

Politechnika Wrocławska

DOCTORAL DISSERTATION

On trees, ideals and cardinal invariants of the reals

Aleksander Cieślak

Supervisor:
Dr. hab. Robert Rałowski
Assistant supervisor:
PhD. Arturo Martínez-Celis

Keywords: Miller trees, Laver trees, ideals, cardinal invariants, closed measure zero, covering, additivity, splitting ideal, destruction of an ideal, infinitely equal real, antichains

Wrocław 2025

Dla mojej babci

Contents

1 Introduction **4**
- 1.1 Summary of the work . . . 4
- 1.2 Preliminaries . . . 6
  - 1.2.1 Reals and trees . . . 6
  - 1.2.2 Forcing . . . 8
  - 1.2.3 Classical cardinal invariants of the reals . . . 10
  - 1.2.4 Ideals on countable sets . . . 12
  - 1.2.5 Relational systems . . . 14

2 Tree types and tree ideals **17**
- 2.1 Tree types and tree ideals . . . 17
- 2.2 Catalogue of tree types . . . 18
  - 2.2.1 Bounded infinitely equal trees . . . 18
  - 2.2.2 Splitting trees . . . 20
  - 2.2.3 Eventually different trees . . . 22
  - 2.2.4 Full-Miller and infinitely-equal trees . . . 24
  - 2.2.5 $\mathcal{J}$-Miller trees . . . 26
  - 2.2.6 $\mathcal{J}$-Laver trees . . . 29
  - 2.2.7 Hurewicz schemas and closed sets of measure zero . . . 34
- 2.3 Tree ideal may not be a σ-ideal . . . 38

3 Cardinal invariants of the tree ideals **40**
- 3.1 Covering numbers . . . 40
- 3.2 Incompatibility Shrinking numbers . . . 44
- 3.3 Antichains of trees . . . 49
- 3.4 Additivities and projections of trees . . . 54
- 3.5 More on additivities . . . 58
- 3.6 Cofinality numbers . . . 66

4 Cardinal invariants of ideals on ω **72**
- 4.1 Catalogue of ideals on ω . . . 72
  - 4.1.1 F_σ-ideals . . . 72
  - 4.1.2 analytic P-ideals . . . 76
  - 4.1.3 Other examples . . . 77
- 4.2 The splitting ideal . . . 80
- 4.3 The infinitely equal ideals . . . 84
- 4.4 More on covering numbers . . . 86
- 4.5 Antichain numbers of $\mathcal{P}(\omega)/\mathcal{J}$'s . . . 94

Acknowledgments

First of all, I would like to thank my beloved wife Ksania for her indulgence and unconditional love. I am grateful for her encouragement and support to quit industry and going back to academia. Without her, I would be lost. I also want to thank my parents for their help and support during my masters studies.

I am grateful to Jacek Cichoń and Bogdan Weglorz for all the inspiring conversations I had as an undergraduate student. It is because of these conversations I knew what I am interested in from the very beginning of my mathematical journey. I would also like to thank the set theory group of Wrocław University of Technology, Szymon Żeberski, Marcin Michalski and my advisor Robert Rałowski for their help and support during the work on this thesis. I still remember a quick conversation I had with Szymon and Robert about descriptive set theory in-between lectures, and later, that same day, I received an email with an invitation to 'tajne komplety z forcingu'.

Furthermore, I thank Janusz Pawlikowski for his patience, guidance, and encouragement during the time when I break from academia for a few years. I would like to thank Piotr Borodulin-Nadzieja and Damian Sobota for their help, support, and the opportunity to visit Vienna twice during my PhD studies. I consider these trips to be a turning point in my PhD thesis and I spent there a very inspiring time for which I am very grateful. I am grateful to Barnabas Farkas and Lyubomyr Zdomskyy for the great time I spent in Vienna October 2024 working on the subject of +-destruction. There are several mathematicians with whom I had a chance to discuss some of the content of this thesis. In particular, I am grateful to Marcin Sabok, to Wolfgang Wohofsky for discussions regarding the projections of trees, and to Yurii Khomskii for his encouragement regarding the subject of the Hurewicz schemas and the ideal $\mathcal{E}$. I thank my friend Tomek Żuchowski for all the inspiring conversations about mathematics, teaching and other common interests (beside the common interests of quitting work at industry and going back to academia at the same time). I would also like to thank Rafał Filipów and Adam Kwela for their hospitality, discussions about the splitting ideal and for suggesting the proof of Proposition 4.2.6, during my visit in Gdańsk on December 2024.

Finally, I would like to thank Arturo Martínez-Celis for countless discussions, for his help and support on the subject of this thesis, and also, for introducing me to the subject of definable ideals on ω and bounded invariants from Cichoń's diagram.

Chapter 1

Introduction

1.1 Summary of the work

The aim of this work is to study the cardinal invariants associated to various types of trees and of definable ideals on countable sets. The work consists of three main chapters. The first chapter contains all types of tree we will be interested in, together with their basic properties and calculation of cardinal invariants related to Borel ideals associated to these trees. These trees are generalizations of Sacks, Miller and Laver trees as well as some pseudo-trees related to the ideal of closed sets of measure zero. The second chapter contains a study of the cardinal invariants of (non-Borel) Marczewski-like ideals related to our tree types. We focus on additivities, covering numbers, cofinalities, incompatibility shrinking numbers and antichain numbers. The third chapter contains a study of cardinal invariants related to ideals on countable sets. We focus on the +-covering numbers and antichain numbers. We also study the splitting ideal. This work continues some investigations of Brendle, Hrušák, Khomskii, Sabok, Spinas, Steprāns, Rosłanowski, Wohofsky and Zapletal.

This thesis is made of two parts, first one about cardinal invariants of (Marczewski-like) ideals related to trees (chapter 2 and 3), second one about cardinal invariants related to ideals on countable sets (chapter 4). Mostly, these two parts can be read independently. However, sometimes each part needs certain invariants, definitions, or some simple facts from the other.

In chapter 1, we review the main classical definitions and notions that appear throughout this thesis. In particular we discuss classical cardinal invariants of the reals and relation systems, basic definitions regarding ideals on countable sets, and, basic notions regarding trees, descriptive set theory or forcing.

Chapter 2, contains basic definitions and properties regarding trees and the ideals related to them. In the first section, we discuss the notion of a tree ideal $\mathcal{I}(\mathbb{T})$, its Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{T})$ and their basic properties. In the second section, we go through the tree types in which we are interested in this thesis.

- two tree types for adding bounded infinitely equal real - $\mathbb{IE}_F$ and $\mathbb{S}_F$,
- two tree types for adding infinitely equal real on ω^ω - $\mathbb{IE}$ and $\mathbb{FM}$,
- Miller and Laver trees which split positively according to an ideal $\mathcal{J}$ - $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$,
- the tree type $\mathbb{SP}$ for adding splitting real and $\mathbb{ED}_F$ for adding bounded eventually different real,
- Hurewicz schemas - a pseudo-tree notion for the ideal $\mathcal{E}$ of closed measure zero sets.

For each of these tree types we discuss fusion, continuous reading of names, related perfect set dichotomy and cardinal invariants of the Borel ideal $\mathcal{I}_{\mathcal{B}}(\mathbb{T})$ related to them.

Chapter 3, contains a study of cardinal invariants of the tree ideals. In the first section, we deal with covering numbers. We prove a general theorem saying that a tree type adds a real of a certain kind (dominating, Cohen etc.) in continuous way, naturally gives an upper bound for the covering number. Later, we discuss which types of reals, each of the tree types add and which they do not. In particular, we show that $\mathbb{L}_{\mathcal{J}}$ never adds random reals and that $\mathbb{L}_{\mathcal{ED}_{fin}}$ has Laver property. We also show that, for a meager ideal $\mathcal{J}$ both $\mathbb{L}_{\mathcal{J}}$ and $\mathbb{M}_{\mathcal{J}}$ increase the groupwise dense number which, as a corollary, gives the consistency of $\mathrm{cov}(\mathbb{T}) < \mathrm{add}(\mathcal{M})$ for the two. In the second section, we discuss the incompatibility shrinking number $is(\mathbb{T})$ i.e. the cardinal invariant which measures how many small sets do we need to intersects every positive set. This number will later play an important role on the discussion of additivities, cofinalities and antichains of tree ideals. In this regard, we show that:

- $is(\mathbb{T}) = \mathrm{add}(\mathcal{M})$ for the tree types that add Cohen reals,
- $is(\mathbb{T}) \geq \mathrm{cov}(\mathcal{M})$ for $\mathbb{IE}_F$ and $\mathbb{S}_F$,
- $is(\mathbb{T}) \geq \min\{\mathfrak{b}, \mathrm{cov}^*_+(\mathcal{J})\}$ for $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ and
- $is(\mathbb{T}) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{T}))\}$ for $\mathbb{M}_{\mathcal{J}}$.

In the third section, we study the antichain numbers $\mathfrak{a}(\mathbb{T})$ for our tree types i.e. the smallest cardinality of an uncountable maximal antichain in $\mathbb{T}$. We prove the following inequalities:

- $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathbb{S}_F), \mathfrak{a}(\mathbb{IE}_F)$,
- $\min\{\mathrm{is}^*(\mathbb{ED}_F), \mathrm{cov}(\mathcal{M})\} \leq \mathfrak{a}(\mathbb{ED}_F)$,
- $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M}_{\mathcal{J}}))\} \leq \mathfrak{a}(\mathbb{M}_{\mathcal{J}})$
- $\mathfrak{p} \leq \mathfrak{a}(\mathbb{P}_{\mathcal{E}})$.

In the fourth section, we discuss the additivities of tree ideals. The results of this section are generalizations of known methods. Under the assumption $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, we prove a general theorem saying that an existence of a projection of a tree onto a relational system $\mathcal{R}$, naturally gives an upper bound for $\mathrm{add}(\mathcal{I}(\mathbb{T}))$ of the form $\mathfrak{b}_\sigma(\mathcal{R})$. Then we show that:

- $\mathbb{IE}_F$ and $\mathbb{S}_F$ have both $\mathfrak{b}$-projection and $\mathfrak{s}$-projection,
- $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ have both $\mathfrak{b}$-projection,
- $\mathbb{M}_{\mathcal{J}}$ has an $\mathfrak{s}$-projection,

We also mention related results from the literature and corollaries. In the fifth section, we establish more bounds regarding the additivities of our tree types. All of them are generalizations of the known arguments for classical tree types like Sacks, Miller or Laver. First, we show that $\mathfrak{t} \leq \mathrm{add}(\mathbb{L}_{\mathcal{J}})$ for F_σ-ideals $\mathcal{J}$. Then we show that nowhere-distributivity of $\mathbb{T}$ gives a natural upper bound for $\mathrm{add}(\mathcal{I}(\mathbb{T}))$ and, we show that

- $\mathbb{IE}_F$ and $\mathbb{S}_F$ are $\mathfrak{b}$-nowhere distributive,
- $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$ are $\mathfrak{h}_{\mathcal{J}}$-nowhere-distributive for a meager ideal $\mathcal{J}$ with a base tree.

We apply these (and results from the third section) to show the consistency of $\mathrm{add}(\mathcal{I}(\mathbb{T})) < \mathrm{cov}(\mathcal{I}(\mathbb{T}))$ for various tree types. In the last, sixth section, we use the methods developed in [BKW17], to analyse the cofinalities of the tree types. In particular, we mention related results, prove the constant or 1-1 property for $\mathbb{L}_{\mathcal{J}}$ where $\mathcal{J}$ is nowhere-maximal P-ideal and, as a corollary, we obtain $\mathrm{cof}(\mathcal{I}(\mathbb{T})) > \mathfrak{c}$ for such trees.

Chapter 4 contains a study of cardinal invariants of ideals on countable sets. In the first section, we gather all ideals important to this work. We will focus on Borel and analytic ideals with a special emphasis on the F_σ-ideals. We will collect known results on cardinal invariants add^*, cov^*, non^*, and cof^* as well as their ω-versions. We also discuss the Katetov order between our examples. In the second and third section, we investigate two ideals on ω, the splitting ideal and the infinitely equal ideal. We study their cardinal invariants and the position in the Katetov order. As a result, we obtain a new upper bound of the covering number of the density zero ideal. In the fourth section, we investigate the cardinal invariant cov^*_+ and continue the study of covering numbers for selected ideals. In the fifth section, we study the antichain numbers of the algebras $\mathcal{P}(\omega)/\mathcal{J}$. As a result, we obtain a strengthening of Steprāns lower bound of the antichain number for the *nwd* ideal.

I must state two comments regarding certain results of other mathematicians, I was not aware of during the work on this thesis. First, when I was working with Arturo Martínez-Celis on the subject of cofinalities of Marczewski-like ideals, we calculated some ω-versions of cardinal invariants of ideal on countable sets. When we finished our work, I realized (by an e-mail conversation with Rafał Filipów) that some of those invariants were already calculated in papers of mathematicians from Gdańsk. Second, in the last year of my Ph.D. studies, I had a pleasure to join Barnabas Farkas and Lyubomyr Zdomskyy on their project on +-covering numbers of ideals (continuation of their work [FZ22]). During my stay in Vienna, in October 2024, I proved a couple of inequalities, which few months later turned out to be already proven in the work [BF17] of Brendle and Flaskova in the dual form. However, I decided to include these results in this thesis as they were part of my Ph.D. studies and I did not know about them at the time.

The main contribution of this work is:

- the isolation and investigation of the incompatibility shrinking numbers, a calculation of which gives a partial answer to the problem of Brendle, Khomskii and Wohofsky,
- the study of the cardinal invariants of the tree types $\mathbb{M}_{\mathcal{J}}$ and $\mathbb{L}_{\mathcal{J}}$,
- the study of the cardinal invariants of Marczewski-like tree ideals,
- the study of ω-invariants of ideals on ω,
- the investigation of antichain numbers of the Boolean algebras $\mathcal{P}(\omega)/\mathcal{J}$ for a wide class of ideal. In particular we show that $\mathfrak{a}(nwd) \geq \mathrm{add}(\mathcal{M})$. This improves a result of Steprāns and give another proof of the inequality than the one of Cancino Manríquez,
- the study of the splitting ideal, which leads to a new upper bound of the covering number of density zero ideal.

1.2 Preliminaries

1.2.1 Reals and trees

A Polish space X is a non-empty, separable, complete metric space X. Elements of such spaces will be revered as reals. There are several important classes of subsets of Polish spaces we will

frequently refer to. The σ-algebra of Borel sets in X (the smallest σ-algebra containing all open sets) will be denoted by Borel(X) and the collection of all analytic subsets of X (the continuous images of Polish spaces) will be denoted by Σ^1_1. A subset M of a Polish space X is nowhere-dense if every non-empty open subset of X contains non-empty open set disjoint from N. A set $M \subseteq X$ is meager if it can be covered by a countable union on nowhere-dense sets. A subset Y of a Polish space is said to have Baire property if it can be expressed as $U \triangle M$ for some open set U and a meager set M. In context of forcing, we will often identify Borel set with its code. For an extensive study of these notions, we refer to Kechris' book [Kec77].
In this thesis, we will mostly work with Polish spaces of the following particular form. Assume that $F = \{F(n) : n \in \omega\}$ is a sequence of non-empty sets. We define

$$\Pi F =: \Pi_{n\in\omega} F(n)$$

For such F, we will use the symbol $Seq(F)$ to denote the set of all finite sequences in ΠF i.e. $Seq(F) = \bigcup_{n\in\omega} \Pi^n_{i=0} F(i)$. We consider ΠF to be a topological space with the product topology with each $F(n)$ treated as a discrete space. In such a case, the sets of the form $[\sigma] := \{x \in \Pi F : \sigma \subseteq x\}$ where $\sigma \in Seq(F)$, form a clopen basis of the topology. When F is a constant sequence with the value c, we will write c^ω instead of ΠF and, $c^{<\omega}$ instead of $Seq(F)$. In particular, ω^ω stands for the standard Baire space and 2^ω for the standard Cantor space. Similarly, $\omega^{<\omega}$ and $2^{<\omega}$ stand for Baire and Cantor tree. Convergence in spaces of this form can be conveniently formulated as follows: $lim_{n\in\omega} x_n = x$ if and only if for every $N \in \omega$, for almost all $n \in \omega$ we have $x|_N \subseteq x_n$. For $\sigma, \tau \in Seq(F)$ we will use the symbol $\sigma^\frown\tau$ to denote the concatenation of the two.
If F is such that every $F(n)$ is finite, we consider the space ΠF with the standard product measure μ: the one such that for $\sigma \in Seq(F)$ we have $\mu([\sigma]) = \Pi_{i<|\sigma|} \frac{1}{|F(i)|}$. In particular, on the Cantor space 2^ω this defines the standard Lebesgue measure.
Other shorthand notation that we will use is $\forall^\infty$ and $\exists^\infty$ to abbreviate "for all but finitely many" and "there are infinitely many", respectively.
Let F be as above. A tree on ΠF is a non-empty subset of $Seq(F)$ closed under taking initial segments. A tree T is pruned, if for every $\sigma \in T$ there is $\tau \in T$ that properly extends σ. For a tree T and $\sigma \in T$ we will use the following standard notation:

- $[T] = \{x \in \Pi F : \forall n \in \omega\ x|_n \in T\}$ is the set of branches of T i.e. the body of T,
- $T|_\sigma = \{\tau \in T : \sigma \subseteq \tau \vee \tau \subseteq \sigma\}$ is the restriction of T to σ,
- $succ_T(\sigma) = \{i \in F(|\sigma|) : \sigma^\frown i \in T\}$ is the set of immediate successors of σ in T,
- the notion $split(T)$ will mean different things for the tree types $\mathbb{S}_F$, $\mathbb{M}_\mathcal{J}$, $\mathbb{L}_\mathcal{J}$ and $\mathbb{FM}$; however, $split_n(T)$ will denote the set $\{\sigma \in split(T) : |split(T) \cap \{\sigma|_i : i < |\sigma|\}| = n\}$,
- if $succ_T(\sigma) = \emptyset$, then we call σ a terminal node of T and denote these as $term(T)$,
- $stem(T)$ be the shortest splitnode of T. Sometimes, when several trees with the same stem are considered, we will simply write $stem$,
- If $F, G \subseteq Seq(F)$, $G \subseteq F$ are finite trees, we will say that F is an end-extension of G if every $\sigma \in F \setminus G$ extends a terminal node of G.

A couple of times in this thesis we will mention multidimensional trees i.e. trees on the set of form A^n where A has at least two elements. Suppose that T is such a tree. If $\sigma \in T$ and $\sigma = \{a_i : i < |\sigma|\}$ where each $a_i \in A^n$, we will treat σ as a sequence $\sigma = \{\sigma_i : i < |\sigma|\}$ where $\sigma_i \in A^{|\sigma|}$ is such that $\sigma_n(i) = a_i(n)$.
If $\mathbb{T}$ is a tree type on ΠF and $n > 1$, then by $\mathbb{T}^n$ we will denote a tree type on $(\Pi F)^n$ consisting of products of $\mathbb{T}$-trees i.e. $S \in \mathbb{T}^n$ if there are $\{S_i : i < n\} \subseteq \mathbb{T}$ such that S is a product of S_i's in the sense that $S = \{\sigma \in Seq(F)^n :$ there are $\{\sigma_i : i < n\} \subseteq Seq(F)$ such that all σ_i's are of the same length, $\sigma_i \in S_i$ and $\sigma = \{\sigma_i : i < |\sigma|\}\}$.

1.2.2 Forcing

We will not reconstruct the entire method of forcing with details here, however, most notations and notions we will going to use are standard and follow the classical text books on the subject like [Ken83], [Jec02] or [BJ95]. Let us briefly recall the most important terminology and notations. Let $\mathbb{P} = (P, \leq)$ be a partially ordered, or in general, a preordered set. We will usualy refer to $\mathbb{P}$ as the underlying set, e.g. we will write $p \in \mathbb{P}$ instead of $p \in P$.

- Elements of $\mathbb{P}$ will be called conditions. If $p \leq q$ then we say that p is stronger than q or that p extends q.
- $p, q \in \mathbb{P}$ are comparable if $p \leq q$ or $q \leq p$, and $p, q \in \mathbb{P}$ are compatible if $r \leq p, q$ for some $r \in \mathbb{P}$. The latter will be denoted by $p||q$. Otherwise the two conditions are incompatible which we will denote as $p \perp q$.
- For $p \in \mathbb{P}$ the $\mathbb{P}|_p$ denote the restriction of $\mathbb{P}$ to p i.e. all $q \in \mathbb{P}$ below p.
- $D \subseteq \mathbb{P}$ is dense if for every $p \in \mathbb{P}$ there is $d \in D$ such that $d \leq p$. $O \subseteq \mathbb{P}$ is open if it is downward closed, that is, if $p \in O$ and $q \leq p$ then $q \in O$.
- $A \subseteq \mathbb{P}$ is an antichain if $p \perp q$ for any two distinct $p, q \in A$. An antichain $A \subseteq \mathbb{P}$ is maximal if every $p \in \mathbb{P}$ is compatible with some $q \in A$.
- Let κ be a cardinal. We will say that $\mathbb{P}$ has κ-c.c. if $|A| < \kappa$ for every antichain $A \subseteq \mathbb{P}$. If $\kappa = \omega_1$ we will say c.c.c. instead of ω_1-c.c.
- $G \subseteq \mathbb{P}$ is a filter in $\mathbb{P}$, if G is nonempty, upward closed ($p \in G$ and $p \leq q$ implies $q \in G$) and for every $p, q \in G$ there is $r \in G$ with $r \leq p, q$.
- Let $\mathcal{D}$ be a family of subsets of $\mathbb{P}$. A filter $G \subseteq \mathbb{P}$ is $\mathcal{D}$-generic if $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. In particular, if $\mathcal{D}$ consists of all open dense subsets of $\mathbb{P}$ of some model of set theory $\mathbf{M}$, such filter will be referred to as $\mathbf{M}$-generic.

A forcing notion $\mathbb{P}$ has axiom A if there is a sequence $\{\leq_n : n \in \omega\}$ of orderings on $\mathbb{P}$ such that:

(1) $\leq_0 = \leq$,

(2) $p \leq_{n+1} q$ implies $p \leq_n q$ for every $n \in \omega$,

(3) If $\{p_n : n \in \omega\} \subseteq \mathbb{P}$ is such that $p_{n+1} \leq_n p_n$ for every $n \in \omega$, then there is a $p \in \mathbb{P}$ such that $p \leq_n p_n$ for every $n \in \omega$.

(4) For every maximal antichain A in $\mathbb{P}$, for every $p \in \mathbb{P}$ and $n \in \omega$, there is a $q \leq_n p$ such that $\{r \in A : r||q\}$ is countable.

All of the forcing notions we are going to use, have the axiom A. It is well-known that, when we force over a model of CH with a forcing notion having axiom A, all cardinal numbers are preserved.

One of the crucial notions in this thesis is the following.

Definition 1.2.1. *Let X be a non-empty set. A collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal on X if $\mathcal{I}$ is closed under subsets and finite unions. A σ-ideal on X is an ideal on X closed under countable unions. We will always assume that ideals are proper ($X \notin \mathcal{I}$) and $\{x\} \in \mathcal{I}$ for every $x \in X$.*

An important role will be played by the idealized forcing approach, an extensive theory developed by Zapletal in [Zap08] and [Zap04].

Definition 1.2.2. *Let $\mathcal{I}$ be a σ-ideal on a Polish space X. Define the forcing notion $\mathbb{P}_\mathcal{I} = \text{Borel}(X) \setminus \mathcal{I}$ of all $\mathcal{I}$-positive Borel subsetes of X ordered by inclusion.*

If G is a $\mathbb{P}_\mathcal{I}$-generic filter, then there is a unique real $\dot{r}_{gen} \in \mathbf{V}[G]$ such that for all Borel sets coded in $\mathbf{V}$, $\dot{r}_{gen} \in B^{\mathbf{V}[G]}$ if and only if $B^\mathbf{V} \in G$. This real will be called the generic real. By previous equivalence, the generic filter can be recovered from the generic real. It follows that $\mathbf{V}[G] = \mathbf{V}[\dot{r}_{gen}]$. Also, every $\mathcal{I}$-positive set B forces that $\dot{r}_{gen} \in B^{\mathbf{V}[G]}$. When $\mathbb{T}$ a tree forcing and $G \subseteq \mathbb{T}$ is a generic filter, the generic real can equivalently be defined as $\bigcup_{T\in} stem(T)$.

Definition 1.2.3. *A forcing notion $\mathbb{P}_\mathcal{I}$ (tree forcing $\mathbb{T}$) is homogenoueus if for each $B \in \mathbb{P}_\mathcal{I}$ ($T \in \mathbb{T}$) there is a homeomorphism $h : \Pi F \to B$ ($h : \Pi F \to [T]$) such that a set $X \subseteq \Pi F$ is $\mathcal{I}$-positive if and only if $h[X]$ is ($X \subseteq \Pi F$ contains body of a $\mathbb{T}$-tree if and only if $h[[T]]$ does).*

The following property will play an important role in this thesis.

Definition 1.2.4. *A forcing notion $\mathbb{P}_\mathcal{I}$ (tree forcing $\mathbb{T}$) has the continuous reading of names if for every $B \in \mathbb{P}_\mathcal{I}$ ($T \in \mathbb{T}$) which forces an $\dot{x}$ to be an element of 2^ω, there is a $C \in \mathbb{P}_\mathcal{I}$, $C \subseteq B$ ($S \in \mathbb{T}$, $S \leq T$) and a continuous function $f : C \to 2^\omega$ ($f : [S] \to 2^\omega$) mapping the generic real onto $\dot{x}$.*

Remark. *If a tree type $\mathbb{T}$ with c.r.n. is homogeneous, then in the definition of c.r.n. the function f may be assumed to be defined on the entire space on which $\mathbb{T}$ lives. The same is holds for $\mathbb{P}_\mathcal{I}$'s.*

Zapletal showed (see [Zap08]) that every proper forcing of form $\mathbb{P}_\mathcal{I}$ has the Borel reading of names. It is worth pointing out that the continuous reading of names depends on the presentation of forcing notion. It might be the case that to forcing notions are forcing equivalent (in the sense that they produce the same forcing extension) yet one has the continuous reading of names while the other does not. An example of such forcing is the Steprāns forcing, on which more details can be found in [Zap08].
For most of the forcing notions of form $\mathbb{P}_\mathcal{I}$ we deal in this thesis with the case that closed sets are dense in $\mathbb{P}_\mathcal{I}$. This will be due to a perfect set style dichotomy:

There is a collection $\mathbb{T}$ of trees such that for every analytic $A \subseteq X$
either $A \in \mathcal{I}$ or there is $T \in \mathbb{T}$ such that $[T] \subseteq A$.

In such a situation, the assignment $T \mapsto [T]$, $T \in \mathbb{T}$ is a dense emending of $\mathbb{T}$ into $\mathbb{P}_\mathcal{I}$. It follows that $\mathbb{T}$ and $\mathbb{P}_\mathcal{I}$ are forcing equivalent. The fact that even compact sets are dense in $\mathbb{P}_\mathcal{I}$ has the following forcing consequences. Recall that a forcing notion $\mathbb{P}$ is ω^ω-bounding if every $\dot{f} \in \omega^\omega$ in $\mathbf{V}^\mathbb{P}$ is dominated by a ground model $g \in \omega^\omega$. We have the following useful result

Theorem 1.2.5. *(Zapletal, [Zap08]) Under the presence of continuous reading of names, the forcing notion $\mathbb{P}_\mathcal{I}$ is ω^ω-bounding if and only if compact sets are dense in $\mathbb{P}_\mathcal{I}$.*

A very important and well-understood class of $\mathbb{P}_\mathcal{I}$'s is the one where the σ-ideal $\mathcal{I}$ is generated by closed sets, meaning that every $Y \in \mathcal{I}$ can be covered by a countable union of closed sets from $\mathcal{I}$. Recall that a forcing notion $\mathbb{P}$ preserves Baire category if $\mathbb{P} \Vdash$ "$2^\omega \cap \mathbf{V} \notin \mathcal{M}$".

Theorem 1.2.6. *(Zapletal, Theorem 4.1.2 in [Zap08]) If the ideals $\mathcal{I}$ is generated be closed sets the forcing notion $\mathbb{P}_{\mathcal{I}}$ is proper, has continuous reading of names and preserves Baire category.*

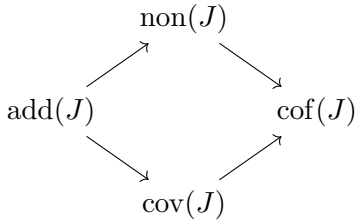
Recall that a forcing notion $\mathbb{P}$ has Laver property if for any $\dot{f} \in \omega^\omega$ in $\mathbf{V}^{\mathbb{P}}$ bounded by a ground model function, there is a slalom $S : \omega \to [\omega]^{<\omega}$, $|S(n)| \leq n$ such that $\dot{f}(n) \in S(n)$ holds for almost all $n \in \omega$.

1.2.3 Classical cardinal invariants of the reals

If $\mathcal{I}$ is a σ-ideal on X, then the standard cardinal characteristics associated with $\mathcal{I}$ are defined as follows:

- $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} \notin \mathcal{I}\}$,
- $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{F} = X\}$,
- $\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \text{ and } X \notin \mathcal{I}\}$,
- $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \text{ and } \forall Y \in \mathcal{I}\ \exists Y' \in \mathcal{F}\ Y \subseteq Y'\}$.

These cardinal invariants are refered to as the additivity, covering, uniformity and cofinality respectively. As $\mathcal{I}$ is close under countable unions, they all are uncountable. The cardinal invariants of any σ-ideal satisfy



where the arrows point towards the greater cardinal invariant. For $f, g \in \omega^\omega$ we write $f \leq^* g$ if g eventually dominates f i.e. $f(n) \leq g(n)$ for almost all $n \in \omega$. A family $\mathcal{F} \subseteq \omega^\omega$ is dominating if for every $g \in \omega^\omega$ there is $f \in \mathcal{F}$ which dominates g. A family $\mathcal{F} \subseteq \omega^\omega$ is unbounded if there is no $g \in \omega^\omega$ which dominates every $f \in \mathcal{F}$. The (un)bounding number is defined as

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is an unbounded family}\}$$

and the dominating number as

$$\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is a dominating family}\}$$

It is well known that both $\mathfrak{b}$ and $\mathfrak{d}$ are uncountable, $\mathfrak{b}$ is regular cardinal and that $\mathfrak{b} \leq cf(\mathfrak{d}) \leq \mathfrak{d}$ holds ([Bla09]). There is another, useful, presentation of the these invariants. Let $\mathbb{IP}$ be the collection of all partitions of ω into finite intervals. If $\overline{I} = (I_n)_{n\in\omega}$, $\overline{J} = (J_k)_{k\in\omega}$ are such partitions we overload the symbol $\leq^*$ and write $\overline{I} \leq^* \overline{J}$ if for almost every $k \in \omega$ there is $n \in \omega$ such that $I_n \subseteq J_k$. Then $\mathfrak{b}$ is equal to the smallest number of interval partitions which are not $\leq^*$ bounded by a single interval partition. Similar, $\mathfrak{d}$ is the smallest number of such partitions needed to dominate any interval partition (see [Bla09]).

Let $A, B \in [\omega]^\omega$. We write $A \subseteq^* B$ if A is almost contained in B i.e. $A \setminus B$ is finite. We say that A splits B is both $A \cap B$ and $B \setminus A$ are infinite. We say that A reaps (unsplit) B if A does not split B i.e. either $B \subseteq^* A$ or $A \cap B$ is finite. A family $\mathcal{F} \subseteq [\omega]^\omega$ is splitting if for every $A \in [\omega]^\omega$ there is a member of $\mathcal{F}$ that splits A. A family $\mathcal{F} \subseteq [\omega]^\omega$ is reaping (independent) if for every $A \in [\omega]^\omega$ there is $B \in \mathcal{F}$ that reaps A. The splitting number is defined as

$$\mathfrak{s} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is a splitting family}\}$$

and the reaping number as

$$\mathfrak{r} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is a reaping family}\}$$

It is well known that $\mathfrak{b} \leq \mathfrak{r}$ and $\mathfrak{s} \leq \mathfrak{d}$.
A set $\mathcal{D} \subseteq [\omega]^\omega$ is dense if for every $A \in [\omega]^\omega$ there is an $B \in [\omega]^\omega$, $B \in \mathcal{D}$ with $B \subseteq^* A$. A set $\mathcal{D} \subseteq [\omega]^\omega$ is open if $A \in \mathcal{D}$ and $B \subseteq^* D$, $B \in [\omega]^\omega$ implies that $B \in \mathcal{D}$. The distributivity number is defined as

$$\mathfrak{h} = \min\{|\mathcal{F}| : \mathcal{F} \text{ consists of open dense subsets of } [\omega]^\omega \text{ and } \bigcap \mathcal{F} = \emptyset\}$$

It is well known that $\mathfrak{p} \leq \mathfrak{h} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$. A tower is a sequence $\mathcal{T} = \{T_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ such that $T_\alpha \subseteq^* T_\beta$ whenever $\alpha \leq \beta$ and there is no $T \in [\omega]^\omega$ such that $T \subseteq T_\alpha$ for all $\alpha < \kappa$. A family $\mathcal{F} \subseteq [\omega]^\omega$ is centered if every finitely many elements of $\mathcal{F}$ have an infinite intersection. The tower number is defined as

$$\mathfrak{t} = \min\{|\mathcal{T}| : \mathcal{T} \text{ is a tower}\}$$

and the pseudointersection number as

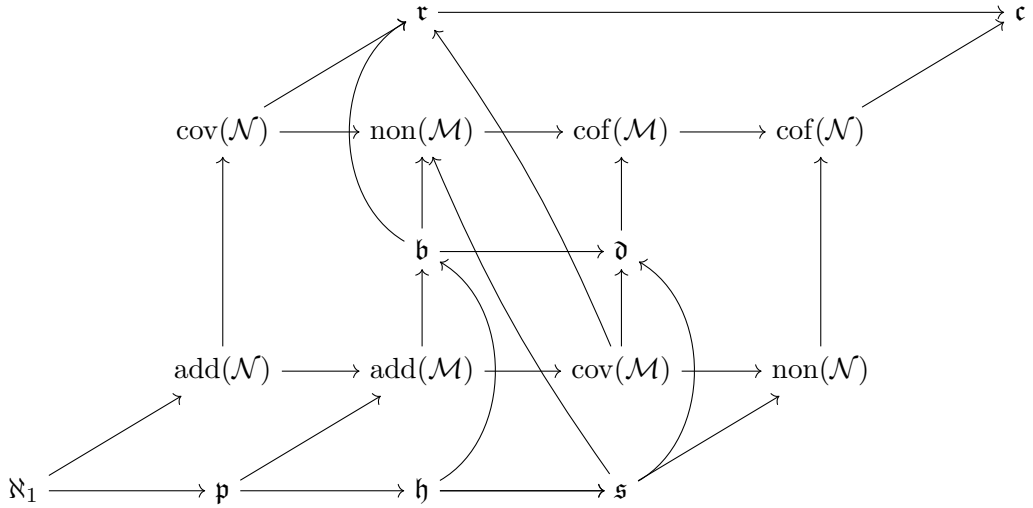
$$\mathfrak{p} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ is centered with no pseudointersection}\}$$

It is a well-known result of Milliaris and Shelah (see [MS16]) that $\mathfrak{p} = \mathfrak{t}$. We will also make use of another characterization of $\mathfrak{p}$ by the means of Martins numbers. Let $\mathcal{P}$ be a class of forcing notions (c.c.c., countable, σ-centered). Then define the martins number of $\mathcal{P}$ as

$$\mathfrak{m}_{\mathcal{P}} = \min\{|\mathcal{D}| : \mathcal{D} \text{ consists of dense subsets of some } \mathbb{P} \in \mathcal{P} \text{ and there is no filter in } \mathbb{P} \text{ intersecting them all}\}$$

It is well-known result of Bell, that $\mathfrak{m}_{\sigma-\text{centered}} = \mathfrak{p}$ and $\mathfrak{m}_{\text{countable}} = \text{cov}(\mathcal{M})$ (see [Bla09]). The invariant $\mathfrak{m}_{\text{c.c.c.}}$ is known as the Martins number and is usually denoted by $\mathfrak{m}$. It is well-known that $\mathfrak{m} \leq \min\{\mathfrak{p}, \text{add}(\mathcal{N})\}$ (see [Bla09]).

As usual, $\mathcal{M}$ stands for the σ-ideal of meager sets in X and - in case X is compact - $\mathcal{N}$ for the σ-ideal of sets of Lebesgue measure zero. The ZFC-provable inequalities between these invariants are summarized in the following extended version of what is widely known as the Cichoń's diagram (the interested reader may consult [BJ95] or [Bla09]). The arrows point towards the larger cardinal invariant.



1.2.4 Ideals on countable sets

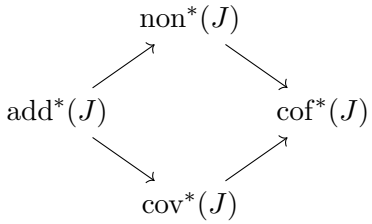
In this thesis, we will often consider ideals on countable sets. If $\mathcal{J}$ is an ideal on ω, then by $\mathcal{I}^*$ we will denote the dual filter i.e. $\mathcal{I}^* = \{\omega \setminus X : X \in \mathcal{I}\}$. Sometimes we will write ω for the countable set on which $\mathcal{J}$ lives even if it is not formally ω. Subsets of ω that are not in $\mathcal{J}$ are referred to as $\mathcal{J}$-positive sets or simply positive if the ideal is known from the context. Similarly, sometimes we will refer to elements of $\mathcal{J}$ as being $\mathcal{J}$-small or simply small. For $A, B \subseteq \omega$ the symbol $A \subseteq_J B$ means that $A \setminus B \in \mathcal{J}$. Since every ideal can be considered as a subset of the Cantor space 2^ω, we can talk about descriptive or topological properties of such ideals.

Definition 1.2.7. *An ideal $\mathcal{J}$ is tall if every infinite $A \subseteq \omega$ contains an infinite $B \in \mathcal{J}$. An ideal $\mathcal{J}$ is a P-ideal if every sequence of $\mathcal{J}$-small sets has a pseudounion in $\mathcal{J}$ i.e. there is set in $\mathcal{J}$, which almost contain every member of the sequence. An ideal $\mathcal{J}$ is a P^+-ideal if, for every almost decreasing sequence of $\mathcal{J}$-positive sets, there is a $\mathcal{J}$-positive pseudointersection i.e. an $A \in [\omega]^\omega \setminus \mathcal{J}$ almost contained in every element of the sequence. An ideal $\mathcal{J}$ is nowhere-maximal is every $\mathcal{J}$-positive set can be decomposed into two $\mathcal{J}$-positive sets.*

Recall the standard cardinal invariants associated to an ideal $\mathcal{J}$:

- $\mathrm{add}^*(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall X \in \mathcal{J}\ \exists F \in \mathcal{F}\ (F \not\subseteq^* X)\}$,
- $\mathrm{cov}^*(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall X \in [\omega]^\omega\ \exists F \in \mathcal{F}\ (|F \cap X| = \omega)\}$,
- $\mathrm{non}^*(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ and } \forall X \subseteq \mathcal{J}\ \exists F \in \mathcal{F}\ (F \cap X =^* \emptyset)\}$,
- $\mathrm{cof}^*(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall X \in \mathcal{J}\ \exists F \in \mathcal{F}\ (X \subseteq^* F)\}$.

