External Examiner Report on the PhD thesis
Perpetual American options with asset-dependent discounting
By Jonas Al-Hadad

This thesis is on pricing American options with asset-dependent discounting. It consists of four chapters. The first chapter is devoted to preliminaries. The second one states the main results of the thesis. The third chapter presents examples where the results of the second chapter are made more explicit either analytically or numerically. The fourth chapter gathers the proofs of the results presented in Chapter 2. It includes figures, plots and a bibliography of 147 references. I will now briefly summarise the content of each chapter.

Chapter 1 is devoted to a literature review and preliminaries on optimal stopping and American options pricing problems when the underlying asset price follows a spectrally negative geometric Lévy process. Classical results on scale functions and first passage problems for this class of one sided Lévy processes are recalled. This chapter contains also the statement, without a proof, of a theorem which will become Theorem 3 in Chapter 2.

Chapter 2 contains the main results of this thesis. It starts with setting the most general jump-diffusion setting which is as follows. Let the stock price $(S_t, t \geq 0)$ be defined on a complete risk neutral filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ by the stochastic differential equation

$$dS_t = \mu(S_{t_-}, t)dt + \sigma(S_{t_-}, t)dB_t + \int_{\mathbb{R}} \gamma(S_{t_-}, t, z)\tilde{v}(dt, dz), \quad t \geq 0, S_0 = s \in \mathbb{R} \quad (1)$$

where $\tilde{\nu}(dt, dz) = (v - q)(dt, dz)$ is a compensated jump martingale random measure of v and v itself is a homogeneous Poisson random measure on $\mathbb{R}_0^+ \times \mathbb{R}$ with intensity measure $q(dt, dz) = dt\Pi(dz)$ where Π is a finite measure. The following assumptions are made.

(A1) μ and $\sigma : \mathbb{R}^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ are continuous functions. $\gamma : \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ is measurable and for each fixed $z \in \mathbb{R}$, the function $(s, t) \to \gamma(s, t, z)$ is continuous.

(A2) We have $\mu^2(s, t) + \sigma^2(s, t) + \gamma^2(s, t, z) \leq Cs^2$ for all $(s, t, z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$, where C is some positive constant.

(A3) We have $|\mu(s_2t) - \mu(s_1, t)| + |\sigma(s_2, t) - \sigma(s_1, t)| + |\gamma(s_2, t, z) - \gamma(s_1, t, z)| \leq C|s_2 - s_1|$ for all $s_1, s_2 \in \mathbb{R}^+$ and $(t, z) \in \mathbb{R}_0^+ \times \mathbb{R}$, where C is some positive constant.

(A4) There exists some $C > -1$ such that $\gamma(s,t,z) > Cs$ for all $(z = s,t,z) \in \mathbb{R}^+ \times \mathbb{R}_0^+ \times \mathbb{R}$.

Assumptions (A1), (A2) and (A3) imply that there exists a unique solution to (1) whereas (A2) and (A4) imply that this solution satisfies $S_t > 0$ a.s. Next, let $g, \omega : \mathbb{R} \to \mathbb{R}$ be a gain and discounting functions, respectively. Define the optimal stopping problem

$$V_A^\omega(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s \left[e^{-\int_0^\tau \omega(S_r)dr} g(S_\tau) \right] \tag{2}$$

$\mathcal{T}$ is the set of stopping times with respect to $(\mathcal{F}_t)_{t \geq 0}$. The authors made the following assumptions on g and the value function V_A^ω.

(A5) The gain function g satisfies $g \in C_{Pol}(\mathbb{R}^+)$, where $C_{Pol}(\mathbb{R}^+)$ is the set of functions of at most polynomial growth.

(A6) $V_A^\omega(s) < \infty$ for all $s \in \mathbb{R}^+$.

These two assumptions insure that $V_A^\omega(s)$ is finite.

In this most general setting, Theorem 1 states the following. Assume that g and $-\omega$ are convex, $(S_t)_{t \geq 0}$ satisfies (1) and assumptions (A1)–(A6) hold. We furthermore suppose that

$$\gamma_{ss}\gamma \geq 0, \quad (\mu_{ss} - 2\omega_s) \frac{\partial}{\partial s} V_E^\omega - \omega_{ss} V_E^\omega \geq 0$$

where

$$V_E^\omega(s,t) = \mathbb{E}_{s,t} \left[e^{-\int_s^t \omega(S_r)dr} g(S_\tau) \right], \quad S_0 = s, t > 0.$$

Then the value function $s \to V_A^\omega(s)$ is convex.

