Report on

On trees, ideals and cardinal invariants of the reals

Doctoral Dissertation by Aleksander Cieślak

The topic of the present thesis are cardinal invariants of ideals on the natural numbers ω or arbitrary countable sets, as well as cardinal invariants of σ-ideals on the real numbers ω^ω. It has been known that there is a strong connection between the two fields, for given a σ-ideal $\mathcal{I}$ in the Borel subsets $\mathcal{B}$ of ω^ω, there is a natural ideal on the countable set $\omega^{<\omega}$, the trace ideal $tr(\mathcal{I})$ consisting of all A whose G_δ-closure belongs to $\mathcal{I}$, such that – under some assumptions – the forcing notion $\mathcal{B}/\mathcal{I}$ is related to the forcing notion $\mathcal{P}(\omega^{<\omega})/tr(\mathcal{I})$, and the cardinal invariants of $\mathcal{I}$ are related to those of $tr(\mathcal{I})$.

More to the point for the present thesis is that many natural ideals on ω^ω can be defined from ideals on ω as follows: let $\mathcal{J}$ be an ideal on ω, and consider Miller forcing $\mathbb{M}_\mathcal{J}$ or Laver forcing $\mathbb{L}_\mathcal{J}$ with $\mathcal{J}$: a condition in $\mathbb{M}_\mathcal{J}$ is a tree in $\omega^{<\omega}$ such that above each node there is a node splitting into a $\mathcal{J}$-positive set of successors; in the case of $\mathbb{L}_\mathcal{J}$ every node above the stem splits into a $\mathcal{J}$-positive set of successors. With any forcing $\mathbb{P}$ whose conditions are subtrees of $\omega^{<\omega}$, one may associate the ideal $\mathcal{I}(\mathbb{P})$ of $\mathbb{P}$-null sets consisting of all $A \subseteq \omega^\omega$ such that for each $T \in \mathbb{P}$ there is $S \leq T$ in $\mathbb{P}$ such that $[S] \cap A = \emptyset$. For non-ccc forcing notions $\mathbb{P}$, these ideals are far from being Borel, and one may then consider the Borel part $\mathcal{I}_\mathcal{B}(\mathbb{P})$ of $\mathcal{I}(\mathbb{P})$ (the collection of Borel sets in $\mathcal{I}(\mathbb{P})$). For many $\mathbb{P}$ of this form (including $\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$), there is a dichotomy saying that an analytic set A either belongs to $\mathcal{I}_\mathcal{B}(\mathbb{P})$ or there is $T \in \mathbb{P}$ such that $[T] \subseteq A$. This means that $\mathbb{P}$ is forcing isomorphic to the quotient algebra $\mathcal{B}/\mathcal{I}_\mathcal{B}(\mathbb{P})$. For $\mathbb{P}$ being either $\mathbb{M}_\mathcal{J}$ or $\mathbb{L}_\mathcal{J}$, one may then compare the cardinal invariants of $\mathcal{J}$, of the Borel ideal $\mathcal{I}_\mathcal{B}(\mathbb{P})$ and of the Marczewski-like ideal $\mathcal{I}(\mathbb{P})$, and such comparisons are one of the main themes of this thesis.

Here is an outline with the most important results of this thesis:

After the introductory Chapter 1, Chapter 2 deals with cardinal invariants of the Borel ideals $\mathcal{I}_\mathcal{B}(\mathbb{P})$ for many choices of $\mathbb{P}$ including $\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$. For the former, a number of inequalities like $\min\{\mathfrak{b}, \mathsf{add}^*_\omega(\mathcal{J})\} \leq \mathsf{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \min\{\mathfrak{d}, \mathsf{add}^*_\omega(\mathcal{J})\}$ are presented (Theorem 2.2.33), while for the latter, the author establishes that the cardinal invariants of $\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})$ can be computed from those of $\mathcal{J}$ and the unbounding and dominating numbers: $\mathsf{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) = \min\{\mathfrak{b}, \mathsf{add}^*_\omega(\mathcal{J})\}$, $\mathsf{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) = \min\{\mathfrak{b}, \mathsf{cov}^*(\mathcal{J})\}$, and their dual results (Theorem 2.2.42).

