{\{s<l\}} + (K-s)\mathbb{1}_{\{s\in[l,u]\}} \\ &+ \frac{h(s)}{h(u)}(K-u)\mathbb{1}_{\{s>u\}}, \end{aligned}$$

where $h(s)$ is a solution to

$$\frac{\sigma^2 s^2}{2}h''(s) + rsh'(s) - \omega(s)h(s) = 0, \tag{2.10}$$

which satisfies

$$\begin{cases} h(s) = K - s, & s \in [l^*, u^*], \\ \lim_{s\to\infty} h(s) = \text{const.} \end{cases} \tag{2.11}$$

Proof of Theorem 4. We prove that for the function h satisfying (2.10), we have

$$\mathbb{E}_s\left[\frac{h(S_{\tau_{l,u}})}{h(s)}e^{-\int_0^{\tau_{l,u}}\omega(S_w)dw}\right] = 1. \tag{4.20}$$

Since the process S_t is continuous in the Black-Scholes model, $S_{\tau_{l,u}}$ equals l or u, depending on the initial state of S_t. We can distinguish three possible scenarios:

1. $s < l$: As the process S_t is a continuous process and starts below the interval $[l,u]$, then $\tau_{l,u} = \tau_l^+$ and $S_{\tau_{l,u}} = l$. Thus, we get

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s,l,u) &= \mathbb{E}_s\left[e^{-\int_0^{\tau_l^+}\omega(S_w)dw}; S_{\tau_l^+} = l\right](K-l) \\ &= \frac{h(s)}{h(l)}(K-l). \end{aligned} \tag{4.21}$$

2. $s \in [l,u]$: If the process S_t starts within the interval $[l,u]$, which is the optimal stopping region, we decide to exercise our option immediately, that is $\tau_{l,u} = 0$. Therefore, we have

$$v^{\omega}_{\mathrm{APut}}(s,l,u) = K - s. \tag{4.22}$$

3. $s > u$: Similarly to the case where $s < l$, the process S_t can enter $[l,u]$ only through u, and thus $\tau_{l,u} = \tau_u^-$ and $S_{\tau_{l,u}} = u$. Hence, we obtain

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s,l,u) &= \mathbb{E}_s\left[e^{-\int_0^{\tau_u^-}\omega(S_w)dw}; S_{\tau_u^-} = u\right](K-u) \\ &= \frac{h(s)}{h(u)}(K-u). \end{aligned} \tag{4.23}$$

Identities (4.21), (4.22) and (4.23) give the first part of the assertion of the theorem. Note that boundary condition (2.11) follows straightforwardly from the definition of the value function of the American put option.

We are left with the proof of (4.20). Therefore, we consider the strictly positive function $h \in C^2(\mathbb{R}^+) \subset D(\mathcal{A})$ that is bounded by some positive constant C. Then by [119, Proposition 3.2, p. 771] the process

$$E^h(t) := \frac{h(S_t)}{h(s)} e^{-\int_0^t \frac{(\mathcal{A}h)(S_w)}{h(S_w)} dw}$$

is a mean-one local martingale, whereas in the case of the Black-Scholes model, we have

$$\mathcal{A}h(s) = \frac{\sigma^2 s^2}{2} h''(s) + rsh'(s). \tag{4.24}$$

Observe that (4.24) is equivalent to (2.10) for

$$\omega(s) = \frac{\mathcal{A}h(s)}{h(s)}.$$

Let

$$\tau_{l,u}^M := \tau_{l,u} \wedge M$$

for some fixed $M > 0$.

Applying the optional stopping theorem for a bounded stopping time, we derive

$$\mathbb{E}_s \left[\frac{h(S_{\tau_{l,u}^M})}{h(s)} e^{-\int_0^{\tau_{l,u}^M} \omega(S_w) dw} \right] = 1. \tag{4.25}$$

We rewrite the left side of (4.25) as the sum of the following two components

$$I_1 := \mathbb{E}_s \left[\frac{h(S_{\tau_{l,u}^M})}{h(s)} e^{-\int_0^{\tau_{l,u}^M} \omega(S_w) dw}; \tau_{l,u} > M \right],$$
$$I_2 := \mathbb{E}_s \left[\frac{h(S_{\tau_{l,u}^M})}{h(s)} e^{-\int_0^{\tau_{l,u}^M} \omega(S_w) dw}; \tau_{l,u} \le M \right].$$