In the rest of the thesis, the focus is on cases where the gain functions are of the form

$$g(x) = (K - x)^+, \quad x \in \mathbb{R}, \tag{3}$$

where K is a real number, i.e., (2) is the value function of a perpetual American put option, with exercise price K, and asset-depending discounting. Theorem 2 states that under the conditions of Theorem 1 and (3), the value function of (2) equals

$$V_{APut}^\omega(s) = \sup_{0 \leq l \leq u \leq K} V_{APut}^\omega(s,l,u)$$

where

$$V_{APut}^\omega(s,l,u) := \mathbb{E}_s \left[e^{-\int_0^{\tau_{l,u}} \omega(S_r)dr} \left(K - S_{\tau_{l,u}}\right)^+ \right] \quad ,$$

where $\tau_{l,u} := \inf\{t \geq 0; S_t \in [l,u]\}$. Furthermore, the optimal stopping rule is τ_{l^*,u^*} where l^* and u^* are the arguments that achieve the supremum above.

Next, the setting is simplified further by assuming that

$$S_t = e^{X_t}, \quad t \geq 0, \tag{4}$$

where $(X_t)_{t\geq 0}$ is a spectrally negative Lévy process started at $\log s$. That corresponds to $\mu(s,t) = \mu s$, $\sigma(s,t) = \sigma s$ and $\gamma(s,t,z) = sz$, where μ and σ are some real numbers, in (1). We refer to this as a spectrally negative geometric Lévy process.

Let us now make some recalls on scale functions for spectrally negative Lévy processes. Fort $q \geq 0$, denote by $W^{(q)}$ the unique right continuous function, vanishing on the negative real half-line, whose Laplace transform is

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

where $\psi(\theta) = \log \mathbb{E}[e^{\theta X_1}]$ is the Laplace exponent of $(X_t)_{t\geq 0}$ and $\Phi(q)$ is the right inverse of $s \to \psi(s)$ at q. We also introduce the ξ-scale functions, for a measurable function ξ, which are the unique solutions to the integral equations

$$\begin{aligned}
\mathcal{W}^{(\xi)}(x) &= W(x) + \int_0^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y)dy; \\
\mathcal{Z}^{(\xi)}(x) &= 1 + \int_0^x W(x-y)\xi(y)\mathcal{Z}^{(\xi)}(y)dy; \\
\mathcal{H}^{(\xi)}(x) &= e^{\Phi(c)x} + \int_0^x W^{(c)}(x-z)(\xi(z)-c)\mathcal{H}^{(\xi)}(z)dz; \\
\mathcal{W}^{(\xi)}(x,z) &= W(x-z) + \int_z^x W(x-y)\xi(y)\mathcal{W}^{(\xi)}(y,z)dz,
\end{aligned}$$

where $W = W^{(0)}$ is the classical scale function. Finally, we set

$$\begin{aligned}
\mathscr{W}^{(\xi)}(s) &= \mathcal{W}^{(\xi \circ exp)}(\log s); \\
\mathscr{F}^{(\xi)}(s) &= \mathcal{Z}^{(\xi \circ exp)}(\log s); \\
\mathscr{H}^{(\xi)}(s) &= \mathcal{H}^{(\xi \circ exp)}(\log s); \\
\mathscr{W}^{(\xi)}(s,z) &= \mathcal{W}^{(\xi \circ exp)}(\log s, z),
\end{aligned}$$

where $\xi \circ exp(x) = \xi(e^x)$, $x > 0$.