Chapter 3 studies cardinal invariants of the Marczewski-like ideals $\mathcal{I}(\mathbb{P})$: first a general theorem (Theorem 3.1.1) relating the covering number of $\mathcal{I}(\mathbb{P})$ to cardinal invariants connected with generic reals added by $\mathbb{P}$ is established and used, e.g., for the inequalities $\mathsf{cov}(\mathcal{I}(\mathbb{M}_\mathcal{J})), \mathsf{cov}(\mathcal{I}(\mathbb{L}_\mathcal{J})) \leq \mathfrak{g}$. Next, the incompatibility shrinking number $\mathsf{is}(\mathbb{P})$ of a forcing $\mathbb{P}$ consisting of trees is introduced and compared to other cardinals; the assumption $\mathsf{is}(\mathbb{P}) = \mathfrak{c}$ later turns out to be crucial for giving upper bounds to the additivity $\mathsf{add}(\mathcal{I}(\mathbb{P}))$ of $\mathcal{I}(\mathbb{P})$. Another important cardinal invariant is the antichain number $\mathfrak{a}(\mathbb{P})$ of $\mathbb{P}$ (the minimal size of an uncountable maximal antichain in $\mathbb{P}$); for example it is proved that $\min\{\mathfrak{b}, \mathsf{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))\} \leq \mathfrak{a}(\mathbb{M}_\mathcal{J})$ for any ideal $\mathcal{J}$ (Theorem 3.3.7). Furthermore, distributivity is used to obtain upper bounds for the additivity $\mathsf{add}(\mathcal{I}(\mathbb{P}))$, while the constant or 1-1 property – or the incompatibility shrinking property – are useful for establishing that the cofinality $\mathsf{cof}(\mathcal{I}(\mathbb{P}))$ is strictly larger than the continuum $\mathfrak{c}$. The main result here is that for a P^+-ideal $\mathcal{J}$ that is not near maximal, $\mathbb{L}_\mathcal{J}$ has the constant or 1-1 property (Theorem 3.6.12).

In Chapter 4, cardinal invariants of ideals on ω are investigated: after a list of the important classical ideals in Subsection 4.1 and the computation of the ω-version of some of their cardinal invariants, the splitting ideal Spl of Sabok and Zapletal is studied in Subsection 4.2, and the new infinitely equal ideals are introduced in Subsection 4.3. Important results of this Chapter are $\mathsf{cov}^*(Spl) = \mathfrak{r}$ (Proposition 4.2.1), which gives a new upper bound for the covering* number of the density ideal $\mathcal{Z}$, namely $\mathsf{cov}^*(\mathcal{Z}) \leq \mathfrak{r}$, results on the +-version of the covering* numbers, like $\mathsf{cov}^*_+(\mathrm{nwd}) = \mathsf{add}(\mathcal{M})$ (Theorem 4.4.2), as well as consistency results about the order relationship of both cov^* and cov^*_+ numbers in Subsection 4.4, and results on antichain numbers in $\mathcal{P}(\omega)/\mathcal{J}$, like the inequality $\mathsf{add}(\mathcal{M}) \leq \mathfrak{a}(\mathrm{nwd})$, which improves a result of Steprāns and was obtained independently by Cancino (Theorem 4.5.5 and Corollary 4.5.8).

Overall, the thesis contains a very long list of new results on cardinal invariants of various ideals on both ω and ω^ω, which is a very active area of research in set theory. Clearly, these results are important and of interest for many people working in the field. It is a good thesis, which obviously satisfies the usual requirements for a PhD, and I strongly recommend that it be accepted as PhD thesis.

My main critique is that the thesis contains many misprints and inaccuracies, none of which are relevant for the main results of the work, but which make it harder to read. Examples are: (1) the wrong definition of F_C in 2.2.12 (it should be $C \subseteq^* x^{-1}[\{1\}]$; also it should be $\mathsf{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{r}$ and not $\mathfrak{r}_\sigma$); (2) the proof of 2.2.33 really shows the stronger $\min\{\mathfrak{d}, \mathsf{cov}^*(\mathcal{J})\} \leq \mathsf{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M}_{\mathcal{J}}))$ and similarly for the dual result; (3) the second part of 2.3.4, i.e. $J_{\subseteq}(\mathbb{P}) = \mathcal{I}_{\mathcal{B}}(\mathbb{P})$, is not proved, and I believe an additional assumption is needed here (namely, a dichotomy like for all of the particular examples) – this problem has some repercussions in Chapter 3 (e.g. 3.4.3); (4) in 4.4.20, $\mathsf{cov}(\mathcal{E})$ and $\mathsf{non}(\mathcal{E})$ should be switched. But, as I said, this does not really affect the importance and quality of this work.

[redacted]

Jörg Brendle
Graduate School of System Informatics
Kobe University
Rokko-dai 1-1, Nada-ku
657-8501 Kobe
Japan
email: brendle@kobe-u.ac.jp

22.09.2025