We now prove that $\lim_{M\to\infty} I_1 = 0$ and $\lim_{M\to\infty} I_2 \in (0, \infty)$. Let us define the last time the value function (3) is positive by

$$\tau_{\text{last}}(K) := \sup\{t \ge 0 : S_t \le K\}.$$

It is easy to see that $\mathbb{P}_s \left(\tau_{l,u}^M \le \tau_{\text{last}}(K) \right) = 1$. Then, from the boundedness of h, the lower boundedness of ω and the Cauchy-Schwarz inequality, we obtain the following

$$\begin{aligned} I_1 &\le \frac{C}{h(s)} \mathbb{E}_s \left[e^{-\underline{\omega} \tau_{\text{last}}(K)}; \tau_{l,u} > M \right] = \frac{C}{h(s)} \mathbb{E}_s \left[e^{-\underline{\omega} \tau_{\text{last}}(K)} \mathbb{1}_{\{\tau_{l,u} > M\}} \right] \\ &\le \frac{C}{h(s)} \sqrt{\mathbb{E}_s \left[e^{-2\underline{\omega} \tau_{\text{last}}(K)} \right] \mathbb{P}_s \left(\tau_{l,u} > M \right)}, \end{aligned}$$

where $\underline{\omega} := \min_{s \in \mathbb{R}^+} \omega(s)$. By [17, Theorem 2, p. 546] we note that $\sqrt{\mathbb{E}_s \left[e^{-2\underline{\omega} \tau_{\text{last}}(K)} \right]} < \infty$. Thus, $\lim_{M\to\infty} I_1 = 0$. Moreover,

$$0 < I_2 \le \frac{C}{h(s)} \mathbb{E}_s \left[e^{-\underline{\omega} \tau_{\text{last}}(K)}; \tau_{l,u} < M \right].$$

Therefore, by (4.25) and the dominated convergence theorem, we get (4.20) as long as h is positive and bounded. Finally, since $S_{\tau_{l,u}}$ equals l or u, the boundedness assumption could be skipped. This completes the proof. $\square$

Theorem 5. *Assume that ω is a non-negative, concave and non-decreasing function. For the exponential Lévy model with X_t given in (2.12), we have $l = 0$. Furthermore,*
(i) if $\sigma = 0$ and $\lambda > 0$ then

$$v^{\omega}_{\mathrm{APut}}(s, 0, u) = \left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right), \tag{2.14}$$

(ii) if $\sigma > 0$ and $\lambda = 0$ then

$$v^{\omega}_{\mathrm{APut}}(s, 0, u) = (K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}}\mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right), \tag{2.15}$$

(iii) if $\sigma > 0$ and $\lambda > 0$ then

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s, 0, u) = &\left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right) \\ &+ (K-u)\left(\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}}\mathscr{W}_{\alpha}^{(\omega_u^{\alpha})}\left(\frac{s}{u}\right)\right)\right), \end{aligned} \tag{2.16}$$

where

$$c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}} = \lim_{z\to\infty}\frac{\mathscr{Z}^{(\omega)}(z)}{\mathscr{W}^{(\omega)}(z)} \quad \text{and} \quad c_{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}/\mathscr{W}_{\alpha}^{(\omega^{\alpha})}} = \lim_{z\to\infty}\frac{\mathscr{Z}_{\alpha}^{(\omega^{\alpha})}(z)}{\mathscr{W}_{\alpha}^{(\omega^{\alpha})}(z)}. \tag{2.17}$$

The optimal boundary u^* in (2.15) and (2.16) can be determined by the smooth fit condition

$$(v^{\omega}_{\mathrm{APut}})'(u^*, 0, u^*) = -1,$$

while the optimal boundary u^* in (2.14) can be determined by the continuous fit condition

$$v^{\omega}_{\mathrm{APut}}(u^*, 0, u^*) = K - u^*.$$

Proof of Theorem 5. From Theorem 1 and Remark 3 it follows that the optimal exercise time is the first entry into the interval $[l, u]$ and by Theorem 2 the value function $V^{\omega}_{\mathrm{APut}}(s)$ is equal to the maximum over l and u of $v^{\omega}_{\mathrm{APut}}(s, l, u)$ defined in (3). We recall the observation that if the discount function ω is non-negative, it is never optimal to wait to exercise the option for small asset prices, that is always $l^* = 0$ in this case, and the stopping region is one-sided. Now we find the function $v^{\omega}_{\mathrm{APut}}(s, l, u)$ in the case of (i) and (ii).