They are called *the additivity*, *the covering*, *the uniformity* and *the cofinality* of the ideal $\mathcal{J}$ respectively. The following diagram contains well-known ZFC relations between these invariants



As usual, the arrows point towards the greater cardinal invariant. This cardinal invariant has been extensively studied. The interested reader can consult [Hru11] or [MA09]. In this thesis, we will be also interested in the ω-versions of these:

- $\mathrm{add}^*_\omega(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J} \text{ and } \forall\{X_n : n \in \omega\} \subseteq \mathcal{J}\ \exists A \in \mathcal{F}\ \forall n \in \omega\ (A \not\subseteq^* X_n)\}$,
- $\mathrm{non}^*_\omega(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \text{ and } \forall \overline{A} \in [\mathcal{J}]^\omega\ \exists F \in \mathcal{F}\ \forall A \in \overline{A}\ (A \cap F =^* \emptyset)\}$,
- $\mathrm{cof}^*_\omega(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\mathcal{J}]^\omega \text{ and } \forall \overline{A} \in [\mathcal{J}]^\omega\ \exists \overline{F} \in \mathcal{F}\ \forall A \in \overline{A}\ \exists F \in \overline{F}\ (A \subseteq^* F)\}$.

Clearly, the relations between the ω-versions are the same as for the original ones and additionally we have that $\mathrm{inv}^*(\mathcal{J}) \leq \mathrm{inv}^*_\omega(\mathcal{J})$ for every $\mathrm{inv} \in \{\mathrm{add}, \mathrm{non}, \mathrm{cof}\}$. The reason for not considering the ω-version of the covering number is the later Theorems 2.2.42 and 2.2.33. We have the following simple observation.

Proposition 1.2.8. *If $\mathcal{J}$ is a P-ideal, then* $\mathrm{inv}^*(\mathcal{J}) = \mathrm{inv}^*_\omega(\mathcal{J})$ *for* $\mathrm{inv} \in \{\mathrm{add}, \mathrm{non}, \mathrm{cof}\}$.

Proof. We only have to prove that the ω-version invariant is lesser or equal to the original one. To prove that $\mathrm{add}^*_\omega(\mathcal{J}) \leq \mathrm{add}^*(\mathcal{J})$ let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}$ with $\kappa < \mathrm{add}^*_\omega(\mathcal{J})$. There are then $\{Y_n : n \in \omega\} \subseteq \mathcal{J}$ such that each X_α is almost-contained in one of the Y_n's. Then, any pseudo-union $Y \in \mathcal{J}$ of the Y_n's is a pseudo-union of the X_α's which finish the proof of the first inequality.
To prove that $\mathrm{non}^*_\omega(\mathcal{J}) \leq \mathrm{non}^*(\mathcal{J})$ let $\mathcal{F} \subseteq [\omega]^\omega$ be a witness for $\mathrm{non}^*(\mathcal{J})$. We claim that it is also a witness for $\mathrm{non}^*_\omega(\mathcal{J})$. Assume towards a contradiction that for some $\{A_n : n \in \omega\} \subseteq \mathcal{J}$ we have that for each $F \in \mathcal{F}$ there is $n \in \omega$ with $|F \cap A_n| = \omega$. Then, any set $C \in \mathcal{J}$ that almost contains all A_n's satisfies $|C \cap F| = \omega$ for all $F \in \mathcal{F}$, meaning, that $\mathcal{F}$ was not a witness for $\mathrm{non}^*(\mathcal{J})$ in the first place.
To prove that $\mathrm{cof}^*_\omega(\mathcal{J}) \leq \mathrm{cof}^*(\mathcal{J})$ let $\{A_\alpha : \alpha < \mathrm{cof}^*(\mathcal{J})\} \subseteq \mathcal{J}$ be a witness for $\mathrm{cof}^*(\mathcal{J})$. We claim that $\{\overline{A}_\alpha : \alpha < \mathrm{cof}^*(\mathcal{J})\}$, where $\overline{A}_\alpha$ is a constant sequence with value A_α, is a witness for $\mathrm{cof}^*_\omega(\mathcal{J})$. To see this let $\{B_n : n \in \omega\} \in [\mathcal{J}]^\omega$. As $\mathcal{J}$ is a P-ideal for some $C \in \mathcal{J}$ we have $B_n \subseteq^* C$, $n \in \omega$, and then $C \subseteq A_\alpha$ for some $\alpha < \mathrm{cof}^*(\mathcal{J})$ which finishes the proof of the last inequality. □

Another invariants (of an ideal $\mathcal{J}$) that will play an important role in this thesis are the following.

Definition 1.2.9. *Let $\mathcal{J}$ be an ideal on ω. The following is the $+$-covering number of an ideal.*

- $\mathrm{cov}^*_+(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J}$ *and every* $X \in \mathcal{J}^+$ *infinitely intersects some* $F \in \mathcal{F}\}$.

We also define the hereditary version of $+$-covering number as

- $h\text{-}\mathrm{cov}^*_+(\mathcal{J}) = \min\{\mathrm{cov}^*_+(\mathcal{J}|_X) : X \in \mathcal{J}^+\}$

The first invariant has been investigated in [FZ22]. Its dual, which we will investigate in the chapter 4 were studied in [BF17]. Because of the applications to incompatibility shrinking numbers and to antichains of $\mathcal{P}(\omega)/\mathcal{J}$, we will be mostly interested in $\mathrm{cov}^*_+(\mathcal{J})$ instead of its dual. Clearly, $\text{h-}\mathrm{cov}^*_+(\mathcal{J}) \leq \mathrm{cov}^*_+(\mathcal{J})$ and the two are equal if $\mathcal{J}$ is homogeneous. A simple observation shows that

Proposition 1.2.10. $\mathrm{add}^*(\mathcal{J}) \leq \mathrm{cov}^*_+(\mathcal{J}) \leq \mathrm{cov}^*(\mathcal{J})$.

The main tool to study relations between ideals is the following notion of Katetov reducibility.

Definition 1.2.11. *Suppose that $\mathcal{J}_0$ and $\mathcal{J}_1$ are ideals on ω. We will write $\mathcal{J}_0 \leq_K \mathcal{J}_1$ and say that $\mathcal{J}_0$ is Katetov reducible to $\mathcal{J}_1$, if there is a function $\pi : \omega \to \omega$ such that $\pi^{-1}[A] \in \mathcal{J}_1$ for any $A \in \mathcal{J}_0$. If such a function is finite-to-one, then we call this Katetov-Blass reduction and write $\mathcal{J}_0 \leq_{KB} \mathcal{J}_1$. The ideals $\mathcal{J}_0$ and $\mathcal{J}_1$ are Katetov equivalent when both $\mathcal{J}_0 \leq_K \mathcal{J}_1$ and $\mathcal{J}_1 \leq_K \mathcal{J}_0$ hold. Similary for Katetov-Blass preorder.*

The following is a well-known consequence of the existence of such reductions.

Proposition 1.2.12. *(folklore, [Hru11]) Suppose that $\mathcal{J}_0$ and $\mathcal{J}_1$ are ideals on ω. Then*

- *$\mathcal{J}_0 \leq_K \mathcal{J}_1$ implies that* $\mathrm{cov}^*(\mathcal{J}_1) \leq \mathrm{cov}^*(\mathcal{J}_0)$ *and* $\mathrm{cov}^*+(\mathcal{J}_1) \leq \mathrm{cov}^*+(\mathcal{J}_0)$,
- *$\mathcal{J}_0 \leq_{KB} \mathcal{J}_1$ implies that* $\mathrm{non}^*(\mathcal{J}_0) \leq \mathrm{non}^*(\mathcal{J}_1)$, $\mathrm{non}^*_\omega(\mathcal{J}_0) \leq \mathrm{non}^*_\omega(\mathcal{J}_1)$ *and* $\mathrm{non}^*_+(\mathcal{J}_0) \leq \mathrm{non}^*_+(\mathcal{J}_1)$.

Definition 1.2.13. *The ideal $\mathcal{J}$ is homogeneous if for every $\mathcal{J}$-positive set X, there is a bijection $\pi : \omega \to X$ such that $Y \subseteq \omega$ is $\mathcal{J}$-positive if and only if $\pi[Y] \subseteq X$ is.*

We will also need the following well-known characterization of ideals with Baire property.

Theorem 1.2.14. *(Talagrand) For an ideal $\mathcal{J}$ on ω the following conditions are equivalent:*

– $\mathcal{J}$ *is meager,*

– $\mathcal{J}$ *has Baire property,*

– *there is a partition of* ω *into finite intervals* $\{I_n : n \in \omega\}$ *such that if* $X \subseteq \omega$ *contains infinitely many of these intervals then* X *is* $\mathcal{J}$*-positive.*

1.2.5 Relational systems

A convenient way to express various cardinal characteristics is via relational systems which we will briefly cover right now.

Definition 1.2.15. *A relation system is a triplet* $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ *where* $\mathcal{X}$ *and* $\mathcal{Y}$ *are nonempty sets and* $\sqsubseteq$ *is a relation between* $\mathcal{X}$ *and* $\mathcal{Y}$*. A set* $X \subseteq \mathcal{X}$ *is* $\sqsubseteq$*-unbounded if there is no single* $y \in \mathcal{Y}$ *such that* $x \sqsubseteq y$ *for all* $x \in X$*. A set* $Y \subseteq \mathcal{Y}$ *is* $\sqsubseteq$*-dominating if for every* $x \in \mathcal{X}$ *there is* $y \in Y$ *such that* $x \sqsubseteq y$*.*

For most systems we are going to consider, both sets $\mathcal{X}, \mathcal{Y}$ are Borel subsets of some uncountable Polish spaces and the relation $\sqsubseteq$ is such that the set $\{x \in \mathcal{X} : x \sqsubseteq y\}$ is Borel for every $y \in \mathcal{Y}$. We call such system Borel. We define the cardinal invariants of relation systems

$$\mathfrak{b}(\mathfrak{R}) = \min\{|X| : X \subseteq \mathcal{X} \text{ is } \sqsubseteq\text{-unbounded }\},$$
$$\mathfrak{d}(\mathfrak{R}) = \min\{|Y| : Y \subseteq \mathcal{Y} \text{ is } \sqsubseteq\text{-dominating }\}$$

Note that $\mathfrak{b}(\mathfrak{R})$ is undefined if and only if $\mathfrak{d}(\mathfrak{R}) = 1$ and $\mathfrak{d}(\mathfrak{R})$ is undefined if and only if $\mathfrak{b}(\mathfrak{R}) = 1$. A useful tool for proving inequalities between invariants of two relation systems, is the notion of Tukey reducibility.

Definition 1.2.16. *Assume that* $\mathfrak{R}_0 = \langle \mathcal{X}_0, \sqsubseteq_0, \mathcal{Y}_0 \rangle$ *and* $\mathfrak{R}_1 = \langle \mathcal{X}_1, \sqsubseteq_1, \mathcal{Y}_1 \rangle$ *are two relation systems. We will write* $\mathfrak{R}_0 \leq_T \mathfrak{R}_1$ *if there are functions* $\phi : \mathcal{X}_0 \to \mathcal{X}_1$ *and* $\psi : \mathcal{Y}_1 \to \mathcal{Y}_0$ *such that for all* $X \subseteq \mathcal{X}_0$ *and* $y \in \mathcal{Y}_1$ *if* $\phi[X] \sqsubseteq_1 y$ *then* $X \sqsubseteq_0 \psi(y)$*.*

The following fact is a well-known consequence of the Tukey reducibility.

Theorem 1.2.17. *The reduction* $\mathfrak{R}_0 \leq_T \mathfrak{R}_1$ *implies that* $\mathfrak{b}(\mathfrak{R}_1) \leq \mathfrak{b}(\mathfrak{R}_0)$ *and* $\mathfrak{d}(\mathfrak{R}_0) \leq \mathfrak{d}(\mathfrak{R}_1)$

If we are given a relation system $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ there is a natural way to define its dual, i.e. the relation system $\mathfrak{R}^\perp = \langle \mathcal{Y}, \sqsubseteq^\perp, \mathcal{X} \rangle$ where the relation $y \sqsubseteq^\perp x$ means that $\neg\, x \sqsubseteq y$. The duality of this is given by the following simple observation.

Proposition 1.2.18. *For a relation system* $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ *:*

– $(\mathfrak{R}^\perp)^\perp = \mathfrak{R}$*,*

– *a set* $X \subseteq \mathcal{X}$ *is* $\sqsubseteq$*-unbounded if and only if it is* $\sqsubseteq^\perp$*-dominating,*

– *a set* $Y \subseteq \mathcal{Y}$ *is* $\sqsubseteq$*-dominating if and only if it is* $\sqsubseteq^\perp$*-unbounded,*

– $\mathfrak{b}(\mathfrak{R}) = \mathfrak{d}(\mathfrak{R}^\perp)$ *and* $\mathfrak{d}(\mathfrak{R}) = \mathfrak{b}(\mathfrak{R}^\perp)$*.*

In this thesis, we will need the σ-versions of all of the notions defined above. The set $Y \subseteq \mathcal{Y}$ is $\sqsubseteq$-σ-dominating is for every $\{x_n : n \in \omega\} \subseteq \mathcal{X}$ there is $y \in \mathcal{Y}$ such that $x_n \sqsubseteq y$ holds for every $n \in \omega$. A set $X \subseteq \mathcal{X}$ is $\sqsubseteq$-σ-unbounded if there are no countably many $\{y_n : n \in \omega\} \subseteq \mathcal{Y}$ such that for every $x \in X$ there is $n \in \omega$ such that $x \sqsubseteq y_n$.

$$\mathfrak{b}_\sigma(\mathfrak{R}) = \min\{|X| : X \subseteq \mathcal{X} \text{ is } \sqsubseteq\text{-}\sigma\text{-unbounded }\},$$
$$\mathfrak{d}_\sigma(\mathfrak{R}) = \min\{|Y| : Y \subseteq \mathcal{Y} \text{ is } \sqsubseteq\text{-}\sigma\text{-dominating }\}.$$

Note that $\mathfrak{b}_\sigma(\mathfrak{R})$ is undefined if and only if $\mathfrak{d}(\mathfrak{R}) = \omega$ and $\mathfrak{d}_\sigma(\mathfrak{R})$ is undefined if and only if $\mathfrak{b}(\mathfrak{R}) = \omega$. A relation system $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ will be called σ-directed if for every $\{y_n : n \in \omega\} \subseteq \mathcal{Y}$ there is $y \in \mathcal{Y}$ such that for every $x \in \mathcal{X}$ we have: if $x \sqsubseteq y_n$ for some $n \in \omega$, then $x \sqsubseteq y$.

Proposition 1.2.19. *If a system $\mathfrak{R}$ is σ-directed, then $\mathfrak{b}(\mathfrak{R}) = \mathfrak{b}_\sigma(\mathfrak{R})$ and $\mathfrak{d}(\mathfrak{R}) = \mathfrak{d}_\sigma\mathfrak{R})$.*

Definition 1.2.20. *Suppose that $\mathbb{P}$ is a forcing notion and $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ is a Borel relational system. We will say that $\mathbb{P}$ adds an $\mathfrak{R}$-unbounded real, if it adds a real $\dot{x} \in \mathcal{X}$ which is not $\sqsubseteq$-dominated by any $y \in \mathcal{Y}$ from the ground model. We also say that $\mathbb{P}$ adds $\mathfrak{R}$-dominating real, if it adds a real $\dot{y}$ such that $x \sqsubseteq \dot{y}$ for any ground model $x \in \mathcal{X}$.*

Several times in this note we will use the notion of composition of two relation systems.

Definition 1.2.21. *(Blass, [Bla09] Section 4) Suppose that $\mathcal{R}_0 = \langle \mathcal{X}_0, \mathcal{X}_0, \sqsubseteq_0 \rangle$ and $\mathcal{R}_1 = \langle \mathcal{X}_1, \mathcal{X}_1, \sqsubseteq_1 \rangle$ are relation systems. The composition of $\mathcal{R}_0$ and $\mathcal{R}_1$ is the system $\mathcal{R}_0 \otimes \mathcal{R}_1 = \langle \mathcal{X}_0 \times \mathcal{X}_1^{\mathcal{Y}_0}, \mathcal{Y}_0 \times \mathcal{Y}_1, \sqsubseteq_\otimes \rangle$ where for $x \in \mathcal{X}_0$, $F \in \mathcal{X}_1^{\mathcal{Y}_0}$, $y \in \mathcal{Y}_0$ and $z \in \mathcal{Y}_1$ the relation $(x, F) \sqsubseteq_\otimes (y, z)$ means that $x \sqsubseteq_0 y$ and $F(y) \sqsubseteq_1 z$.*

The invariants of the composition can be directly calculated from the coefficients of the factors

Proposition 1.2.22. *(Blass, [Bla09] Theorem 4.10) If $\mathcal{R}_0 = \langle \mathcal{X}_0, \mathcal{X}_0, \sqsubseteq_0 \rangle$ and $\mathcal{R}_1 = \langle \mathcal{X}_1, \mathcal{X}_1, \sqsubseteq_1 \rangle$ are relation systems, then $\mathfrak{b}(\mathcal{R}_0 \otimes \mathcal{R}_1) = \min\{\mathfrak{b}(\mathcal{R}_0), \mathfrak{b}(\mathcal{R}_1)\}$ and $\mathfrak{d}(\mathcal{R}_0 \otimes \mathcal{R}_1) = \mathfrak{d}(\mathcal{R}_0) \cdot \mathfrak{d}(\mathcal{R}_1)$.*

Below we state the relation systems that will be important to us:

- The bounding/dominating systems:
  Consider the relational systems $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$ and $\langle \mathbb{IP}, \leq^*, \mathbb{IP} \rangle$. Both systems is σ-directed. Its cardinal characteristics are $\mathfrak{b}(\omega^\omega, \leq^*, \omega^\omega) = \mathfrak{b}(\mathbb{IP}, \leq^*, \mathbb{IP}) = \mathfrak{b}$ and $\mathfrak{d}(\omega^\omega, \leq^*, \omega^\omega) = \mathfrak{d}(\mathbb{IP}, \leq^*, \mathbb{IP}) = \mathfrak{d}$. The $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-unbounded and $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-dominating reals are simply unbounded and dominating reals in the classical sense (see [BJ95] or [Bla09]). Similarly, $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-unbounded and $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-dominating subsets of ω^ω are simply unbounded and dominating set in the classical sense.

- The reaping and splitting systems:
  Consider the relational systems $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$ and $\langle [\omega]^\omega, splits, [\omega]^\omega \rangle$. The relation $'A$ reaps B' means $A \cap B = \emptyset$ or $A \subseteq B$ and $'A$ splits B' means that both $A \cap B$ and $B \subseteq A$ are infinite. Clearly, $\mathfrak{b}([\omega]^\omega, reaps, [\omega]^\omega) = \mathfrak{d}([\omega]^\omega, splits, [\omega]^\omega) = \mathfrak{s}$ and $\mathfrak{d}([\omega]^\omega, reaps, [\omega]^\omega) = \mathfrak{b}([\omega]^\omega, splits, [\omega]^\omega) = \mathfrak{r}$. The characteristics $\mathfrak{b}_\sigma([\omega]^\omega, reaps, [\omega]^\omega) = \mathfrak{d}_\sigma([\omega]^\omega, splits, [\omega]^\omega)$ and $\mathfrak{d}_\sigma([\omega]^\omega, reaps, [\omega]^\omega) = \mathfrak{b}_\sigma([\omega]^\omega, splits, [\omega]^\omega)$ are known in literature as $\mathfrak{s}_\sigma$ and $\mathfrak{r}_\sigma$ respectively. It is an old open problem whether $\mathfrak{s} < \mathfrak{s}_\sigma$ or $\mathfrak{r}_\sigma < \mathfrak{r}$ is consistent. For a real $r \in [\omega]^\omega$ being $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-unbounded and $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-dominating reals are in literature known as being a reaping (unsplit) and splitting real respectively. The notion of being $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-unbounded coincide with being $\langle [\omega]^\omega, splits, [\omega]^\omega \rangle$-dominating and is usually referred to being a 'splitting' family. Dually, the notion of being $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-dominating coincide with being $\langle [\omega]^\omega, splits, [\omega]^\omega \rangle$-unbounded and is usually referred as being a 'reaping' (unsplit) family.

Next, we list couple of relation systems related to the combinatorics of measure and category. For more information the reader may consult [AA23].

- The systems related to infinitely equal and eventually different system:
  Consider the relational systems $\langle \omega^\omega, =^\infty, \omega^\omega \rangle$ and $\langle \omega^\omega, \neq^*, \omega^\omega \rangle$. Here the relation $f =^\infty g$ means that $f(n) = g(n)$ for infinitely many $n \in \omega$ and $f \neq^* g$ means that for some $N \in \omega$, $f(n) \neq g(n)$ for all $n > N$. A well-known Theorem of Bartoszyński-Miller ([BJ95]) says that $\mathfrak{b}(\omega^\omega, =^\infty, \omega^\omega) = \mathfrak{d}(\omega^\omega, \neq^*, \omega^\omega) = \mathrm{cov}(\mathcal{M})$ and $\mathfrak{d}(\omega^\omega, =^\infty, \omega^\omega) = \mathfrak{b}(\omega^\omega, \neq^*, \omega^\omega) = \mathrm{non}(\mathcal{M})$. The $\langle \omega^\omega, \neq^*, \omega^\omega \rangle$-unbounded real (which is also $\langle \omega^\omega, =^\infty, \omega^\omega \rangle$-dominating) is referred to as infinitely-equal real.

- The bounded inf. equal and event. diff. systems:
  Consider the relational systems $\langle \Pi F, =^\infty, \Pi F\rangle$ and $\langle \Pi F, \neq^*, \Pi F\rangle$. The relations $=^\infty$ and $\neq^*$ have the same meaning as in the previous relation systems but below the function F. It is known (see [AA23]) that $\mathfrak{d}(\Pi F, =^\infty, \Pi F) = \mathfrak{b}(\Pi F, \neq^*, \Pi F) \leq \mathrm{non}(\mathcal{M})$ and both are greater-equal $\mathrm{cov}(\mathcal{N})$ for sufficiently fast F. Dually, $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{b}(\Pi F, =^\infty, \Pi F) = \mathfrak{d}(\Pi F, \neq^*, \Pi F)$ and both are less than or equal $\mathrm{non}(\mathcal{N})$ for sufficiently fast F. The $\langle \Pi F, \neq^*, \Pi F\rangle$-unbounded real (which is also $\langle \Pi F, =^\infty, \Pi F\rangle$-dominating) is refereed to as F-bounded infinitely-equal real. The $\langle \Pi F, \neq^*, \Pi F\rangle$-dominating real (which is also $\langle \Pi F, =^\infty, \Pi F\rangle$-unbounded) is refereed to as F-bounded eventually-different real.

In the last chapter, in the section regarding covering and +-coverings numbers we will consider two more relational systems related to slaloms. The cardinal invariants of ideals on ω and σ-ideals of reals also can be presented in this fashion. For $\mathcal{I}$ being a σ-ideal of reals and $\mathcal{J}$ being an ideal on ω we consider the following relation systems

- The systems related to σ-ideals on ΠF:
  Let $\mathcal{I}$ be a σ-ideal on ΠF. Consider the relational systems $\langle \mathcal{I}, \subseteq, \mathcal{I}\rangle$ and $\langle \Pi F, \in, \mathcal{I}\rangle$. Clearly, $\mathfrak{b}(\mathcal{I}, \subseteq, \mathcal{I}) = \mathrm{add}(\mathcal{I})$, $\mathfrak{b}(\Pi F, \in, \mathcal{I}) = \mathrm{non}(\mathcal{I})$, $\mathfrak{d}(\Pi F, \in, \mathcal{I}) = \mathrm{cov}(\mathcal{I})$ and $\mathfrak{d}(\mathcal{I}, \subseteq, \mathcal{I}) = \mathrm{cof}(\mathcal{I})$. As $\mathcal{I}$ is closed under countable unions, the ω-versions are equal to the original ones for both systems. A $\langle \Pi F, \in, \mathcal{M}\rangle$-unbounded real is usually called a Cohen real.

- The systems related to the ideals on countable sets:
  Let $\mathcal{J}$ be an ideal on ω. Consider the relational systems $\langle \mathcal{J}, \subseteq^*, \mathcal{J}\rangle$ and $\langle [\omega]^\omega, \cap^\infty, \mathcal{J}\rangle$. The relation $A \cap^\infty B$ means that the intersection $A \cap B$ is infinite. Clearly, $\mathfrak{b}(\mathcal{J}, \subseteq^*, \mathcal{J}) = \mathrm{add}^*(\mathcal{J})$ and $\mathfrak{d}(\mathcal{J}, \subseteq^*, \mathcal{J}) = \mathrm{cof}^*(\mathcal{J})$. Also $\mathfrak{b}([\omega]^\omega, \cap^\infty, \mathcal{J}) = \mathrm{non}^*(\mathcal{J})$ and $\mathfrak{d}([\omega]^\omega, \cap^\infty, \mathcal{J}) = \mathrm{cov}^*(\mathcal{J})$. Adding a $\langle [\omega]^\omega, \cap^\infty, \mathcal{J}\rangle$-unbounded real is referred in literature as destruction of an ideal. Therefore, to destroy an ideal $\mathcal{J}$ means to add an $\dot{A} \in [\omega]^\omega$ which is almost disjoint with every ground model $X \in \mathcal{J}$. Consider also the relational system $\langle \mathcal{J}^+, \cap^\infty, \mathcal{J}\rangle$ where for $A \in \mathcal{J}^+$ and $B \in \mathcal{J}$, the relation $A \cap^\infty B$ again means that the intersection $A \cap B$ is infinite. Clearly, $\mathfrak{d}(\mathcal{J}^+, \cap^\infty, \mathcal{J}) = \mathrm{cov}^*_+(\mathcal{J})$. Adding a $\langle \mathcal{J}^+, \cap^\infty, \mathcal{J}\rangle$-unbounded real is referred in [FZ22] as +-destruction of an ideal.

Chapter 2

Tree types and tree ideals

2.1 Tree types and tree ideals

Definition 2.1.1. *A collection of pruned trees $\mathbb{T}$ on ΠF is a **tree forcing** (a tree type) if:*

- *for every $T \in \mathbb{T}$, $T|_\sigma \in \mathbb{T}$ for all $\sigma \in T$.*

*Additionally a tree type $\mathbb{T}$ will be called a **combinatorial tree forcing** if:*

- *for every $T \in \mathbb{T}$ there is $\{S_\alpha \subseteq T : \alpha < \mathfrak{c}\} \subseteq \mathbb{T}$ such that $[S_\alpha] \cap [S_\beta] = \emptyset$ for $\alpha \neq \beta$.*

The first property is sometimes referred to as arboreal tree type. The second one asserts that below any tree there is a large disjoint antichain. In particular, a c.c.c. tree type cannot be combinatorial. A tree type $\mathbb{T}$ will always be considered as a forcing notion with the order: $S \leq T$ if $S \subseteq T$.
With every tree type $\mathbb{T}$ there are two associated notions of smallness.

Definition 2.1.2. *A set $X \subseteq \Pi F$ is a $\mathbb{T}$-nwd set if for any $T \in \mathbb{T}$ there is $S \leq T$, $S \in \mathbb{T}$ such that $[S] \cap X = \emptyset$. The **tree ideal** associated to $\mathbb{T}$, denoted by $\mathcal{I}(\mathbb{T})$, is the σ-ideal generated by the $\mathbb{T}$-nwd sets. The ideal $\mathcal{I}_\mathcal{B}(\mathbb{T})$ is the **Borel part** of $\mathcal{I}(\mathbb{T})$, that is the σ-ideal generated by $\mathcal{I}(\mathbb{T}) \cap \mathrm{Borel}(\Pi F)$.*

In general, the $\mathbb{T}$-nwd sets may not form σ-ideal for a combinatorial tree type. Notice also that if a tree type $\mathbb{T}$ is combinatorial, then $\mathrm{non}(\mathcal{I}(\mathbb{T})) = \mathfrak{c}$: if $X \subseteq \Pi F$ is of cardinality lesser than $\mathfrak{c}$, then the disjoint antichain below any $T \in \mathbb{T}$ must contain an element disjoint with X. It is worth pointing out that in general $\mathcal{I}(\mathbb{T})$ may not have a Borel basis (more on that in the section about the cofinalities of tree ideals). However, a basis of $\mathcal{I}(\mathbb{T})$ can be conviniently described using maximal antichains in $\mathbb{T}$.

Proposition 2.1.3. *If $\mathbb{T}$ is a combinatorial tree forcing and $\mathcal{A} \subseteq \mathbb{T}$, we define the following set:*

$$\mathcal{X}(\mathcal{A}) := \Pi F \setminus \bigcup_{T \in \mathcal{A}} [T]$$

Then the following observations hold:

- *if $\mathcal{A}$ is a maximal antichain (or open dense) in $\mathbb{T}$, then $\mathcal{X}(\mathcal{A})$ is in $\mathbb{T}$-nwd,*
- *every set in $\mathcal{I}(\mathbb{T})$ can be covered by countably many sets of form $\mathcal{X}(\mathcal{A})$ for some maximal antichain (open dense) $\mathcal{A} \subseteq \mathbb{T}$.*

Moreover,

- *if $\mathcal{A}_1$ refines $\mathcal{A}_0$, then $\mathcal{X}(\mathcal{A}_0) \subseteq \mathcal{X}(\mathcal{A}_1)$,*
- *If $\mathbb{T}'$ is a dense subset of $\mathbb{T}$, then the tree ideals of the two are equal.*

Proof. For (1) let $T \in \mathbb{T}$. As $\mathcal{A}$ is maximal there are $R \in \mathcal{A}$ and $S \in \mathbb{T}$ such that $S \leq R$ and $S \leq T$. But then $[S] \subseteq \bigcup_{R\in\mathcal{A}}[R]$ so $[S] \subseteq \mathcal{X}(\mathcal{A}) = \emptyset$. The (2) follows directly from the definition and (1). To see the last part note that if $\mathcal{A}_1$ refines $\mathcal{A}_0$[1], then $\bigcup_{T\in\mathcal{A}_1}[T] \subseteq \bigcup_{T\in\mathcal{A}_0}[T]$ and thus $\mathcal{X}(\mathcal{A}_0) \subseteq \mathcal{X}(\mathcal{A}_1)$. □

It follows that the sets of form $\mathcal{X}(\mathcal{A})$ where $\mathcal{A} \subseteq \mathbb{T}$ is either maximal antichain, dense or predense, form a basis of the ideal $\mathcal{I}(\mathbb{T})$.

Corollary 2.1.4. *If a combinatorial tree forcing $\mathbb{T}$ is a c.c.c. tree type, then $\mathcal{I}(\mathbb{T}) = \mathcal{I}_{\mathcal{B}}(\mathbb{T})$.*

Proof. If an antichain $\mathcal{A}$ is countable, the set $\mathcal{X}(\mathcal{A})$ is Borel as a countable intersection of open sets. □

2.2 Catalogue of tree types

We list here all the tree types which are related to this work. For each tree type, we discuss satisfying the definition of being the combinatorial tree forcing with c.r.n., fusion argument and the corresponding perfect set style dichotomy. We also mention related results and calculate the cardinal invariants of the Borel part. We start with the compact trees.

2.2.1 Bounded infinitely equal trees

Throughout this section we will consider two generalizations of Sacks forcing. Both forcings will be parametrized by a function $F \in \omega^\omega$. For such function, we will always assume that F is increasing and strictly above zero in all considerations in this thesis.

Definition 2.2.1. *A tree $T \subseteq Seq(F)$ is F-**bounded Sacks** if every $\sigma \in T$ has an extension $\tau \in T$ such that $succ_T(\tau) = F(|\tau|)$. The collection of F-Sacks trees will be denoted by $\mathbb{S}_F$.*

Definition 2.2.2. *A tree $T \subseteq Seq(F)$ is F-**bounded infinitely equal tree** if every $\sigma \in T$ there is $N_T(\sigma) > |\sigma|$ such that for every $i \in F(N_T(\sigma))$ exists $\tau \in T$, $\sigma \subseteq \tau$, $|\tau| > N_T(\sigma)$ with $\tau(N_T(\sigma)) = i$. The collection of all F-infinitely equal trees will be denoted by $\mathbb{IE}_F$.*

We will always assume that, in the definition of such tree, the value $N_T(\sigma)$ is the smallest one with the required property. Notice that if F is constant with the value 2, the $\mathbb{S}_F$-trees and $\mathbb{IE}_F$-trees are simply Sacks (perfect) trees on 2^ω. Notice also that in $\mathbb{S}_F$ dense is the set of all trees $T \in \mathbb{S}_F$ that split only when they split fully i.e. for all $\sigma \in T$ we have $|succ_T(\sigma)| \in \{1, F(|\sigma|)\}$. Similarly, in $\mathbb{IE}_F$ dense is the set of all $T \in \mathbb{IE}_F$ such that the 'full-splits' are realized without repetition i.e. for all $\sigma \in T$ and every $i \in F(N_T(\sigma))$ there is unique $\tau \in T|_\sigma$ such that $\tau(N_T(\sigma)) = i$. For $T \in \mathbb{S}_F$, by $split(T)$ we will denote the set $\{\sigma \in T : |succ_T(\sigma)| = F(|\sigma|)\}$. The following proposition gathers the basic properties of tree types $\mathbb{S}_F$ and $\mathbb{IE}_F$.

Proposition 2.2.3. *The following hold:*

- *both $\mathbb{S}_F$ and $\mathbb{IE}_F$ are combinatorial,*
- *both $\mathbb{S}_F$ and $\mathbb{IE}_F$ have continuous reading of names,*
- *both $\mathbb{S}_F$ and $\mathbb{IE}_F$ have axiom A,*
- *both $\mathbb{S}_F$-nwd sets and $\mathbb{IE}_F$-nwd sets are closed under countable unions.*

Proof. • We prove the first item only for the tree type $\mathbb{IE}_F$, as the argument for $\mathbb{S}_F$ is almost identical. Let $T \in \mathbb{IE}_F$. For every $x \in 2^\omega$ we construct $T_x \in \mathbb{IE}_F$, $T_x \subseteq T$ as follows: inductively construct a sequence $\{F^x_n : n \in \omega\}$ such that for every $n \in \omega$ we have:

[1] $\mathcal{A}_1$ refines $\mathcal{A}_0$ is every element of $\mathcal{A}_1$ is below some element of $\mathcal{A}_0$

$(1)_n$ F^x_n is a finite subtree of T,

$(2)_n$ F^x_{n+1} is an end-extension of F^x_n,

$(3)_n$ for every $\sigma \in term(F^x_n)$ there is unique $\tau \in F^x_{n+1}$ such that $\sigma \subseteq \tau$, $|\tau| = N_T(\sigma) + 1$ and $\tau(N_T(\sigma)) = x(n)$,

$(4)_n$ for each $\sigma \in term(F^x_n)$ exists $N > |\sigma|$ such that for every $i \in F(N)$ there is $\tau \in F^x_{n+1}$, with $|\tau| = N + 1$ such that $\tau(N) = i$.

Start the induction with $F^x_0 = \{\emptyset\}$. Assume that F^x_n has been constructed. For every $\sigma \in term(F^x_n)$ pick $\tau_\sigma \in T$, $\sigma \subseteq \tau_\sigma$, $|\tau_\sigma| = N_T(\sigma) + 1$ such that $\tau_\sigma(N_T(\sigma)) = x(n)$. Let then F^x_{n+1} be the tree generated by the set $\{\rho \in T : \rho \text{ extends some } \sigma \in term(F^x_n) \text{ and } |\rho| = N_T(\tau_\sigma)\} + 1$. When the induction is finished, define $T_x = \bigcup_n F^x_n$. Because of the conditions $(4)_n$, $n \in \omega$ we have that $T_x \in \mathbb{IE}_F$. We claim that for $x \neq y$ in 2^ω we have that $[T_x] \cap [T_y] = \emptyset$. Let $z \in [T_x] \cap [T_y]$ and let $n \in \omega$ be the smallest such that $x(n) \neq y(n)$. Note that $F^x_i = F^y_i$ for every $i \leq n$. Let $\sigma \in term(F^x_i) = term(F^y_i)$ be such that $\sigma \subseteq z$ but then by the condition $(3)_n$ we get that $y(n) = z(N_T(\sigma)) = x(n)$, which is impossible.

• Both have the continuous reading of names because of the Theorem 1.2.6 and the fact that the Borel parts $\mathcal{I}_\mathcal{B}(\mathbb{S}_F)$ and $\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)$ are generated by closed sets (see the Theorem 2.2.4 and the Theorem 2.2.5).

• As for $\mathbb{S}_F$ this has been shown in [NRa93], we focus only on $\mathbb{IE}_F$. For $T \in \mathbb{IE}_F$ define an increasing sequence $\{k^T_n : n \in \omega\}$ of natural numbers as follows: let $k^T_0 = 0$ and if k^T_n has been defined let $k^T_{n+1} = \max\{N_T(\sigma) : \sigma \in T, |\sigma| = k^T_n\} + 1$. Define now $T \leq_n S$ if $T \leq S$ and $T \cap \omega^{\leq k^S_n} = S \cap \omega^{\leq k^S_n}$. In particular $T \leq_n S$ implies $k^T_i = k^S_i$ for $i \leq n$. It is easy to see that for $n \in \omega$ and $T \in \mathbb{IE}_F$ with $T \Vdash$ "$\dot{x} \in \mathbf{V}$" there is $S \leq_n T$ and a countable X such that $S \Vdash$ "$\dot{x} \in X$": for every $\tau \in T$ of length $|\tau| = k^T_n$ find $S_\tau \leq T|_\tau$ such that $S_\tau \Vdash$ "$\dot{x} = x_\tau$" for some x_τ. Then $S = \bigcup_{\tau \in T \cap \omega^{k^T_n}} S_\tau$ is such that $S \leq_n T$ and $S \Vdash$ "$\dot{x} \in X$" for $X = \{x_\tau : \tau \in T \cap \omega^{k^T_n}\}$. We will check that for a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{IE}_F$ such that $T_{n+1} \leq_n T_n$ for all $n \in \omega$, there is $T \in \mathbb{IE}_F$ with $T \leq_n T_n$ for all $n \in \omega$. Define $T = \bigcap_n T_n$. First we will check that $T \in \mathbb{IE}_F$. Let $\sigma \in T$. We want to show that there is $M > |\sigma|$ and $\{\tau_i : i < F(M)\} \subseteq T$ such that $\tau_i(M) = i$ for every $i < F(M)$. Let $N \in \omega$ be such that $|\sigma| < k^{T_N}_N$. As $\sigma \in T_N$, there is $M > |\sigma|$, $M < k^{T_N}_{N+1}$ for which there is $\{\tau_i : i < F(M)\} \subseteq T_N|_\sigma \cap \omega^{M+1}$ such that for all $i < F(M)$, $\tau_i(M) = i$. On one hand for every $n \geq N$, $T_{n+1} \leq_n T_n$ implies that $\{\tau_i : i < F(M)\} \subseteq T_{n+1}$. On the other hand, if $n < N$, then $T_N \subseteq T_n$, so also $\{\tau_i : i < F(M)\} \subseteq T_n$. It follows that $T \in \mathbb{IE}_F$. The relation $T \leq_n T_n$ holds by the definition of T.