Theorem 3 states that if the stock price is a spectrally negative geometric Lévy process and ω is a measurable, bounded from below, concave and non-decreasing function such that $\omega(s) = c$ for all $s \in (0,1]$, for some $c \in \mathbb{R}$, then we have

$$\begin{aligned} V^{\omega}_{A^{Put}}(s,l,u) &= \frac{\mathscr{H}^{(\omega)}(s)}{\mathscr{H}^{(\omega)}(l)}(K-l)\mathbf{1}_{\{s<l\}} + (K-s)\mathbf{1}_{\{s\in[l,u]\}} \\ &+ \int_0^\infty\int_0^\infty \frac{\mathscr{H}^{(\omega_u)}((ue^{-y})\wedge l)}{\mathscr{H}^{(\omega_u)}(l)}(K-e^{\log u - y})r(s,u,z)\Pi(-z-dy)dz\mathbf{1}_{\{s>u\}} \\ &+ (K-u)\lim_{\alpha\to\infty}\left(\frac{s}{u}\right)^{\alpha}\left(\mathscr{Z}^{(\omega_u^\alpha)}\left(\frac{s}{u}\right) - c_{\mathscr{Z}_\alpha^{(\omega^\alpha)}/\mathscr{W}_\alpha^{(\omega^\alpha)}}\mathscr{W}_\alpha^{(\omega_u^\alpha)}\left(\frac{s}{u}\right)\right)\mathbf{1}_{s>u} \end{aligned}$$

where $\omega_u(s) = \omega(us)$, $s > 0$,

$$c_{\mathscr{Z}_\alpha^{(\omega^\alpha)}/\mathscr{W}_\alpha^{(\omega^\alpha)}} = \lim_{z\to\infty}\frac{\mathscr{Z}_\alpha^{(\omega^\alpha)}(z)}{\mathscr{W}_\alpha^{(\omega^\alpha)}(z)}$$

and

$$r(s,u,z) = \mathscr{W}^{(\omega_u)}(\log s - \log u)c_{\mathscr{Z}_\alpha^{(\omega^\alpha)}/\mathscr{W}_\alpha^{(\omega^\alpha)}} - \mathscr{W}^{(\omega_u)}(\log s - \log u, z).$$

Theorem 4 pushes further the calculations of Theorem 3 for the case when the stock price is a geometric Brownian motion with drift, i.e., the Black-Scholes model. Theorem 5 gives a more explicit formula of the value function of Theorem 3 in case where $(S_t, t \geq 0)$ is a spectrally negative geometric Lévy process with downward exponential jumps. In the same setting, Theorem 6 gives a characterisation of the ξ-scale functions $\mathcal{Z}^{(\xi)}$ and $\mathcal{W}^{(\xi)}$ in terms of differential equations with boundary conditions. In the setting of Theorem 3, Theorem 7 gives a characterisation of the value function as a solution to Hamilton-Jacobi-Bellman system while Theorem 8 gives the value function for an American call option through the put-call parity formula.

Chapter 3 consists of examples where the formulae of the theorems of Chapter 2 are made more explicit either analytically or numerically. Section 3.2 looks at the Black and Scholes model with discounting function given by $\omega(s) = -D - C(s+1)^{-1}$, $s \geq 0$, where D and C are positive real numbers. Section 3.3 is devoted to examples for the spectrally negative geometric Lévy process with downward exponential jumps. The case of constant, linear and power discount functions are treated in Section 3.3.1, Section 3.3.2 and Section 3.3.3, respectively. This chapter includes several plots of the different value functions of the examples and and corresponding payoffs, against the stock prices, for different values of the parameters of the models.

Chapter 4 gathers the proofs of the results presented in Chapter 2. It also includes some intermediate results with their proofs. Note that he proofs are well detailed, rigorous and clear.

Up to my knowledge, the candidate is the first to study optimal stopping problems with discounting in the general jump-diffusion setting considered here; examples of such models include the class of spectrally negative geometric Lévy processes which are considered in more details throughout this thesis. This work includes a good set of interesting and original results. The thesis is well written and I did not find any mathematical gap/errors in the statements or their proofs. There are few minor typos which I am not reporting here. The presentation is good although sometimes the text is a little hard to read/follow (for example notations used in Chapter 2 were introduced in the second chapter and there is no reference to where one could find the notations, chapter 3 is a list of successive examples which needs to be read together with Chapter 2 at the same time). I also noticed repetition of some statements, in particular, I do not know why the exact same statements that are given in Chapter 2 are reproduced in Chapter 4 before their proofs are given therein. The candidate has already a co-authored publication in the Journal of risk and financial management.

I believe that the candidate presents general theoretical knowledge in the field and the ability to independently conduct scientific work to advance knowledge in the area. I recommend Mr Jonas Al-Hadad be awarded the PhD in Mathematics.

Dr Larbi Alili

Coventry, Monday 8^{th} May, 2023.