If $\sigma = 0$, due to the lack of memory of exponential random variable and using a similar analysis to that used in the proof of Theorem 3, we have

$$\begin{aligned} v^{\omega}_{\mathrm{APut}}(s, 0, u) &= \mathbb{E}\left(K - e^{\log u - Y}\right)^{+}\mathbb{E}_s\left[e^{-\int_0^{\tau_u^-}\omega(S_w)dw}; \tau_u^- < \infty\right] \\ &= \left(K - \frac{u\rho}{\rho+1}\right)\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; \sigma_0^- < \infty\right] \\ &= \left(K - \frac{u\rho}{\rho+1}\right)\left(\mathcal{Z}^{(\eta_u)}(x - \log u) - c_{\mathcal{Z}^{(\eta)}/\mathcal{W}^{(\eta)}}\mathcal{W}^{(\eta_u)}(x - \log u)\right) \\ &= \left(K - \frac{u\rho}{\rho+1}\right)\left(\mathscr{Z}^{(\omega_u)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}^{(\omega)}/\mathscr{W}^{(\omega)}}\mathscr{W}^{(\omega_u)}\left(\frac{s}{u}\right)\right). \end{aligned}$$

It completes the proof of part (i).

If $\sigma > 0$, then

$$v^{\omega}_{\mathrm{A^{Put}}}(s,0,u) = \mathbb{E}\left(K - e^{\log u - Y}\right)^{+}\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; \sigma_0^- < \infty; X_{\sigma_0^-} < 0\right]$$
$$+ (K-u)\mathbb{E}_{(x-\log u)}\left[e^{-\int_0^{\sigma_0^-}\eta_u(X_w)dw}; \sigma_0^- < \infty; X_{\sigma_0^-} = 0\right].$$

The first increment can be analysed as in the case of $\sigma = 0$. The expression for the second component follows from (4.18).

Finally, the smooth fit condition follows from Theorem 7. $\square$

Theorem 6. *We assume that the function ξ is continuously differentiable. For the exponential Lévy model with X_t given in* (2.12) *we have*
(i) If $\sigma = 0$ and $\lambda > 0$ or $\lambda = 0$ and $\sigma > 0$, then $\mathcal{W}^{(\xi)}(x)$ solves

$$\mathcal{W}^{(\xi)\prime\prime}(x) = \left((\Upsilon_1+\Upsilon_2)\xi(x)+\gamma_2\right)\mathcal{W}^{(\xi)\prime}(x) + \left((\Upsilon_1+\Upsilon_2)\xi'(x) - \gamma_2\Upsilon_1\xi(x)\right)\mathcal{W}^{(\xi)}(x) \qquad (2.20)$$

with

$$\begin{cases}\mathcal{W}^{(\xi)}(0) = \Upsilon_1+\Upsilon_2,\\ \mathcal{W}^{(\xi)\prime}(0) = (\Upsilon_1+\Upsilon_2)^2\xi(0)+\Upsilon_2\gamma_2.\end{cases} \qquad (2.21)$$

Moreover, the function $\mathcal{Z}^{(\xi)}(x)$ solves the same equation (2.20) *with*

$$\begin{cases}\mathcal{Z}^{(\xi)}(0) = 1,\\ \mathcal{Z}^{(\xi)\prime}(0) = (\Upsilon_1+\Upsilon_2)\xi(0).\end{cases} \qquad (2.22)$$

(ii) If $\sigma > 0$ and $\lambda > 0$, then the function $\mathcal{W}^{(\xi)}(x)$ solves

$$\begin{aligned}\mathcal{W}^{(\xi)\prime\prime\prime}(x) &= (\gamma_2+\gamma_3)\,\mathcal{W}^{(\xi)\prime\prime}(x)\\ &\quad + \left(\Upsilon_2(\gamma_2-\gamma_3)\xi(x) - \gamma_2\gamma_3 - \gamma_3\Upsilon_1\xi(x)\right)\mathcal{W}^{(\xi)\prime}(x)\\ &\quad + \left(\Upsilon_2(\gamma_2-\gamma_3)\xi'(x) + \gamma_2\gamma_3\Upsilon_1\xi(x) - \gamma_3\Upsilon_1\xi'(x)\right)\mathcal{W}^{(\xi)}(x)\end{aligned} \qquad (2.23)$$

with

$$\begin{cases}\mathcal{W}^{(\xi)}(0) = 0,\\ \mathcal{W}^{(\xi)\prime}(0) = \Upsilon_2\gamma_2+\Upsilon_3\gamma_3,\\ \mathcal{W}^{(\xi)\prime\prime}(0) = \Upsilon_2{\gamma_2}^2+\Upsilon_3{\gamma_3}^2.\end{cases} \qquad (2.24)$$