• Let $\{X_n : n \in \omega\} \subseteq \mathbb{IE}_F$-nwd and $T \in \mathbb{IE}_F$. Inductively, we construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{IE}_F$ such that for every $n \in \omega$ we have:

$(1)_n$ $T_{n+1} \leq_n T_n$,

$(2)_n$ $[T_{n+1}] \cap X_n = \emptyset$,

Start the induction with $T_0 = T$. Assume now that T_n has been built. For every $\sigma \in T_n \cap \omega^{k^{T_n}_n}$ find $S_\sigma \leq T_n|_\sigma$ such that $[S_\sigma] \cap X_n = \emptyset$. Let then $T_{n+1} = \bigcup\{S_\sigma : \sigma \in T_n \cap \omega^{k_n}\}$. By the conditions $(1)_n$, $n \in \omega$ we get that $S \in \mathbb{IE}_F$. By the conditions $(2)_n$, $n \in \omega$ we get that $[S] \cap \bigcup_n X_n = \emptyset$. □

For these two tree types there are known an appropriate perfect-set style dichotomies. The first one is from a work of Rosłanowski and Newelski.

Theorem 2.2.4. *(Rosłanowski and Newelski, [NRa93]) For every analytic $A \subseteq \Pi F$, either there is $T \in \mathbb{S}_F$ such that $[T] \subseteq A$, or for some $\phi : Seq(F) \to \omega$ with $\phi(\sigma) \in F(|\sigma|)$, A is contained in the set of form*

$$D_\phi := \{x \in \Pi F : \forall^\infty n \in \omega \; x(n) \neq \phi(x|_n)\}.$$

Theorem 2.2.5. *(Khomskii and Laguzzi, [KL17]) For every analytic $A \subseteq \Pi F$, either there is $T \in \mathbb{IE}_F$ such that $[T] \subseteq A$, or for some $f \in \Pi F$, A is contained in the set of form*

$$E_f := \{x \in \Pi F : \forall^\infty n \in \omega \; x(n) \neq f(n)\}.$$

It follows that the tree types $\mathbb{S}_F$ and $\mathbb{IE}_F$ are forcing equivalent with $\text{Borel}(\Pi F) \setminus \mathcal{I}_\mathcal{B}(\mathbb{S}_F)$ and $\text{Borel}(\Pi F) \setminus \mathcal{I}_\mathcal{B}(\mathbb{IE}_F)$ respectively. Clearly $\mathcal{I}_\mathcal{B}(\mathbb{IE}_F) \subseteq \mathcal{I}_\mathcal{B}(\mathbb{S}_F)$. When F is constant with the value 2 both dichotomies simplify to the classical perfect set theorem for analytic sets. Notice, that the Borel parts of $\mathbb{S}_F$'s and $\mathbb{IE}_F$'s, are σ-generated by closed sets and are properly contained in the σ-ideal of meager sets.
The following proposition collects the cardinal invariants of the Borel parts.

Proposition 2.2.6. *The following holds:*

- $\text{add}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) = \text{add}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) = \omega_1$,
- $\text{cof}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) = \text{cof}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) = \mathfrak{c}$,
- $\text{cov}(\mathcal{M}) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) = \mathfrak{b}(\Pi F, =^\infty, \Pi F)$,
- $\mathfrak{d}(\Pi F, =^\infty, \Pi F) = \text{non}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) \leq \text{non}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) \leq \text{non}(\mathcal{M})$.

Proof. The additivities and cofinalities of the $\mathcal{I}_\mathcal{B}(\mathbb{S}_F)$ have been calculated in [NRa93]. The same arguments work for $\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)$ as abserved in [KL17]. The inequalities $\text{cov}(\mathcal{M}) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F))$ and $\text{non}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) \leq \text{non}(\mathcal{M})$ hold because $\mathcal{I}_\mathcal{B}(\mathbb{S}_F) \subseteq \mathcal{M}$. The inequalities $\text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F)) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F))$ and $\text{non}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) \leq \text{non}(\mathcal{I}_\mathcal{B}(\mathbb{S}_F))$ hold because $\mathcal{I}_\mathcal{B}(\mathbb{IE}_F) \subseteq \mathcal{I}_\mathcal{B}(\mathbb{S}_F)$. The equalities $\text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F)) = \mathfrak{b}(\Pi F, =^\infty, \Pi F)$ and $\mathfrak{d}(\Pi F, =^\infty, \Pi F) = \text{non}(\mathcal{I}_\mathcal{B}(\mathbb{IE}_F))$ follows from the definitions. □

2.2.2 Splitting trees

Another tree type which is gaining some popularity in recent years is the Splitting tree forcing. This forcing was independently discovered by Shelah ([She92]), Spinas ([Spi04]) and Schilhan ([Sch22]) for the perpouse of adding splitting reals in a minimal way (see also [LMSR23], [HS20]).

Definition 2.2.7. *A tree $T \subseteq 2^\omega$ is a **splitting tree** if for every $\sigma \in T$ there is $N_T(\sigma) > |\sigma|$ such that for every $k > N_T(\sigma)$ and $i \in 2$ there is $\tau \in T \cap 2^k$, $\sigma \subseteq \tau$ satisfying $\tau(k-1) = i$. The collection of all splitting trees is denoted by $\mathbb{SP}$.*

Proposition 2.2.8. *The following hold:*

- $\mathbb{SP}$ *is combinatorial,*
- $\mathbb{SP}$ *have the continuous reading of names,*
- $\mathbb{SP}$*-nwd sets are closed under countable unions.*

Proof. • The fact that $\mathbb{SP}$ is combinatorial has been shown in [LMSR23] (For a different proof the reader may also see the Proposition 2.2.54).
• $\mathbb{SP}$ has the continuous reading of names by the Theorem 1.2.6 and the Theorem 2.2.12.
• Tho show that $\mathbb{SP}$-nwd sets are closed under countable unions we will need the following claim

Claim 2.2.9. *For every $T \in \mathbb{SP}$, $m \in \omega$ and $X \in \mathbb{SP}$-nwd, there is $S \leq T$, $S \in \mathbb{SP}$ such that:*

- $[S] \cap X = \emptyset$,
- *for each $\sigma \in T \cap 2^m$ we have $N_T(\sigma) = N_S(\sigma)$.*

proof of the claim. Let $T \in \mathbb{SP}$, $m \in \omega$ and $X \in \mathbb{SP}$-nwd. Fix $\sigma \in T \cap 2^m$. Pick any $\sigma* \in T \cap 2^{N_T(\sigma)}$. For every $\tau \in T|_\sigma \cap 2^{N_T(\sigma*)}$ such that $\sigma* \subseteq \tau$ is not true, find a tree $R^\sigma_\tau \leq T|_\tau$, $R^\sigma_\tau \in \mathbb{SP}$ with $[R^\sigma_\tau] \cap X = \emptyset$. Let $k_\sigma = \min\{N_{R^\sigma_\tau}(\tau) : \tau \in T|_\sigma \cap 2^{N_T(\sigma*)}$ and it is not the case that $\sigma* \subseteq \tau\}$. For every $\rho \in T|_{\sigma*} \cap 2^{k_\sigma}$ find $R^\sigma_\rho \leq T|_\rho$, $R^\sigma_\rho \in \mathbb{SP}$ such that $[R^\sigma_\rho] \cap X\emptyset$. Let then R^σ be the union of all R^σ_τ's and R^σ_ρ's defined above. Unfix $\sigma \in T \cap 2^m$ and let $S = \bigcup_{\sigma \in T \cap 2^m} R^\sigma$. Clearly $[S]$ is disjoint with X as each R^σ is. We will now show that for every $\sigma \in T \cap 2^m$ we have $N_T(\sigma) = N_S(\sigma)$. Clearly $N_T(\sigma) \leq N_S(\sigma)$ because of $S \leq T$. To show the opposite inequality, let $\sigma \in T \cap 2^m$ and $i \geq N_S(\sigma)$. We want to show that we can find $\rho'_0, \rho'_1 \in S \cap 2^{i+1}$ such that $\rho'_0(i) = 0$ and $\rho'_1(i) = 1$. If $i \in [N_T(\sigma), N_T(\sigma*))$, then there are required ρ'_0, ρ'_1 because no changes has been made to the tree $T_\sigma \cap 2^{\leq N_T(\sigma*)}$. If $i \in [N_T(\sigma*), k_\sigma)$, then there are required ρ'_0, ρ'_1 because of the tree $T_{\sigma*}$. If $i \in [k_\sigma, \infty)$, then there are required ρ'_0, ρ'_1 because of the trees R^σ_τ for $\tau \in T|_\sigma \cap 2^{N_T(\sigma*)}$. This finishes the proof. □

When the claim has been proven, let $\{X_n : n \in \omega\} \subseteq \mathbb{SP}$-nwd and $T \in \mathbb{SP}$. We will inductively construct $\{T_n : n \in \omega\} \subseteq \mathbb{SP}$ and $\{m_n : n \in \omega\} \in \omega^\omega$ such that for every $n \in \omega$ we have:

$(1)_n$ $T_{n+1} \leq T_n$ and $m_{n+1} > m_n$,

$(2)_n$ for every $\sigma \in T_n \cap 2^{m_n}$ we have $N_{T_n}(\sigma) = N_{T_{n+1}}(\sigma)$,

$(3)_n$ $[T_{n+1}] \cap X_n = \emptyset$.

The inductive construction is straightforward. Start with $T_0 = 0$, arbitrarily m_0, and at each step of the induction use the claim to get another T_n. When induction is over, we define $S = \bigcap_n T_n$. Because of the conditions $(1)_n$, $n \in \omega$ and $(2)_n$, $n \in \omega$ we get that $S \in \mathbb{SP}$. By the conditions $(3)_n$, $n \in \omega$ we have $[S] \cap \bigcup_n X_n = \emptyset$. □

The following problem regarding homogeneity seems to be open.

Question 2.2.10. *Is the tree type $\mathbb{SP}$ homogeneous?*

Similar argument as in the proposition above, exposes the fusion technique for splitting trees.

Proposition 2.2.11. *(Laguzzi, Mildenberger, Stuber-Rousselle, [LMSR23]) $\mathbb{SP}$ has axiom A.*

The following perfect-set style dichotomy comes from the work Spinas.

Theorem 2.2.12. *(Spinas, [Spi04]) For every analytic $A \subseteq 2^\omega$, either there is $T \in \mathbb{SP}$ such that $[T] \subseteq A$, or A is not ω-splitting i.e. there is a countable collection $\{C_n : n \in \omega\} \subseteq [\omega]^\omega$ such that A is covered by the sets F_{C_n} where for an infinite $C \subseteq \omega$*

$$F_C =: \{x \in 2^\omega : x^{-1}[\{1\}] \subseteq^* C \text{ or } x^{-1}[\{1\}] \cap C =^* \emptyset\}$$

It follows that the tree type $\mathbb{SP}$ is forcing equivalent with $\mathrm{Borel}(2^\omega) \setminus \mathcal{I}_\mathcal{B}(\mathbb{SP})$. Notice also that the Borel part $\mathcal{I}_\mathcal{B}(\mathbb{SP})$ is generated by a collection of closed sets and is properly contained in the σ-ideal of meager sets. Note also that $A \subseteq B$ implies $F_B \subseteq F_A$. The cardinal invariants of the σ-ideal of not ω-splitting sets behave as follows.

Proposition 2.2.13. *The following holds:*

- $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{SP})) = \omega_1$,

– $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{r}_\sigma$,

– $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{s}_\sigma$,

– $\mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{c}$.

Proof. The equalities $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{r}_\sigma$ and $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{s}_\sigma$ follow directly from the definitions. To show the remaining equalities let $\mathcal{A} \subseteq [\omega]^\omega$ be an maximal almost disjoint family of size $\mathfrak{c}$. The $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \omega_1$ will come from the fact that for every $\mathcal{A}' \subseteq \mathcal{A}$ of size ω_1 we have $\bigcup_{A\in\mathcal{A}'} F_A \notin \mathcal{I}_{\mathcal{B}}(\mathbb{SP})$. Suppose that $\bigcup_{A\in\mathcal{A}'} F_A \subseteq \bigcup_n F_{B_n}$ for some countably many infinite B_n's. Without loss of generality, we may assume that each B_n is a subset of some $A_n \in \mathcal{A}'$. Let $A \in \mathcal{A}' \setminus \{A_n : n \in \omega\}$. We claim that F_A is not covered by $\bigcup_{n\in\omega} F_{B_n}$. Notice that if $x \in 2^\omega$ is the characteristic function of $\omega \setminus A$, then $x \in F_A \setminus \bigcup_{n\in\omega} F_{B_n}$. To show that $\mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{SP})) = \mathfrak{c}$ assume that $\kappa < \mathfrak{c}$ and every element of $\mathcal{I}_{\mathcal{B}}(\mathbb{SP})$ can be covered by a countable union of sets F_{C_α} where $\{C_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$. Again, we may assume that each C_α is contained in some $A_\alpha \in \mathcal{A}$. As $\kappa < \mathfrak{c}$ we may pick $A \in \mathcal{A} \setminus \{A_\alpha : \alpha < \kappa\}$. We claim that for any $\{\alpha_n : n \in \omega\} \subseteq \kappa$ the set F_A is not contained in the set $\bigcup_n F_{C_{\alpha_n}}$. The argument is identical as in the case of the additivity. □

2.2.3 Eventually different trees

We will briefly discuss a certain tree type for adding bounded eventually different reals. For these trees, we will always assume that F is increasing and strictly above zero for all considerations in this thesis.

Definition 2.2.14. *A tree $T \subseteq Seq(F)$ is an **F-bounded eventually different tree** if for every $N \in \omega$, almost all $\sigma \in T$ satisfies $|succ_T(\sigma)| > N$. The collection of all F-bounded eventually different trees is denoted by $\mathbb{ED}_F$.*

It is easy to see that dense in $\mathbb{ED}_F$ is the set of all trees such that longer node in the tree has greater (or equal) number of successors. Without loss of generality, we may work with such trees only. For any $T \in \omega$ we define an increasing sequence of natural numbers $\{M_T(n) : n \in \omega\}$ as follows: let $M_T(0) = 0$ and let $M_T(n+1)$ be the smallest number $m > M_T(n)$ such that for every $\sigma \in T$, if $|\sigma| \geq m$, then $|succ_T(\sigma)| > n+1$. Let also for $n \in \omega$ and $S, T \in \mathbb{ED}_F$ we will write $S \leq_n T$ if $S \leq T$, $M_T(i) = M_S(i)$ for every $i \leq n$ and $S \cap \omega^{M_T(n)} = T \cap \omega^{M_T(n)}$.

Proposition 2.2.15. *$\mathbb{ED}_F$ has axiom A.*

Proof. The axiom A of $\mathbb{ED}_F$ has been shown by Hernández-Hernández and Hrušák in [HHH07] and by Żuchowski, Khomskii in [KZ]. It is also proved along the lines of the next proposition. □

We gather the basic properties of the tree type $\mathbb{ED}_F$.

Proposition 2.2.16. *The following hold:*

– *$\mathbb{ED}_F$ are combinatorial,*

– *$\mathbb{ED}_F$ have the continuous reading of names,*

– *$\mathbb{ED}_F$-nwd sets are closed under countable unions,*

Proof. • Let $T \in \mathbb{ED}_F$. Let $\{N_n : n \in \omega\}$ be such an $N_n = M_T(2n)$ for every $n \in \omega$. For each $x \in 2^\omega$ let $T_x \in \mathbb{ED}_F$, $T_x \subseteq T$ be the unique tree such that $stem(T) \subseteq stem(T_x)$ and for $\sigma \in T_x$ with $n \in \omega$ we have that: if $|\sigma| \in [N_n, N_{n+1})$, then $succ_{T_x}(\sigma)$ consists of first half of the elements of $succ_T(\sigma)$ if $x(n) = 0$ or, $succ_{T_x}(\sigma)$ consists of second half of the elements of $succ_T(\sigma)$ if $x(n) = 1$. The first and the second half we mean in the increasing order of $succ_T(\sigma)$. Notice that because of the choice of $\{N_n : n \in \omega\}$ we have that for every $n \in \omega$ and $\sigma \in T_x$

with $|\sigma| \in [N_n, N_{n+1})$, $|succ_{T_x}(\sigma)| \geq n$. It follows that $T_x \in \mathbb{ED}_F$ for every $x \in 2^\omega$. We claim that for $x \neq y$ in 2^ω, $[T_x] \cap [T_y] = \emptyset$. Let $z \in [T_x] \cap [T_y]$ and let $n \in \omega$ be the smallest such that $x(n) \neq y(n)$. Notice that $T_x \cap \omega^{<N_n} = T_y \cap \omega^{<N_n}$. By the construction of T_x, the $z(n)$ should belong to the first half of the elements of $succ_T(z|_n)$. Similarly, because of the construction of T_y, the $z(n)$ should belong to the second half of the elements of $succ_T(z|_n)$. As the two sets are disjoint, we get a contradiction.

- The continuous reading of names has been shown by Khomskii and Żuchowski in [KZ].

- Next we prove that $\mathbb{ED}_F$-nwd sets are closed under countable unions. We will need the following lemma.

Lemma 2.2.17. *For every number $n \in \omega$, set $X \in \mathbb{ED}_F$-nwd and a tree $T \in \mathbb{ED}_F$ there is $S \leq_n T|_\sigma$ with the stem σ, such that $[S] \cap X = \emptyset$.*

proof of the lemma. We will say that a tree $T \in \mathbb{ED}_F$ an n-tree with the stem $\sigma \in T$ if $|succ_T(\tau)| \geq n$ for every τ which extends the $stem(T) = \sigma$. For $n \in \omega$ and $T \in \mathbb{ED}_F$ define rank function $rank_n : T \to \omega$ as follows:

$$rank(\sigma) = 0 \text{ iff there is } n\text{-tree } S \leq T|_\sigma, \, stem(S) = \sigma \text{ such that } [S] \cap X = \emptyset,$$
$$rank(\sigma) \leq k+1 \text{ iff not } rank(\sigma) \leq k \text{ and } |\{i \in succ_T(\sigma) : rank^T(\sigma^\frown i) \leq k\}| \geq \tfrac{1}{2}|succ_T(\sigma)|.$$

We will show that $rank$ is defined on every element of T. Assume that for some $\sigma \in T$ the $rank(\sigma)$ is not defined. Note that the set $\{i \in succ_T(\sigma) : rank(\sigma^\frown i) \text{ is not defined}\}$ has to be of cardinality greater-equal $\frac{1}{2}|succ_T(\sigma)|$. Using this observation, we can inductively construct a tree $S \leq T|_\sigma$ such that $stem(S) = \sigma$, for every $\tau \in S$ extending σ, $rank_n(\tau)$ is not defined and $|succ_S(\tau)| \geq \frac{1}{2}|succ_T(\tau)|$. As $X \in \mathbb{ED}_F$-nwd, there us $R \leq S$ with $[R] \cap X = \emptyset$. By the definition of $\mathbb{ED}_F$ there is $\rho \in R$ such that $R|_\rho$ is an n-tree. This contradicts the fact that $rank_n(\rho)$ is undefined.
For any $\sigma \in T \cap \omega^{M_T(2n)}$ we will construct an n-tree $R_\sigma \subseteq T$ with the stem σ. Fix $\sigma \in T \cap \omega^{M_T(2n)}$. We will consider two cases:
First, that $rank(\sigma) = 0$. Then by the definition of rank, there is n-tree R with the stem σ such that $[R] \cap X = \emptyset$. Let then R_σ be such R.
Second, that $rank(\sigma) > 0$. Then, one can find a finite tree F with that stem σ such that:

- $\forall \tau \in term(F_\sigma)\ rank(\tau) = 0$,
- $\forall \tau \in F \setminus term(F)$, if $\sigma \subseteq \tau$, then $|succ_F(\tau)| > \frac{1}{2}|succ_F(\tau)|$

Note that in the second condition $|succ_F(\tau)| > \frac{1}{2}|succ_F(\tau)| \geq \frac{1}{2}2n = n$. For every $\tau \in term(F)$, as $rank(\tau) = 0$, we find an n-tree R^τ with the stem τ such that $[R^\tau] \cap X = \emptyset$. Define then $R_\sigma = \bigcup_{\tau \in term(F)} R^\tau$.
Unfix $\sigma \in T \cap \omega^{M_T(2n)}$ and let $S = \bigcup_{\sigma \in T \cap \omega^{M_T(2n)}} R_\sigma$. Clearly, $S \cap X = \emptyset$, S is an $\mathbb{ED}_F$-tree and $S \leq_n T$. This finishes the proof of the lemma. □

To show that $\mathbb{ED}_F$-nwd sets are closed under countable unions, let $\{X_n : n \in \omega\} \subseteq \mathbb{ED}_F$-nwd and $T \in \mathbb{ED}_T$. We will inductively construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{ED}_F$ such that for every $n \in \omega$ we have:

$(1)_n$ $T_{n+1} \leq_n T_n$,

$(2)_n$ $[T_{n+1}] \cap X_n = \emptyset$.

Start the induction with $T_0 = T$ and at each step use the Lemma 2.2.17 to obtain T_{n+1} from T_n. When the induction is over let $S = \bigcap_n T_n$. Clearly we get that $[S] \cap \bigcup_n X_n = \emptyset$ so we check that $S \in \mathbb{ED}_F$. To see this notice that for every $n \in \omega$, if $\sigma \in T$ is such that $|\sigma| > M_{T_n}(n)$, then $|succ_S(\sigma)| > n$. This means that only for finitely many σ's in T we have that $|succ_S(\sigma)| < n$. □

The homogeneity of $\mathbb{ED}_F$ is not clear to me. Consider now an ideal on ΠF, σ-generated by the sets of form

$$E_\phi = \{x \in \Pi F : \exists^\infty_{n\in\omega} x(n) = \phi(x|_n)\}$$

where $\phi : Seq(F) \to \omega$ is such that $\phi(\sigma) \in F(|\sigma|)$ for every $\sigma \in Seq(F)$. Denote this σ-ideal by $\mathcal{I}_{\mathbb{ED}_F}$. Despite some effort, the following basic question remains open.

Question 2.2.18. *Does $\mathbb{ED}_F$ stand with the ideal $\mathcal{I}_{\mathbb{ED}_F}$ in a perfect set-style dichotomy?*

In case of cardinal invariant of the ideal $\mathcal{I}_{\mathbb{ED}_F}$ the following result is known.

Theorem 2.2.19. *(Żuchowski, [KZ]) The following holds:*

- $\mathrm{add}(\mathcal{I}_{\mathbb{ED}_F}) = \omega_1$,
- $\mathfrak{b}(\Pi F, =^\infty, \Pi F) \leq \mathrm{non}(\mathcal{I}_{\mathbb{ED}_F}) \leq \mathrm{non}(\mathcal{N})$ *for such $F \in \omega^\omega$ that $\Sigma_n \frac{1}{F(n)} < \infty$,*
- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{I}_{\mathbb{ED}_F}) \leq \mathfrak{d}(\Pi F, =^\infty, \Pi F)$ *for such $F \in \omega^\omega$ that $\Sigma_n \frac{1}{F(n)} < \infty$,*
- $\mathrm{cof}(\mathcal{I}_{\mathbb{ED}_F}) = \mathfrak{c}$.

In [HHH07] the authors showed that $\mathbb{ED}_F$ is ω^ω-bounding. Zapletal ([Zap08]) showed that such trees always preserve $\mathfrak{r}$.

Question 2.2.20. *Is there a tree type which stands in a perfect set like dichotomy with the ideal on ΠF generated by the sets of form $E_f = \{x \in \Pi F : \exists^\infty_{n\in\omega} x(n) = f(n)\}$ for $f \in \Pi F$?*

2.2.4 Full-Miller and infinitely-equal trees

The following two tree types has been investigated in the work of Khomskii and Laguzzi [KL17]. The first has been also investigated in the work [NRa93] of Rosłanowski and Newelski.

Definition 2.2.21. *A **full-Miller tree** is a tree on $\omega^{<\omega}$ such that for every $\sigma \in T$ there is $\tau \in T$, $\sigma \subseteq \tau$ such that τ splits fully in T, i.e. $succ_T(\tau) = \omega$. The collection of Full-Miller trees will be denoted by $\mathbb{FM}$.*

Definition 2.2.22. *An **infinitely-equal tree** is a tree on $\omega^{<\omega}$ such that for every $\sigma \in T$ there is $N > |\sigma|$ such that for any $i \in \omega$ there is $\tau \in T$, $\sigma \subseteq \tau$, $|\tau| = N + 1$ with $\tau(N) = i$. The collection of all infinitely-equal trees will be denoted by $\mathbb{IE}$.*

Dense in $\mathbb{FM}$ is the set of those $\mathbb{FM}$-trees such that $succ_T(\sigma)$ is either a singleton or entire ω for every $\sigma \in T$. We will always assume that we work with such trees. For $T \in \mathbb{FM}$, by $split(T)$ we will denote the set $\{\sigma \in T : succ_T(\sigma) = \omega\}$. The following proposition gathers the basic properties of tree types $\mathbb{FM}$ and $\mathbb{IE}$.

Proposition 2.2.23. *The following hold:*

- *both $\mathbb{FM}$ and $\mathbb{IE}$ are combinatorial,*
- *both $\mathbb{FM}$ and $\mathbb{IE}$ hove the continuous reading of names,*
- *$\mathbb{FM}$ is homogeneous,*
- *both $\mathbb{FM}$ and $\mathbb{IE}$ have axiom A,*
- *both $\mathbb{FM}$-nwd sets and $\mathbb{IE}$-nwd sets are closed under countable unions.*

Proof. The proof that both tree types are combinatorial is almost identical to the one for $\mathbb{M}_{\mathcal{J}}$-trees (see the Proposition 2.2.30). Both have the continuous reading of names because of the Theorem 1.2.6 and the fact that the Borel parts $\mathcal{I}_{\mathcal{B}}(\mathbb{FM})$ and $\mathcal{I}_{\mathcal{B}}(\mathbb{IE})$ are generated by closed sets (see the Theorem 2.2.25 and the Theorem 2.2.26). The fact that the tree type $\mathbb{FM}$ is homogeneous has been proved in [KL17] and the argument is similar to the one for $\mathcal{J}$-Miller trees (for $\mathbb{IE}$ see the question below). The fact that $\mathbb{FM}$ has axiom A has been shown been proven in [NRa93]. The fact that $\mathbb{IE}$ has axiom A is similar to the one for $\mathcal{J}$-Miller trees and $\mathbb{IE}_F$-trees. Because of this we skip it. The argument that $\mathbb{FM}$-nwd sets and $\mathbb{IE}$-nwd sets are closed under countable unions is also almost identical to the argument for $\mathbb{M}_{\mathcal{J}}$-trees (see the Proposition 2.2.30). □

Question 2.2.24. *Is the tree type $\mathbb{IE}$ homogeneous?*

Also, the tree type $\mathbb{FM}$ adds Cohen reals (see [NRa93]) while $\mathbb{IE}$ adds Cohen reals below certain conditions. It is worth noticing that $\mathbb{IE}$ adds infinitely equal real on ω^ω (see [KL17]). However, it is not clear whether there is a tree $T \in \mathbb{IE}$ such that the restriction $\mathbb{IE}|_T$ does not add Cohen reals. In such case, this would lead to another solution of long standing problem called 'half-Cohen' problem: is there a forcing notion $\mathbb{P}$ which adds infinitely-equal real but no Cohen reals. This question has been answered in positive by Zapletal using forcing of form $\mathrm{Borel}(X) \setminus \mathcal{I}$ for some Polish space X and a peculiar σ-ideal $\mathcal{I}$. However, this σ-ideal comes from infinitely-dimensional topology, making it natural to ask if there is such forcing with a more combinatorial definition. In this context, it would be nice to know the answer for the tree type $\mathbb{IE}$.
We have the following two perfect-set style dichotomies.

Theorem 2.2.25. *(Rosłanowski and Newelski, [NRa93]) For every analytic $A \subseteq \omega^\omega$, either there is $T \in \mathbb{FM}$ such that $[T] \subseteq A$, or for some $\phi : \omega^{<\omega} \to \omega$, A is contained in the set of form*

$$F_\phi := \{x \in \omega^\omega : \forall^\infty n \in \omega\ x(n) \neq \phi(x|_n)\}.$$

It follows that the sets F_ϕ, $\phi : \omega^{<\omega} \to \omega$ form a basis of the Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{FM})$. The second dichotomy comes from Spinas [Spi08] with a correction from Khomskii and Laguzzi [KL17].

Theorem 2.2.26. *(Spinas, [Spi08]; Khomskii and Laguzzi, [KL17]) For every analytic $A \subseteq \omega^\omega$, either there is $T \in \mathbb{IE}$ such that $[T] \subseteq A$, or for some $f \in \omega^\omega$, A is contained in the set of form*

$$E_f := \{x \in \omega^\omega : \forall^\infty n \in \omega\ x(n) \neq f(n)\}.$$

It follows that the sets E_ϕ, $\phi : \omega^{<\omega} \to \omega$ form a basis of the Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{FM})$. Clearly, both Borel parts are generated by closed sets and contained in the ideal of meager sets.
In the following proposition, we collect the values of cardinal invariants.

Proposition 2.2.27. *(Rosłanowski and Newelski, [NRa93]; Khomskii and Laguzzi, [KL17]) The following holds:*

- $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{FM})) = \mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{IE})) = \omega_1$,
- $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{FM})) = \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{IE})) = \mathrm{cov}(\mathcal{M})$,
- $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{FM})) = \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{IE})) = \mathrm{non}(\mathcal{M})$,
- $\mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{FM})) = \mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{IE})) = \mathfrak{c}$.

2.2.5 $\mathcal{J}$-Miller trees

The following generalization of Miller trees has been considered by Sabok and Zapletal in the article [SZ11], in the context of σ-ideals generated by closed subsets of Polish spaces.

Definition 2.2.28. *Suppose that $\mathcal{J}$ is an ideal on ω. A $\mathcal{J}$-**Miller tree** is a tree on $\omega^{<\omega}$ such that for every $\sigma \in T$ there is $\tau \in T$, $\sigma \subseteq \tau$ such that $succ_T(\tau) \in \mathcal{J}^+$. The collection of all $\mathcal{J}$-Miller trees will be denoted by $\mathbb{M}_\mathcal{J}$.*

Note that $\mathbb{M}_{\mathcal{F}in}$ is just standard Miller forcing, sometimes called superperfect or rational-perfect forcing. Note also that dense in $\mathbb{M}_\mathcal{J}$ is the set all those trees that split $\mathcal{J}$-positively whenever they split i.e. for every $\sigma \in T$ we have $succ_T(\sigma)$ is either a singleton or $\mathcal{J}$-positive. We will always assume that we work with such trees. For $T \in \mathbb{M}_\mathcal{J}$, by $split(T)$ we will denote the set $\{\sigma \in T : succ_T(\sigma) \in \mathcal{J}^+\}$.

Proposition 2.2.29. *$\mathbb{M}_\mathcal{J}$ has axiom A for any ideal $\mathcal{J}$.*

Proof. The proof is identical as the one for classical Miller trees: for $S, T \in \mathbb{M}_\mathcal{J}$ and $n \in \omega$ we define the relation $T \leq_n S$ if $split_i(T) = split_i(S)$ for every $i \leq n$. □

Proposition 2.2.30. *The following holds:*

- *$\mathbb{M}_\mathcal{J}$ is combinatorial for any ideal $\mathcal{J}$,*
- *$\mathbb{M}_\mathcal{J}$ has the continuous reading of names no matter the ideal $\mathcal{J}$,*
- *$\mathbb{M}_\mathcal{J}$ is homogeneous whenever $\mathcal{J}$ is,*
- *$\mathbb{M}_\mathcal{J}$-nwd sets are closed under countable unions for any ideal $\mathcal{J}$.*

Proof. • Let $T \in \mathbb{M}_\mathcal{J}$. For any $x \in 2^\omega$ we we define a tree $T_x \in \mathbb{M}_\mathcal{J}$, $T_x \subseteq T$ to be the unique tree such that $stem(T_x) = stem(T)$ and we have that: if $\sigma \in T_x$ is such that $succ_T(\sigma) \in \mathcal{J}^+$ and the number $N_\sigma = |\{m < |\sigma| : succ_T(\sigma|_m) \in \mathcal{J}^+\}|$ is odd, then $succ_{T_x}(\sigma) = succ_T(\sigma)$; if $\sigma \in T_x$ is such that $succ_T(\sigma) \in \mathcal{J}^+$ and the number N_σ is even, then $succ_{T_x}(\sigma)$ consists only of the first element of $succ_T(\sigma)$ if $x(\frac{N_\sigma}{2}) = 0$, and only of the second element of $succ_T(\sigma)$ if $x(\frac{N_\sigma}{2}) = 1$. We will show that for $[T_x] \cap [T_y] = \emptyset$ for $x \neq y$ in 2^ω. Assume that $z \in [T_x] \cap [T_y]$ and let $n \in \omega$ be the smallest such that $x(n) \neq y(n)$. Let $m \in \omega$ be such that $N_{z|_m}$ is even and $\frac{N_{z|_m}}{2} = n$. Then $z(m) \in succ_{T_x}(z|_m) \cap succ_{T_y}(z|_m)$. As $x(n) \neq y(n)$, $z(m)$ should be equal to both the first and the second element of the set $succ_T(z|_m)$ in the same time, which is impossible.
• Continuous reading of names follows from the Theorem 1.2.6 and the fact that $\mathcal{I}_\mathcal{B}(\mathbb{M}_J)$ is generated by the closed sets by the Theorem 2.2.31.
• To prove that $\mathbb{M}_\mathcal{J}$ is homogeneous, assume that $T \in \mathbb{M}_\mathcal{J}$ and that $\mathcal{J}$ is homogeneous. For any $\sigma \in T$ denote by $SuccSplit_T(\sigma)$ the shortest $\mathcal{J}$-splitnode of T that extends σ. For any $\sigma \in split(T)$ fix a bijection $b_\sigma : \omega \to succ_T(\sigma)$ such that $X \subseteq \omega$ is $\mathcal{J}$-positive iff $b_\sigma[X]$ is $\mathcal{J}$-positive. Recursively we define $h' : \omega^{<\omega} \to split(T)$ as follows: $h'(\emptyset) = stem(T)$ and $h'(\sigma^\frown i) = SuccSplit_T(h'(\sigma)^\frown b_{h'(\sigma)}(i))$. Note that, as every b_σ is a bijection, h' is as well a bijection. Define now $h : \omega^\omega \to [T]$ such that for $x \in \omega^\omega$ we have $h(x) = \bigcup_n h'(x|_n)$. As h' is a bijection, h is also a bijection. Note also that for $\sigma \in \omega^{<\omega}$ we have that $h[[\sigma]] = [T] \cap [h'(\sigma)]$. It follows that h is a homeomorphism between ω^ω and $[T]$. We will show that if $S \in \mathbb{M}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$, then $h[[S]] = [R]$ for some $R \in \mathbb{M}_\mathcal{J}$, $R \subseteq T$. Because of the similar argument, if $R \subseteq T$, $R \in \mathbb{M}_\mathcal{J}$, then $h^{-1}[[R]] = [S]$ for some $S \in \mathbb{M}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$. This will finish the proof. Let then $S \in \mathbb{M}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$. Define $R \subseteq T$ to be the unique $\mathbb{M}_\mathcal{J}$-tree such that $split(R) = h'[split(S)]$. Then $R \in \mathbb{M}_\mathcal{J}$ and $h[[S]] = [R]$.
• To show that $\mathbb{M}_\mathcal{J}$-nwd sets are closed under countable unions let $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathbb{M}_\mathcal{J})$ and $T \in \mathbb{M}_\mathcal{J}$. We will construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{M}_\mathcal{J}$ such that for each $n \in \omega$ we have:

$(1)_n$ $T_{n+1} \leq T_n$ and $[T_n] \cap X_n = \emptyset$,

$(2)_n$ $split_n(T_n) = split_n(T_{n+1})$.

Let $T_{-1} = T$ and assume that T_n has been constructed. For every $\sigma \in split_n(T_n)$ we find $S_\sigma \in \mathbb{M}_\mathcal{J}$, $S_\sigma \leq T_n|_\sigma$, $[S_\sigma] \cap X_{n+1} = \emptyset$ and we let $T_{n+1} = \bigcup_\sigma S_\sigma$. Having $\{T_n : n \in \omega\}$ constructed we define $R = \bigcap_{n\in\omega} T_n$. Clearly $R \leq T$. By $(1)_n$, $n \in \omega$ we have that $[R] \cap \bigcup_{n\in\omega} X_n = \emptyset$ and by $(2)_n$, $n \in \omega$ we have that $R \in \mathbb{M}_\mathcal{J}$. This finishes the proof. □

The Borel part $\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$ admits a neat combinatorial characterization due to Sabok and Zapletal:

Theorem 2.2.31. *(Sabok and Zapletal, [SZ11]) For every analytic $A \subseteq \omega^\omega$, either there is $T \in \mathbb{M}_\mathcal{J}$ such that $[T] \subseteq A$, or for some $\phi : \omega^{<\omega} \to \mathcal{J}$, A is contained in the set of form*

$$M_\phi := \{x \in \omega^\omega : \forall^\infty n \in \omega\ x(n) \in \phi(x|_n)\}.$$

This implies that the Borel part $\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$ is the σ-ideal generated by the sets F_σ-sets M_ϕ for $\phi : \omega^{<\omega} \to \mathcal{J}$ and the forcing notion $\mathbb{M}_\mathcal{J}$ is forcing equivalent to $\mathrm{Borel}(\omega^\omega) \setminus \mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. By the Theorem 1.2.6 the tree type $\mathbb{M}_\mathcal{J}$ has continuous reading of names for any ideal $\mathcal{J}$. A well-known instance of the above is when $\mathcal{J} = \mathcal{F}in$. In such case the dichotomy is a well-known result of Kechris where the Borel part consists of σ-compact sets. In such case it is well-known that $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in})) = \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in})) = \mathfrak{b}$ and $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in})) = \mathfrak{d}$.

Proposition 2.2.32. *Suppose that $\mathcal{J}$, $\mathcal{J}_0$ and $\mathcal{J}_1$ are ideals on ω. Then the following holds:*

- $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathfrak{d}$,
- $\mathfrak{b} \leq \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{non}(\mathcal{M})$,
- $\mathcal{J}_0 \leq_K \mathcal{J}_1$ *implies that* $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{J}_1})) \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{J}_0}))$.

Proof. • $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathfrak{d}$
The first inequality follows from the fact that M_ϕ's are meager. For the second notice that if $\{f_\alpha : \alpha \leq \mathfrak{d}\} \subseteq \omega^\omega$ is a dominating family, then the sets M_{ϕ_α}, where ϕ_α is defined as $\phi_\alpha(\sigma) = f_\alpha(|\sigma|)$, covers ω^ω.

• $\mathfrak{b} \leq \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{non}(\mathcal{M})$
This is just a dualization of the argument above. A family $\{f_\alpha : \alpha \leq \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))\}$ which witnesses for $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$ then it must be unbounded, otherwise it would belong to the ideal $\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. If $\{f_\alpha : \alpha \leq \mathrm{non}(\mathcal{M})\}$ witnesses for $\mathrm{non}(\mathcal{M})$, then it witnesses for $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$ because sets M_ϕ are meager.

• $\mathcal{J}_0 \leq_K \mathcal{J}_1$ implies that $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{J}_1})) \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{J}_0}))$
Let $\pi : dom(\mathcal{J}_1) \to dom(\mathcal{J}_0)$ be Katetov reduction and let $\kappa < \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{J}_1}))$. Given a family $\{\phi_\alpha : \alpha \in \kappa\}$ such that $\phi_\alpha : dom(\mathcal{J}_0)^{<\omega} \to \mathcal{J}_0$, define $\psi_\alpha : dom(\mathcal{J}_1)^{<\omega} \to \mathcal{J}_1$ as $\psi_\alpha(\sigma) = \pi^{-1}[\phi_\alpha(\pi(\sigma))] \in \mathcal{J}_1$. Now, if $x \notin \bigcup_\alpha M_{\psi_\alpha}$, then $\pi \circ x \notin \bigcup_\alpha M_{\phi_\alpha}$ which finish the proof. □

Theorem 2.2.33. *Suppose that $\mathcal{J}$ is an arbitrary ideal on ω. Then*

- $\min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\} \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \min\{\mathfrak{d}, \mathrm{add}^*_\omega(\mathcal{J})\}$,
- $\min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\} \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$,
- $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{non}^*_\omega(\mathcal{J})\}$,
- $\max\{\mathfrak{b}, \mathrm{cof}^*_\omega(\mathcal{J})\} \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathcal{J})\}$.

Proof. Before the proof, let us introduce the following notation. If $\phi : \omega^{<\omega} \to \mathcal{J}$ is a constant function with the value $B \in \mathcal{J}$, then we will write M_B for the set M_ϕ from the dichotomy for $\mathbb{M}_\mathcal{J}$ trees (Theorem 2.2.31).

• $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$

The comparisons with $\mathfrak{b}$ and $\mathfrak{d}$ follows from the proposition 2.2.32 above.

• $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{add}^*_\omega(\mathcal{J})$,

Let $\{B_\alpha : \alpha \in \kappa\}$ be a witness for $\mathrm{add}^*_\omega(\mathcal{J})$. We claim that $\bigcup_{\alpha\in\kappa} M_{B_\alpha} \notin \mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. Aiming for a contradiction, assume that for some $\phi : \omega^{<\omega} \to \mathcal{J}$ we have and $\bigcup_{\alpha\in\kappa} M_{B_\alpha} \subseteq M_\phi$. Find $\alpha \in \kappa$ such that for all $\sigma \in \omega^{<\omega}$ the set $B_\alpha \setminus \phi(\sigma)$ is infinite. Recursively construct $x : \omega \to \omega$ such that, for all $n \in \omega$ we have $x(n) \in B_\alpha \setminus \phi(x|_n)$. Clearly, then $x \in M_{B_\alpha} \setminus M_\phi$.

• $\mathrm{cof}^*_\omega(\mathcal{J}) \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$,

Let $\{M_{\phi_\alpha} : \alpha < \kappa\}$ be a basis of $\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. For each $\alpha \in \kappa$ define $\overline{B}_\alpha = \{\phi_\alpha(\sigma) : \sigma \in \omega^{<\omega}\} \in [\mathcal{J}]^\omega$. We claim that the collection $\{\overline{B}_\alpha : \alpha < \kappa\}$ witnesses $\mathrm{cof}^*_\omega(\mathcal{J})$. Towards a contradiction, assume that for some $\overline{A} = \{A_i : i \in \omega\} \in [\mathcal{J}]^\omega$ we have that for each $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ there is an $i_\alpha \in \omega$ such that for all $\sigma \in \omega^{<\omega}$ the set $A_{i_\alpha} \setminus \phi_\alpha(\sigma)$ is infinite. We will show that $\bigcup_{i\in\omega} M_{A_i}$ is not contained in any M_{ϕ_α}. To see this fix $\alpha \in \kappa$ and inductively construct an increasing $x \in \omega^\omega$ such that for each $n \in \omega$ we have $x(n) \in A_{i_\alpha} \setminus \phi_\alpha(x|_n)$. Then we have $x \in M_{A_{i_\alpha}} \setminus M_{\phi_\alpha}$.

• $\min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\} \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$,

Let $\kappa < \min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\}$ and assume that we are given a collection $\{M_{\phi_\alpha} : \alpha \in \kappa\}$ such that $\phi_\alpha : \omega^{<\omega} \to \mathcal{J}$. For each $\sigma \in \omega^{<\omega}$, as $\kappa < \mathrm{add}^*_\omega(\mathcal{J})$, there is a sequence $\{C^\sigma_n : n \in \omega\} \subseteq \mathcal{J}$ such that for each $\alpha < \kappa$ there is $n^\sigma_\alpha \in \omega$ with $\phi_\alpha(\sigma) \subseteq^* C^\sigma_{n^\sigma_\alpha}$. We may assume that each such sequence is increasing. For each $\alpha < \kappa$ we define functions $g_\alpha : \omega^{<\omega} \to \omega$ and $f_\alpha : \omega^{<\omega} \to \omega$ as follows: for $\sigma \in \omega^{<\omega}$ let $g_\alpha(\sigma) = n^\sigma_\alpha$ and let $f_\alpha(\sigma) \in \omega$ be such that $\phi_\alpha(\sigma) \subseteq C^\sigma_{n^\sigma_\alpha} \cup f_\alpha(\sigma)$. As $\kappa < \mathfrak{b}$ we may find $g : \omega^{<\omega} \to \omega$ and $f : \omega^{<\omega} \to \omega$ such that g dominates all g_α's and f dominates all f_α's. Let $\phi : \omega^{<\omega} \to \mathcal{J}$ be defined as $\phi(\sigma) = C^\sigma_{g(\sigma)} \cup f(\sigma)$ for $\sigma \in \omega^{<\omega}$. We will finish the inequality by showing that $\bigcup_\alpha M_{\phi_\alpha} \subseteq M_\phi$. Let $\alpha \in \kappa$ and let $x \in M_{\phi_\alpha}$. Then for almost all n's we have $x(n) \in \phi_\alpha(x|_n) \subseteq C^{x|_n}_{n^{x|_n}_\alpha} \cup f_\alpha(x|_n)$. But for almost all out of those almost all n's we have that $C^{x|_n}_{n^{x|_n}_\alpha} \cup f_\alpha(x|_n) \subseteq C^{x|_n}_{g(x|_n)} \cup f(x|_n) = \phi(x|_n)$ because $C^{x|_n}_k$'s, for $k \in \omega$, are increasing. This in fact shows that $x \in M_\phi$.

• $\min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\} \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$,

Let $\kappa < \min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\}$ and assume that we are given $\{M_{\phi_\alpha} : \alpha < \kappa\} \subseteq \mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. As $\kappa < \mathrm{cov}^*(\mathcal{J})$ there is an infinite set A almost disjoint from all of the sets $\phi_\alpha(\sigma)$ for $\sigma \in \omega^{<\omega}$ and $\alpha < \kappa$. For $\alpha < \kappa$ define $f_\alpha : \omega^{<\omega} \to \omega$ in a way that $A \cap \phi_\alpha(\sigma) \subseteq f_\alpha(\sigma)$ for every $\sigma \in \omega^{<\omega}$. As $\kappa < \mathfrak{b}$, there is $g : \omega^{<\omega} \to \omega$ that dominates all f_α's. Inductively construct an increasing $x \in \omega^\omega$ such that for all $n \in \omega$ we have $x(n) \in A \setminus g(x|_n)$. This is possible as A is infinite. We will finish the proof of the inequality by showing that $x \notin M_{\phi_\alpha}$ for each $\alpha < \kappa$. If for some $\alpha < \kappa$ we would have $x \in M_{\phi_\alpha}$, then for almost all n's we have $x(n) \in A \cap \phi_\alpha(x|_n) \subseteq f_\alpha(x|_n)$. But then for almost all of those n's we have $x(n) \in g(x|_n)$ which contradicts the definition of the x.

• $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{non}^*_\omega(\mathcal{J})\}$,

Let $\{f_\gamma : \gamma < \mathfrak{d}\}$ be a dominating family of functions from $\omega^{<\omega}$ to ω and let $\{F_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ be a collection of sets that witnesses for $\mathrm{non}^*_\omega(\mathcal{J})$. For every $\alpha < \kappa$ and $\gamma < \mathfrak{d}$ we inductively construct a point $x^{\alpha,\gamma} \in \omega^\omega$ such that $x^{\alpha,\gamma}(n) \in F_\alpha \setminus f_\gamma(x^{\alpha,\gamma}|_n)$ for every $n \in \omega$. This is possible because F_α is infinite. We will show that the set $\{x^{\alpha,\gamma} : \alpha < \kappa,\ \gamma < \mathfrak{d}\}$ witnesses for $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$. To see this, let $\phi : \omega^{<\omega} \to \mathcal{J}$. Then there exists $\alpha < \kappa$ such that $F_\alpha \cap \phi(\sigma)$ is finite for every $\sigma \in \omega^{<\omega}$. Let then $g : \omega^{<\omega} \to \omega$ be defined in a way that $F_\alpha \cap \phi(\sigma) \subseteq g(\sigma)$ for all $\sigma \in \omega^{<\omega}$. Then there is $\gamma < \kappa$ such that $g \leq^* f_\gamma$. We claim that $x^{\alpha,\gamma} \notin M_\phi$. Assume towards a contradiction that this is not the case, i.e. $x^{\alpha,\gamma}(n) \in \phi(x^{\alpha,\gamma}|_n)$ for almost all n's. Choose $n \in \omega$ large enough so that $g(x^{\alpha,\gamma}|_n) \leq f_\gamma(x^{\alpha,\gamma}|_n)$. But then $x^{\alpha,\gamma}(n) \in F_\alpha \cap \phi(x^{\alpha,\gamma}|_n) \subseteq g(x^{\alpha,\gamma}|_n) \subseteq f_\gamma(x^{\alpha,\gamma}|_n)$ which contradicts the definition of $x^{\alpha,\gamma}(n)$.

• $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathcal{J})\}$,

Let $\{f_\alpha : \gamma < \mathfrak{d}\}$ be a dominating family of functions from $\omega^{<\omega}$ to ω and let $\{\overline{A}_\alpha : \alpha < \mathrm{cof}^*_\omega(\mathcal{J})\}$ be a witness for $\mathrm{cof}^*_\omega(\mathcal{J})$. Assume that for each $\alpha \in \mathrm{cof}^*_\omega(\mathcal{J})$ we have $\overline{A}_\alpha = \{A^\alpha_i : i \in \omega\} \in [\mathcal{J}]^\omega$. We may also assume that each f_α is increasing along branches i.e. $\sigma \subseteq \tau$ implies that $f_\alpha(\sigma) \leq f_\alpha(\tau)$. For every $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ and $\gamma < \mathfrak{d}$ define $\phi_{\alpha,\gamma} : \omega^{<\omega} \to \mathcal{J}$ as $\phi_{\alpha,\gamma}(\sigma) = \bigcup_{i=0}^{f_\gamma(\sigma)} A^\alpha_i \cup f_\gamma(\sigma)$ for $\sigma \in \omega^{<\omega}$. We will finish the proof by showing that the collection $\{M_{\phi_{\alpha,\gamma}} : \alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ and $\gamma < \mathfrak{d}\}$ is a basis of the ideal $\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})$. Take any set M_ψ where $\psi : \omega^{<\omega} \to \mathcal{J}$. Let $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ be such that for each $\sigma \in \omega^{<\omega}$ there is $i_\sigma \in \omega$ with $\psi(\sigma) \subseteq^* A^\alpha_{i_\sigma}$. Define then functions $g : \omega^{<\omega} \to \omega$ and $h : \omega^{<\omega} \to \omega$ as follows: for $\sigma \in \omega^{<\omega}$ let $g(\sigma) = i_\sigma$ and let $h(\sigma) \in \omega$ be such that $\psi(\sigma) \subseteq A^\alpha_{i_\sigma} \cup h(\sigma)$. There is a $\gamma < \mathfrak{d}$ such that f_γ dominates both g and h. We claim that $M_\psi \subseteq M_{\phi_{\alpha,\gamma}}$. If $x \in M_\psi$ then $x(n) \in \psi(x|_n) \subseteq A^\alpha_{i_{x|_n}} \cup h(x|_n)$ for almost all n's. But then for almost all of those almost all n's we have that $A^\alpha_{i_{x|_n}} \subseteq \bigcup_{i=0}^{f_\gamma(x|_n)} A^\alpha_i \cup h(x|_n) = \phi_{\alpha,\gamma}(x|_n)$ which in fact shows that $x \in M_{\phi_{\alpha,\gamma}}$. □

Let us now summarize the values of discussed cardinal invariants for for ideals.

Corollary 2.2.34. *The following holds:*

- $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{ED}})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{ED}_{fin}})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{R})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{S})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{conv})) =$
  $= \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{W})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{S}pl})) = \omega_1$,
- $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{ED}})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{ED}_{fin}})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{R})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{S})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{conv})) =$
  $= \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{W})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{S}pl})) = \mathfrak{c}$,
- $\mathfrak{p} \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{nwd})) \leq \mathrm{add}(\mathcal{M})$ *and* $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{b}$,
- $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{nwd}))$ *and* $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{d}$,
- $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathrm{cof}(\mathcal{N})$ *for all analytic P-ideals,*
- $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) = \mathrm{cof}(\mathcal{N})$ *for the ideals* $\mathcal{I}_{\frac{1}{n}}$, $tr(\mathcal{N})$ *and* $\mathcal{Z}$.

2.2.6 $\mathcal{J}$-Laver trees

Definition 2.2.35. *Suppose that $\mathcal{J}$ is and ideal on ω. The $\mathcal{J}$-**Laver tree** is a tree on $\omega^{<\omega}$ such that for every $\sigma \in T$ which extends the stem(T), we have that $succ_T(\sigma) \in \mathcal{J}^+$. The collection of all $\mathcal{J}$-Laver trees will be denoted by $\mathbb{L}_\mathcal{J}$.*

Note that $\mathbb{L}_{\mathcal{F}in}$ is simply the standard Laver forcing for Borel conjecture. Let $split(T) = \{\sigma \in T : succ_T(\sigma) \in \mathcal{J}^+\}$.

Proposition 2.2.36. *$\mathbb{L}_\mathcal{J}$ has axiom A for any ideal $\mathcal{J}$.*

Proof. The argument is identical to the one for classical Laver trees. We only sketch the proof. For $T, S \in \mathbb{L}_\mathcal{J}$ and $n \in \omega$ define: $T \leq_n S$ if $T \leq S$ and $split_n(T) = split_n(S)$. In particular $T \leq_0 S$ means that $T \leq S$ and both trees have the same stem. □

Before establishing the basic properties of the tree type $\mathbb{L}_\mathcal{J}$ we will need the following important property of $\mathcal{J}$-Laver trees.

Proposition 2.2.37. *(Pure decision property) If $T \in \mathbb{L}_\mathcal{J}$ is such that $T \Vdash \dot{x} \in 2$ then there is $S \leq_0 T$ which decides the value of $\dot{x}$ i.e. either $S \Vdash$ "$\dot{x} = 0$" or $S \Vdash$ "$\dot{x} = 1$".*

Proof. The proof is almost identical to the original one. Because of this we only sketch the proof. Assume towards a contradiction that there is no such $S \leq_0 T$ and let $T_0 = T$. Inductively, we construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{J}$ such that for every $n \in \omega$ we have $T_{n+1} \leq_n T_n$ and for every $\sigma \in split_n(T)$ there is no $S \leq T_n|\sigma$ deciding the value of $\dot{x}$. Assume that T_n has been constructed. For each $\sigma \in split_n(T)$ the set $A_\sigma =: \{i \in succ_{T_n}(\sigma) : \exists S \leq_0 T_n|_{\sigma^\frown i} \text{ deciding } \dot{x}\}$ is in $\mathcal{J}$. We then construct T_{n+1} by letting $succ_{T_{n+1}}(\sigma) = succ_{T_n}(\sigma) \setminus A_\sigma$ for $\sigma \in split_n(T)$. Finally, if S is the fusion of T_n's then there is no $R \leq S$ deciding $\dot{x}$. T. □

Proposition 2.2.38. *The following holds:*

- $\mathbb{L}_\mathcal{J}$ *is combinatorial for nowhere-maximal* $\mathcal{J}$*,*
- $\mathbb{L}_\mathcal{J}$ *has the continuous reading of names no matter the ideal* $\mathcal{J}$*,*
- $\mathbb{L}_\mathcal{J}$ *is homogeneous whenever* $\mathcal{J}$ *is,*
- $\mathbb{L}_\mathcal{J}$*-nwd sets are closed under countable unions for any* $\mathcal{J}$*.*

Proof. • Suppose that $\mathcal{J}$ is nowhere-maximal and $T \in \mathbb{L}_\mathcal{J}$. For each $\sigma \in T$ with $stem(T) \subseteq \sigma$, because of nowhere-maximality of $\mathcal{J}$, there are disjoint sets $A^0_\sigma, A^1_\sigma \in \mathcal{J}^+$ such that $succ_T(\sigma) = A^0_\sigma \cup A^1_\sigma \in \mathcal{J}^+$. For each $x \in 2^\omega$ let $T_x \subseteq T$ be the unique $\mathbb{L}_\mathcal{J}$-tree such that $stem(T_x) = stem(T)$ and for every $\sigma \in T_x$, $succ_{T_x}(\sigma) = A^{x(|\sigma|)}_\sigma$. We will show that for $X \neq y$ in 2^ω we have $[T_x] \cap [T_y] = \emptyset$. Let $z \in [T_x] \cap [T_y]$ and let $n \in \omega$ be the smallest such that $x(n) \neq y(n)$. Then $z(n) \in succ_{T_x}(z|_n) \cap succ_{T_y}(z|_n) = A^{x(n)}_{z|_n} \cap A^{x(y)}_{z|_n} = \emptyset$. A contradiction.
• To show that $\mathbb{L}_\mathcal{I}$ has the continuous reading of names let $T \Vdash \dot{x} \in 2^\omega$. Inductively we construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{I}$ such that $T_0 = T$, for every $n \in \omega$ $T_{n+1} \leq_n T_n$ and for every $\sigma \in split_n(T)$, there is x_σ such that $T_n|\sigma \Vdash$ "$\dot{x}|_n = x_\sigma$". Then, if S is the fusion of T_n's we define $f : [S] \to 2^\omega$ as $f(y) = \bigcup x_{y|_n}$. Clearly, f is continuous and maps generic real into $\dot{x}$.
• Now the continuous reading of names. Let $T \Vdash \dot{x} \in 2^\omega$. Inductively, we construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{J}$ such that $T_0 = T$, for every $n \in \omega$ $T_{n+1} \leq_n T_n$ and for every $\sigma \in split_n(T_n)$, $T_n|\sigma \Vdash \dot{x}|_n = x_\sigma$. Then, if S is the fusion of T_n's we define $f : [S] \to 2^\omega$ as $f(y) = \bigcup x_{y|_n}$. The function f is clearly continuous and maps generic real onto the $\dot{x}$.
• We will now show that $\mathbb{L}_\mathcal{J}$ is homogeneous whenever $\mathcal{J}$ is. Let $T \in \mathbb{L}_\mathcal{J}$. As $\mathcal{J}$ is homogeneous, for every $\sigma \in T$ with $stem(T) \subseteq \sigma$, there is a bijection $b_\sigma : \omega \to succ_T(\sigma)$ such that $X \subseteq \omega$ is $\mathcal{J}$-positive iff $b_\sigma[X]$ is. Define a function $h' : \omega^{<\omega} \to \{\sigma \in T : stem(T) \subseteq \sigma\}$ such that for $\sigma \in \omega^{<\omega}$ the value $h'(\sigma)$ is defined inductively such that

$$h'(\sigma) = stem(T)^\frown b_{\tau_0}(\sigma(0))^\frown b_{\tau_1}(\sigma(1))^\frown ... ^\frown b_{\tau_{|\sigma|-1}}(\sigma(|\sigma|-1))$$

where for $i < |\sigma|$, $\tau_i = h'(\sigma)|_{||stem(T)|+i}$. Note that, as every b_σ is a bijection, h' is a bijection. Define now $h : \omega^\omega \to [T]$ such that for $x \in \omega^\omega$ we have

$$h(x) = \bigcup_n h'(x|_n)$$

As h' is a bijection, h is also a bijection. Note also that for $\sigma \in \omega^{<\omega}$ we have that $h[[\sigma]] = [T] \cap [h'(\sigma)]$. It follows that h is a homeomorphism between ω^ω and $[T]$. We will show that if $S \in \mathbb{L}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$, then $h[[S]] = [R]$ for some $R \in \mathbb{L}_\mathcal{J}$, $R \subseteq T$. Because of the similar argument, if $R \subseteq T$, $R \in \mathbb{L}_\mathcal{J}$, then $h^{-1}[[R]] = [S]$ for some $S \in \mathbb{L}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$. This will finish the proof. Let then $S \in \mathbb{L}_\mathcal{J}$, $S \subseteq \omega^{<\omega}$. Define $R \subseteq T$ to be the unique tree such that $stem(R) = h'(stem(S))$ and for every $\sigma \in R$ that extends the stem, $succ_R(\sigma) = b_{h'^{-1}(\sigma)}[succ_S(h'^{-1}(\sigma))]$. Then $R \in \mathbb{L}_\mathcal{J}$ and $h[[S]] = [R]$.
• We now show that $\mathbb{L}_\mathcal{J}$-nwd sets are closed under countable unions for any $\mathcal{J}$. The identical argument as in the proof of pure decision property shows that, if $T \in \mathbb{L}_\mathcal{J}$ and $X \in \mathcal{I}(\mathbb{L}_\mathcal{J})$, there is $S \leq_0 T$, $S \in \mathbb{L}_\mathcal{J}$ such that $[S] \cap X = \emptyset$. To prove the last item let $\{X_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{J}$-nwd and $T \in \mathbb{L}_\mathcal{J}$. We will inductively construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{J}$ such that for every $n \in \omega$ the following two conditions hold:

– $T_n \cap \omega^{\leq |stem(T)|+n} = T_{n+1} \cap \omega^{\leq |stem(T)|+n}$,

– $[T_{n+1}] \cap X_n = \emptyset$.

Assuming that T_n's has been constructed let S be a fusion of T_n's i.e. $S = \bigcap_n T_n$. Clearly $S \subseteq T$, also, the first condition implies that $S \in \mathbb{L}_{\mathcal{J}}$ and the second that $[S] \cap \bigcup_n X_n = \emptyset$. The induction proceed as follows: let $T_0 = T$ and assume that T_n has been constructed. For every $\sigma \in T_n$ of length $|stem(T)| + n$ using the observation find $S_\sigma \leq T_n|_\sigma$ with $stem(S_\sigma) = \sigma$ and that $[S_\sigma] \cap X_n = \emptyset$. To finish the induction step define $T_{n+1} = \bigcup\{S_\sigma : \sigma \in T_n \text{ and } |\sigma| = |stem(T)| + n\}$. □

Note that if an ideal $\mathcal{J}$ is maximal, then $\mathbb{L}_{\mathcal{J}}$ is σ-centered and thus a c.c.c.forcing notion. The Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$ admits a neat combinatorial characterization due to Miller:

Theorem 2.2.39. *(Miller, [Mil12]) For every analytic $A \subseteq \omega^\omega$, either there is $T \in \mathbb{L}_{\mathcal{J}}$ such that $[T] \subseteq A$, or for some $\phi : \omega^{<\omega} \to \mathcal{J}$, A is contained in the set of form*

$$L_\phi := \{x \in \omega^\omega : \exists^\infty n \in \omega \; x(n) \in \phi(x|_n)\}.$$

Let us make here a couple of observations. First, note that L_ϕ's are G_δ-sets. However, as $\mathbb{L}_{\mathcal{J}}$ always adds a dominating real (see the Proposition below), the ideal $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$ is not Σ^1_1-on-Π^1_1 (see [Zap08]) and thus cannot be σ-generated by closed sets. Second, from Miller's dichotomy it follows that the Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$ is a σ-ideal has a basis made of the sets L_ϕ for $\phi : \omega^{<\omega} \to \mathcal{J}$ and the forcing notion $\mathbb{L}_{\mathcal{J}}$ is forcing equivalent Borel $\setminus \mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$.
If $\mathcal{J} = \mathcal{F}in$, then the Borel part $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$ is known as the σ-ideal of not strongly dominating subsets of ω^ω, investigated in [DR13]. In this paper, the authors prove that $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \mathfrak{b}$ and $\mathfrak{d} = \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$.

Proposition 2.2.40. *Suppose that $\mathcal{J}$, $\mathcal{J}_0$ and $\mathcal{J}_1$ are ideals on ω. Then the following holds:*

– $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathfrak{b}$,

– $\mathfrak{d} \leq \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$,

– $\mathcal{J}_0 \leq_K \mathcal{J}_1$ *implies* $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}_1})) \leq \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}_0}))$.

Proof. To show that $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathfrak{b}$ let $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family. then the sets L_{ϕ_α}, where ϕ_α is defined as $\phi_\alpha(\sigma) = f_\alpha(|\sigma|)$, covers ω^ω.
To show that $\mathfrak{d} \leq \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$ let $\{f_\alpha : \alpha < \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))\} \subseteq \omega^\omega$ be a witness for $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$. Then it must form a dominating family. This is because if f is not bounded by any f_α, then $\{f_\alpha : \alpha < \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))\} \subseteq L_{\phi_f}$ where ϕ_f is defined as in the previous point. The proof of the last point is analogous to the version for $\mathcal{J}$-Miller trees. See Proposition 2.2.32. □

Before the next theorm we will need the following simple observation about basis of the Borel part.

Proposition 2.2.41. *For every $\{\phi_n : n \in \omega\}$, $\phi_n : \omega^{<\omega} \to \mathcal{J}$, there is $\phi : \omega^{<\omega} \to \mathcal{J}$ with $\bigcup_n L_{\phi_n} \subseteq L_\phi$*

Proof. For $\sigma \in \omega^\omega$ define $\phi(\sigma) = \bigcup_{i \leq |\sigma|} \phi_i(\sigma)$. Clearly $\phi(\sigma) \in \mathcal{J}$ and also if for some $x \in \omega^\omega$ and $n \in \omega$ the $x(k) \in \phi_n(x|_k)$ holds for infinitely many k's, then for almost all of these k's we have $x(k) \in \phi(x|_k)$, which proves the inclusion. □

Theorem 2.2.42. *Suppose that $\mathcal{J}$ is an arbitrary ideal on ω. Then*

– $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\}$,

– $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\}$,

– $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \max\{\mathfrak{d}, \mathrm{non}^*_\omega(\mathcal{J})\}$,

– $\mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) = \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathcal{J})\}$.

Proof. Before the proof, let us introduce the following notation. If $\phi : \omega^{<\omega} \to \mathcal{J}$ is a constant function with the value $B \in \mathcal{J}$, then we will white L_B for the set L_ϕ from the dichotomy for $\mathbb{L}_{\mathcal{J}}$ trees (2.2.39).

• $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathfrak{b}$ and $\mathfrak{d} \leq \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$

The comparisons with $\mathfrak{b}$ and $\mathfrak{d}$ follows from the proposition 2.2.40 above.

• $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathrm{add}^*_\omega(\mathcal{J})$,

Let $\{B_\alpha : \alpha \in \kappa\}$ be a witness for $\mathrm{add}^*_\omega(\mathcal{J})$. We claim that $\bigcup_{\alpha\in\kappa} L_{B_\alpha} \notin \mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$. Aiming for a contradiction, assume that for some $\phi : \omega^{<\omega} \to \mathcal{J}$ we have and $\bigcup_{\alpha\in\kappa} L_{B_\alpha} \subseteq L_\phi$. Find $\alpha \in \kappa$ such that for all $\sigma \in \omega^{<\omega}$ the set $B_\alpha \setminus \phi(\sigma)$ is infinite. Recursively construct $x : \omega \to \omega$ such that, for all $n \in \omega$ we have $x(n) \in B_\alpha \setminus \phi(x|_n)$. Clearly, then $x \in L_{B_\alpha} \setminus L_\phi$.

• $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})) \leq \mathrm{cov}^*(\mathcal{J})$,

Let $\{B_\alpha : \alpha \in \kappa\} \subseteq \mathcal{J}$ be a witness for $\mathrm{cov}^*(\mathcal{J})$. Let $C_n = \{x \in \omega^\omega : rng(x) \subseteq n\}$. Note that each C_n is a compact set, and thus belongs to the ideal $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$. We claim that $\omega^\omega = \bigcup_\alpha L_{B_\alpha} \cup \bigcup_n C_n$. To see this let $x \in \omega^\omega$. If $rng(x)$ is bounded, then $x \in C_n$ for some $n \in \omega$. If not, then $rng(x)$ as an infinite subset of ω must have infinite intersection with some B_α i.e. there is $A \in [\omega]^\omega$ such that $x(n) \in B_\alpha$ for every $n \in A$. But this means that $x \in L_{B_\alpha}$.

• $\mathrm{non}^*_\omega(\mathcal{J}) \leq \mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$,

Let $\{x_\alpha : \alpha \in \kappa\} \subseteq \omega^\omega$ be a collection of points that witnesses for $\mathrm{non}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$. Without loss of generality we may assume that for each x_α we have $rng(x_\alpha) \in [\omega]^\omega$. We claim that $\{rng(x_\alpha) : \alpha \in \kappa\} \subseteq [\omega]^\omega$ witnesses for $\mathrm{non}^*_\omega(\mathcal{J})$. To see this let $\overline{B} = \{B_i : i \in \omega\} \in [\mathcal{J}]^\omega$ and let $\alpha < \kappa$ be such that $x_\alpha \notin \bigcup_i L_{B_i}$. Then, by the definition of L_{B_i}, we have that the intersection $B_i \cap rng(x_\alpha)$ is finite for every $i \in \omega$ which finishes the proof of the inequality.

• $\mathrm{cof}^*_\omega(\mathcal{J}) \leq \mathrm{cof}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$,

Let $\{L_{\phi_\alpha} : \alpha < \kappa\}$ be a basis of $\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$. For each $\alpha \in \kappa$ define $\overline{B}_\alpha = \{\phi_\alpha(\sigma) : \sigma \in \omega^{<\omega}\} \in [\mathcal{J}]^\omega$. We claim that the collection $\{\overline{B}_\alpha : \alpha < \kappa\}$ witnesses $\mathrm{cof}^*_\omega(\mathcal{J})$. Towards a contradiction, assume that for some $\overline{A} = \{A_i : i \in \omega\} \in [\mathcal{J}]^\omega$ we have that for each $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ there is an $i_\alpha \in \omega$ such that for all $\sigma \in \omega^{<\omega}$ the set $A_{i_\alpha} \setminus \phi_\alpha(\sigma)$ is infinite. We will show that $\bigcup_{i\in\omega} L_{A_i}$ is not contained in any L_{ϕ_α}. To see this fix $\alpha \in \kappa$ and inductively construct an increasing $x \in \omega^\omega$ such that for each $n \in \omega$ we have $x(n) \in A_{i_\alpha} \setminus \phi_\alpha(x|_n)$. Then we have that $x \in L_{A_{i_\alpha}} \setminus L_{\phi_\alpha}$.

• $\min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\} \leq \mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$,

Let $\kappa < \min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathcal{J})\}$ and assume that we are given a collection $\{L_{\phi_\alpha} : \alpha \in \kappa\}$ such that $\phi_\alpha : \omega^{<\omega} \to \mathcal{J}$. For each $\sigma \in \omega^{<\omega}$, as $\kappa < \mathrm{add}^*_\omega(\mathcal{J})$, there is a sequence $\{C^\sigma_n : n \in \omega\} \subseteq \mathcal{J}$ such that for each $\alpha < \kappa$ there is $n^\sigma_\alpha \in \omega$ with $\phi_\alpha(\sigma) \subseteq^* C^\sigma_{n^\sigma_\alpha}$. We may assume that each such sequence is increasing. For each $\alpha < \kappa$ we define functions $g_\alpha : \omega^{<\omega} \to \omega$ and $f_\alpha : \omega^{<\omega} \to \omega$ as follows: for $\sigma \in \omega^{<\omega}$ let $g_\alpha(\sigma) = n^\sigma_\alpha$ and let $f_\alpha(\sigma) \in \omega$ be such that $\phi_\alpha(\sigma) \subseteq C^\sigma_{n^\sigma_\alpha} \cup f_\alpha(\sigma)$. As $\kappa < \mathfrak{b}$ we may find $g : \omega^{<\omega} \to \omega$ and $f : \omega^{<\omega} \to \omega$ such that g dominates all g_α's and f dominates all f_α's. Let $\phi : \omega^{<\omega} \to \mathcal{J}$ be defined as $\phi(\sigma) = C^\sigma_{g(\sigma)} \cup f(\sigma)$ for $\sigma \in \omega^{<\omega}$. We will finish the inequality by showing that $\bigcup_\alpha L_{\phi_\alpha} \subseteq L_\phi$. Let $\alpha \in \kappa$ and let $x \in L_{\phi_\alpha}$. Then for infinitely many n's we have $x(n) \in \phi_\alpha(x|_n) \subseteq C^{x|_n}_{n^{x|_n}_\alpha} \cup f_\alpha(x|_n)$. But for almost all out of those infinitely many n's we have that $[C^{x|_n}_{n^{x|_n}_\alpha} \cup f_\alpha(x|_n)] \subseteq [C^{x|_n}_{g(x|_n)} \cup f(x|_n)] = \phi(x|_n)$ because $C^{x|_n}_k$'s, for $k \in \omega$, are increasing. This in fact shows that $x \in L_\phi$.

• $\min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\} \leq \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}}))$,

Let $\kappa < \min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{J})\}$ and assume that we are given $\{L_{\phi_\alpha} : \alpha < \kappa\} \subseteq \mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$. As $\kappa < \mathrm{cov}^*(\mathcal{J})$ there is an infinite set A almost disjoint from all of the sets $\phi_\alpha(\sigma)$ for $\sigma \in \omega^{<\omega}$ and $\alpha < \kappa$. For $\alpha < \kappa$ define $f_\alpha : \omega^{<\omega} \to \omega$ in a way that $A \cap \phi_\alpha(\sigma) \subseteq f_\alpha(\sigma)$ for every $\sigma \in \omega^{<\omega}$. As $\kappa < \mathfrak{b}$, there is $g : \omega^{<\omega} \to \omega$ that dominates all f_α's. Inductively construct an increasing $x \in \omega^\omega$ such that for all $n \in \omega$ we have $x(n) \in A \setminus g(x|_n)$. This is possible as A is infinite. We will

finishe the proof of the inequality by showing that $x \notin L_{\phi_\alpha}$ for each $\alpha < \kappa$. If for some $\alpha < \kappa$ we would have $x \in L_{\phi_\alpha}$, then for infinitely many n's we have $x(n) \in A \cap \phi_\alpha(x|_n) \subseteq f_\alpha(x|_n)$. But then for almost all of those n's we have $x(n) \in g(x|_n)$ which contradicts the definition of the x.

• $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{non}^*_\omega(\mathcal{J})\}$,

Let $\{f_\gamma : \gamma < \mathfrak{d}\}$ be a dominating family of functions from $\omega^{<\omega}$ to ω and let $\{F_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ be a collection of sets that witnesses for $\mathrm{non}^*_\omega(\mathcal{J})$. For every $\alpha < \kappa$ and $\gamma < \mathfrak{d}$ we inductively construct a point $x^{\alpha,\gamma} \in \omega^\omega$ such that $x^{\alpha,\gamma}(n) \in F_\alpha \setminus f_\gamma(x^{\alpha,\gamma}|_n)$ for every $n \in \omega$. This is possible because F_α is infinite. We will show that the set $\{x^{\alpha,\gamma} : \alpha < \kappa, \gamma < \mathfrak{d}\}$ witnesses for $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J}))$. To see this let $\phi : \omega^{<\omega} \to \mathcal{J}$. Then there exists $\alpha < \kappa$ such that $F_\alpha \cap \phi(\sigma)$ is finite for every $\sigma \in \omega^{<\omega}$. Let then $g : \omega^{<\omega} \to \omega$ be defined in a way that $F_\alpha \cap \phi(\sigma) \subseteq g(\sigma)$ for all $\sigma \in \omega^{<\omega}$. Then there is $\gamma < \kappa$ such that $g \leq^* f_\gamma$. We claim that $x^{\alpha,\gamma} \notin L_\phi$. Assume towards a contradiction that this is not the case, i.e. $x^{\alpha,\gamma}(n) \in \phi(x^{\alpha,\gamma}|_n)$ for infinitely many n's. Choose $n \in \omega$ large enough so that $g(x^{\alpha,\gamma}|_n) \leq f_\gamma(x^{\alpha,\gamma}|_n)$. But then $x^{\alpha,\gamma}(n) \in F_\alpha \cap \phi(x^{\alpha,\gamma}|_n) \subseteq g(x^{\alpha,\gamma}|_n) \subseteq f_\gamma(x^{\alpha,\gamma}|_n)$ which contradicts the definition of $x^{\alpha,\gamma}(n)$.

• $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathcal{J})\}$,

Let $\{f_\alpha : \gamma < \mathfrak{d}\}$ be a dominating family of functions from $\omega^{<\omega}$ to ω and let $\{\overline{A}_\alpha : \alpha < \mathrm{cof}^*_\omega(\mathcal{J})\}$ be a witness for $\mathrm{cof}^*_\omega(\mathcal{J})$. Assume that for each $\alpha \in \mathrm{cof}^*_\omega(\mathcal{J})$ we have $\overline{A}_\alpha = \{A^\alpha_i : i \in \omega\} \in [\mathcal{J}]^\omega$. We may also assume that each f_α is increasing along branches i.e. $\sigma \subseteq \tau$ implies that $f_\alpha(\sigma) \leq f_\alpha(\tau)$. For every $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ and $\gamma < \mathfrak{d}$ define $\phi_{\alpha,\gamma} : \omega^{<\omega} \to \mathcal{J}$ as $\phi_{\alpha,\gamma}(\sigma) = \bigcup_{i=0}^{f_\gamma(\sigma)} A^\alpha_i \cup f_\gamma(\sigma)$ for $\sigma \in \omega^{<\omega}$. We will finish the proof by showing that the collection $\{L_{\phi_{\alpha,\gamma}} : \alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ and $\gamma < \mathfrak{d}\}$ is a basis of the ideal $\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})$. Take any set L_ψ where $\psi : \omega^\omega \to \mathcal{J}$. Let $\alpha < \mathrm{cof}^*_\omega(\mathcal{J})$ be such that for each $\sigma \in \omega^{<\omega}$ there is $i_\sigma \in \omega$ with $\psi(\sigma) \subseteq^* A^\alpha_{i_\sigma}$. Define then functions $g : \omega^{<\omega} \to \omega$ and $h : \omega^{<\omega} \to \omega$ as follows: for $\sigma \in \omega^{<\omega}$ let $g(\sigma) = i_\sigma$ and let $h(\sigma) \in \omega$ be such that $\psi(\sigma) \subseteq A^\alpha_{i_\sigma} \cup h(\sigma)$. There is a $\gamma < \mathfrak{d}$ such that f_γ dominates both g and h. We claim that $L_\psi \subseteq L_{\phi_{\alpha,\gamma}}$. If $x \in L_\psi$, then $x(n) \in \psi(x|_n) \subseteq A^\alpha_{i_{x|_n}} \cup h(x|_n)$ for infinitely many n's. But then for almost all of those infinitely many n's we have $A^\alpha_{i_{x|_n}} \subseteq \bigcup_{i=0}^{f_\gamma(x|_n)} A^\alpha_i \cup h(x|_n) = \phi_{\alpha,\gamma}(x|_n)$ which in fact shows that $x \in L_{\phi_{\alpha,\gamma}}$. □

Let us now summarize the values of discussed cardinal invariants for selected ideals.