Moreover, the function $\mathcal{Z}^{(\xi)}(x)$ solves the same equation (2.23) *with*

$$\begin{cases}\mathcal{Z}^{(\xi)}(0) = 1,\\ \mathcal{Z}^{(\xi)\prime}(0) = 0,\\ \mathcal{Z}^{(\xi)\prime\prime}(0) = \Upsilon_2(\gamma_2-\gamma_3)\xi(0) - \gamma_3\Upsilon_1\xi(0).\end{cases} \qquad (2.25)$$

Proof of Theorem 6. Assume first that $\sigma = 0$. Then

$$W(x) = \Upsilon_1 e^{\gamma_1 x} + \Upsilon_2 e^{\gamma_2 x} \qquad (4.26)$$

with $\gamma_1 = 0$. To produce the ordinary differential equation for $\mathcal{W}^{(\xi)}(x)$ we start from equation (1.14). Putting (4.26) there gives

$$\mathcal{W}^{(\xi)}(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_1 \int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy + \Upsilon_2 \int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy. \tag{4.27}$$

Taking the derivative of both sides gives

$$\mathcal{W}^{(\xi)\prime}(x) = \Upsilon_2\gamma_2 e^{\gamma_2 x} + \Upsilon_1\xi(x)\mathcal{W}^{(\xi)}(x) + \Upsilon_2\left(\xi(x)\mathcal{W}^{(\xi)}(x) + \gamma_2\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy\right). \tag{4.28}$$

From (4.27) we have

$$\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy = \frac{1}{\Upsilon_2}\left(\mathcal{W}^{(\xi)}(x) - \Upsilon_1 - \Upsilon_2 e^{\gamma_2 x} - \Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy\right).$$

We put it in (4.28) and derive

$$\mathcal{W}^{(\xi)\prime}(x) = ((\Upsilon_1+\Upsilon_2)\xi(x) + \gamma_2)\mathcal{W}^{(\xi)}(x) - \gamma_2\Upsilon_1 - \gamma_2\Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy.$$

We again take the derivative of both sides to get (2.20). From (1.14), (4.26) and (4.28) we derive both initial conditions (2.21).

Similar analysis can be performed for the function $\mathcal{Z}^{(\xi)}(x)$ that produces equation (2.20) and its initial conditions (2.22). It completes the proof of case (i).

In the case where $\sigma > 0$, observe that

$$W(x) = \Upsilon_1 e^{\gamma_1 x} + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x} \tag{4.29}$$

with $\gamma_1 = 0$. Thus, from (1.14) $\mathcal{W}^{(\xi)}(x)$ satisfies the following equation

$$\mathcal{W}^{(\xi)}(x) = \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x} + \int_0^x (\Upsilon_1 + \Upsilon_2 e^{\gamma_2(x-y)} + \Upsilon_3 e^{\gamma_3(x-y)})\xi(y)\mathcal{W}^{(\xi)}(y)dy.$$

We simplify it by deriving

$$\begin{aligned}\mathcal{W}^{(\xi)}(x) &= \Upsilon_1 + \Upsilon_2 e^{\gamma_2 x} + \Upsilon_3 e^{\gamma_3 x} + \Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy \\ &\quad + \Upsilon_2\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy + \Upsilon_3\int_0^x e^{\gamma_3(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy.\end{aligned} \tag{4.30}$$

In the next step, we take the derivative of both sides to get

$$\begin{aligned}\mathcal{W}^{(\xi)\prime}(x) &= \Upsilon_2\gamma_2 e^{\gamma_2 x} + \Upsilon_3\gamma_3 e^{\gamma_3 x} + \Upsilon_1\xi(x)\mathcal{W}^{(\xi)}(x) + \Upsilon_2\Big(\xi(x)\mathcal{W}^{(\xi)}(x) \\ &\quad + \gamma_2\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy\Big) + \Upsilon_3\Big(\xi(x)\mathcal{W}^{(\xi)}(x) \\ &\quad + \gamma_3\int_0^x e^{\gamma_3(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy\Big).\end{aligned} \tag{4.31}$$

From (4.30), we have

$$\begin{aligned}\int_0^x e^{\gamma_3(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy &= \frac{1}{\Upsilon_3}\Big(\mathcal{W}^{(\xi)}(x) - \Upsilon_1 - \Upsilon_2 e^{\gamma_2 x} - \Upsilon_3 e^{\gamma_3 x} \\ &\quad -\Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy - \Upsilon_2\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy\Big).\end{aligned}$$