Corollary 2.2.43. *The following holds:*

- $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}_{fin}})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{R})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{S})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{conv})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{W})) = \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{S}pl})) = \omega_1$,
- $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}_{fin}})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{R})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{S})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{conv})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{W})) = \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{S}pl})) = \mathfrak{c}$,
- $\mathfrak{p} \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{nwd})) \leq \mathrm{add}(\mathcal{M})$ *and* $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{b}$,
- $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{nwd}))$ *and* $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{d}$,
- $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \mathrm{cof}(\mathcal{N})$ *for all analytic P-ideals,*
- $\mathrm{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) = \mathrm{cof}(\mathcal{N})$ *for the ideals* $\mathcal{I}_{\frac{1}{n}}$, $tr(\mathcal{N})$ *and* $\mathcal{Z}$.

Corollary 2.2.44. *The following holds:*

- $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}})) = \mathfrak{b}$ *and* $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}})) = \mathfrak{d}$,
- $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}_{fin}})) = \min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\}$ *and* $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{ED}_{fin}})) = \max\{\mathfrak{d}, \mathrm{non}^*(\mathcal{ED}_{fin})\}$,
- $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{b}$ *and* $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in})) = \mathfrak{d}$,
- $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$ *for analytic P-ideals,*
- $\mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{nwd})) = \mathrm{add}(\mathcal{M})$ *and* $\mathrm{non}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{nwd})) = \mathrm{cof}(\mathcal{M})$.

2.2.7 Hurewicz schemas and closed sets of measure zero

In this section we will investigate the 'tree type' related to σ-ideals generated by closed sets, in particular to the ideal $\mathcal{E}$ of closed sets of measure zero in 2^ω. We will say that an ideal $\mathcal{I}$ on a Polish space X is of external nature if there is an ideal $\mathcal{I}'$ such that $\mathcal{I} \subseteq \mathcal{I}'$, for every $B \in \mathcal{I}^+$ there is a Borel $C \subseteq B$ with $C \in \mathcal{I}' \setminus \mathcal{I}$ and that $\mathcal{I}' \cap F_\sigma$ forms a basis of $\mathcal{I}$. In particular, $\mathcal{E}$ is of external nature with $\mathcal{N}$ as a witness. First, we observe the following simple fact saying that there is no tree type $\mathbb{T}$, that stands with the ideal $\mathcal{E}$ in a perfect-set-style dichotomy.

Proposition 2.2.45. *If the ideal $\mathcal{I}$ is of external nature, then closed sets are not dense in $\mathbb{P}_\mathcal{I}$.*

Proof. Notice that if $B \in \mathcal{I}' \setminus \mathcal{I}$, then all closed subsets of B are in $\mathcal{I}$. □

In particular, we get an observation of [SZ11] Sabok and Zapletal, that the ideal of countable subsets of 2^ω or the ideal of σ-compact subsets of ω^ω are not of external nature.

Corollary 2.2.46. *If $\mathcal{I}$ is of external nature, then the forcing $\mathbb{P}_\mathcal{I}$ adds an unbounded real.*

Proof. By Theorem 1.2.5, $\mathbb{P}_\mathcal{I}$ is ω^ω-bounding if and only if compact sets are dense in $\mathbb{P}_\mathcal{I}$. But by the previous proposition, even closed sets are not dense in $\mathbb{P}_\mathcal{I}$. □

We will now generalize the inequalities $\mathrm{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ and $\min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}(\mathcal{E})$ obtained by Bartoszyński and Shelah (see [BJ95]) for any σ-ideal of external nature.

Proposition 2.2.47. *Suppose that $\mathcal{I}$ is of external nature with $\mathcal{I}'$ as a witness and that $\mathcal{I}'$ has a Borel basis. Then*

- $\max\{\mathrm{cov}(\mathcal{I}'), \mathrm{cov}(\mathcal{M})\} \leq \mathrm{cov}(\mathcal{I}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{I}')\}$,
- $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{I}')\} \leq \mathrm{non}(\mathcal{I}) \leq \min\{\mathrm{non}(\mathcal{I}'), \mathrm{non}(\mathcal{M})\}$.

In fact, the Tukey reduction $(\mathbb{R}, \mathcal{I}, \in) \leq_T (\mathbb{R}, \mathcal{I}', \in) \otimes (\omega^\omega, \omega^\omega, \leq^*)$ holds.

Proof. The inequalities $\max\{\mathrm{cov}(\mathcal{I}'), \mathrm{cov}(\mathcal{M})\} \leq \mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I}) \leq \min\{\mathrm{non}(\mathcal{I}'), \mathrm{non}(\mathcal{M})\}$ simply follows from the fact that $\mathcal{I} \subseteq \mathcal{M} \cap \mathcal{I}'$. To prove the remaining inequalities we will show that the Tukey reduction $(\mathbb{R}, \mathcal{I}, \in) \leq_T (\mathbb{R}, \mathcal{I}', \in) \otimes (\omega^\omega, \omega^\omega, \leq^*)$ holds. Then the result will follow from the Proposition 1.2.22. As $\mathcal{I}'$ has Borel basis and every Borel set is continuous 1-1 image of the Baire space for each $X \in \mathcal{I}$ we fix continuous $h_X : \omega^\omega \to \mathbb{R}$ such that $X \subseteq h_X[\omega^\omega] \in \mathcal{I}'$ We define the functions $\phi : \mathbb{R} \to \mathbb{R} \times (\omega^\omega)^{\mathcal{I}'}$ and $\psi : \mathcal{I}' \times \omega^\omega \to \mathcal{I}$ as follows: for $r \in \mathbb{R}$ let $\phi(r) = (r, F_r)$ where $F_r : \mathcal{I}' \to \omega^\omega$ is defined as $F_r(X) = h_X^{-1}(r)$; for $X \in \mathcal{I}'$ and $f \in \omega^\omega$ let $\psi(X, f) = h_X[\{g \in \omega^\omega : g \leq^* f\}]$. First, notice that $\psi(X, f) \in \mathcal{I}$ because it is a countable union of continuous images of compact sets. To see that ϕ and ψ provide the Tukey reduction assume that for r, X, f as above we have $\phi(r) \sqsubseteq_\otimes (X, f)$. Then $r \in X$ and $h^{-1}[r] = F_r(X) \leq^* f$ thus $r \in \psi(X, f)$. □

There are many interesting examples of σ-ideals generated by closed sets. Some of them we already discussed in the Section 2.2 (as the Borel parts of some of our tree types). These are: the bounded infinitely equal ideals related to the tree types $\mathbb{S}_F$, $\mathbb{IE}_F$; the σ-ideal generated by not countably splitting sets on 2^ω (related to the tree type $\mathbb{SP}$); the Borel ideals related to $\mathbb{M}_\mathcal{J}$, in particular the ideal of σ-compact sets on the Baire space ω^ω; the infinitely equal ideals related to the trees $\mathbb{FM}$ and $\mathbb{IE}$. Also, in [La96] Łabedzki investigated the σ-ideal related to c.c.c. eventually different forcing[2] and its restriction to F_σ-sets. For more interesting examples, the reader may consult the survey article [MZ07]. We will mention three more examples of such ideals, starting with the following, well-known one.
Let $\mathcal{E}$ be the σ-ideal generated by closed sets of measure zero on 2^ω. Clearly, $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$, the

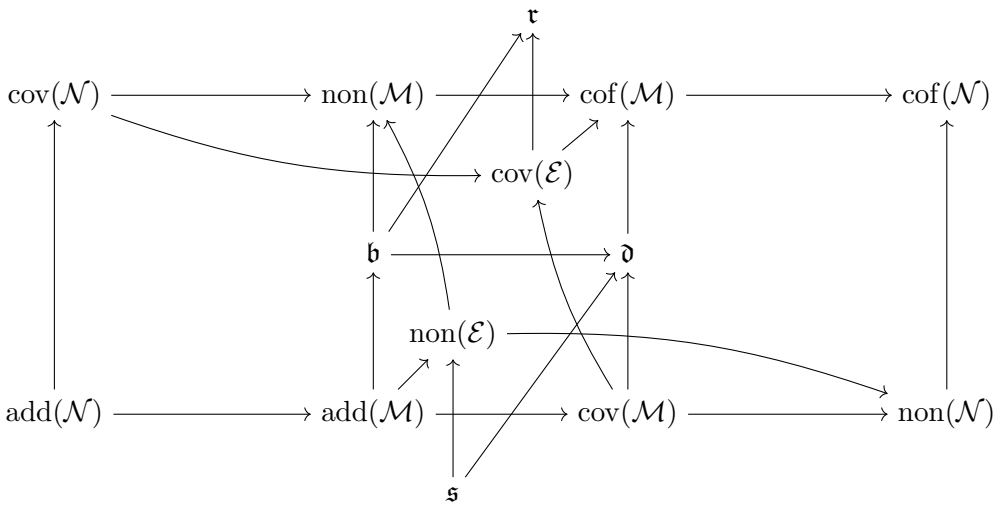
[2] this forcing is not related to $\mathbb{ED}_F$'s

inclusion is proper (see [BJ95]). As $\mathcal{E}$ is of external nature, by the Proposition 2.2.45 closed sets are not dense in the forcing $\mathcal{B}or(2^\omega) \setminus \mathcal{E}$. In particular there is no tree which can stand with $\mathcal{E}$ in a perfect set like dichotomy. The cardinal invariants are as follows

Theorem 2.2.48. *(Bartoszyński and Shelah [BJ95]) The following equalities hold:*

- $\mathrm{add}(\mathcal{E}) = \mathrm{add}(\mathcal{M})$,
- $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$,
- $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{r}$ *and* $\mathfrak{s} \leq \mathrm{non}(\mathcal{E})$,
- $\max\{\mathrm{cov}(\mathcal{N}), \mathrm{cov}(\mathcal{M})\} \leq \mathrm{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$,
- $\min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}(\mathcal{E}) \leq \min\{\mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M})\}$.

For more information about the cardinal invariants of $\mathcal{E}$ as well as related consistency results, the reader may consult [AC23] or [AC24]. The following diagram summarizes how the cardinal invariants of $\mathcal{E}$ place themselves in-between those of measure and category.



We already know that there is no tree type associated to the σ-ideal $\mathcal{E}$. However, the following dichotomy provides a pseudo-tree type for this ideal and will be crucial in several considerations.

Theorem 2.2.49. *(Solecki [Sol94]) If $\mathcal{I}$ is a σ-ideal generated by closed subsets of a Polish space X, then the following dichotomy holds: any analytic set $A \subseteq X$ is either in $\mathcal{I}$ or contains $\mathcal{I}$-positive Π^0_2-set.*

Notice that if on some $B \in \mathbb{P}_\mathcal{I}$ the ideals of meager subsets of X and $\mathcal{I}$ coincide, then for such B the dichotomy is clear as B has Baire property. Thus, we focus on the remaining case.

The proof of the Theorem 2.2.49 distinguishes between two cases: either there is an $\mathcal{I}$-positive Borel set on which the $\mathcal{I}$-small sets and the meager sets coincide, or every such Borel set contains an $\mathcal{I}$-positive meager set. In the first case the requierd Π^0_2-set exists because of the Baire property. This shows that the second case is the crucial one which motivates the following definition.

Definition 2.2.50. *Suppose that $\mathcal{I}$ is generated by closed sets. We will say that $\mathbb{P}_\mathcal{I}$ is nowhere meager if every $\mathcal{I}$-positive Borel set contains a closed, nowhere dense, but $\mathcal{I}$-positive subset.*

Assuming that $\mathbb{P}_\mathcal{I}$ is nowhere meager, the construction of the Π^0_2-set from Solecki's theorem relies on the following notion of special Luzin schema.

Definition 2.2.51. *A collection $\mathcal{U} = \{U_\sigma, N_\sigma : \sigma \in \omega^{<\omega}\}$ is a Hurewicz schema if*

1. *$\{U_\sigma : \sigma \in \omega^{<\omega}\}$ is a Luzin schema of open subsets of X,*
2. *each N_σ is closed, nowhere-dense and $\mathcal{I}$-positive subset of U_σ,*
3. *the set $lim(\{U_{\sigma^\frown n} : n \in \omega\})$, defined as $\overline{\bigcup_n U_{\sigma^\frown n}} \setminus \bigcup_{n\in\omega} \overline{U}_{\sigma^\frown n}$, is equal N_σ.*

Let $\mathbb{HS}$ be the collection of all Hurewicz schemas. A projection of a Hurewicz schema $\mathcal{U}$ is defined as

$$[\mathcal{U}] = \bigcup_{x\in\omega^\omega} \bigcap_n U_{x|_n} = \bigcap_n \bigcup_{|\sigma|=n} U_\sigma.$$

For $\mathcal{U}$ is a Hurewicz schema, then let $\phi_\mathcal{U} : \omega^\omega \to [\mathcal{U}]$ be the homeomorphism associated with $\mathcal{U}$ i.e. $\{\phi_\mathcal{U}(x)\} = \bigcap_n U_{x|_n}$.

In this way the Solecki's dichotomy can be stated as follows:

Theorem 2.2.52. *(Solecki [Sol94]) If $\mathcal{I}$ is a σ-ideal generated by closed subsets of a Polish space X, then the following dichotomy holds: any analytic set $A \subseteq X$ is either in $\mathcal{I}$ or there is Hurewicz schema $\mathcal{U}$ such that $[\mathcal{U}] \subseteq A$. Every set $[\mathcal{U}]$ is positive according to $\mathcal{I}$. In particular, the sets $[\mathcal{U}]$ where $\mathcal{U} \in \mathbb{HS}$, form a dense subset of $\mathbb{P}_\mathcal{I}$.*

For technical reasons, we will make some additional assumptions about the Hurewicz schemas. For every $\mathcal{U} \in \mathbb{HS}$ there is $\mathcal{V} \in \mathbb{HS}$ with $[V] \subseteq [U]$ such that the following conditions hold:

4. for $n \neq m$ the distances $dist(U_{\sigma^\frown n}, N_\sigma)$ and $dist(U_{\sigma^\frown m}, N_\sigma)$ are not equal,
5. $diam(U_\sigma) < \frac{1}{2^{\max\{rng(\sigma)\}+|\sigma|}}$,
6. each U_σ contains open set disjoint from every $U_{\sigma^\frown n}$ for $n \in \omega$.

The first assumption implies that the map $\sigma \mapsto U_\sigma$ is the unique order-isomorphism between $\omega^{<\omega}$ and $\mathcal{U}$ (both ordered by inclusion) such that for every $\sigma \in \omega^{<\omega}$ we have $dist(U_{\sigma^\frown n}, N_\sigma) < dist(U_{\sigma^\frown m}, N_\sigma)$ for all $m < n$. When $\mathcal{U}$ is clear from the context, we will shorten N_{U_τ} to N_τ.

Proposition 2.2.53. *For every Hurewicz schema $\mathcal{U}$ the set $[\mathcal{U}]$ is meager.*

Proof. By the condition 6. stated above, the closure of $[\mathcal{U}]$ is nowhere dense. □

The following proposition asserts that Hurewicz schemas treated as trees are combinatorial.

Proposition 2.2.54. *For a nowhere meager σ-ideal $\mathcal{I}$, the forcing notion $\mathbb{P}_\mathcal{I}$ has disjoint antichains of size $\mathfrak{c}$ below any condition.*

Proof. Let $\mathcal{U} = \{U_\sigma : \sigma \in \omega^{<\omega}\}$ be a Hurewicz schema. For each $\sigma \in \omega^{<\omega}$ decompose ω into two infinite disjoint sets, A^0_σ and A^1_σ, in such a way that for $i \in 2$ we have

$$lim(\{U_{\sigma^\frown n} : n \in A^i_\sigma\}) = N^\mathcal{U}_\sigma$$

For each $x \in 2^\omega$ define $\mathcal{V}_x \subseteq \mathcal{U}$, $\mathcal{V}_x = \{V_\sigma : \sigma \in \omega^{<\omega}\}$ such that for each $\sigma \in \omega^{<\omega}$ the successors of $V_\sigma = U$ in $\mathcal{V}_x$ are equal to the $succ_{\mathcal{V}_x}(U) = A^{x(|\#(U,\mathcal{U})|)}_U$. We claim that for all $x, y \in 2^\omega$ if $x \neq y$, then $[\mathcal{V}_x] \cap [\mathcal{V}_y] = \emptyset$. To see this let $n \in \omega$ be such that $x|_n = y|_n$ and $x(n) \neq y(n)$. Then $(\bigcup_{|\sigma|=n} V^x_\sigma) \cap (\bigcup_{|\sigma|=n} V^y_\sigma) = \emptyset$ and we have $[\mathcal{V}_x] \cap [\mathcal{V}_y] = \emptyset$. □

Remark. *It follows from the above proposition that a nowhere meager ideal $\mathcal{I}$ must be far from having Fubini property, in the sense that the set $B = \bigcup_{x\in 2^\omega} \{x\} \times [\mathcal{U}_x]$ is a Borel subset of $X \times X$ with $\mathcal{I}$-positive vertical sections, and with horizontal sections consisting of at most one element.*

We now consider another two examples of σ-ideals generated by closed sets. First, for $F \in (\omega \setminus 2)^\omega$, consider the σ-ideal $\mathcal{H}_F$ on ΠF, generated by the sets of form

$$H_{x,f} = \{y \in \Pi F : \forall^\infty_n \ \exists i \in [f(n), f(n+1)) \ x(i) = y(i)\}$$

where $x \in \omega^\omega$ and $f \in \omega^\omega$ is strictly increasing. The cardinal invariants $\mathrm{non}(\mathcal{H}_F)$ and $\mathrm{cov}(\mathcal{H}_F)$ (in form of bounding and dominating numbers of appropriate relation systems) were investigated in [CMRM24] in the context of null-additive and meager-additive sets.

Proposition 2.2.55. *Compact subsets of* ΠF *are not dense in the forcing* $\mathcal{B}or(\Pi F) \setminus \mathcal{H}_F$.

Proof. Let $G = \{y \in \Pi F : \exists^\infty_n y(n) = 0\}$. We will show that G is $\mathcal{H}_F$-positive but any closed subset of G is in $\mathcal{H}_F$. To see the former fix a set $H_{x,f}$. Without loss of generality we may assume that each interval $[f(n), f(n+1))$ consists of at least two elements. To see that $G \setminus H_{x,f} \neq \emptyset$ let $y \in \Pi F$ be such that for every $n \in \omega$, $y(i_n) = x(i_n)$ for some $i_n \in [f(n), f(n+1))$ and $y(i_n + 1) = 0$. Then $y \in G \setminus H_{x,f}$.
To see the latter let $D \subseteq G$ be closed and let $D = [T]$ for some pruned tree $T \subseteq Seq(F)$. As D is compact, there are $A_n \subseteq T$ such that for every $n \in \omega$:

1. A_n is a finite maximal antichain in $(T, \subseteq)$,
2. $\max\{|\tau| : \tau \in A_n\} + 1 < \min\{|\tau| : \tau \in A_{n+1}\}$,
3. for every $\tau \in A_n$ we have $\tau(|\tau| - 1) = 0$.

Let now $f \in \omega^\omega$ be such that $f(n) = \max\{|\tau| : \tau \in A_n\}$. We claim that $D \subseteq H_{x,f}$ where $x \in \Pi F$ in constant sequence with the value 0. Clearly, if $x \in [T]$ then there is an increasing sequence $(k_n)_{n\in\omega}$ such that $x|_{k_n} \in A_n$ so $x(k_n - 1) = 0$ and $k_n \in [f(n), f(n+1))$, which means that $x \in H_{x,f}$. □

We finish this section with one more example of a σ-ideal generated by closed sets. In the paper [MMR4] the authors investigated the chopped real on the Baire space ideal $\mathcal{M}_-$ i.e. the σ-ideal generated by the sets of form

$$H_{x,\overline{I}} = \{y \in \omega^\omega : \forall^\infty_n \ y|_{I_n} \neq x|_{I_n}\}$$

where $x \in \omega^\omega$ and $\overline{I} = (I_n)_{n\in\omega}$ is an interval partition. Clearly $\mathcal{M}_-$ is generated by closed sets. The authors showed that $\mathcal{M}_-$ is properly contained in $\mathcal{M}$.

Proposition 2.2.56. *The ideal* $\mathcal{M}_-$ *does not have c.c.c.*

Proof. Enumerate $\omega^{<\omega}$ as $\{\sigma_n : n \in \omega\}$. Define then $F_2 = \{n^\frown\sigma_n^\frown i : n \in \omega, \ i \in 2\}$ and for $i \in 2$ let $F_i = \{n^\frown\sigma_n^\frown i : n \in \omega\}$. Let $T \subseteq \omega^{<\omega}$ be the tree generated by finite concatenations of elements of F_2.

Claim 2.2.57. $[T]$ *is positive according to* $\mathcal{M}_-$.

Proof. Assume that a set $H_{x,\overline{I}}$ is given. We want to construct a point $z \in [T] \setminus H_{x,\overline{I}}$. Inductively construct an increasing sequence $\{k_n : n \in \omega\}$ of natural numbers and a sequence $\{\tau_n : n \in \omega\} \subseteq T$ with $\tau_n \subseteq \tau_{n+1}$ for every $n \in \omega$. Start with $\tau_0 = \emptyset$. Assume now that τ_n has been constructed. Extending τ_n, if necessary, we may assume that $succ_T(\tau_n) = \omega$. Choose then $i_{n+1} \in \omega$ with $i_{n+1} > i_n$ such that $|\tau_n| < \min I_{i_{n+1}}$. Then, by the definition of T there must be $\sigma \in \omega^{<\omega}$ and $m \in \omega$ such that $\tau_n^\frown m^\frown\sigma \in T$ and $\sigma|_{I_{i_{n+1}}} = x|_{I_{i_{n+1}}}$. Define then τ_{n+1} to be $\tau_n^\frown m^\frown\sigma$. When the induction is over, the point $z = \bigcup_{n\in\omega} \tau_n$ is not in the set $H_{x,\overline{I}}$ because for every $n \in \omega$ we have $z|_{I_{i_n}} = x|_{I_{i_n}}$. □

We continue the proof of the proposition. For any $x \in 2^\omega$ consider the tree $T_x \subseteq T$ generated by the sequences of the form $\rho_0^\frown \rho_1^\frown ... ^\frown \rho_m$ where $m \in \omega$ and $\rho_i \in F_{x(i)}$ for every $i \in m$. The identical argument as the one used in the claim above shows that $[T_x]$ is positive according to $\mathcal{M}_-$ for every $x \in 2^\omega$. It is enough to show that for $x \neq y$, $x, y \in 2^\omega$ we have $[T_x] \cap [T_y] = \emptyset$. Towards a contradiction, assume that there is $z \in [T_x] \cap [T_y]$. Let $n \in \omega$ be such that $x|_n = y|_n$ but $x(n) \neq y(n)$. But then, for some $m \in \omega$ we have $z|_m = j_0^\frown \rho_0^\frown x(0)^\frown ... ^\frown j_n^\frown \rho_n^\frown x(n)$ and $z|_m = j_0^\frown \rho_0^\frown y(0)^\frown ... ^\frown j_n^\frown \rho_n^\frown y(n)$ which is a contradiction as $x(n) \neq y(n)$. □

I do not know, if trees of the form described in the proposition above (or similar), can be put with $\mathcal{M}_-$ into a perfect-set-style dichotomy. This motivates the following question

Question 2.2.58. *Are closed sets dense in $\mathcal{B}or(\Pi F) \setminus \mathcal{M}_-$?*

2.3 Tree ideal may not be a σ-ideal

Let us discuss certain phenomenon related to adding Cohen reals by a tree forcing. Suppose that $\mathbb{T}$ is a combinatorial tree type and for the sake of simplisity assume that $\mathbb{T}$ lives on ω^ω. Consider the collection:

$$J_\subseteq(\mathbb{T}) = \{B \in \text{Borel}(\omega^\omega) : \neg\exists T \in \mathbb{T}\ [T] \subseteq B\}$$

For the tree types like $\mathbb{S}$, $\mathbb{IE}_F$, $\mathbb{SP}$, $\mathbb{M}_J$, $\mathbb{L}_J$, because of the appropriate perfect set dichotomy, we have that $J_\subseteq(\mathbb{T})$ is proper σ-ideal and in fact $J_\subseteq(\mathbb{T}) = \mathcal{I}_\mathcal{B}(\mathbb{T})$. However, this is not the case for several tree types adding Cohen reals. Let us define the following $(*)$-property

$$\exists \phi_\mathbb{T} : \omega^\omega \to 2^\omega \text{ continuous } \forall T \in \mathbb{T}\ int(\phi_\mathbb{T}[[T]]) \neq \emptyset$$

Below we will show consequences of $(*)$-property, but first we mention the following result.

Theorem 2.3.1. *(Khomskii and Laguzzi, [KL17]) $\mathbb{IE}$ does not have the $(*)$-property.*

Below, using a different approach, we will obtain the same result for a wide class of tree types, which will include both $\mathbb{IE}$ and $\mathbb{FM}$.

Proposition 2.3.2. *If $\mathbb{T}$ has the $(*)$-property, then $\mathbb{T}$ adds Cohen reals, $J_\subseteq(\mathbb{T})$ is not a proper ideal and $\mathbb{T}$-nwd sets are not closed under countable unions.*

Proof. To prove the first fact, note that $\Vdash$ "$\phi_\mathbb{T}(\dot{r}_{gen})$ is a Cohen real". This is because if $T \in \mathbb{T}$ and N is nowhere dense set from the ground model, there must be an open U such that $U \subseteq \phi_\mathbb{T}[[T]]$ and $U \cap N = \emptyset$. Then, as $\mathbb{T}$ is arboreal and $\phi_\mathbb{T}$ is continuous, $S = T \cap \phi_\mathbb{T}^{-1}[U]$ is an element of $\mathbb{T}$ and clearly $S \Vdash$ "$\phi_\mathbb{T}(\dot{r}_{gen}) \notin N$". For the second part just note that if we present 2^ω as a disjoint union of two Borel, dense sets, then $\phi_\mathbb{T}^{-1}[A], \phi_\mathbb{T}^{-1}[B] \in J_\subseteq(\mathbb{T})$ and union of these two is entire ω^ω. To see that $\mathbb{T}$-nwd sets are not closed under countable unions let $\{N_n : n \in \omega\}$ be a sequence of nowhere dense subsets of the 2^ω such that $\bigcup_n N_n$ is not nowhere dense. Note that because of the previous argument, $\phi^{-1}[N_n]$ is $\mathbb{T}$-nwd set and then, if $\bigcup_n \phi^{-1}[N_n]$ would be $\mathbb{T}$-nwd, the set $\bigcup_n N_n$ would be nowhere dense set. □

The following result relies on well-known fact that $\mathbb{L}^2$, $\mathbb{M}^3$ and $\mathbb{B}^2$ add Cohen reals. Beacause of this we only sketch the proof. In each case, the way that the Cohen real is produced out of the generic real, is actually the continuous reading that witnesses the $(*)$-property. As these are known results we only sketch the proof.

Proposition 2.3.3. *The tree types $\mathbb{L}^2$, $\mathbb{M}^3$, $\mathbb{B}^2$ have the $(*)$-property.*

Proof. For the case of $\mathbb{L}^2$ define $\phi : \omega^\omega \times \omega^\omega \to 2^\omega$ as $\phi(x,y)(n) = 0$ iff $x(n) < y(n)$. It is easy to see that ϕ witness for $(*)_{\mathbb{L}^2}$. Let $T, S \in \mathbb{L}$ and let $x \in [T]$, $y \in [S]$. Then $[\phi(x,y)|_{\max\{|stem(T)|,|stem(S)|\}}]\subseteq\phi[[T]\times[S]]$. For tree type $\mathbb{B}^2$ let $\phi : 2^\omega \times 2^\omega \to 2^\omega$ be defined as $\phi_{\mathbb{B}^2}(x,y)(n) = x(n) + y(n)\ mod 2$. It is an old theorem of Steinhaus that for any two sets of positive measure, its algebraic sum contains an interval that witnesses $(*)_{\mathbb{B}}$. Finally, for the tree type $\mathbb{M}^3$, let $\phi_{\mathbb{M}^3} : \omega^\omega \times \omega^\omega \times \omega^\omega \to \omega^\omega$ be defined in the following way. Let $\langle \sigma_n : n < \omega\rangle$ be canonical enumeration $\omega^{<\omega}$. We write $\#\sigma = n$ when $\sigma = \sigma_n$. For $x, y, z \in \omega^\omega$ inductively build three sequences $\{a_n : n < \omega\}, \{b_n : n < \omega\} \in \omega^\omega$, $\{c_n : n < \omega\} \in 2^\omega$ to set $\phi_{\mathbb{M}^3}(x,y,z) = \{c_n : n < \omega\}$. Start with $a_0 = 7$ and assume that a_{n+1}, b_n, c_n has been defined. Let $b_{n+1} = \#(z|_{a_{n+1}})$, $c_{n+1} = 0$ if $(\#x|_{b_{n+1}} < \#y|_{b_{n+1}})$ and let $a_{n+2} = a_{n+1} + \#x|_{b_{n+1}}$. It the similar way a for $\mathbb{L}^2$-trees one may show that $\phi_{\mathbb{M}^3}$ witnesses $(*)_{\mathbb{M}^3}$. □

Next, we want to show that the $(*)$-property does not hold for the tree types $\mathbb{FM}$, $\mathbb{M}^{nwd}$ or $\mathbb{L}^{nwd}$ even they do add Cohen reals. We have the following result

Proposition 2.3.4. *If $\mathbb{T}$ has the continuous reading of names, then $J_\subseteq(\mathbb{T})$ is a σ-ideal, it is equal to the Borel part $\mathcal{I}_\mathcal{B}(\mathbb{T})$ and $(*)$-property does not hold for $\mathbb{T}$.*

Proof. Assume that $\{B_n : n \in \omega\} \subseteq J_\subseteq(\mathbb{T})$ but for some $T \in \mathbb{T}$ we have $[T] \subseteq \bigcup_n B_n$. By absoluteness $T \Vdash$ "$\dot{r}_{gen} \in \bigcup_n B_n$". Strengthening condition T to T' we find $n \in \omega$ such that $T' \Vdash$ "$\dot{r}_{gen} \in B_n$". Assume now that $D \subseteq \omega^\omega \times \omega^\omega$ is closed set that projects onto B_n. Strengthening T' to S we find a $\mathbb{T}$-name $\dot{z}$ and continuous function $f : [S] \to \omega^\omega$ such that $S \Vdash$ "$(\dot{r}_{gen}, \dot{z}) \in D \wedge f(\dot{r}_{gen}) = \dot{z}$". We claim that $[S] \subseteq B_n$. Assume towards contradiction that there is $x \in [S] \setminus B_n$. We have than that $(x, f(x)) \notin D$. As D is closed and f continuous there are $\sigma, \tau \in \omega^{<\omega}$ such that $\sigma \subseteq x$, $\tau \subseteq f(x)$, $f[[\sigma]] \subseteq [\tau]$ and $([\sigma] \times [\tau]) \cap D = \emptyset$. Then we have that $R =: S|_\sigma \in \mathbb{T}$ $R \Vdash$ "$\dot{r}_{gen} \in [\sigma] \wedge \dot{z} = f(x) \in [\tau]$". This contradicts $R \Vdash$ "$(\dot{r}_{gen}, \dot{z}) \in D$. □

Chapter 3

Cardinal invariants of the tree ideals

3.1 Covering numbers

The main upper bounds for the covering number of $\mathcal{I}(\mathbb{T})$ comes from the types of reals (dominating, splitting etc) the forcing $\mathbb{T}$ adds. We start this section with the following theorem

Theorem 3.1.1. *Suppose that $\mathcal{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y} \rangle$ is a Borel relation system and $\mathbb{T}$ is a combinatorial tree type with a continuous reading of names. Then, if $\mathbb{T}$ adds $\mathcal{R}$-unbounded real, then $\mathrm{cov}(\mathcal{I}(\mathbb{T})) \leq \mathfrak{d}(\mathcal{R})$. If additionally $\mathbb{T}$ is homogenoueus, then the inequality holds also for the Borel part, i.e. $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{T})) \leq \mathfrak{d}(\mathcal{R})$.*

Remark. *An example of non-homogeneous forcing for which $\mathrm{cov}(\mathcal{I}(\mathbb{T})) \leq \mathfrak{d}$ holds but the inequality $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{T})) \leq \mathfrak{d}$ is not provable in ZFC, is the forcing notion $\mathrm{Borel}(2^\omega) \setminus \mathcal{E}$. In fact, because of $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{E})$, the inequality $\mathfrak{d} < \mathrm{cov}(\mathcal{E})$ holds in the random model.*

Proof. Suppose that $\mathbb{T}$ lives on ω^ω. Let $\dot{X}$ be a $\mathbb{T}$-name for an $\mathcal{R}$-unbounded real and let $\mathcal{A} \subseteq \mathbb{T}$ be a maximal antichain in $\mathbb{T}$ such that for each $T \in \mathcal{A}$ there is a continuous function $f_T : [T] \to \mathcal{X}$ with $T \Vdash$ "$f_T(\dot{r}_{gen}) = \dot{X}$". For $Y \in \mathcal{Y}$ define $Z_Y = \bigcup_{T \in \mathcal{A}} \{z \in [T] : f_T(z) \sqsubseteq Y\}$. Note that if $\mathcal{Y}' \subseteq \mathcal{Y}$ is an $\mathcal{R}$-dominating set, then $\omega^\omega = \mathcal{X}(\mathcal{A}) \cup \bigcup_{Y \in \mathcal{Y}'} Z_Y$. By the Proposition 2.1.3 the set $\mathcal{X}(\mathcal{A}) \in \mathcal{I}(\mathbb{T})$ so the proof will be finished by showing that $Z_Y \in \mathcal{I}(\mathbb{T})$ for every $Y \in \mathcal{Y}'$. Let $S \in \mathbb{T}$ and fix $Y \in \mathcal{Y}'$. As $\mathcal{A}$ is maximal, we may assume that $S \leq T$ for some $T \in \mathcal{A}$. As $\dot{X}$ is $\mathcal{R}$-unbounded real there is $R \leq S$ such that $R \Vdash$ "$f_T(\dot{r}_{gen})$ is not $\sqsubseteq$-dominated by Y". This means that the Borel set $[R] \cap f_T^{-1}[\{X \in \mathcal{X} : X \sqsubseteq Y\}]$ cannot contain branches of a tree of type $\mathbb{T}$. By the Proposition 2.3.4 this means that $[R] \cap f_T^{-1}[\{X \in \mathcal{X} : X \sqsubseteq Y\}] \in \mathcal{I}_{\mathcal{B}}(\mathbb{T})$. Clearly then we can find $R' \leq R$ of type $\mathbb{T}$ such that $[R'] \cap f_T^{-1}[\{X \in \mathcal{X} : X \sqsubseteq Y\}] = \emptyset$ and then $[R'] \cap Z_Y = \emptyset$. This finishes the proof of the first inequality.
To prove the second inequality, note that by the Remark 1.2.2 we may assume that $\dot{X}$ is produced out of $\dot{r}_{gen}$ by a map f which is defined on the entire ω^ω. Then, similar argument as above shows that the sets $\{z \in \omega^\omega : f(z) \sqsubseteq Y\}$, where $Y \in \mathcal{Y}'$, are in $\mathcal{I}_{\mathcal{B}}(\mathbb{T})$ and covers entire ω^ω. □

For the rest of this section we will discuss what types of reals are added by the tree types form the catalog. Starting with the compact ones, we have

Proposition 3.1.2. *Both $\mathbb{S}_F$ and $\mathbb{IE}_F$ add F-bounded infinitely equal real.*

Proof. We will give the proof for the tree type $\mathbb{IE}_F$ only. We will show that the generic real is infinitely equal below F. Let $T \in \mathbb{IE}_F$ and $f \in \Pi F$. We will construct $S \leq T$ s.t. $S \Vdash$ "$|\dot{r}_{gen} \cap f| = \omega$". Inductively we construct a sequence $\{T_n : n \in \omega\}$ of $\mathbb{IE}_F$-trees and a sequence $\{k_n : n \in \omega\}$ of natural numbers such that for every $n \geq 1$ we have:

$(1)_n$ $T_{n+1} \leq T_n$ and $T_{n+1} \cap \omega^{\leq k_n} = T_n \cap \omega^{\leq k_n}$,

$(2)_n$ $k_n \leq k_{n+1}$,

$(3)_n$ $T_n \Vdash$ "$\exists i > n\ \dot{r}_{ger}(i) = f(i)$",

$(4)_n$ $\forall \sigma \in T_n \cap \omega^{k_n} \exists k \in (|\sigma|, k_{n+1}]\ \forall i \in F(k)\ \exists \tau \in T_{n+1},\ |\tau| = k,\ \sigma \subseteq \tau,\ \tau(k) = i.$

The induction goes as follows: we start with $T_0 = T$ and $k_0 = 0$. Assume that T_n and k_n has been constructed. As $T_n \in \mathbb{IE}_F$, for each $\sigma \in T_n \cap \omega^{k_n}$ there is $N_\sigma \in \omega$ and $\tau_n \in T_n$ with $\sigma \subseteq \tau_\sigma$, $|\sigma| = N_\sigma + 1$ such that $\tau_\sigma(N_\sigma) = f(N_\sigma)$. Then, for each $\sigma \in T_n \cap \omega^{k_n}$ there is $M_\sigma > |\tau_\sigma|$ such that for every $i \in F(M_\sigma)$ there is $\rho \in T_n$, $\tau_\sigma \subseteq \rho$, $|\rho| = M_\sigma + 1$ such that $|\rho(M_\sigma)| = i$. Let $k_{n+1} = \max\{M_\sigma : \sigma \in T_n \cap \omega^{k_n}\} + 1$ and let

$$T_{n+1} = \{\tau \in T_n : \exists \sigma \in T_n \cap \omega^{k_n} \exists \rho \in T_n, |\rho| > k_{n+1}, \tau \subseteq \rho, \tau_\sigma \subseteq \rho\}.$$

Clearly, T_{n+1} and k_{n+1} satisfies the items above. This finishes the inductuctive step.
When the sequences are constructed we define S as $\bigcap_n T_n$. Notice that S is as required: by $(1)_n$ and $(4)_n$ we have $S \in \mathbb{IE}_F$ and, by $(2)_n$ and $(3)_n$ we have S forces generic real to be infinitely equal to f. □

By the Theorem 1.2.5 both of the tree types $\mathbb{S}_F$ and $\mathbb{IE}_F$ are ω^ω-bounding and by the Theorem 1.2.6 both preserves Baire cathegory. It follows that in the model obtained by c.s.i. of length ω_2 over a model of CH, with $\mathbb{S}_F$ or $\mathbb{IE}_F$ standing as each iterand, we obtain model of $\mathrm{cof}(\mathcal{M}) < \mathfrak{b}(\Pi F, =^\infty, \Pi F)$.
Similarly to the Proposition above, the forcing notions $\mathbb{FM}$ and $\mathbb{IE}$ adds infinitely equal real on the Baire space

Proposition 3.1.3. *The following holds:*

- *The tree type $\mathbb{FM}$ adds Cohen reals,*
- *The tree type $\mathbb{IE}$ adds infinitely equal reals,*
- *There is $T \in \mathbb{IE}$ such that $\mathbb{IE}|_T$ adds Cohen reals.*

Proof. The first item has been shown by Rosłanowski and Newelski in [NRa93]. The second one, by Khomskii and Laguzzi in [KL17] (the proof is similar to the one for $\mathcal{ED}_F$). The last on is an unpublished result of Goldstern and Shelah (see the discussion in [KL17]). □

By the Theorem 1.2.6 both $\mathbb{FM}$ and $\mathbb{IE}$ preserves Baire cathegory. It is well-known that if $\mathbb{P}$ adds infinitely equal real then two-step iteration of $\mathbb{P}$ adds a Cohen real. It follows that in the model obtained by c.s.i. of length ω_2 over a model of CH, with $\mathbb{FM}$ or $\mathbb{IE}$ standing as each iterand, we obtain model of $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$.
Moving towards $\mathcal{J}$-Miller and $\mathcal{J}$-Laver trees we have the following

Proposition 3.1.4. *The following holds:*

- *$\mathbb{L}_\mathcal{J}$ adds dominating real no matter the ideal $\mathcal{J}$,*
- *$\mathbb{M}_\mathcal{J}$ adds unbounded real no matter the ideal $\mathcal{J}$,*
- *For any ideal $\mathcal{J}$, the tree type $\mathbb{L}_\mathcal{J}$ destroys $\mathcal{J}$ i.e. $\mathbb{L}_\mathcal{J}$ adds $\mathfrak{R}^\flat_\mathcal{J}$-unbounded real.*

Proof. The argument for the first and the second item is identical to the one for the classical Laver and Miller trees. In this cases the generic real is dominating or unbounded respectively. The last item is a folklore resutl: Let $\dot{X}$ be a $\mathbb{L}_\mathcal{J}$-name for the set $rng(\dot{r}_{gen})$. Then for every $T \in \mathbb{L}_\mathcal{J}$ and ground model $A \in \mathcal{J}$, if $S \leq T$ is such that $succ_S(\sigma) \cap A = \emptyset$ for each $\sigma \in S$ extending the stem of S, then $S \Vdash$ "$|\dot{X} \cap A| < \omega$". □

For $\mathcal{J}$-Miller trees, characterization of what types of reals these trees add is much more difficult task.

Theorem 3.1.5. *(Sabok and Zapletal, [SZ11]) Suppose that $\mathcal{J}$ is a tall Borel ideal on omega:*

- *$\mathbb{M}_{\mathcal{J}}$ does not add Cohen reals iff $\mathcal{J}|_X \not\geq_K nwd$ for each $X \in \mathcal{J}^+$,*
- *$\mathbb{M}_{\mathcal{J}}$ does not add splitting reals iff $\mathcal{J}|_X \not\geq_K SPL$ for each $X \in \mathcal{J}^+$,*
- *$\mathbb{M}_{\mathcal{J}}$ does not preserve Lebesgue measure iff $\mathcal{J}|_X \not\geq_K \mathcal{S}$ for each $X \in \mathcal{J}^+$.*

As $\mathbb{L}_{\mathcal{J}}$ always adds dominating real, it always adds splitting real, thus in this context the ideal $\mathcal{S}pl$ play no important role for characterization of adding splitting reals by $\mathbb{L}_{\mathcal{S}pl}$. The following result has been obtained by Osvaldo Guzmán and Francisco Santiago during the work on this thesis.

Theorem 3.1.6. *(Guzmán and Santiago, [GS24]) If $\mathcal{J}$ is a tall Borel ideal on ω, then $\mathbb{L}_{\mathcal{J}}$ does not add Cohen reals iff $\mathcal{J}|_X \not\geq_K nwd$ for each $X \in \mathcal{J}^+$.*

In the context of adding Cohen - or random - reals, we will focus on the Laver property of the forcing notion $\mathbb{L}_{\mathcal{J}}$ in two particular cases. The general case seems to be open.

Question 3.1.7. *It possible to characterize these ideals (F_σ ideals, fragmented ideals) $\mathcal{J}$ such that $\mathbb{L}(\mathcal{J}^+)$ has Laver property?*

Proposition 3.1.8. *The forcing $\mathbb{L}_{\mathcal{ED}_{fin}}$ has Laver property.*

Proof. We need to show that for any $\mathbb{L}_{\mathcal{ED}_{fin}}$-name $\dot{x}$ for a real bounded by some ground model $f \in \omega^\omega$ there is a slalom $S : \omega \to [\omega]^{<\omega}$, size of which is bounded by F, which almost covers $\dot{x}$. First, a couple of observations: define a function $F \in \omega^\omega$ such that $F(0) = 1$ and $F(n+1) = (n+1)F(n)$. Also, notice that any $\mathcal{ED}_{fin}$-positive set contains a set of form $E = \bigcup_n E_n$ such that $|E_n| = n$, each E_n contains elements of ω^2 with the same first coordinate, say e_n, and that $e_n < e_{n+1}$ for every $n \in \omega$. In such a case, call E_n the n'th chunk of E. A simple density argument shows that any $T \in \mathbb{L}_{\mathcal{ED}_{fin}}$ contains a $\mathbb{L}_{\mathcal{ED}_{fin}}$-tree which splits into the sets exactly of this form. Now let $T \Vdash$ "$\dot{x} \leq f$". Inductively we construct a sequence $\{T_n : n \in \omega\}$ of $\mathbb{L}_{\mathcal{ED}_{fin}}$-subtrees of T, a sequence $\{H_n : n \in \omega\}$ of finite trees and a sequence $S = \{S_n : n \in \omega\}$ such that for every $n \in \omega$ we have:

$(a)_n$ H_n is a finite subtree T_n,

$(b)_n$ $T_{n+1} \leq T_n$ and $H_n \subseteq H_{n+1}$,

$(c)_n$ $S_n \subseteq f(n)$ and $|S_n| \leq F(n)$,

$(d)_n$ $T_n \Vdash$ "$\dot{x}(n) \in S_n$".

Start with $T_0 = T$, $H_0 = \{stem(T)\}$ and $S(0) = \emptyset$. Assume that T_n, H_n and S_n have been constructed. Let $H_{n+1} = \{\sigma^\frown i : \sigma \in H_n \text{ and } i \text{ is in } n\text{'th chunk of } succ_T(i)\}$. For every tree $\sigma \in H_n + 1$ let T_n^σ consists of these $t \in T_n$ such that the only elements of H_{n+1} which t extends are the ones contained in σ. Notice that there are $F(n+1)$ many such trees. For every $\sigma \in H_{n+1}$ use pure decision to find $R^\sigma \leq_0 T_n^\sigma$ which decides the value of $\dot{x}(n+1)$. Then let T_{n+1} be an amalgamation of R^σ's.
When inductive construction is over define $T' = \bigcap_n T_n$. is an $\mathbb{L}(\mathcal{ED}_{fin})$-tree . By the conditions $(d)_n$, $n \in \omega$ the tree T' forces that $\dot{x}$ is almost covered by the slalom S. □

Remark. *The same proof can be adapted for the van der Wearden ideal (by defining the n'th chunk as arithmetical sequence of length n).*

Let us now discuss the the question wheatear $\mathbb{L}_{\mathcal{J}}$ - even when it doesn't have Laver property - may add random reals. We have the following.

Proposition 3.1.9. *$\mathbb{L}_J$ does not add a random real for any ideal $\mathcal{J}$.*

Proof. Let $T \in \mathbb{L}(\mathcal{J})$ and $T \Vdash$ "$\dot{r} \in 2^\omega$". We will construct $S \leq T$ and a measure zero set $N \subseteq 2^\omega$ such that $S \Vdash$ "$\dot{r} \in N$". Fix a bijection $b : \omega^{<\omega} \to \omega$. Inductively we will build a sequence $\{T_n : n \in \omega\}$ of $\mathbb{L}_\mathcal{J}$-trees, and $\{\tau_\sigma : \sigma \in \omega^{<\omega}\} \subseteq 2^{<\omega}$ of different lengths, such that for each $n \in \omega$ we have:

$(1)_n$ $T_n \cap \omega^{|stem(T)|+n} = T_{n+1} \cap \omega^{|stem(T)|+n}$,

$(2)_n$ for every $\sigma \in T_n \cap \omega^{|stem(T)|+n}$ we have $T_{n+1}|_\sigma \Vdash$ "$\dot{r}|_{b(\sigma)} = \tau_\sigma$".

Start with $T_0 = T$. Assume that T_n has been constructed. For each $\sigma \in T_n \cap \omega^{|stem(T)|+n}$ use pure decision (the Proposition 2.2.37) to find $R_\sigma \leq_0 T_n|_\sigma$ which decides the value of $\dot{r}|_{b(\sigma)}$ to be certain $\tau \in 2^{b(\sigma)}$ which we declare as the τ_σ. Let then $T_{n+1} := \bigcup\{R_\sigma : \sigma \in T_n \cap \omega^{|stem(T)|+n}\}$. This finishes the induction step. When T_n's are built, let $S = \bigcap_n T_n$ and let $N = \{x \in 2^\omega : \exists^\infty_\sigma \tau_\sigma \subseteq x\}$. We claim that S and N are as required. For each $l \in \omega$ there is at most one τ_σ of length l, and then it is well known that the set N is of measure zero (see Lemma 2.5.1 in [BJ95]). By the conditions $(1)_n$, $n \in \omega$ we have that $S \in \mathbb{L}_\mathcal{J}$ and, by the conditions $(2)_n$, $n \in \omega$ we have that $S \Vdash$ "$\forall_{n\in\omega}\ \tau_{\dot{r}_{gen}|_n} \subseteq \dot{r}$", so in particular $S \Vdash$ "$\exists^\infty_\sigma\ \tau_\sigma \subseteq \dot{r}$". □

For the next proposition, we will use the $\mathcal{IE}_F$ ideal on the countable set $\omega^{<\omega}$. This ideal will be discussed in the section 4.3.

Proposition 3.1.10. *Suppose that $F \in \omega$ is increasing. If $\mathcal{IE}_F \leq_K \mathcal{J}$, then $\mathbb{L}_\mathcal{J}$ and $\mathbb{M}_\mathcal{J}$ add F-bounded infinitely equal real.*

Proof. We prove the proposition only for $\mathbb{M}_\mathcal{J}$ as the Laver case is identical. Fix $F \in \omega^\omega$. Let $\pi : \omega \to Seq(F)$ be a function which witnesses $\mathcal{IE}_F \leq_K \mathcal{J}$. In $V^{\mathbb{M}_\mathcal{J}}$ define a real

$$c = \pi(\dot{r}_{gen}(0))^\frown\pi(\dot{r}_{gen}(1))^\frown...$$

Let $\dot{c}$ be a name for this real. We claim that $\dot{c} \in \Pi F$ and that $\mathbb{M}_\mathcal{J}$ forces $\dot{c}$ to be infinitely equal to any ground model $g \in \Pi F$. To see the former, denote by l_n the length of the $\pi(\dot{r}_{gen}(0))^\frown...^\frown\pi(\dot{r}_{gen}(n))$. Let $i \in [l_n, l_{n+1})$ and let j be such that $i = l_n + j$. Then because of $j < i$ and the fact that F is increasing, we have

$$\dot{c}(i) = \dot{c}(l_n + j) = \pi(\dot{r}_{gen}(n+1))(j) \leq F(j) \leq F(i)$$

To prove the latter, it will suffice to show that for every g below F and every $N \in \omega$ the set

$$D_{g,N} = \{T \in \mathbb{M}_\mathcal{J} : T \Vdash \text{"}\dot{c}(i) = g(i) \text{ for some } i > N\text{"}\}$$

is dense in $\mathbb{M}_\mathcal{J}$. Let $S \in \mathbb{M}_\mathcal{J}$. Let M be length of $\dot{c}$ already decided by S, i.e. $M = |\pi(stem(S)(0)| + ... + |\pi(stem(S)(|stem(S)| - 1))|$. Without loss of generality we may assume that $M > N$. Because π is Katetov reduction the set $\pi[succ_S(stem(S))]$ is $\mathcal{IE}_F$-positive. Thus it contains a $\sigma \in \pi[succ_S(stem(S))]$ such that $\sigma \notin E_{h,0}$ where $h \in \omega^\omega$ is defined by $h(i) = g(M+i)$ for $i \in \omega$. Let then $T = S|_{stem(S)^\frown j}$ where $j \in \omega$ is such that $\pi(j) = \sigma$. Then $T \Vdash$ "$\dot{c}(i) = g(i)$ for some $i \in [M, M + |\sigma|)$" and $i > N$. □

Question 3.1.11. *For increasing $F \in \omega$, does $\mathcal{IE}_F \leq_K \mathcal{J}$ characterize whether tree types $\mathbb{L}_\mathcal{J}$ and $\mathbb{M}_\mathcal{J}$ add F-bounded infinitely equal real?*

We will now show that $\mathbb{L}_\mathcal{J}$ and $\mathbb{M}_\mathcal{J}$ increase the groupwise density number if the ideal $\mathcal{J}$ is meager. We say that a set $\mathcal{D} \subseteq [\omega]^\omega$ is groupwise dense if it is open ($A \in \mathcal{D}$ and $B \subseteq^* D$, $B \in [\omega]^\omega$ implies that $B \in \mathcal{D}$) and for every partition $\{I_n : n \in \omega\}$ of ω into finite intervals, there is an infinite $B \in [\omega]^\omega$ such that $\bigcup_{n\in B} I_n \in \mathcal{D}$. Denote by $\mathbb{GD}$ the collection of all groupwise dense sets. We recall the definition of the groupwise density number

$$\mathfrak{g} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{GD} \text{ and } \bigcap \mathcal{F} = \emptyset\}$$

It is known that $\mathfrak{h} \leq \mathfrak{g} \leq \mathfrak{d}$ (see [Bla09]). Consider the relational system $\langle [\omega]^\omega, \sqsubseteq, \mathbb{GD}\rangle$ where for $A \in [\omega]^\omega$ and $D \in \mathbb{GD}$ the relation $A \sqsubseteq D$ means that $A \notin D$. Notice, that by the definition, $\mathfrak{d}([\omega]^\omega, \sqsubseteq, \mathbb{GD}) = \mathfrak{g}$. A forcing notion $\mathbb{P}$ adds a $\langle [\omega]^\omega, \sqsubseteq, \mathbb{GD}\rangle$-unbounded real (also called groupwise dense real) if in $\mathbf{V}^\mathbb{P}$ there is a $A \in [\omega]^\omega$ such that for every groud model groupwise dense set D, $A \in D$. In [Eit] Todd Eisworth proved that both $\mathbb{M}$ and $\mathbb{L}$ add a groupwise dense real. The following generalizes his result for a wide class of $\mathbb{L}_\mathcal{J}$'s and $\mathbb{M}_\mathcal{J}$'s.

Proposition 3.1.12. *If an ideal $\mathcal{J}$ is meager, then $\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$ add a groupwise dense real.*

Proof. We show this only for Laver trees as Miller's case is identical. Let $\dot{x}$ be the canonical name for $rng(\dot{r}_{gen}) \in [\omega]^\omega$. We will show that for any groupwise dense set $D \subseteq [\omega]^\omega$ the set of these $S \in \mathbb{L}_\mathcal{J}$ such that $S \Vdash$ "$\dot{x} \in D$" is dense in $\mathbb{L}_\mathcal{J}$. By Talagrand's theorem, for each $\sigma \in split(T)$ there is an interval partition $(I^\sigma_k)_{k\in\omega}$ such that $succ_T(\sigma) \cap \bigcup_{k\in A} I^\sigma_k \in \mathcal{J}^+$ for every infinite $A \subseteq \omega$. Enumerate $split(T)$ as $\{\sigma_i : i \in \omega\}$. Now, let $(P_n)_{n\in\omega}$ be an interval partition such that each P_n contains some element of $(I^{\sigma_i}_k)_{k\in\omega}$ for each $i \leq n$. As D is groupwise dense, there is $A \in [\omega]^\omega$ such that $\bigcup_{n\in A} P_n \in D$. Let now $S \leq T$ be the unique tree such that $stem(S) = stem(T)$ and $succ_S(\sigma) = succ_T(\sigma) \cap \bigcup_{n\in A} P_n$ for each $\sigma \in split(S)$. Clearly $S \in \mathbb{L}_\mathcal{J}$ as $(P_n)_{n\in\omega}$ diagonalizes $(I^\sigma_k)_{k\in\omega}$. Also $S \Vdash$ "$\dot{x} \subseteq \bigcup_{n\in A} P_n$" so $S \Vdash$ "$\dot{x} \in D$" which finishes the density argument. □

Corollary 3.1.13. $\mathrm{cov}(\mathcal{I}(\mathbb{M}_\mathcal{J})), \mathrm{cov}(\mathcal{I}(\mathbb{L}_\mathcal{J})) \leq \mathfrak{g}$ *for all $\mathcal{J}$ with Baire property.*

Proof. The inequalities follow from the Theorem 3.1.1 and the Proposition 3.1.12. □

As a corollary, we get the following simple consistency result.

Theorem 3.1.14. *The inequalities* $\mathrm{cov}(\mathcal{I}(\mathbb{M}_\mathcal{J})) < \mathrm{add}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{I}(\mathbb{L}_\mathcal{J})) < \mathrm{add}(\mathcal{M})$ *is consistent with ZFC for any meager ideal $\mathcal{J}$.*

Proof. The inequality holds in Hechlers model i.e. finite support iteration of length ω_2 over a CH model with Hechler forcing as each iterand. On one hand, it is well-known that $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ because of Cohen and dominating reals added by Hechler forcing. On the other hand, the inequality $\mathrm{cov}(\mathcal{I}(\mathbb{T})) \leq \mathfrak{g}$ holds for both $\mathbb{L}_\mathcal{J}$ and $\mathbb{M}_\mathcal{J}$, by the above corollary. Recall now the result of Yorioka ([Yor03]) saying that $\mathfrak{g} = \omega_1$ holds in Hechlers model. □

One may ask for strengthening the above propositions and show that no cardinal invariant from Cichoń's diagram stands as a lower bound for the $\mathrm{cov}(\mathcal{I}(\mathbb{T}))$ for our tree types.

Question 3.1.15. *Is* $\mathrm{cov}(\mathcal{I}(\mathbb{T})) < \mathrm{add}(\mathcal{N})$ *consistent for the tree types considered in this tesis?*

3.2 Incompatibility Shrinking numbers

In this section we will introduce the incompatibility shrinking number. This cardinal invariant will play an important role in the study of additivities and cofinalities of tree ideals as well as the antichain numbers. Our main usage of this invariant is to assume that the incompatibility shrinking number is equal to the continuum. To justify this assumption, it is worth mentioning the paper [SS23] of Spinas and Shelah, in which this number has been inscribed in the main framework, under their paper is written.

Definition 3.2.1. *The* ***incompatibility shrinking number*** *of a tree type $\mathbb{T}$ is defined as*

$$\mathrm{is}(\mathbb{T}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I}_\mathcal{B}(\mathbb{T}) \wedge \forall T \in \mathbb{T}\ [T] \cap \bigcup \mathcal{F} \neq \emptyset\}$$

The hereditary version of incompatibility shrinking number is defined as follows:

$$\mathrm{is}_h(\mathbb{T}) = \min\{\mathrm{is}(\mathbb{T}|_T) : T \in \mathbb{T}\}$$

It is clear that $\mathrm{is}_h(\mathbb{T}) \leq \mathrm{is}(\mathbb{T})$ and when $\mathbb{T}$ is homogeneous, then the two are equal. The assumption that $\mathrm{is}_h(\mathbb{T}) = \mathfrak{c}$ has been used in [BKW17]. In [GRSS95] and [JMS92] the authors implicitly used it to estimate the additivity of $\mathcal{I}(\mathbb{S})$, $\mathcal{I}(\mathbb{M})$ and $\mathcal{I}(\mathbb{L})$ to then consistently distinguish the additivities and covering numbers of the tree ideals.

Proposition 3.2.2. $\mathrm{add}(\mathcal{I}_{\mathcal{B}}(\mathbb{T})) \leq \mathrm{is}(\mathbb{T}) \leq \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{T}))$ *and* $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \mathrm{is}_h(\mathbb{T})$.

Proof. If a family $\mathcal{F} \subseteq \mathcal{I}_{\mathcal{B}}(\mathbb{T})$ intersects every set of form $\{x\}$ for $x \in \omega^\omega$, then it intersects every $[T]$ for $T \in \mathbb{T}$, and then it intersects every set of form $\omega^\omega \setminus X$ for $X \in \mathcal{I}_{\mathcal{B}}(\mathbb{T})$. To prove the last inequality note that if $\{B_\alpha : \alpha < \mathrm{is}(\mathbb{T})\}$ witnesses for $\mathrm{is}(\mathbb{T})$ then $\bigcup_\alpha B_\alpha \notin \mathcal{I}(\mathbb{T})$ because otherwise there would exist $T \in \mathbb{T}$ disjoint with all B_α's. □

First, we calculate the incompatibility shrinking numbers for classical σ-ideals of measure and category (which are homogeneous by the 2.3.3 and 2.3.8 of [Zap04])

Proposition 3.2.3. *The following holds:*[1]

- $\mathrm{is}(\mathbb{C}) = \mathrm{add}(\mathcal{M})$,
- $\mathrm{is}(\mathbb{B}) = \mathrm{add}(\mathcal{N})$.

Proof. We will show this only in the case of $\mathbb{B}$ as the other one follows from identical argument. As $\mathcal{I}(\mathbb{B}) = \mathcal{I}_{\mathcal{B}}(\mathbb{B}) = \mathcal{N}$, ($\mathcal{I}(\mathbb{C}) = \mathcal{I}_{\mathcal{B}}(\mathbb{C}) = \mathcal{M}$) by the Proposition 3.2.2 it is enough to show that $\mathrm{is}(\mathbb{B}) \leq \mathrm{add}(\mathcal{N})$. Assume that $\{N_\alpha : \alpha < \kappa\} \subseteq \mathcal{N}$ and $\kappa < \mathrm{is}(\mathbb{B})$. Without loss of generality, we may assume that $N_\alpha + \mathbb{Q} = N_\alpha$. If $B \in \mathrm{Borel} \setminus \mathcal{N}$ is such that $B \cap \bigcup_\alpha N_\alpha = \emptyset$, then $\bigcup_\alpha N_\alpha \subseteq 2^\omega \setminus (B + \mathbb{Q}) \in \mathcal{N}$. As, by Stainhaus theorem, $B + \mathbb{Q} \in \mathcal{N}$, we are done. □

It is clear that for classical Sacks trees we have $\mathrm{is}(\mathbb{S}) = \mathfrak{c}$. In case of classical Miller trees $\mathbb{M}$ this has been calculated by Spinas. For the sake of completeness and the fact that $\mathrm{is}^*(\mathbb{M})$ is equal the covering number of $\mathcal{I}_{\mathcal{B}}(\mathbb{M})$ (this does not follow from the results regarding tree types $\mathbb{M}_{\mathcal{J}}$) we include the proof

Proposition 3.2.4. *(Spinas, [Spi95])* $\mathrm{is}_h(\mathbb{M}) = \mathfrak{d}$.

Proof. By the Proposition 3.2.2 and the well-known fact that $\mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M})) = \mathfrak{d}$, it is enough to prove the inequality $\mathfrak{d} \leq \mathrm{is}(\mathbb{M})$. Assume then that $T \in \mathbb{M}$, and a collection $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$, $\kappa < \mathfrak{d}$ is given. For each $\alpha < \kappa$ define $h_\alpha \in \omega^\omega$ such that

$$h_\alpha(n) = \max\{f_\alpha(\sigma) : \sigma \in \mathit{split}_i(T \cap \mathit{Seq}^{\downarrow}(f_\alpha)) \text{ and } i \leq n\}$$

As $\kappa < \mathfrak{d}$ there must be $g \in \omega^\omega$ which is not $\leq^*$-bounded by any h_α. Let $S \leq T$ be a constructed in a way that we trim successors of σ's in $\mathit{split}_n(T)$ using $g(n)$ i.e let S be the unique Miller tree such that $\mathit{stem}(S) = \mathit{stem}(T)$ and for each $n \in \omega$ and $\sigma \in \mathit{split}_n(S)$ we have $\mathit{succ}_S(\sigma) = \mathit{succ}_T(\sigma) \setminus g(n)$. We claim that $[S] \cap [T_\alpha]$ is empty for every $\alpha < \kappa$. Assume towards contradiction that there is $x \in [S] \cap [T_\alpha]$ for some $\alpha < \kappa$. On the one hand we have $x \leq^* f_\alpha$. On the other, if $\{n_i : i < \omega\}$ is an increasing sequence of natural numbers such that $\{x|_{n_i} : i \in \omega\} = \{x|_j : j \in \omega\} \cap \mathit{split}(S)$, then $g(i) \leq x(n_i)$ for every $i \in \omega$. Then for almost all such i's we have $g(i) \leq x(n_i) \leq f_\alpha(n_i) \leq h_\alpha(j)$ contradicting the choice of g over h_α's. □

Proposition 3.2.5. *The following holds:*

- $\mathrm{cov}(\mathcal{M}) \leq \mathrm{is}_h(\mathbb{S}_F)$,

[1] The Cohen forcing $\mathbb{C}$ can be we considered as the forcing with trees of form $\{\tau \in 2^{<\omega} : \sigma \text{ is comparable } \tau\}$ for $\sigma \in 2^{<\omega}$. The random forcing $\mathbb{B}$ can be thought of as the forcing with trees $T \subseteq 2^{<\omega}$ such that $\mathit{lim}_n \frac{|T \cap 2^n|}{2^n} > 0$.

– $\mathrm{cov}(\mathcal{M}) \leq \mathrm{is}_h(\mathbb{IE}_F)$.

Proof. As the proofs of the two inequalities follows the same ideal we will show this only for slightly more difficult case, which is the latter. Recall that $\mathcal{I}_{\mathcal{B}}(\mathbb{IE}_F)$ has a basis made of the sets of form $E_f := \{x \in \Pi F : \forall^\infty n \in \omega\ x(n) \neq f(n)\}$ where $f \in \Pi F$. Suppose that $\kappa < \mathrm{cov}(\mathcal{M})$, we are given a collection of sets $\{E_{f_\alpha} : \alpha < \kappa\} \subseteq \mathcal{I}_{\mathcal{B}}(\mathbb{IE}_F)$ where $\{f_\alpha : \alpha < \kappa\} \subseteq \Pi F$ and that $T \in \mathbb{IE}_F$. Consider the following notion of forcing $\mathbb{P} = \{F \subseteq T : F \text{ is a finite tree}\}$. The order is the end-extension i.e. $F \leq H$ if every $\sigma \in F \setminus H$ extends a terminal node of H. Note that $\mathbb{P}$ is countable. For $\alpha < \kappa$ and $N \in \omega$ define the sets

$$C^N := \{F \in \mathbb{P} : \forall \sigma \in F \cap 2^N \neq \emptyset\ \exists n > |\sigma|\ \forall i \in F(n)\ \exists \tau \in T|_\sigma,\ |\tau| = n+1,\ \tau(n) = i\}$$

and

$$D^N_\alpha := \{F \in \mathbb{P} : \forall \sigma \in F \cap 2^N \neq \emptyset\ \exists \tau \in F|_\sigma\ \tau(|\tau|-1) \neq \phi_\alpha(\sigma|_{|\sigma|-1})\}$$

It is easy to extend any $F \in \mathbb{P}$ to the one which satisfies the formulas in C^N's and D^N_α's. It follows, that the C^N's and D^N_α's are dense in $\mathbb{P}$. Recall that the Martins number of countable notions of forcing is equal $\mathrm{cov}(\mathcal{M})$. Assume that $G \subseteq \mathbb{P}$ is filter generic over the collection of our dense sets. Define $S = \bigcup_{F \in G} F$. As the sets C^N, $N \in \omega$ are dense, $S \in \mathbb{IE}_F$. As the sets D^N_α, $N \in \omega$, $\alpha < \kappa$ are dense, $[S] \cap E_{f_\alpha} = \emptyset$ for every $\alpha < \kappa$ which finishes the proof. □

However, the following is unknown for me.

Question 3.2.6. *Can we consistently get* $\mathrm{is}^*(\mathbb{S}_F) < \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{S}_F))$ *or* $\mathrm{is}^*(\mathbb{IE}_F) < \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{IE}_F))$*?*

The following estimate is a result from the work [HS20] of Hein and Spinas. We include the sketch of the proof as we will need it later on.

Proposition 3.2.7. *(Hein and Spinas, [HS20])* $\mathrm{cov}(\mathcal{M}) \leq \mathrm{is}_h(\mathbb{SP})$

Proof. Recall that $\mathcal{I}_{\mathcal{B}}(\mathbb{SP})$ has a basis made of countable unins of the sets of form $F_C =: \{x \in 2^\omega : x^{-1}[1] \subseteq^* C \text{ or } x^{-1}[1] \cap C =^* \emptyset\}$ where $C \in [\omega]^\omega$. Let $T \in \mathbb{SP}$, $\kappa < \mathrm{cov}(\mathcal{M})$ and let $\{X_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ be parameters for sets from the basis of $\mathcal{I}_{\mathcal{B}}(\mathbb{SP})$. Define the following incarnation of Cohen forcing: let $\mathbb{P}$ consist of all finite subtrees F of T all terminal nodes of which are of the same length $h_F \in \omega$. The order $F_0 \leq F_1$ means that F_0 is an end-extension of F_1 and $N_{T|_{F_0}}(\sigma) = N_{T|_{F_1}}(\sigma)$ for each $\sigma \in F_1$. One may show ([HS20]) that the following sets are dense in $\mathbb{P}$ for $n \in \omega$ and $\alpha \in \kappa$:

$$D_n = \{F \in \mathbb{P} : h_F > n\},$$
$$E_\alpha = \{F \in \mathbb{P} : \forall \sigma \in term(F) \exists n_0, n_1 \in X_\alpha \cap h_F \text{ such that } \sigma(n_0) = 0 \text{ and } \sigma(n_1) = 1\}$$

It follows that if $G \subseteq \mathbb{P}$ is a filter intersecting all D_n's and all E_α's, then the set $S \subseteq 2^\omega$ defined as $\bigcup G$ is a splitting tree, $S \leq T$ and $[S] \cap \bigcup_{\alpha<\kappa} F_{X_\alpha}$ (see [HS20] for details). □

Theorem 3.2.8. *Suppose that $\mathcal{J}$ is an arbitrary ideal on ω. Then*

– $\min\{\mathfrak{b}, h\text{-}\mathrm{cov}^*_+(\mathcal{J})\} \leq \mathrm{is}_h(\mathbb{M}_{\mathcal{J}})$,

– $\min\{\mathfrak{b}, h\text{-}\mathrm{cov}^*_+(\mathcal{J})\} \leq \mathrm{is}_h(\mathbb{L}_{\mathcal{J}})$.

Proof. To show that $\min\{\mathfrak{b}, \text{h-}\mathrm{cov}^*_+(\mathcal{J})\} \leq \mathrm{is}^*(\mathbb{L}_{\mathcal{J}})$ assume that $\kappa < \min\{\mathfrak{b}, \text{h-}\mathrm{cov}^*_+(\mathcal{J})\}$, $T \in \mathbb{L}_{\mathcal{J}}$ and we are given a collection $\{L_{\phi_\alpha} : \alpha < \kappa\} \subseteq \mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$ such that $\phi_\alpha : \omega^{<\omega} \to \omega$ for each $\alpha < \kappa$. For each $\sigma \in \omega^{<\omega}$, as $\kappa <$h-$\mathrm{cov}^*_+(\mathcal{J})$ there is an $\mathcal{J}$-positive set $C_\sigma \subseteq succ_T(\sigma)$ such that for each $\alpha < \kappa$ the intersection $C_\sigma \cap \phi_\alpha(\sigma)$ is finite. Let $f_\alpha \in \omega^\omega$ be such that $C \cap \phi_\alpha(\sigma) \subseteq f_\alpha(\sigma)$. As $\kappa < \mathfrak{b}$ there is a function $g \in \omega^\omega$ which dominates all f_α's. Let $S \in \mathbb{L}_{\mathcal{J}}$ be a unique subtree of T defined as follows: $stem(T) = stem(S)$ and if $\sigma \in split(S)$ then $succ_S(\sigma) = C_\sigma \setminus g(\sigma)$. We claim that $[S] \cap \bigcup_\alpha L_{\phi_\alpha} = \emptyset$. □

Remark. *In case of $\mathcal{J}$-Laver trees it is not possible to prove* $\min\{\mathfrak{b}, h\text{-cov}^*_+(\mathcal{J})\} = \text{is}_h(\mathbb{L}_\mathcal{J})$. *For example* $h\text{-cov}^*_+(\mathcal{F}in\times\mathcal{F}in) = \text{cov}^*_+(\mathcal{F}in\times\mathcal{F}in) = \omega$ *which would lead to* $\text{is}^*(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in}) = \omega$, *but we know that the number* $\text{is}^*_h(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in})$ *is greater than* $\text{add}(\mathcal{I}_\mathcal{B}(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in}))$ *which is uncountable.*

Remark. *In case of the upper bounds one may show that the following inequalities hold:*

- $\text{is}(\mathbb{M}_\mathcal{J}) \leq \min\{\mathfrak{d}, \omega\text{-cov}^*_+(\mathcal{J})\}$,
- $\text{is}(\mathbb{L}_\mathcal{J}) \leq \min\{\mathfrak{b}, \omega\text{-cov}^*_+(\mathcal{J})\}$

Here, for an ideal $\mathcal{J}$ on ω, the number $\omega\text{-cov}^*_+(\mathcal{J})$ is defined as the smallest cardinality of a family $\mathcal{F}\subseteq\mathcal{J}$ such that for every sequence $\{A_n : n\in\omega\}$ of $\mathcal{J}$-positive sets there is $F\in\mathcal{F}$ such that $F\cap A_n$ is infinite for every $n\in\omega$.

Proof. On the one hand we have $\text{is}(\mathbb{L}_\mathcal{J}) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})) \leq \mathfrak{b}$ and $\text{is}(\mathbb{M}_\mathcal{J}) \leq \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})) \leq \mathfrak{d}$ because of the Propositions 2.2.40 and 2.2.32. To show that $\text{is}(\mathbb{L}_\mathcal{J}) \leq \omega\text{-cov}^*_+(\mathcal{J})$ let $\{A_\alpha : \alpha < \kappa\}\subseteq\mathcal{J}$ be a witness for $\omega\text{-cov}^*_+(\mathcal{J})$. Let $L_\alpha := \{x\in\omega^\omega : \exists^\infty_{n\in\omega} x(n)\in A_\alpha\}$. By the Theorem 2.2.39 each L_α is $\mathcal{I}_\mathcal{B}(\mathbb{L}_\mathcal{J})$-set. We will show that the sets L_α witness for $\text{is}(\mathbb{L}_\mathcal{J})$. To see this let $T\in\mathbb{L}_\mathcal{J}$. Then there is $\alpha<\kappa$ such that for every $\sigma\in\{\tau\in T : succ_T(\tau)\in\mathcal{J}^+\}$ the intersection $A_\alpha\cap succ_T(\sigma)$ is infinite. Then construct inductively $x\in[T]$ such that $stem(T)\subseteq x$ and for each $n\geq|stem(T)|$ we have $x(n)\in succ_T(x|_n)\cap A_\alpha$. Clearly, then $x\in[T]\cap L_\alpha$. The $\mathcal{J}$-Miller case is analogous. □

For Miller-like trees, we have stronger lower bound in terms of the covering of the Borel part.

Theorem 3.2.9. *The inequality* $\min\{\mathfrak{b}, \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{T}))\} \leq \text{is}_h(\mathbb{T})$ *holds for the tree types $\mathbb{M}_\mathcal{J}$ (with any ideal $\mathcal{J}$) and $\mathbb{FM}$.*

Proof. We will focus only on the tree type $\mathbb{M}_\mathcal{J}$ as the proof for $\mathbb{FM}$ is almost identical. Assume that $\kappa < \min\{\mathfrak{b}, \text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))\}$ and that we are given $T\in\mathbb{M}_\mathcal{J}$ and a family $\{\phi_\alpha : \alpha<\kappa\}$ where $\phi_\alpha : \omega^{<\omega}\to\mathcal{J}$. In this proof let $M'_{\phi_\alpha} = \{x\in\omega^\omega : \forall n\in\omega\ x(n)\notin\phi_\alpha(x|_n)\}$. Inductively we will construct a sequence $\{F_n : n\in\omega\}$ of well-founded subtrees of T such that for every $n\in\omega$ we have:

(1) F_{n+1} is an end-extension of F_n and $term(F_n)\subseteq split(T)$,

(2) for every $\sigma\in term(F_n)$, if $n\in succ_T(\sigma)$ then there is unique $\tau\in term(F_{n+1})$ such that $\sigma^\frown n\subseteq\tau$,

(3) for every $\sigma\in term(F_n)$ for almost all $\tau\in term(F_{n+1})$ extending σ, we have $[\tau]\cap M'_{\phi_\alpha}=\emptyset$.

Assume that F_n has been constructed and fix $\sigma\in term(F_n)$. For each $n\in succ_T(\sigma)$ we can choose a point $x^\sigma_n\in[\sigma^\frown n]\setminus\bigcup_\alpha M'_{\phi_\alpha}$ as $\kappa<\text{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))$. For every $\alpha<\kappa$ there is $f_\alpha\in\omega^\omega$ such that

$$\forall n\in succ_T(\sigma)\ x^\sigma_n|_{f_\alpha(n)}\notin M'_{\phi_\alpha}$$

As $\kappa<\mathfrak{b}$ let $g\in\omega^\omega$ dominate all of f_α's. For $n\in\omega\wedge\sigma\in term(F_n)$ let ρ^n_σ be the shortest splitnode of T which extends $x^\sigma_n|_{g(n)}$. Let then F_{n+1} be the tree generated by the set $\{\rho^n_\sigma : n\in\omega\wedge\sigma\in term(F_n)\}$. Let $S=\bigcup_n F_n\in\mathbb{M}_\mathcal{J}$. The condition (3) implies that $[S]\cap M'_{\phi_\alpha}$ is compact and therefore bounded by some $h_\alpha\in\omega^\omega$. Again, let $h\in\omega^\omega$ dominate all of h_α's and let $R\in\mathbb{M}_\mathcal{J}$ be such that $[R]$ consists of branches of $[S]$ which are not dominated by h. Then, clearly $[R]\cap\bigcup_\alpha M'_\alpha=\emptyset$ for all $\alpha<\kappa$. □

For the types of trees that add Cohen reals, we are able to show that this lower bound is precise. To prove this, we will use the following result.