We put it in (4.31) deriving

$$\begin{aligned}
\mathcal{W}^{(\xi)'}(x) &= \Upsilon_2(\gamma_2-\gamma_3)e^{\gamma_2 x} + (\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi(x)\mathcal{W}^{(\xi)}(x) \\
&\quad + \Upsilon_2(\gamma_2-\gamma_3)\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy + \gamma_3\mathcal{W}^{(\xi)}(x) - \gamma_3\Upsilon_1 \\
&\quad - \gamma_3\Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy.
\end{aligned} \tag{4.32}$$

Taking again the derivative of both sides, we obtain

$$\begin{aligned}
\mathcal{W}^{(\xi)''}(x) &= \Upsilon_2(\gamma_2-\gamma_3)\gamma_2 e^{\gamma_2 x} + (\Upsilon_1+\Upsilon_2+\Upsilon_3)(\xi'(x)\mathcal{W}^{(\xi)}(x) + \xi(x)\mathcal{W}^{(\xi)'}(x)) \\
&\quad + \Upsilon_2(\gamma_2-\gamma_3)\left(\xi(x)\mathcal{W}^{(\xi)}(x) + \gamma_2\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy\right) + \gamma_3\mathcal{W}^{(\xi)'}(x) \\
&\quad - \gamma_3\Upsilon_1\xi(x)\mathcal{W}^{(\xi)}(x).
\end{aligned} \tag{4.33}$$

From (4.32) we have

$$\begin{aligned}
\int_0^x e^{\gamma_2(x-y)}\xi(y)\mathcal{W}^{(\xi)}(y)dy = \frac{1}{\Upsilon_2(\gamma_2-\gamma_3)}\Big(&\mathcal{W}^{(\xi)'}(x) - \Upsilon_2(\gamma_2-\gamma_3)e^{\gamma_2 x} \\
&- (\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi(x)\mathcal{W}^{(\xi)}(x) - \gamma_3\mathcal{W}^{(\xi)}(x) + \gamma_3\Upsilon_1 \\
&+\gamma_3\Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy\Big).
\end{aligned}$$

We put it in (4.33) to get

$$\begin{aligned}
\mathcal{W}^{(\xi)''}(x) &= (\Upsilon_1+\Upsilon_2+\Upsilon_3)(\xi'(x)\mathcal{W}^{(\xi)}(x) + \xi(x)\mathcal{W}^{(\xi)'}(x)) \\
&\quad + \Upsilon_2(\gamma_2-\gamma_3)\xi(x)\mathcal{W}^{(\xi)}(x) \\
&\quad + \gamma_2\Big(\mathcal{W}^{(\xi)'}(x) - (\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi(x)\mathcal{W}^{(\xi)}(x) - \gamma_3\mathcal{W}^{(\xi)}(x) \\
&\quad +\gamma_3\Upsilon_1 + \gamma_3\Upsilon_1\int_0^x \xi(y)\mathcal{W}^{(\xi)}(y)dy\Big) \\
&\quad + \gamma_3\mathcal{W}^{(\xi)'}(x) - \gamma_3\Upsilon_1\xi(x)\mathcal{W}^{(\xi)}(x).
\end{aligned}$$

Taking again the derivative and simplifying, we have the following

$$\begin{aligned}
\mathcal{W}^{(\xi)'''}(x) &= \Big((\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi(x) + \gamma_2 + \gamma_3\Big)\mathcal{W}^{(\xi)''}(x) \\
&\quad + \Big(2(\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi'(x) + \Upsilon_2(\gamma_2-\gamma_3)\xi(x) - (\Upsilon_1+\Upsilon_2+\Upsilon_3)\gamma_2\xi(x) - \gamma_2\gamma_3 \\
&\quad - \gamma_3\Upsilon_1\xi(x)\Big)\mathcal{W}^{(\xi)'}(x) + \Big((\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi''(x) + \Upsilon_2(\gamma_2-\gamma_3)\xi'(x) \\
&\quad - \gamma_2(\Upsilon_1+\Upsilon_2+\Upsilon_3)\xi'(x) + \gamma_2\gamma_3\Upsilon_1\xi(x) - \gamma_3\Upsilon_1\xi'(x)\Big)\mathcal{W}^{(\xi)}(x).
\end{aligned}$$

Ultimately, taking into account the fact that $W(0) = 0$, we conclude that $\Upsilon_1+\Upsilon_2+\Upsilon_3 = 0$. Therefore, we obtain the equation we wanted to prove. From (4.29) and (1.14) we have

$$\mathcal{W}^{(\xi)}(0) = \Upsilon_1+\Upsilon_2+\Upsilon_3 = 0.$$

From (4.32) it follows that

$$\mathcal{W}^{(\xi)\prime}(0) = \Upsilon_2\gamma_2 + \Upsilon_3\gamma_3 + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)^2\xi(0) = \Upsilon_2\gamma_2 + \Upsilon_3\gamma_3.$$