Theorem 3.2.10. *(Brendle, [Bre99]) If* $\mathbf{M}$ *is submodel of* $\mathbf{V}$ *and* $\mathbf{V}$ *contains a Miller tree of Cohen real over* $\mathbf{M}$*, then* $\mathbf{V}$ *contains a dominating real over* $\mathbf{M}$*.*

Theorem 3.2.11. *Assume that* $\mathbb{T}$ *is a combinatorial tree type that adds Cohen reals and has the continuous reading of names, then* $\mathrm{is}_h(\mathbb{T}) \leq \mathfrak{b}$*.*

Proof. Let $T \in \mathbb{T}$ and let $f : [T] \to \omega^{<\omega}$ be a continuous function such that $T \Vdash$ "$f(\dot{r}_{gen})$ is a Cohen real". We will need the following simple observation

Claim 3.2.12. *For every closed nowhere dense set* $N \subseteq \omega^\omega$*, we have that* $f^{-1}[N] \in \mathcal{I}(\mathbb{T})$*.*

Proof. Let $S \leq T$. Find $\sigma \in \omega^{<\omega}$ and $S' \leq S$ such that $S' \Vdash \sigma \subseteq f(\dot{r}_{gen})$ and $\langle\sigma\rangle \cap N = \emptyset$. As the function f is continuous, we can find $\tau \in S'$ such that $f[\langle\tau\rangle] \subseteq \langle\sigma\rangle$. Let $S'' = S'|_\tau$. Clearly we then have that $S'' \cap f^{-1}[N] = \emptyset$ which finishes the proof of the claim. □

To prove that $\mathrm{is}(\mathbb{T}) \leq \mathfrak{b}$ assume towards a contradiction that $\mathrm{is}(\mathbb{T}|_\mathbb{T}) > \mathfrak{b}$ for some $T \in \mathbb{T}$. Let $\mathbf{M}$ be an elementary substructure of some $H(\kappa)$ such that $f, T, \mathbb{T} \in \mathbf{M}$, $\mathbf{M} \cap \omega^\omega$ is unbounded and $|\mathbf{M}| = \mathfrak{b}$. Then, since $|\mathbf{M}| < \mathrm{is}(\mathbb{T}|_T)$ there must be an $R \in \mathbb{T}$, $R \leq T$ such that $[R] \cap X = \emptyset$ for every $X \in t^0_\mathcal{B} \cap \mathbf{M}$. By Proposition 3.2.12, if $N \subseteq \omega^\omega$ is a nowhere-dense set and $N \in \mathbf{M}$, then $[R] \cap f^{-1}[N] = \emptyset$ and therefore $f[[R]] \cap N = \emptyset$. But then, this means that $f[[R]]$ contains exclusively Cohen reals over $\mathbf{M}$. Since σ-compact sets are meager, and $R \Vdash f(\dot{r}_{gen})$ is Cohen, then $f[[R]]$ cannot be σ-compact, so by Theorem 2.2.31 for Miller trees, $f[[R]]$ must contain the branches of a Miller tree. Therefore, by Theorem 3.2.10, there must be a real $d \in \omega^\omega$ such that d dominates every $y \in \omega^\omega \cap \mathbf{M}$, which is impossible since $\mathbf{M}$ contains an unbounded family. □

We will summarize results regarding the incompatibility shrinking number fot the tree types $\mathbb{FM}$, $\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$ with the following

Corollary 3.2.13. *The following holds:*

- $\mathrm{is}^*(\mathbb{FM}) = \mathrm{is}^*(\mathbb{L}_{nwd}) = \mathrm{is}^*(\mathbb{M}_{nwd}) = \mathrm{add}(\mathcal{M})$*,*
- $\mathrm{is}^*(\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in}) = \mathfrak{b}$ *and* $\mathfrak{b} \leq \mathrm{is}^*(\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in}) \leq \mathfrak{d}$*,*
- $\mathrm{is}^*(\mathbb{L}_{\mathcal{ED}}) = \mathfrak{b}$ *and* $\mathfrak{b} \leq \mathrm{is}^*(\mathbb{M}_{\mathcal{ED}}) \leq \mathfrak{d}$*,*
- $\mathrm{is}^*(\mathbb{L}_{\mathcal{ED}_{fin}}) = \min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\}$ *and* $\min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\} \leq \mathrm{is}^*(\mathbb{M}_{\mathcal{ED}_{fin}}) \leq \mathfrak{d}$*,*
- $\mathrm{is}^*(\mathbb{L}_\mathcal{S}) = \min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$ *and* $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{is}^*(\mathbb{M}_\mathcal{S}) \leq \mathfrak{d}$*,*
- $\mathrm{is}^*(\mathbb{L}_\mathcal{J}) = \mathfrak{b}$ *and* $\mathfrak{b} \leq \mathrm{is}^*(\mathbb{M}_\mathcal{J}) \leq \mathfrak{d}$ *for both* $\mathcal{R}$ *and conv.*

I do not know the answer to the following question.

Question 3.2.14. *What are the values of* $\mathrm{is}(\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in})$*,* $\mathrm{is}(\mathbb{L}_\mathcal{S})$ *and* $\mathrm{is}(\mathbb{L}_{\mathcal{ED}_{fin}})$*?*

Theorem 3.2.15. $\mathrm{is}(\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in}) < \mathfrak{d}$ *is consistent with ZFC.*

Proof. We will show that the inequality holds in the model obtained by adding ω_2 many Cohen reals (simultaneously) to a model of CH. Let $\dot{T}$ be a $\mathbb{C}(\omega_2)$-name to an $\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in}$-tree. By the standard arguments with good/nice names there is a countable $W \subseteq \omega_2$ such that $\dot{T}$ is in fact $\mathbb{C}(W)$-name. As $\mathbb{C}(W)$ is forcing equivalent to the single step Cohen forcing $\mathbb{C}$, the proof of the theorem will be finished by showing the following

Claim 3.2.16. *For every* $\mathbb{C}$*-name for an* $\mathbb{M}_{\mathcal{F}in\times\mathcal{F}in}$*-tree* $\dot{T}$ *there is* $\phi : (\omega\times\omega)^{<\omega} \to \mathcal{F}in \times \mathcal{F}in$ *such that* $\Vdash$ "$[\dot{T}] \cap M_\phi \neq \emptyset$"*.*

Without loss of generality we may assume that $\Vdash$ "$stem(\dot{T}) = \rho_\emptyset$" for some $\rho_\emptyset \in \omega^{<\omega}$. Let also $c_\emptyset = \emptyset$ and let $(a_\emptyset, b_\emptyset) = \rho_\emptyset(0)$. We will inductively construct sequences $\{c_\sigma : \sigma \in \omega^{<\omega}\} \subseteq \mathbb{C}$, $\{(a_\sigma, b_\sigma) : \sigma \in \omega^{<\omega}\} \subseteq \omega \times \omega$, $\{\rho_\sigma : \sigma \in \omega^{<\omega}\} \subseteq \omega^{<\omega}$ such that for all $\sigma, \sigma' \in \omega^{<\omega}$ we have

a) $\{c_{\sigma^\frown n} : n \in \omega\}$ is dense below c_σ,

b) $a_{\sigma^\frown n} < a_{\sigma^\frown n+1}$,

c) $c_{\sigma^\frown n} \Vdash$ "$(a_{\sigma^\frown n}, b_{\sigma^\frown n}) \in succ_{\dot{T}}(\rho_\sigma)$",

d) $c_{\sigma^\frown n}$ decides the value of $Succ_{\dot{T}}(\rho_\sigma^\frown(a_{\sigma^\frown n}, b_{\sigma^\frown n}))$ to be $\rho_{\sigma^\frown n} \in \mathbb{V}$.

The construction of such sequences is straightforward. Notice that b) implies $\{(a_{\sigma^\frown n}, b_{\sigma^\frown n}) : n \in \omega\} \in \mathcal{F}in \times \mathcal{F}in$ and that $\rho_{\sigma^\frown n}(|\rho_\sigma|) = (a_{\sigma^\frown n}, b_{\sigma^\frown n})$. Define now $\phi : (\omega \times \omega)^{<\omega} \to \mathcal{F}in \times \mathcal{F}in$ such that for $\tau \in (\omega \times \omega)^{<\omega}$ we have

- $\phi(\tau) = \emptyset$ if τ is not contained in any ρ_σ,
- $\phi(\tau) = \{(a_{\sigma^\frown n}, b_{\sigma^\frown n}) : n \in \omega\}$ if $\tau = \rho_\sigma$ for some $\sigma \in \omega^{<\omega}$,
- $\phi(\tau) = \{\rho_\sigma(|\tau|)\}$ if ρ_σ is the shortest that contains τ.

To show that $\Vdash$ "$[\dot{T}] \cap M_\phi \neq \emptyset$" let $\dot{r}_{gen}$ be a $\mathbb{C}$-generic real. Because of a) there is $\dot{y} \in \omega^\omega$ such that $\dot{r}_{gen} = \bigcup_k c_{y|_k}$. Let $\dot{x} = \bigcup_k \rho_{\dot{y}|_k}$. We will see that $\Vdash$ "$\dot{x} \in [\dot{T}] \cap M_\phi$". Notice that $c_{y|_k} \Vdash$ "$\rho_{\dot{y}|_k}(n) \in \dot{T}$" for every $k \in \omega$ so $\Vdash$ "$\dot{x} \in [\dot{T}]$". Next for $n > 0$ we consider two cases: if $n \notin \{|\rho_{\dot{y}|_k}| : k \in \omega\}$ let $k \in \omega$ be the smallest natural number such that $n < |\rho_{\dot{y}|_k}|$. Then $\dot{x}(n) = \rho_{\dot{y}|_k}(n) \in \{\rho_{\dot{y}|_k}(n)\} = \phi(\dot{x}|_n)$. If $n = |\rho_{\dot{y}|_k}|$ for some $k \in \omega$, then because of d), we have that $\dot{x}(n) = \rho_{\dot{y}|_{k+1}}(n) \in \{(a_{\dot{y}|_{k+1}}, b_{\dot{y}|_{k+1}})\} \subseteq \phi(\dot{x}|_n)$. It follows from the two cases, that $\Vdash$ "$\dot{x} \in M_\phi$", which finishes the proof. □

3.3 Antichains of trees

Suppose that $\mathbb{P}$ is a non-c.c.c. preorder. Another cardinal invariant related to a notion of forcing is the *antichain number*, defined as follows.

$$\mathfrak{a}(\mathbb{P}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathbb{P} \text{ is an uncountable maximal antichain}\}$$

Such numbers were investigated for several classical tree forcings.

Theorem 3.3.1. *(Spinas and Wyszkowski [SW15])* $\mathfrak{d} \leq \mathfrak{a}(\mathbb{S}i)$[2]

Theorem 3.3.2. *(Spinas [Spi97])* $\mathfrak{b} \leq \mathfrak{a}(\mathbb{M}a)$[3]

Theorem 3.3.3. *(Spinas and Hein [HS20]) The following inequalities hold:*

- $\mathfrak{d} \leq \mathfrak{a}(\mathbb{S})$,
- $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathbb{SP})$,
- $\mathfrak{b} \leq \mathfrak{a}(\mathbb{M})$.

We will investigate the antichain numbers of the forcings $\mathbb{S}_F$, $\mathbb{IE}_F$, $\mathbb{M}_\mathcal{J}$ and $\mathbb{P}_\mathcal{E}$. The case of Laver forcing remains open (as asked in [HS20]).The following lemma is crucial for our constructions.

Lemma 3.3.4. *(Shelah and Spinas [HS20]) For every $\{A_\alpha : \alpha < \omega_1\} \subseteq [\omega]^\omega$ there is a countable $C \subseteq \omega_1$ such that for all $n \in C$ and $F \in [A_n]^{<\omega}$ there are infinitely many $m \in C$ with $F \subseteq A_m$*

Theorem 3.3.5. $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathbb{IE}_F)$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{a}(\mathbb{S}_F)$

[2] $\mathbb{S}i$ stands for the standard Silver tree forcing

[3] $\mathbb{M}a$ stands for the standard Mathias tree forcing

Proof. We give the proof for $\mathbb{IE}_F$ as the one for $\mathbb{S}_F$ is almost identical. Let $\kappa < \mathrm{cov}(\mathcal{M})$ and let $\{T_\alpha : \alpha < \kappa\}$ be an antichain in $\mathbb{IE}_F$. By the Theorem 3.2.5 we have that $\kappa < \mathrm{cov}(\mathcal{M}) \leq \mathrm{is}_h(\mathbb{IE}_F)$. It follows that there exists a collection $\{S_\alpha : \alpha < \kappa\} \subseteq \mathbb{IE}_F$ such that for every $\alpha, \beta < \kappa$ we have:

- $S_\alpha \leq T_\alpha$,
- $[S_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$.

By the Lemma 3.3.4, without loss of generality we may assume that

$$\forall n \in \omega \forall F \in [S_n]^{<\omega} \exists^\infty m \in \omega F \subseteq S_m \tag{3.1}$$

Consider the following notion of forcing

$$\mathbb{P} = \{F \subseteq Seq(F) : F \text{ is a finite tree contained in } S_n \text{ for some } n \in \omega\}$$

with the following order: $F' \leq F$ if F' is an end-extension of F. Note that $\mathbb{P}$ is countable, non-atomic order. For $\alpha < \kappa$ and $m \in \omega$ consider the following sets:

- $D_\alpha = \{F \in \mathbb{P} : term(F) \cap T_\alpha = \emptyset\}$,
- $E_m = \{F \in \mathbb{P} : \forall \sigma \in F \cap \omega^m \exists N > |\sigma| \forall i \in F(N) \exists \tau \in F, \sigma \subseteq \tau, |\tau| > N \text{ and } \tau(N) = i\}$.

We will check that these sets are dense in $\mathbb{P}$:
To see that D_α is dense let $F \in \mathbb{P}$ and let $n \in \omega$ be such that $F \subseteq S_n$. Consider two cases: First, that $\alpha \neq n$. Because $[T_\alpha] \cap [S_n] = \emptyset$, the intersection $T_\alpha \cap S_n$ is finite and there is $N \in \omega$ such that no $\sigma \in S_n$ longer than N belongs to the T_α. Let the $F' \subseteq S_n$ be an end-extension of F such that every terminal node of F' is longer than N. It follows that $F' \in D_\alpha$. For the second case assume that $\alpha = n$. By the condition 3.1, there is $m > n$ such that $F \subseteq S_m$. Again, as $S_m \cap T_n$ is finite, we may find an end-extension $F' \subseteq S_m$ of F whose terminal nodes are not in T_n. It follows that $F' \in D_\alpha$.
To see that E_m is dense let $F \in \mathbb{P}$ and let $n \in \omega$ be such that $F \subseteq S_n$. As $S_n \in \mathbb{IE}_F$ for every $\sigma \in term(F)$ there is N_σ such that for each $i \in F(N_\sigma)$ there is $\tau_i^\sigma \in S_n$, $\sigma \subseteq \tau_i^\sigma$, $|\tau_i^\sigma| > N_\sigma$ such that $\tau_i^\sigma(N_\sigma) = i$. Let then F' be the tree generated by all such τ_i^σ's. Clearly, F' is an end-extension of F and $F' \in E_m$.
Now, as $\kappa < \mathrm{cov}(\mathcal{M}) = \mathfrak{m}_{ctbl}$, there is a filter $G \subseteq \mathbb{P}$ intersecting all D_α and E_m for $\alpha < \kappa$ and $m \in \omega$. Define then a tree T as $\bigcup_{F \in G} F$. As $G \cap E_m \neq \emptyset$ for every $m \in \omega$, we get that $T \in \mathbb{IE}_F$. As $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$, we get that $T \cap T_\alpha$ is finite, so $[T] \cap [T_\alpha] = \emptyset$. This means that the antichain $\{T_\alpha : \alpha < \kappa\} \subseteq \mathbb{IE}_F$ is not maximal. □

Theorem 3.3.6. $\min\{\mathrm{is}_h(\mathbb{ED}_F), \mathrm{cov}(\mathcal{M})\} \leq \mathfrak{a}(\mathbb{ED}_F)$.

Proof. Let $\kappa < \min\{\mathrm{is}_h(\mathbb{ED}_F), \mathrm{cov}(\mathcal{M})\}$ and let $\{T_\alpha : \alpha < \kappa\} \subseteq \mathbb{ED}_F$ be an antichain. As $\kappa < \mathrm{is}_h(\mathbb{ED}_F)$, there is a collection $\{S_\alpha : \alpha < \kappa\} \subseteq \mathbb{ED}_F$ such that for every $\alpha, \beta < \kappa$ we have:

- $S_\alpha \leq T_\alpha$,
- $[S_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$.

By the Lemma 3.3.4, without loss of generality we may assume that

$$\forall n \in \omega \forall F \in [S_n]^{<\omega} \exists^\infty m \in \omega F \subseteq S_m \tag{3.2}$$

Consider the following notion of forcing $\mathbb{P}$: $F \in \mathbb{P}$ if F is a finite tree contained in some S_n and for $\sigma \subseteq \tau \in F \setminus term(F)$ we have that $|succ_F(\sigma)| \leq |succ_F(\tau)|$. The order is the following: $F' \leq F$ if F' is an end-extension of F. Note that $\mathbb{P}$ is a countable, non-atomic order. For $\alpha < \kappa$ and $m \in \omega$ consider the following sets:

– $D_\alpha = \{F \in \mathbb{P} : term(F) \cap T_\alpha = \emptyset\}$,

– $E_m = \{F \in \mathbb{P} : \forall \sigma \in term(F)(|\sigma| > m \text{ and } |succ_F(\sigma|_{|\sigma|-1})| > m)\}$.

We will check that these sets are dense in $\mathbb{P}$:
The argument that D_α's are dense is identical to the one in the previous theorem, thus we will only show that E_m's are dense. Let $F \in \mathbb{P}$ and let $n \in \omega$ be such that $F \subseteq S_n$. As $S_n \in \mathbb{ED}_F$, there is $N \in \omega$ such that $|succ_{S_n}(\sigma)| > m$ for every $\sigma \in S_n$ longer than N. Let then $F' \subseteq S_n$, $F' \in \mathbb{P}$ be such an end-extension of F that for every $\sigma \in term(F')$ we have $|\sigma| > \max\{m, N\}$ and $|succ_{F'}(\sigma|_{|\sigma|-1})| > m$. Finding such F' is possible because of the choice of N. Then $F' \in E_m$.
As $\kappa < \mathrm{cov}(\mathcal{M}) = \mathfrak{m}_{ctbl}$ there is a filter $G \subseteq \mathbb{P}$ that intersects all sets D_α, $\alpha < \kappa$ and all sets E_m, $m \in \omega$. Define T as $\bigcup_{F \in G} F$. As $G \cap D_\alpha \neq \emptyset$, we have that $[T] \cap [T_\alpha] = \emptyset$ for every $\alpha < \kappa$. We will check that $T \in \mathbb{ED}_F$. This will show that the antichain $\{T_\alpha : \alpha < \kappa\}$ is not maximal which will finish the proof. To see this, notice that if $F \in G \cap E_m$, then, by the definition of $\mathbb{P}$, for every $\sigma \in T$ with $|\sigma| > m$ we have that $|succ_T(\sigma)| \geq |succ_T(\tau)| > m$ where $\tau \in term(F)$ is such that $\tau \subseteq \sigma$. □

Theorem 3.3.7. $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J}))\} \leq \mathfrak{a}(\mathbb{M}_\mathcal{J})$ *for any ideal* $\mathcal{J}$.

Proof. Let $\kappa < \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_\mathcal{B}(\mathbb{M}_\mathcal{J})\}$ and let $\{T_\alpha : \alpha < \kappa\} \subseteq \mathbb{M}_\mathcal{J}$ be an antichain. By the Proposition 3.2.9 we have that $\kappa < \mathrm{is}_h(\mathbb{M}_\mathcal{J})$. It follows there is a collection $\{S_\alpha : \alpha < \kappa\} \subseteq \mathbb{M}_\mathcal{J}$ such that for every $\alpha, \beta < \kappa$ we have:

– $S_\alpha \leq T_\alpha$,

– $[S_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$.

By the Lemma 3.3.4 without loss of generality we may assume that

$$\forall n \in \omega \forall F \in [S_n]^{<\omega} \exists^\infty m \in \omega F \subseteq S_m \tag{3.3}$$

By induction, we build an increasing sequence $\{F_n : n \in \omega\}$ of well-founded subtrees of $\omega^{<\omega}$ such that for every $n \in \omega$ the following condition hold:

$(1)_n$ for every $\sigma \in term(F_n)$, $succ_{F_{n+1}}(\sigma) \in \mathcal{J}^+$,

$(2)_n$ for every $\sigma \in term(F_n)$, $\sigma \notin \bigcup_{i<n} T_i$ and $\sigma \in S_N$ for some $N \in \omega$,

$(3)_n$ for every $\sigma \in term(F_n)$ and every $i \in succ_{F_{n+1}}(\sigma)$
there is unique $\tau(\sigma, i) \in term(F_{n+1})$ such that $\sigma \subseteq \tau(\sigma, i)$,

$(4)_n$ for every $\sigma \in term(F_n)$ and every $\alpha \in \kappa \setminus \omega$ the set $\{i \in succ_{F_{n+1}}(\sigma) : \tau(\sigma, i) \in T_\alpha\}$ is finite.

Start the induction by setting F_0 to be the tree generated by the stem of S_0. Assume now that F_n has been constructed. Fix $\sigma \in term(F_n)$ and let $N_\sigma \in \omega$ be such that $\sigma \in S_{N_\sigma}$. For every $i \in succ_{S_{N_\sigma}}(\sigma)$ pick any point $x_i^\sigma \in [S_{N_\sigma}]$, $\sigma^\frown i \subseteq x_i^\sigma$. For every $\alpha \in \kappa \setminus \omega$ define a function $f_\alpha : succ_{S_{N_\sigma}}(\sigma) \to \omega$ as follows: for $i \in succ_{S_{N_\sigma}}(\sigma)$, as $x_i^\sigma \notin [T_\alpha]$, let $f_\alpha(i) \in \omega$ be such that $x_i^\sigma|_{f_\alpha(i)} \notin T_\alpha$. As $\kappa < \mathfrak{b}$, there is a function $f : succ_{S_{N_\sigma}}(\sigma) \to \omega$ that almost dominates all f_α's. By the condition (3.3), for each $i \in succ_{S_{N_\sigma}}(\sigma)$ there is $N' > n$ such that $x_i^\sigma|_{f(i)} \in S_{N'}$. For every such i, extend $x_i^\sigma|_{f(i)}$ to a $\tau(\sigma, i) \in S_{N'} \setminus \bigcup_{i<n+1} T_i$ such that $succ_{S_{n'}}(\tau(\sigma, i)) \in \mathcal{J}^+$. Unfix σ and define F_{n+1} to be the tree generated by all such $\tau(\sigma, i)$'s where $\sigma \in term(F_n)$ and $i \in succ_{S_{N_\sigma}}(\sigma)$.
When the induction is over, let $T = \bigcup_n F_n$. By the conditions $(1)_n$, $n \in \omega$, we have that $T \in \mathbb{M}_\mathcal{J}$. For every $\alpha \in \kappa \setminus \omega$, by the conditions $(4)_n$, $n \in \omega$, we have that $T \cap T_\alpha$ is a finitely splitting tree on $\omega^{<\omega}$. Thus T is incompatible with every such T_α. For every $N \in \omega$, by the conditions $(2)_n$, $n \in \omega$, we have that $[T] \cap [T_n] = \emptyset$. It follows that the antichain $\{T_\alpha : \alpha \in \kappa\}$ is not maximal. □

Corollary 3.3.8. $\mathrm{add}(\mathcal{M}) \leq \mathfrak{a}(\mathbb{M}_{\mathcal{J}})$ *for any ideal* $\mathcal{J}$.

Proof. By the Proposition 2.2.32 we have that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M}_{\mathcal{J}}))$. By the previous theorem we have $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} \leq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{I}_{\mathcal{B}}(\mathbb{M}_{\mathcal{J}}))\} \leq \mathfrak{a}(\mathbb{M}_{\mathcal{J}})$ □

Regarding the σ-ideals generated by closed subsets of a Polish space X we have the following estimate for the antichain number. For this result we will use the notion of Hurewicz schema from the section 2.2.7.

Theorem 3.3.9. $\mathfrak{p} \leq \mathfrak{a}(\mathbb{P}_{\mathcal{E}})$.

Proof. Let $\kappa < \mathfrak{p}$ and let $\{G_\alpha : \alpha < \kappa\}$ be an uncountable antichain in $\mathbb{P}_{\mathcal{E}}$. By the Solecki's dichotomy 2.2.49, there is $\{\mathcal{U}_\alpha : \alpha < \kappa\}$, $\mathcal{U}_\alpha = \{U^\alpha_\sigma, N^\alpha_\sigma : \sigma \in \omega^{<\omega}\}$, a collection of Hurewicz schemas for the ideal $\mathcal{E}$ such that $[\mathcal{U}_\alpha] \subseteq G_\alpha$ for every $\alpha < \kappa$. By the Lemma 3.3.4 we may assume that:

$$\forall n \in \omega\ \forall F \in [\mathcal{U}_n]^{<\omega},\ \exists^\infty m \in \omega\ F \subseteq \mathcal{U}_m \tag{3.4}$$

We will construct an $\mathcal{E}$-positive set which will be incompatible with every G_α. Recall that $\mathfrak{p}$ is the Martins number of σ-centered forcings. Define an order $\mathbb{Q}$ consisting of conditions $p = (F^p, H^p, U^p, \delta^p)$ such that

$1_{\mathbb{Q}}$: F^p is a finite subtree of $\omega^{<\omega}$,

$2_{\mathbb{Q}}$: H^p is a finite subset of $\kappa \setminus \omega$,

$3_{\mathbb{Q}}$: $U^p : F^p \to \mathbb{B}$ such that $\sigma \subset \tau$ implies $\overline{U^p(\tau)} \subseteq U^p(\sigma)$,

$4_{\mathbb{Q}}$: $\delta^p : F^p \to \omega \times \omega^{<\omega}$ is such that if $\delta^p(\sigma) = (n, \tau)$, then for each $i \in succ_{F^p}(\sigma)$ we have $U^p(\sigma^\frown i) \in \mathcal{U}_n$, $U^p(\sigma^\frown i) \subseteq U^n_\tau$, $dist(U^p(\sigma^\frown i), N^n_\tau) < \frac{1}{|\sigma|+i}$ and $n \geq |\sigma|$.

The order $p \leq q$ is defined as

$1_{\leq}$: $F^q \subseteq F^p$, $H^q \subseteq H^p$, $U^q \subseteq U^p$ and $\delta^q \subseteq \delta^p$,

$2_{\leq}$: $\forall \sigma \in F^p \setminus F^q\ \forall \alpha \in H^q\ U^p(\sigma) \cap G_\alpha = \emptyset$

$3_{\leq}$: $\forall \sigma \in F^p \setminus F^q\ \forall i < |\sigma|\ U^p(\sigma) \cap G_i = \emptyset$.

Note at this point that the order $(\mathbb{Q}, \leq)$ is σ-centered: if $p, q \in \mathbb{Q}$ are such that $F^p = F^q$, $U^p = U^q$, $\delta^p = \delta^q$ and there are countably many such possibilities, then the condition r defined as $(F^p, H^p \cup H^q, U^p, \delta^p)$ is stronger then both p and q. Define now the following sets: for $\alpha \in \kappa \setminus \omega$, $\sigma \in \omega^{<\omega}$ let

$$A_\alpha = \{p \in \mathbb{Q} : \alpha \in H^p\}$$
$$B_\sigma = \{p \in \mathbb{Q} : \sigma \in F^p\}$$

and for $m \in \omega$, $\sigma \in \omega^{<\omega}$ let

$$\mathrm{C}_{\sigma,n} = \{p \in B_\sigma : \text{if } \delta^p(\sigma) = (k, \tau) \text{ then for every } x \in N^k_\tau \text{ there is } i \in succ_{F_p}(\sigma) \text{ with } dist(U^p(\sigma^\frown i), x) < \tfrac{1}{m}\}.$$

Claim 3.3.10. *The sets* A_α, B_σ, $C_{\sigma,n}$ *are dense in* $\mathbb{Q}$.

proof of the claim. Clearly, the A_α's are dense. To see that B_σ is dense let $q \in \mathbb{Q}$. Let σ' be the longest initial part of σ which belongs to F^q and assume that $\delta^q(\sigma') = (n, \tau)$. Use the condition (3.4) to find $m > |\sigma|$ such that $U^q(\sigma') \in \mathcal{U}_m$. Find an open set W such that

– $W \in \mathcal{U}_m$, $W \subseteq U^q(\sigma')$,

– $W \cap U^q(\sigma'^\frown i) = \emptyset$ for each $i \in succ_{F^q}(\sigma')$,

– $dist(W, N^n_\tau) < \frac{1}{|\sigma'|+\sigma(|\sigma'|)}$,

– $W \cap G_\alpha = \emptyset$ for each $\alpha \in H^q$,

– $W \cap G_i = \emptyset$ for each $i < |\sigma|$.

Finding a W satisfying the last two conditions is possible by the Proposition 2.2.53. Choose arbitrary sequence $\{\rho_j : j = 1, 2, ..., |\sigma| - |\sigma'|\} \subseteq \omega^{<\omega}$ such that $W = U^m_{\rho_1}$ and $\rho_j \subsetneq \rho_{j+1}$. Define the condition $q \in \mathbb{Q}$ as follows: $H^p = H^q$, $U^p|_{F^q} = U^q$, $\delta^p|_{F^q} = \delta^q$, $F^p = F^q \cup \{\sigma|_i : i \in |\sigma| \setminus |\sigma'|\}$ and let $U^p(\sigma|_{|\sigma'|+i}) = U^m_{\rho_i}$ for $i = 1, ..., |\sigma| \setminus |\sigma'|$, and for $i = 1, ..., |\sigma| \setminus |\sigma'|$ let $\delta^p(\sigma|_{|\sigma'|+i}) = (m, \tau)$ is such that $W_i = U^m_\tau$. Note that $p \in \mathbb{Q}$, $p \leq q$ and $p \in B_\sigma$.
To see that $C_{\sigma,m}$ is dense let $q \in B_\sigma$. Assume that $\delta^q(\sigma) = (n, \tau)$ and $j^* = max\ succ_{F^q}(\sigma)$. Because the set N^n_τ is compact there is a finite set of points $\{x_i : i < k\} \subseteq N^n_\tau$ such that $N^n_\tau \subseteq \bigcup_{i<k} B(x_i, \frac{1}{m})$. Because $N^n_\tau = lim(succ_{\mathcal{U}_n}(U^n_\tau))$ holds, by lemma 1 there is a collection of open sets $\{W_i : i < k\}$ such that for each $i < k$ the following conditions hold:

– $W_i \in \mathcal{U}_n$, $W_i \subseteq U^n_\tau$,

– $W_i \cap U^q(\sigma^\frown j) = \emptyset$ for each $j \in succ_{F^q}(\sigma)$,

– $W_i \subseteq B(x_i, \frac{1}{m})$,

– $dist(W_i, N^n_\tau) < \frac{1}{|\sigma|+j^*+i+1}$,

– $W_i \cap G_\alpha = \emptyset$ for each $\alpha \in H^q$,

– $W_i \cap G_j = \emptyset$ for each $j \leq |\sigma|$.

Define then the condition $q \leq p$ as follows: $H^q = H^p$, $F^p = F^q \cup \{\sigma^\frown l : l = j^* + i + 1, i < k\}$, $U^p|_{F^q} = U^q$, $\delta^p|_{F^q} = \delta^q$, $U^p(\sigma^\frown l) = W_i$ where $l = j^* + i + 1$ and $\delta^p(\sigma^\frown l) = (n, \tau)$. Notice that $p \in C_{\sigma,m}$ and $p \leq q$. □

As $\kappa < \mathfrak{p}$ there exists a filter $G \subseteq \mathbb{Q}$ intersecting all the sets A_α, B_σ, $C_{\sigma,n}$. Define $\mathcal{U}^{gen} = \{U^{gen}(\sigma) : \sigma \in \omega^{<\omega}\} := \bigcup\{U^p : p \in G\}$ together with a function $\delta^{gen} : \omega^{<\omega} \to \omega \times \omega^{<\omega}$, $\delta^{gen} := \bigcup\{\delta^p : p \in G\}$. The proof of the theorem will be finished with the following two claims

Claim 3.3.11. *$\mathcal{U}^{gen}$ is a Hurewicz schema.*

proof of the claim. Because the sets B_σ are dense, the functions U^{gen} and δ^{gen} are defined on the entire $\omega^{<\omega}$. The remaining part is to show that for each $\sigma \in \omega^{<\omega}$ the set N^{gen}_σ defined as $lim\{U^{gen}_{\sigma^\frown i} : i \in \omega\}$ is closed, nowhere-dense and $\mathcal{E}$-positive. We will prove this by showing that $N^{gen}_\sigma = N^n_\tau$ where $\delta^{gen}(\sigma) = (n, \tau)$. We will consider two cases:

Case 1 - there is $z \in N^{gen}_\sigma \setminus N^n_\tau$
Let $k \in \omega$ be such that $[z|_k] \cap N^n_\tau = \emptyset$ and choose a sequence $\{i_l : l \in \omega\}$ of natural numbers such that for each $l \in \omega$ we have $U^{gen}_{\sigma^\frown i_l} \subseteq [z|_k]$. For each $l \in \omega$ choose also $p_l \in G$ such that $\sigma^\frown i_l \in F^{p_l}$. Note that in particular there is $\epsilon > 0$ such that $dist(U^{p_l}(\sigma^\frown i_l), N^n_\tau) > \epsilon$ for each $l \in \omega$. This is a contradiction as by $(4_\mathbb{Q})$ we have that $dist(U^{p_l(\sigma^\frown i_l)}, N^n_\tau) < \frac{1}{|\sigma|+i_l}$.

Case 2 - there is $z \in N^n_\tau \setminus N^{gen}_\sigma$:
Le $k \in \omega$ be such that $[z|_k] \cap N^{gen}_\sigma = \emptyset$. For each $l \in \omega$ choose $p_l \in G \cap C_{\sigma,l}$. Then for each $l \in \omega$ there is $i_l \in succ_{F^{p_l}}(\sigma)$ such that $dist(U^{p_l}(\sigma^\frown i_l), z) < \frac{1}{l}$. This means that for almost all $l \in \omega$ we have $U^{p_l}(\sigma \cap i_l) \subseteq [z|_k]$. As $U^{p_l}(\sigma^\frown i_l) = U^{gen}(\sigma^\frown i_l)$ we reach a contradiction. □

Claim 3.3.12. *$[\mathcal{U}^{gen}] \cap G_\alpha = \emptyset$ for each $\alpha < \kappa$.*

proof the claim. To see that $[\mathcal{U}^{gen}]\cap G_n = \emptyset$ assume that for some $n \in \omega$ we have $x \in [\mathcal{U}^{gen}]\cap G_n$. Then there is $y \in \omega^\omega$ such that $\{x\} = \bigcap_m U^{gen}_{y|_m}$. If we pick $p, q \in G$ such that $y|_{n+1} \in F^p \setminus F^q$, then by $(3_\leq)$ we have that $U^p_{y|_{n+1}} \cap G_n = \emptyset$ which contradicts $U^p_{y|_{n+1}} = U^{gen}_{y|_{n+1}}$. To see that for $\alpha \in \kappa \setminus \omega$ we have $[\mathcal{U}^{gen}] \cap G_\alpha = \emptyset$ assume that $x \in [\mathcal{U}^{gen}] \cap G_\alpha$. Let $p \in G \cap A_\alpha$ and let $\sigma \in \omega^{<\omega} \setminus F^p$ be such that $x \in U^{gen}_\sigma$. Then, by the condition $2_\leq$ we have that for every $q \leq p$, $q \in G \cap B_\sigma$ we have that $U^{gen}_\sigma = U^q_\sigma$ is disjoint from G_α. A contradiction. □

We finish this section with the following questions.

Question 3.3.13. *Is* $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{E})\} \leq \mathfrak{a}(\mathcal{E})$ *provable in ZFC?*

The case of reasonable lower bound for Laver tree (which has been asked in [HS20]) remains open

Question 3.3.14. *Does Martins Axiom (or Proper Forcing Axiom) implies* $\mathfrak{a}(\mathbb{L}) = \mathfrak{c}$*?*

3.4 Additivities and projections of trees

The following definition will be crucial for finding upper bounds of additivities of tree ideals

Definition 3.4.1. *Suppose that* $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y}\rangle$ *is a relation system and* $\mathbb{T}$ *is a combinatorial tree type. A function* $\pi : \mathbb{T} \to \mathcal{Y}$ *will be called a* $\mathfrak{R}$*-projection if*

- *if* $S \leq T$ *and* $x \sqsubseteq \pi(T)$ *then* $x \sqsubseteq \pi(S)$
- *for every* $x \in \mathcal{X}$ *the set* $\{T \in \mathbb{T} : x \sqsubseteq \pi(T)\}$ *is dense in* $\mathbb{T}$

Before the main theorem of this section, we need to prove the following simple lemma.

Lemma 3.4.2. *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ *and that* $\mathcal{A}$ *is a maximal antichain in* $\mathbb{T}$*. If* $[T] \subseteq \bigcup_{S\in\mathcal{A}}[S]$ *where* $|\mathcal{A}| < \mathfrak{c}$*, then there is* $\sigma \in T$ *and* $S \in \mathcal{A}$ *such that* $[T|_\sigma] \subseteq [S]$*.*

Proof. Note that the sets $[T]\cap[S]$, $S \in \mathcal{A}$ are closed and cover $[T]$. As $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$, one of these sets is not relatively nowhere dense on $[T]$. It follows that there is $\sigma \in T$ and $S \in \mathcal{A}$ such that $[T|_\sigma] \subseteq [S]$. □

However, to show this we need the following technical lemma from [GRSS95]. The lemma in the original form was formulation for Sacks, Miller and Laver trees but the proof can be rewritten completely. Because of this we only include sketch of the proof of the general case.