Finally, from (4.33) we have

$$\begin{aligned}\mathcal{W}^{(\xi)\prime\prime}(0) &= \Upsilon_2\gamma_2(\gamma_2 - \gamma_3) + (\Upsilon_1 + \Upsilon_2 + \Upsilon_3)(\xi'(0)\mathcal{W}^{(\xi)}(0) + \xi(0)\mathcal{W}^{(\xi)\prime}(0)) \\ &\quad + \Upsilon_2(\gamma_2 - \gamma_3)\xi(0)\mathcal{W}^{(\xi)}(0) + \gamma_3\mathcal{W}^{(\xi)\prime}(0) - \gamma_3\Upsilon_1\xi(0)\mathcal{W}^{(\xi)}(0) \\ &= \Upsilon_2{\gamma_2}^2 + \Upsilon_3{\gamma_3}^2.\end{aligned}$$

The analysis of $\mathcal{Z}^{(\xi)}(x)$ can be done in the same way. It completes the proof. □

Theorem 7. *Assume that the asset price is a spectrally negative exponential Lévy process* (2.7). *Let ω be a concave function bounded from below with the opposite monotonicity to the payoff function g. Assume that $V_{\mathrm{A}}^{\omega}(s) \in D(\mathcal{A})$ and $g(s) \in C^1(\mathbb{R}^+)$. Then $V_{\mathrm{A}}^{\omega}(s)$ uniquely solves the following HJB system*

$$\begin{cases}\mathcal{A}V_{\mathrm{A}}^{\omega}(s) - \omega(s)V_{\mathrm{A}}^{\omega}(s) = 0, & s \notin [l^*, u^*], \\ V_{\mathrm{A}}^{\omega}(s) = g(s), & s \in [l^*, u^*].\end{cases} \tag{2.26}$$

Moreover, if 1 *is regular for* $(0,1)$ *and for the process S_t, then there is a smooth fit at the right end of the stopping region*

$$(V_{\mathrm{A}}^{\omega})'(u^*) = g'(u^*).$$

Similarly, if 1 *is regular for* $(1,\infty)$ *and for the process S_t then there is a smooth fit at the left end of the stopping region*

$$(V_{\mathrm{A}}^{\omega})'(l^*) = g'(l^*).$$

Proof of Theorem 7. From the fact that $V_{\mathrm{A}}^{\omega}(s) \in D(\mathcal{A})$ and that the Lévy process X_t is right-continuous and left-continuous over stopping times, we can conclude, using classical arguments, that $V_{\mathrm{A}}^{\omega}(s)$ solves uniquely equation (2.26), see [120, Theorem 2.4, p. 37] and [52] for details. More formally, our function as a convex function is continuous in a whole domain. Since our boundary is sufficiently regular, we know that the Dirichlet/Poisson problem can be solved uniquely in $D(\mathcal{A})$. This solution can then be identified with the value function $V_{\mathrm{A}}^{\omega}(s)$ itself using stochastic calculus or infinitesimal generator techniques in the continuation set; see [120, p. 131] for further details. Similar considerations have been made only for a local operator $\mathcal{A}$ in [137, Theorem 1, p. 1022]. Note that we can handle the non-local case of $\mathcal{A}$ only due to proving the convexity of the value function first. We are left with the proof of the smoothness at the boundary of the stopping set. We prove this at u^*. The proof at the lower end l^* follows exactly in the same way. We choose to follow the idea given in [102], although one can also apply [52] or arguments similar to those given in [53].

Suppose then that 1 is regular for $(0,1)$. Since $V_{\mathrm{A}}^{\omega}(s) \geq g(s)$ and $V_{\mathrm{A}}^{\omega}(u^*) = g(u^*)$, we have

$$\frac{V_{\mathrm{A}}^{\omega}(u^* + h) - V_{\mathrm{A}}^{\omega}(u^*)}{h} \geq \frac{g(u^* + h) - g(u^*)}{h}.$$

Hence

$$\liminf_{h\to 0^+} \frac{V_{\mathrm{A}}^{\omega}(u^* + h) - V_{\mathrm{A}}^{\omega}(u^*)}{h} \geq g'(u^*).$$

To get the opposite inequality, we introduce

$$\tau_h := \inf\{t \geq 0 : S_t \in [l^*, u^*] | S_0 = u^* + h\}.$$

From the assumed regularity, we have $\tau_h \to 0$ a.s. as $h \to 0^+$. Furthermore, from the Markov property, we have

$$V_{\mathrm{A}}^{\omega}(u^*) \geq \mathbb{E}_{u^*}\left[e^{-\int_0^{\tau_h} \omega(S_w)dw} g\left(S_{\tau_h}\right)\right]$$

Then by (A6) and the space homogeneity of $\log S_t$,

$$\begin{aligned}
\frac{V_{\mathrm{A}}^{\omega}(u^*+h) - V_{\mathrm{A}}^{\omega}(u^*)}{h} &\leq \frac{\mathbb{E}_{u^*+h}\left[e^{-\int_0^{\tau_h} \omega(S_w)dw} g\left(S_{\tau_h}\right)\right] - \mathbb{E}_{u^*}\left[e^{-\int_0^{\tau} \omega(S_w)dw} g\left(S_{\tau}\right)\right]}{h} \\
&\leq \frac{\mathbb{E}_1\left[e^{-\int_0^{\tau_h} \omega(S_w)dw} g\left((u^*+h)S_{\tau_h}\right)\right] - \mathbb{E}_1\left[e^{-\int_0^{\tau} \omega(S_w)dw} g\left(u^* S_{\tau}\right)\right]}{h}
\end{aligned}$$

and

$$\limsup_{h \to 0^+} \frac{V_{\mathrm{A}}^{\omega}(u^*+h) - V_{\mathrm{A}}^{\omega}(u^*)}{h} \leq g'(u^*),$$

where we use the fact that g is continuously differentiable at u^* in the last step. This completes the proof. $\square$

Theorem 8. *Assume that $\psi(1) < \infty$. Let $0 \leq l \leq u \leq K$. Then we have the following*

$$v_{\mathrm{A\,Call}}^{\omega}(s, K, \zeta, \sigma, \Pi, l, u) = v_{\mathrm{A\,Put}}^{\vartheta^{(1)}}\left(K, s, -\zeta, \sigma, \hat{\Pi}, \frac{lK}{s}, \frac{uK}{s}\right), \tag{2.27}$$

where

$$\begin{aligned}
\hat{\Pi}(dx) &= e^{-x}\Pi(-dx), \\
\vartheta^{(1)}(\cdot) &= \omega\left(\frac{1}{\cdot}\frac{s}{K}\right) - \psi(1).
\end{aligned} \tag{2.28}$$

Moreover, if the assumptions of Theorem 1 hold for the function $\vartheta^{(1)}$ then the American call option admits a double continuation region with optimal stopping boundaries l_c^* and u_c^* such that

$$\frac{l^*}{l_c^*} = \frac{u^*}{u_c^*} = \frac{K}{s}, \tag{2.29}$$

where l^* and u^* are the stopping limits for the put option.

Proof of Theorem 8. We recall that

$$\begin{aligned}
v_{\mathrm{A\,Call}}^{\omega}(s, K, \zeta, \sigma, \Pi, l, u) &= \mathbb{E}_s\left[e^{-\int_0^{\tau_{l,u}} \omega(S_w)dw}(S_{\tau_{l,u}} - K)^+\right] \\
&= \mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log l, \log u}} \eta(X_w)dw}(e^{X_{\sigma_{\log l, \log u}}} - K)^+\right],
\end{aligned}$$

where $x = \log s$ and $\sigma_{\log l, \log u} = \inf\{t \geq 0 : X_t \in [\log l, \log u]\}$.

By our assumption for the general Lévy process X_t, we can define a new measure $\mathbb{P}^{(1)}_{(0)}$ via

$$\left.\frac{d\mathbb{P}^{(1)}_{(0)}}{d\mathbb{P}_{(0)}}\right|_{\mathcal{F}_t} = e^{X_t - \psi(1)t},$$

see (1.3). Then

$$\begin{aligned}
&\mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log l, \log u}} \eta(X_w)dw}(e^{X_{\sigma_{\log l,\log u}}} - K)^+\right] \\
&= \mathbb{E}_{(x)}\left[e^{-\int_0^{\sigma_{\log l,\log u}}(\eta(X_w)-\psi(1))dw} e^{X_{\sigma_{\log l,\log u}} - \psi(1)\sigma_{\log l,\log u}}(1 - Ke^{-X_{\sigma_{\log l,\log u}}})^+\right] \\
&= \mathbb{E}^{(1)}_{(0)}\left[e^{-\int_0^{\sigma_{\log l - x,\log u - x}}(\eta(X_w+x)-\psi(1))dw}(e^x - Ke^{-X_{\sigma_{\log l-x,\log u-x}}})^+\right] \\
&= \mathbb{E}^{(1)}_K\left[e^{-\int_0^{\tau_{\frac{lK}{s},\frac{uK}{s}}}\left(\omega(\frac{s}{\hat{S}_w K}-\psi(1)\right)dw}(s - \hat{S}_{\tau_{\frac{lK}{s},\frac{uK}{s}}})^+\right] \\
&= v^{\vartheta(1)}_{\mathrm{APut}}\left(K, s, -\zeta, \sigma, \hat{\Pi}, \frac{lK}{s}, \frac{uK}{s}\right),
\end{aligned} \tag{4.34}$$

where $\hat{S}_t = e^{\hat{X}_t}$ and $\hat{X}_t = -X_t$ is the dual process for X_t and from [53, 74, 119] it follows that under $\mathbb{P}^{(1)}_{(0)}$ it is again the Lévy process with the triple $(-\zeta, \sigma, \hat{\Pi})$ for $\hat{\Pi}$ defined in (2.28). It completes the proof of identity (2.27). From general stopping theory, we know that the optimal stopping region for the call option is of the form $\tau_c = \inf\{t \geq 0 : S_t \in D_c\}$ for some stopping set D_c, see [120, Theorem 2.4, p. 37]. Performing the same transformation as in (4.34) with τ_c instead of $\tau_{l,u}$, we can conclude that the optimal stopping time for the call option is the same as the stopping time $\tau^* = \inf\{t \geq 0 : \hat{S}_t \in D\}$ for the put option (replacing $\tau_{\frac{lK}{s},\frac{uK}{s}}$ in this transformation on the right side), where $D = \{\frac{xK}{s} \quad \text{and} \quad x \in D_c\}$. However, from Theorem 2, we know that the optimal stopping time for the put option is the first entry time to some optimal interval. Thus, from the above considerations, it follows that the stopping region for the call option is of the same type as for the put option. Therefore, (2.29) is a consequence of (2.27). □

Conclusions

In this thesis, we provided an analysis of a perpetual American option with asset-dependent discounting. We focused mainly on the put option and derived several results related to this instrument. Our findings extend results known from the classical theory of option pricing. The foundation of our work is the assumption of a robust and functional dependence between the discount rate and the asset price. The optimal stopping problem considered in our research could provide a field of application in many branches of mathematics and other sciences, not only related to financial applications.

Our main goal of this dissertation was to derive a closed-form expression of the value function for the option analysed in the case where the asset price is modelled by a spectrally negative exponential Lévy process. To this end, we proved a number of auxiliary theorems and lemmas, starting with the convexity theorem of the value function. Thanks to the assumptions on the payoff function and the discount function, the convexity of the value function is satisfied, which allowed us to infer the form of the optimal stopping time in our problem. This key step enabled us to obtain the closed-form of the function $v^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s, l, u)$, whose maximisation with respect to the parameters l and u leads us to the value function we are looking for, denoted as $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$. In the second part of the work, we obtained specific cases for the value function in the case of the Black-Scholes model and the exponential Lévy process with downward exponential jumps. The final theoretical results in the dissertation are those related to the Hamilton-Jacobi-Bellman system and the put-call parity.

Lastly, we presented several examples in which we analytically or numerically determine the value function $V^{\omega}_{\mathrm{A}^{\mathrm{Put}}}(s)$ for different discount functions. For the Black-Scholes model, we considered the interesting case of a negative discount function. This scenario generates an untypical double continuation region. In turn, for the exponential Lévy process with downward exponential jumps, we took into account a few positive discount functions and presented analytical formulas of the value function together with its figures. These cases required solving differential equations to obtain generalised scale functions, which are contained in the formula for the value function. Due to the complexity of these equations, we presented two approaches to solving them, analytic and numerical. We verified that they return the same expected figures.

The results of this dissertation open up many directions for the future research. We would like to highlight here the most prospective or expansive ones. It is tempting to analyse other discount functions, for example, when ω is a random function or just a random variable dependent on the asset process S_t. One can take other processes as a discount rate where the dependence is introduced not only via correlation between Gaussian components but via a common jump structure. This jump dependence is crucial, since crashes in the market affect a large portion of business at the same time, see e.g. [38]. Another direction to extend our problem is to take a Poisson version of American options, where exercise is possible only at independent Poisson epochs. The first attempt at classical perpetual American options has been already made in

[118]. We believe that the present analysis can be generalised to this set-up. Moreover, it would be good to work out details for different payoff functions, hence for various options, like barrier, Russian, Israeli or Swing options. Another concept for research is to take into account Markov switching markets and to use omega scale matrices introduced in [49]. We expect that in this setting the optimal exercise time is also the first entrance time to the interval which ends depend on the governing Markov chain.

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