Lemma 3.4.3. *Let* $\mathbb{T}$ *be a combinatorial tree type with continuous reading of names. Assume that* $\mathrm{is}_h(\mathbb{T}) = \mathfrak{c}$*. Then any open dense* $D \subseteq \mathbb{T}$ *contains a disjoint maximal antichain* $\mathcal{A} \subseteq D$ *such that for every* $T \in \mathbb{T}$ *we have*

$$\text{if } [T] \subseteq \bigcup_{S\in\mathcal{A}}[S], \text{ then } \exists \mathcal{B} \in [\mathcal{A}]^{<\mathfrak{c}}, [T] \subseteq \bigcup_{S\in\mathcal{B}}[S]$$

Proof. Assume that $\mathbb{T}$ lives on ΠF. Enumerate $\mathbb{T}$ as $\{T_\gamma : \gamma < \mathfrak{c}\}$. We will inductively build two sequences, $\{S_\gamma : \gamma < \mathfrak{c}\} \subseteq \mathbb{T}$ and $\{r_\gamma : \gamma < \mathfrak{c}\} \subseteq \Pi F$ (assuming that $\mathbb{T}$ lives on ΠF). We will allow S_γ or r_γ to be undefined. So, at the end of the proof, whenever we refer to S_γ or r_γ we mean that the referred object is defined. Assuming that $\{S_\gamma : \gamma < \beta\}$ and $\{r_\gamma : \gamma < \beta\}$ has been constructed, we proceed as follows. First, we construct r_β. If $[T_\beta] \setminus \bigcup_{\gamma<\beta}[S]$ is non-empty, let $r_\beta \in [T_\beta] \setminus \bigcup_{\gamma<\beta}[S]$. If not, let r_β be undefined. Next we construct S_β. If T_β is compatible with some S_γ for $\gamma < \beta$, let S_β be undefined. Otherwise, for every $\gamma < \beta$, the intersection $[T_\beta] \cap [S_\gamma]$ does not contain branches of a $\mathbb{T}$-tree. By the Proposition 2.3.4, $[T_\beta] \cap [S_\gamma] \in \mathcal{I}_\mathcal{B}(\mathbb{T})$ for every $\gamma < \beta$. As $\mathrm{is}_h(\mathbb{T}) = \mathfrak{c}$ and $\mathrm{non}(\mathcal{I}(\mathbb{T})) = \mathfrak{c}$, there is $S \in \mathbb{T}$, $S \leq T$ such that

$$[S] \cap \{r_\gamma : \gamma \leq \beta\} = \emptyset \text{ and } [S] \cap [S_\gamma] = \emptyset \text{ for every } \gamma < \beta.$$

Let then $S_\beta \in \mathbb{T}$ be such that $S_\beta \leq S$ and $S_\beta \in D$. This finishes the induction.
Let $\mathcal{A} = \{S_\gamma : \gamma < \mathfrak{c}\}$. Clearly, by the construction, $\mathcal{A}$ forms an antichain. As T_γ's enumerated all $\mathbb{T}$-trees, the antichain $\mathcal{A}$ is maximal. We will now show that, if $[T] \subseteq \bigcup_{S\in\mathcal{A}}[S]$, then $\exists \mathcal{B} \in [\mathcal{A}]^{<\mathfrak{c}}, [T] \subseteq \bigcup_{S\in\mathcal{B}}[S]$. Assume that $T \in \mathbb{T}$ is such that $[T] \subseteq \bigcup_{S\in\mathcal{A}}[S]$. Let $\beta < \mathfrak{c}$ be such that $T_\beta = T$. We claim that $[T] \subseteq \bigcup_{\gamma\leq\beta}[S_\gamma]$. If not, then $r_\beta \in [T_\beta] \setminus \bigcup_{\gamma<\beta}[S_\gamma]$ has been defined. On the other hand, by the construction of S_γ's, $\gamma > \beta$, we have $r_\beta \notin \bigcup_{\gamma\geq\beta}[S_\gamma]$. It follows that $[T] = [T_\beta]$ is not contained in $\bigcup_{\gamma<\mathfrak{c}}[S_\gamma]$. □

We will now prove the main result of this section. For this result we will need the notion of ω-Tukey reduction discussed in section 1.2.5.

Theorem 3.4.4. *Suppose that $\mathbb{T}$ is a combinatorial tree forcing with continuous reading of names, $\mathfrak{R} = \langle \mathcal{X}, \sqsubseteq, \mathcal{Y}\rangle$ is a relational system and assume that $\mathbb{T}$ has an $\mathfrak{R}$-projection, $\pi : \mathbb{T} \to \mathcal{Y}$. Assume that $\mathrm{is}_h(\mathbb{T}) = \mathfrak{c}$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. Then we have*

$$\mathfrak{R} \leq^\omega_T \langle \mathcal{I}(\mathbb{T}), \subseteq, \mathcal{I}(\mathbb{T})\rangle$$

In particular, the inequality $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \mathfrak{b}_\sigma(\mathfrak{R})$ holds.

Proof. Assume that $\mathbb{T}$ lives on ΠF. First, we will define $\phi : \mathcal{X} \to \mathcal{I}(\mathbb{T})$. Let $X \in \mathcal{X}$. Use the Lemma 3.4.3 to find a maximal antichain $\mathcal{A}_X$ in the open dense set $D_X = \{T \in \mathbb{T} : X \sqsubseteq \pi(T)\}$ and define $\phi(X) = \mathcal{X}(\mathcal{A}_X)$. By the Proposition 2.1.3, we have that $\phi(X) \in \mathcal{I}(\mathbb{T})$. We will now define $\{\psi_\sigma : \mathcal{I}(\mathbb{T}) \to \mathcal{Y}, \sigma \in Seq(F)\}$. Let $Z \in \mathcal{I}(\mathbb{T})$ and let $\sigma \in Seq(F)\}$. Pick a tree $T^Z \in \mathbb{T}$ such that $[T^Z] \cap Z = \emptyset$ and define $\psi_\sigma(Z) = \pi(T^Z|_\sigma) \in \mathcal{Y}$ if $\sigma \in T^Z$. If $\sigma \notin T^Z$ define the value $\psi_\sigma(Z)$ to be arbitrary element of $\mathcal{Y}$. To show that these define the required Tukey reduction we will show that for $X \in \mathcal{X}$ and $Z \in \mathcal{I}(\mathbb{T})$, if $\phi(X) \subseteq Z$, then $X \sqsubseteq \psi_\sigma(Z)$ for some $\sigma \in Seq(F)$. If $\phi(X) \subseteq Z$ then $[T^Z] \subseteq \bigcup_{T\in\mathcal{A}_X}[T]$. By the Lemma 3.4.3 there is $\mathcal{B}_X \subseteq \mathcal{A}_X$, $|\mathcal{B}_X| < \mathfrak{c}$ such that $[T^Z] \subseteq \bigcup_{T\in\mathcal{B}_X}[T]$. Then, by the Lemma 3.4.2 there is $\sigma \in T^Z$ such that $[T^Z|_\sigma] \subseteq [T]$ for some $T \in \mathcal{B}_X$. On the one hand $X \sqsubseteq \pi(T)$ (because of $T \in \mathcal{A}_X \subseteq D_X$) and on the other $T^Z|_\sigma \leq T$. By the properties of projection we conclude that $X \sqsubseteq \pi(T^Z|_\sigma) = \psi_\sigma(Z)$. This finishes the proof. □

In the rest of this section we will discuss which of the tree types we are interested in have which types of projections. We will need the following simple lemma.

Lemma 3.4.5. *If $S \in \mathbb{S}_F$ and $color : split(S) \to \omega$, then there is $R \leq S$, $R \in \mathbb{S}_F$ such that $color|_{split(R)}$ is either constant or increasing along R.*

Proof. Consider the following two cases:
• Case 1: for every $\sigma \in S$ the set $split(S_\sigma)$ is colored with infinitely many colors
Inductively construct an increasing sequence of finite subtrees of S such that $F_0 = \{\emptyset\}$ and for each $n \in \omega$ we have:

1) F_{n+1} is an end-extension of F_n,

2) $term(F_n) \subseteq split(S)$,

3) for every $\sigma \in term(F_n)$ we have $succ_{F_{n+1}}(\sigma) = succ_S(\sigma)$,

4) for all $\sigma \in term(F_n)$ and $\tau \in term(F_{n+1})$, if $\sigma \subseteq \tau$, then $color(\sigma) < color(\tau)$.

Assuming that F_n has been built, for each $\sigma \in term(F_n)$ and $i \in succ_S(\sigma)$ use the assumption to find $\tau_{\sigma,i} \in split(S_{\sigma^\frown i})$ such that $color(\sigma) < color(\tau_{\sigma,i})$. Then let F_{n+1} to be the tree generated by all $\tau_{\sigma,i}$'s for $\sigma \in term(F_n)$ and $i \in succ_S(\sigma)$. When the construction is over, let $R = \bigcup_n F_n$. Because of 3) the tree R is a $\mathbb{S}_F$-tree and because of 4) the color is increasing along R.
• Case 2: there is $\sigma \in S$ such that $split(S_\sigma)$ is colored with finitely many colors
It will be enough to show that if $split(S_\sigma)$ is colored with only two colors, then the monochromatic

subtree exists. Assume for simplicity that $rng(color|_{split(S_\sigma)}) = \{0,1\}$. Consider two subcases: first, that there is $\sigma \subseteq \sigma'$ such that $split(S_\sigma)$ is colored with 0 only. Then we are done. Second, that every $\sigma' \in S_\sigma$ can be extended to $\sigma'' \in split(S)$ of color 1. In such a (sub)case continue with a construction of F_n's as in the first case but choose all $\tau_{\sigma,i}$'s to be of color 1. Then $R = \bigcup_n F_n$ is be monochromatic of color 1. □

Regarding the projections of compact tree types we have the following.

Theorem 3.4.6. *The following holds:*

- $\mathbb{S}_F$ *and* $\mathbb{IE}_F$ *have the* $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$*-projection,*
- $\mathbb{S}_F$ *and* $\mathbb{IE}_F$ *have the* $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$*-projection,*

Proof. • $\mathbb{S}_F$ and $\mathbb{IE}_F$ have the $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-projection
Recall that for $T \in \mathbb{IE}_F$ and $\sigma \in T$ the $N_T(\sigma) > |\sigma|$ is the smallest such that for all $i \in F(N_T(\sigma))$ there is $\tau \in T$, $\sigma \subseteq \tau$, $|\tau| > N_T(\sigma)$ with $\tau(N_T(\sigma)) = i$. We overload the symbol $N_F(\sigma)$ so it works (if such value exists) also over finite trees $F \subseteq Seq(F)$. Define $\pi : \mathbb{IE}_F \to \omega^\omega$ such that for $T \in \mathbb{IE}_F$ the $\pi(T)$ is defined inductively as follows: let $\pi(T)(0) = N_T(\emptyset)$ and $\pi(T)(n+1) = \max\{N_T(\sigma) : \sigma \in T \text{ and } |\sigma| = \pi(T)(n)\} + 1$. We will show that π is as required. First, we claim that for every $T \in \mathbb{IE}_F$ and every $f \in \omega^\omega$ there is $S \leq T$ such that $f \leq \pi(S)$. To see this we inductively build a sequence $\{F_n : n \in \omega\}$ of finite subtrees of T such that for every $n \in \omega$ the following two conditions hold:

$(a)_n$ for every $\sigma \in term(F_n)$ the value $N_{F_{n+1}}(\sigma)$ is defined,

$(b)_n$ for every $\sigma \in term(F_n)$ the value $N_{F_{n+1}}(\sigma) > f(n)$.

Let $F_0 = \{\emptyset\}$ and assume that F_n has been constructed. For every $\sigma \in term(F_n)$ pick $\tau_\sigma \in T$, $\sigma \subseteq \tau_\sigma$ such that $|\tau_\sigma| > f(n)$. Let then H_n be the tree generated by all such τ_σ's. Let then F_{n+1} be the tree generated by the set $\{\rho \in T : \exists \tau \in term(H_n),\ \tau \subseteq \rho,\ |\rho| = N_T(\sigma) + 1\}$. Clearly, the conditions $(a)_n$ and $(b)_n$ are satisfied. When the induction is over, we let S to be $\bigcup_{n\in\omega} F_n$. Because of the condition $(a)_n$, $n \in \omega$ we have that $T \in \mathbb{IE}_F$ and because of the conditions $(b)_n$, $n \in \omega$ we get that $f \leq \pi(S)$. Second, we claim that $S \leq T$ implies $\pi(T) \leq \pi(S)$. Clearly $\pi(T)(0) \leq \pi(S)(0)$. Inductively we show that, if $\pi(T)(n) \leq \pi(S)(n)$, then by $S \leq T$, we have $\pi(T)(n+1) \leq \pi(S)(n+1)$.
• $\mathbb{S}_F$ has the $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-projection
For $T \in \mathbb{S}_F$ let $\pi(T)$ be the infinite set $\{|\sigma| : succ_T(\sigma) = F(|\sigma|)\}$. Clearly, $S \leq T$ implies that $\pi(S) \subseteq \pi(T)$. We claim that for every $T \in \mathbb{S}_F$ and every $A \in [\omega]^\omega$ there is $S \leq T$ such that $\pi(S) \subseteq A$ or $\pi(S) \cap A = \emptyset$. Fix $A \in [\omega]^\omega$ and $T \in \mathbb{S}_F$. For $\sigma \in T$ such that $succ_T(\sigma) = F(|\sigma|)$ define

$$color(\sigma) = \begin{cases} 0 & \text{if } |\sigma| \in A, \\ 1 & \text{if } |\sigma| \notin A. \end{cases}$$

Now, by the Lemma 3.4.5 there is a monochromatic tree. This finishes the argument. □

With respect to the projection of the splitting tree, the following is known.

Theorem 3.4.7. *(Hein, Spinas, [HS20]) The splitting tree* $\mathbb{SP}$ *has* $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$*-projection.*

Theorem 3.4.8. $\mathbb{ED}_F$ *has the* $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$*-projection.*

Proof. Define $\pi : \mathbb{ED}_F \to \omega^\omega$ in a way that for $T \in \mathbb{ED}_F$ we have:

$$\pi(T)(n) = \min\{N \in \omega : \forall \sigma \in T(|\sigma| > N \to succ_T(\sigma) > n)\}$$

By the definition of T, the set over the minimum is taken is non-empty, thus π is well-defined. To show that $S \leq T$ implies $\pi(T) \leq \pi(S)$, notice that for $n \in \omega$ we have $\{N \in \omega : \forall \sigma \in S(|\sigma| > N \to succ_S(\sigma) > n)\} \subseteq \{N \in \omega : \forall \sigma \in T(|\sigma| > N \to succ_T(\sigma) > n)\}$. It follows that $\pi(T)(n) \leq \pi(S)(n)$.
We will now show that for $T \in \mathbb{ED}_F$ and $f \in \omega^\omega$ there is $S \leq T$ with $f \leq^* \pi(S)$. We will inductively construct $\{F_n : n \in \omega\}$ and $\{N_n : n \in \omega\}$ such that for every $n \in \omega$ we have:

$(1)_n$ F_n is finite subtree of T and $N_n \in \omega$,

$(2)_n$ F_{n+1} is an end-extension of F_n,

$(3)_n$ $f(n) < N_n < N_{n+1}$,

$(4)_n$ for every $\sigma \in term(F_n)$ we have $|\sigma| = N_{n+1}$,

$(5)_n$ for every $\sigma \in F_n$, if $|\sigma| \in [N_n, N_{n+1})$, then $|succ_{F_{n+1}}(\sigma)| = n + 1$,

$(6)_n$ N_n is such that every $\sigma \in T$ splits into at least $n + 1$ successors whenever $|\sigma| > N_n$.

Start the induction with $N_0 = 0$ and $F_{-1} = \{\emptyset\}$. Assume now that for some $n \in \omega$ the N_n and F_{n-1} has been defined. Let N_{n+1} be such that $(3)_n$ and $(6)_n$ is satisfied. Then let F_n be a tree which satisfies $(1)_n$, $(2)_n$, $(4)_n$ and $(5)_n$.
When the induction is over, define $S = \bigcup_n F_n$. As for every $n \in \omega$ the conditions $(3)_n$ and $(5)_n$ hold, we have that $S \in \mathbb{ED}_F$. As for every $n \in \omega$ the conditions $(5)_n$ and $(6)_n$ hold, we have that $f(n) < \pi(S)(n)$ for each $n > 0$. □

We will now discuss the projections of the $\mathcal{J}$-Miller and $\mathcal{J}$-Laver trees. However, first we will need the following lemma

Lemma 3.4.9. *(Miller, [Mil84]) Suppose that $T \in \mathbb{M}_\mathcal{J}$ and $c : split(T) \to 2$ is a coloring of splitnodes of T into two colors. Then there is $S \leq T$ with all splitnodes of the same color, i.e. $c|_{split(S)}$ is constant.*

Now, we have the following.

Theorem 3.4.10. *The following holds:*

- *$\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$ have the $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-projection for every $\mathcal{J}$,*
- *$\mathbb{M}_\mathcal{J}$ has the $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-projection no matter the ideal $\mathcal{J}$,*

Proof. • $\mathbb{M}_\mathcal{J}$ has the $\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$-projection
For $T \in \mathbb{M}_\mathcal{J}$ let $\pi(T)$ be the infinite set $\{|\sigma| : succ_T(\sigma) \in \mathcal{J}^+\}$. Clearly, $S \leq T$ implies that $\pi(S) \subseteq \pi(T)$. We claim that for every $T \in \mathbb{M}_\mathcal{J}$ and every $A \in [\omega]^\omega$ there is $S \leq T$ such that $\pi(S) \subseteq A$ or $\pi(S) \cap A = \emptyset$. Fix $A \in [\omega]^\omega$ and $T \in \mathbb{M}_\mathcal{J}$. For $\sigma \in T$ such that $succ_T(\sigma) \in \mathcal{J}^+$ define

$$color(\sigma) = \begin{cases} 0 & \text{if } |\sigma| \in A, \\ 1 & \text{if } |\sigma| \notin A. \end{cases}$$

Now, by the Lemma 3.4.9 there is a monochromatic tree. This finishes the argument.
• $\mathbb{M}_\mathcal{J}$ has the $\langle \omega^\omega, \leq^*, \omega^\omega \rangle$-projection no matter the ideal $\mathcal{J}$
For $T \in \mathbb{M}_\mathcal{J}$ let $\pi(T)$ be the increasing enumeration of the set $\{|\sigma| : \sigma \in split(T)\}$. Clearly, $S \leq T$ implies that $\{|\sigma| : \sigma \in split(S)\} \subseteq \{|\sigma| : \sigma \in split(T)\}$ which implies that $\pi(T) \leq \pi(S)$. We will show the second condition. Let $\{s_n : n \in \omega\}$ be an ordering of $\omega^{<\omega}$ in the type ω such that $s_n \subseteq s_m$ implies $n \leq m$. We will inductively build $\{\sigma_n : n \in \omega\} \subseteq split(T)$ such that

- the tree generated by $\{\sigma_n : n \in \omega\}$, say S, is a $\mathbb{M}_\mathcal{J}$-tree and $split(S) = \{\sigma_n : n \in \omega\}$,

– $|\sigma_n| < |\sigma_{n+1}|$ for each $n \in \omega$,

– $f(n) < |\sigma_{n+1}|$ for each $n \in \omega$,

– the assignment $s_n \longmapsto \sigma_n$ is an order isomorphism between $(\omega^\omega, \subseteq)$ and $(\{\sigma_n : n \in \omega\}, \subseteq)$.

Clearly, $\pi(S)$ dominates f point-wisely. Let $\sigma_0 \in split(T)$ be such that $|\sigma_0| > f(0)$ and assume that σ_n has been constructed. To define σ_{n+1} let $m \in \omega$, $k \in \omega$ be such that $s_{n+1} = s_m^\frown k$. Let $\sigma_{n+1} \in split(T)$ be such that $\sigma_n \subseteq \sigma_{m+1}$, $|\sigma_n| < |\sigma_{n+1}|$, $f(n+1) < |\sigma_{n+1}|$ and $\sigma_{n+1}(|\sigma_m|)$ is the smallest element of the set $succ_T(\sigma_m) \setminus \{\sigma_i(|\sigma_m|) : \sigma_m \subseteq \sigma_i\}$. This finishes the induction step and thus the induction process. Notice that the last condition implies that $succ_T(\sigma_m) = succ_S(\sigma_m)$ for each $m \in \omega$, and thus $S \in \mathbb{M}_\mathcal{J}$. □

Question 3.4.11. *Does $\mathbb{L}_\mathcal{J}$ have $\langle [\omega]^\omega, reaps, [\omega]^\omega\rangle$-projection fo every ideal $\mathcal{J}$? Does $\mathbb{M}_{tr(\mathcal{N})}$, or $\mathbb{L}_{tr(\mathcal{N})}$, has a $\mathfrak{R}_\mathcal{N}$-projection?*

We will finish this section with mentioning a couple of results regarding projections of Silver trees. Recall that a tree $T \subseteq 2^{<\omega}$ is a Silver tree if there is a coinfinite $a \subseteq \omega$ and $x : a \to 2$ such that $T = \{\sigma \in 2^{<\omega} : \sigma|_a \subseteq b\}$. Denote by $\mathbb{S}i$ the collection of all Silver tres. It is well-known that $\mathbb{S}i$ type is a combinatorial tree forcing and has continuous reading of names. Consider the following relational system: $\langle [\omega]^\omega, \hookrightarrow_{sel}, macs\rangle$ where $macs = \{\mathcal{A} \subseteq \mathcal{P} : \mathcal{A}$ is a maximal almost disjoint family$\}$. The relation $A \hookrightarrow_{sel} \mathcal{A}$ means that the set $A \in [\omega]^\omega$ selects from $\mathcal{A}$ i.e. $A \subseteq^* B$ for some $B \in \mathcal{A}$. Clearly, $\mathfrak{b}([\omega]^\omega, \hookrightarrow_{sel}, macs) = \mathfrak{h}$. The following is a result of Otmar Spinas.

Theorem 3.4.12. *(Spinas) The following holds:*

– *$\mathbb{S}i$ has a $\langle \omega^\omega, \leq^*, \omega^\omega\rangle$-projection,*

– *$\mathbb{S}i$ has a $\langle \mathcal{M}, \subseteq, \mathcal{M}\rangle$-projection,*

– *$\mathbb{S}i$ has a $\langle [\omega]^\omega, \hookrightarrow_{sel}, macs\rangle$-projection.*

Proof. The first projection has been proven in [SW15], The second one in [Spi16]. The third one in [Spi22]. □

It is well known that Silver trees does not add Cohen reals because it has Sacks property. This motivates the question, whether there is such an example among the types of trees we are interested in? In particular, we have

Question 3.4.13. *Is there an ideal $\mathcal{J}$ such that $\mathbb{M}_\mathcal{J}$ or $\mathbb{L}_\mathcal{J}$ has $\langle \mathcal{M}, \subseteq, \mathcal{M}\rangle$-projection, but does not add Cohen reals?*

3.5 More on additivities

We will start with the lower estimate $\mathfrak{t} \leq \mathrm{add}(\mathcal{I}(\mathbb{L}_\mathcal{J}))$ for $_\sigma$-ideals. Suppose that $\mathcal{J}$ is an ideal on ω. Recall that for $A, B \subseteq \omega$ we will write $A \subseteq_\mathcal{J} B$ if $A \setminus B \in \mathcal{J}$. We will say that a sequence $\{T_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}^+$ is $\mathcal{J}$-decreasing, if for $\alpha < \beta < \kappa$ we have $T_\beta \subseteq_J T_\alpha$.

Definition 3.5.1. *Suppose that $\mathcal{J}$ is an ideal on ω. The $\mathcal{J}$-tower number $\mathfrak{t}(\mathcal{J})$ is defined as the smallest length of a $\mathcal{J}$-decreasing sequence $\{T_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}^+$ such that there is no $T \in \mathcal{J}^+$ such that $T \subseteq_\mathcal{J} T_\alpha$ for all $\alpha < \kappa$.*

Clearly, for the ideal of finite sets, the number $\mathfrak{t}_{\mathcal{F}in}$ is the standard tower number $\mathfrak{t} = \mathfrak{p}$. We will need the following well-known fact.

Proposition 3.5.2. *(folklore) If $\mathcal{J}$ is F_σ ideal then $\mathfrak{t}(\mathcal{J}) \geq \mathfrak{t}$.*

Proof. By the Mazurs Theorem 4.1.1, we find ϕ, a l.s.c.s.m. such that $\mathcal{J} = Fin(\phi)$. Let also $\kappa < \mathfrak{t} = \mathfrak{p}$ and let $\{T_\alpha : \alpha < \kappa\}$ be $\mathcal{J}$-decreasing sequence of $\mathcal{J}$-positive sets.
Define a partial order $\mathbb{Q}$ consisting of pairs (A, H) such that $A \in [\omega]^{<\omega}$ and $H \in [\kappa]^{<\omega}$. The order $(A, H) \leq (B, F)$ is given by $B \subseteq A$, $F \subseteq H$ and $A \setminus B \subseteq T_\alpha$ where $\alpha = \max F$. Clearly $\mathbb{Q}$ is σ-centered as witnessed by the first coordinate.
For $n \in \omega$ and $\alpha \in H_\alpha$ define the sets

$$D_n = \{(A, H) \in \mathbb{Q} : \phi(A) > n\},$$
$$E_\alpha = \{(A, H) \in \mathbb{Q} : \alpha \in H\}$$

We will check that D_n's and E_α's are dense. To see that D_n is dense let $(B, F) \in \mathbb{Q}$. Let $\alpha = \max F$. By the semi-continuity of ϕ there is $B' \subseteq T_\alpha \setminus B$ such that $\phi(B') > n$. But then $(B \cup B', F) \leq (B, F)$ and $(B \cup B', F) \in D_n$. To see that E_α is dense notice that for $(B, F) \in \mathbb{Q}$, the condition $(B, F \cup \{\alpha\}) \in E_\alpha$ and $(B, F \cup \{\alpha\}) \leq (B, F)$.
As $\kappa < \mathfrak{p}$ there must be a filter $G \subseteq \mathbb{Q}$ which intersects all D_n's and all E_α's. But then the set $T \in [\omega]^\omega$ defined as $\bigcup\{A : \exists H\ (A, H) \in G\}$ is $\mathcal{J}$-positive (because of D_n's) and is almost contained (even in the modulo finite sense) in each T_α (because of E_α's). □

Theorem 3.5.3. *For any F_σ-ideal $\mathcal{J}$, the inequality $\mathfrak{t} \leq \mathrm{add}(\mathcal{I}(\mathbb{L}_\mathcal{J}))$ holds.*

Proof. Let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{I}(\mathbb{L}_\mathcal{J})$, $\kappa < \mathfrak{t}$ and let $T \in \mathbb{L}_\mathcal{J}$. We want to construct $S \leq T$ with $[S] \cap \bigcup_\alpha X_\alpha = \emptyset$. To simplify the construction, we will assume that $T = \omega^{<\omega}$ but the entire argument can be adapted inside of the tree T. Consider the following partial order:

$$\mathbb{P} = \{p = \{A^p_\sigma : \sigma \in \omega^{<\omega}\} : A^p_\sigma \in \mathcal{J}^+ \text{ for each } \sigma \in \omega^{<\omega}\}$$

For $p, q \in \mathbb{P}$ we have $q \leq p$ if $A^q_\sigma \subseteq^* A^p_\sigma$ for all $\sigma \in \omega^{<\omega}$ and $A^q_\sigma \subseteq A^p_\sigma$ for almost all $\sigma \in \omega^{<\omega}$. Also, for $p \in \mathbb{P}$ define sequence $\{S_p(\sigma) : \sigma \in \omega^{<\omega}\}$ of $\mathbb{L}_\mathcal{J}$-trees such that $S_p(\sigma)$ is the unique tree with stem σ and such that $succ_{S_p(\sigma)}(\tau) = A^p_\tau$ for every $\tau \in S^p(\sigma)$ extending the stem. We will say that $p \in \mathbb{P}$ is disjoint with a given set $X \subseteq \omega^\omega$ if $[S^p(\sigma)] \cap X = \emptyset$ for every $\sigma \in \omega^{<\omega}$. We will need the following three claims:

Claim 3.5.4. *If p is disjoint with $X \subseteq \omega^\omega$ and $q \leq p$, then q is disjoint with X.*

proof of the claim. Let $\sigma \in \omega^{<\omega}$. Let N be a natural number large enough so that the set $\{\sigma \in \omega^{<\omega} : A^q(\sigma) \setminus A^p(\sigma) \neq \emptyset\}$ is contained in $\omega^{\leq N}$. We will show that for every $\sigma \in \omega^{<\omega}$ we have

$$[S_q(\sigma)] \subseteq \bigcup_{|\tau|=N} [S_p(\tau)]$$

Let $x \in [S_q(\sigma)]$ and assume that $|\sigma| < N$. Let $\tau = x|_N$. We have that for all $n \geq |\sigma|$, $x(n) \in A^q(x|_n)$. Then for all $n \geq |\tau|$ we have $x(n) \in A^q(x|_n) \subseteq A^q(x|_n)$ which means that $x \in [S_p(\tau)] \subseteq \bigcup_{|\tau|=N} [S_p(\tau)]$. On the other hand, if $|\sigma| \geq N$, then $x(n) \in A^q(x|_n) \subseteq A^q(x|_n)$ above the stem, and then $x \in [S_p(\tau)] \subseteq \bigcup_{|\tau|=N} [S_p(\tau)]$. This finishes the proof of the first claim. □

Claim 3.5.5. *For every $p \in \mathbb{P}$ and $X \in \mathcal{I}(\mathbb{L}_\mathcal{J})$ there is a $q \in \mathbb{P}$, $q \leq p$ disjoint with X.*

proof of the claim. Inductively we will construct sequences $\{W^n : n \in \omega\}$ and $\{R^n_m : m, n \in \omega\}$ such that the following conditions hold:

- R^n_m is a $\mathbb{L}_\mathcal{J}$-tree for every $n, m \in \omega$,
- $W^n = \bigcup_m R^n_m$ for every $n \in \omega$,
- for every $\sigma \in \omega^{<\omega}$ there are unique $n, m \in \omega$ with $\sigma \in \{\tau \in R^n_m : stem(R^n_m) \subseteq \tau\}$,
- for every $n \in \omega$ and $m_0 \neq m_1$ in ω, $stem(R^n_{m_0})$ and $stem(R^n_{m_1})$ are incompatible.

In fact, for every $n \in \omega$ the trees R^n_m, will be indexed not by $m \in \omega$ but by a different countable set than ω. In the first step of the induction, use pure decision and find $W_0 \leq_0 S_p(\emptyset)$ such that $[W_0] \cap X = \emptyset$. Assume now that W_n has been built. For every $\tau \in \{\sigma \in \omega^{<\omega} : \sigma \notin W_n \text{ but } \sigma|_{|\sigma|-1} \in W_n\}$ we use pure decision to find $R^{n+1}_\tau \leq_0 S_p(\tau)$ such that $[R^{n+1}_\tau] \cap X = \emptyset$. Let then $W_{n+1} = \bigcup_\tau R^{n+1}_\tau$. This finishes the induction step. When the induction is over, define $q \in \mathbb{P}$ as follows: let $\sigma \in \omega^{<\omega}$ and let $n \in \omega$ and τ be unique values such that $\sigma \in R^n_\tau$. Let then $A^q(\sigma) = succ_{R^n_\tau}(\sigma)$. In such a case we have $S_q(\sigma) \subseteq R^n_\tau$ and $[R^n_\tau] \cap X = \emptyset$ which shows that q is disjoint with X. This finishes the proof of the second claim. □

Claim 3.5.6. *For every $\gamma < \kappa$ and every sequence $\{p_\alpha : \alpha < \gamma\} \subseteq \mathbb{P}$, such that for all $\alpha < \beta < \gamma$ we have $p_\beta \leq p_\alpha$, there is $q \in \mathbb{P}$ such that $q \leq p_\alpha$ for all $\alpha < \gamma$. In other words, $\mathbb{P}$ is $\mathfrak{t}$-closed.*

proof of the claim. Fix $\sigma \in \omega^{<\omega}$. As $\{A^p_\sigma : \alpha < \gamma < \kappa\}$ forms a $\subseteq_{\mathcal{J}}$-decreasing sequence in $\mathcal{P}(\omega) \setminus \mathcal{J}$ and $\gamma < \kappa < \mathfrak{t}$, by the Proposition 3.5.2, there is $C_\sigma \in \mathcal{J}^+$ such that for all $\alpha < \gamma$ we have $C_\sigma \setminus A^{p_\alpha}_\sigma \in \mathcal{J}$. As, by Proposition 4.4.12 $\gamma < \mathfrak{t} = \mathfrak{p} \leq \mathrm{cov}^*_+(\mathcal{J})$, there is $D_\sigma \in \mathcal{J}^+$, $D_\sigma \subseteq C_\sigma$ such that $D_\sigma \setminus A^{p_\alpha}_\sigma$ is finite for all $\alpha < \gamma$. Unfix σ. For every $\alpha < \gamma$ define $f_\alpha : \omega^{<\omega} \to \omega$ such that $D_\sigma \setminus A^{p_\alpha}_\sigma \subseteq f_\alpha(\sigma)$ for all $\sigma \in \omega^{<\omega}$. As $\gamma < \mathfrak{t} \leq \mathfrak{b}$ there must be $f : \omega^{<\omega} \to \omega$ which dominates all f_α's. Define $q \in \mathbb{P}$ such that $A^q(\sigma) = D_\sigma \setminus f(\sigma)$ for all $\sigma \in \omega^{<\omega}$. We claim that $q \leq p_\alpha$ for all $\alpha < \gamma$. □

Having the claims proven we continue the proof of the theorem as follows: inductively construct a sequence $\{p_\alpha : \alpha \leq \kappa\}$ in $\mathbb{P}$ such that for all $\alpha < \beta \leq \kappa$ we have $p_\beta \leq p_\alpha$ and p_α is disjoint from X_α. At the successor step, if p_α has been constructed, use the claim 3.5.5 to obtain $p_{\alpha+1}$ disjoint with $X_{\alpha+1}$. At limit step, if $\{p_\gamma : \gamma < \alpha\}$ has been constructed, first use the claim 3.5.6 to obtain q and then use the claim 3.5.5 to obtain p_α disjoint with X_α.
When the sequence $\{p_\alpha : \alpha \leq \kappa\}$ is constructed, by the claim 3.5.4 the condition p_κ must be disjoint with all X_α's, $\alpha < \kappa$. It follows that $[S_{p_\kappa}(\emptyset)] \cap \bigcup_{\alpha<\kappa} X_\alpha = \emptyset$ which finishes the proof of the theorem. □

The remaining estimates for the additivities of tree ideal are based on the notion of distributivity of a forcing notion. Recall the following well-known definition.

Definition 3.5.7. *A forcing notion $\mathbb{P}$ is (κ, λ, μ)-nowhere distributive if there exist maximal antichains $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ in $\mathbb{P}$, where $\mathcal{A}_\alpha = \{p^\alpha_\gamma : \gamma < \lambda\} \subseteq \mathbb{P}$, such that*

$$\forall p \in \mathbb{P}\ \exists \alpha < \kappa\ |\{\gamma < \lambda :\ p^\alpha_\gamma || p\}| = \mu \tag{✍}$$

In [Pet93] Simon proved that Sacks forcing is $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$-nowhere distributive and in [HS20] Hein and Spinas proved that splitting tree forcing is $(\mathfrak{d}, \mathfrak{c}, \mathfrak{c})$-nowhere distributive. Also, in [GRSS95] Goldstern, Repicky, Shelah and Spinas proved that both Miller and Laver forcings are $(\mathfrak{h}, \mathfrak{c}, \mathfrak{c})$-nowhere distributive.
The reason why we are interested in nowhere-distributivity is the it gives a natural bound for the additivity of the forcing ideal. Recall that for a maximal antichain $\mathcal{A} \subseteq \mathbb{T}$, the set $\mathcal{X}(\mathcal{A})$ was defined as $\omega^\omega \setminus \bigcup_{T \in \mathcal{A}}[T]$ (assuming that $\mathbb{T}$ lives on ω^ω).

Proposition 3.5.8. *Suppose that $\mathbb{T}$ is a tree type and* $\mathrm{is}(\mathbb{T}) = \mathfrak{c}$. *If $\mathbb{T}$ is $(\kappa, \mathfrak{c}, \mathfrak{c})$-nowhere distributive, then* $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \kappa$.

Proof. Let $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ be a collection of maximal antichains in $\mathbb{T}$ which witnesses for the $(\kappa, \mathfrak{c}, \mathfrak{c})$-nowhere distributivity of $\mathbb{T}$. By the Lemma 3.4.3 for every $\alpha < \kappa$ we can find a maximal antichain $\mathcal{A}'_\alpha$ in the open dense set $\{T \in \mathbb{T} : \exists S \in \mathcal{A}_\alpha\ T \subseteq S\}$. We claim that the sets $\{\mathcal{X}(\mathcal{A}'_\alpha) : \alpha < \kappa\}$ (which are in $\mathcal{I}(\mathbb{T})$ by the Proposition 2.1.3) witness for the $\mathrm{add}(\mathcal{I}(\mathbb{T}))$. Towards a contradiction, assume that $\bigcup_{\alpha<\kappa} \mathcal{X}(\mathcal{A}'_\alpha) \in \mathcal{I}(\mathbb{T})$. Then there is a $T \in \mathbb{T}$ such that $[T] \cap \bigcup_{\alpha<\kappa} \mathcal{X}_\alpha(\mathcal{A}'_\alpha) = \emptyset$, which means that $[T] \subseteq \bigcup_{S \in \mathcal{A}'_\alpha}[S]$ for every $\alpha < \kappa$. By the claim of the

Lemma 3.4.3, for each $\alpha < \kappa$ there is $\mathcal{B}_\alpha \in [\mathcal{A}'_\alpha]^{<\mathfrak{c}}$ such that $[T] \subseteq \bigcup_{S\in\mathcal{B}_\alpha}[S]$. As $\mathcal{A}'_\alpha$ refines $\mathcal{A}_\alpha$, for every $S \in \mathcal{A}'_\alpha$ there is $R_S \in \mathcal{A}_\alpha$ with $S \subseteq R_S$. Then for every $\alpha < \kappa$ we have

$$[T] \subseteq \bigcup_{S\in\mathcal{B}_\alpha}[S] \subseteq \bigcup_{S\in\mathcal{B}_\alpha}[R_S]$$

This however, is impossible. As $\{\mathcal{A}_\alpha : \alpha < \kappa\}$ witnesses for nowhere-distributivity, there must be an $\alpha < \kappa$ such that T is compatible with $\mathfrak{c}$-many elements of $\mathcal{A}_\alpha$, so it cannot be contained in less than $\mathfrak{c}$ elements of it. □

The proof of the following theorem follows the idea of Petr Simon from [Pet93] in which he showed that Sacks forcing is $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$-nowhere distributive.

Theorem 3.5.9. *The tree types $\mathbb{IE}_F$ and $\mathbb{S}_F$ are $(\mathfrak{b}, \mathfrak{c}, \mathfrak{c})$-nowhere distributive.*

Proof. We will give the proof only for the tree type $\mathbb{IE}_F$ as the one for $\mathbb{S}_F$ is analogous. Let $\mathcal{F} \subseteq \omega^\omega$ be an unbounded family of cardinality $\mathfrak{b}$. For each $T \in \mathbb{IE}_F$ choose $f_T \in \omega^\omega$ such that for all $n \in \omega$ we have: for every $\sigma \in T$ with $|\sigma| = f_T(n)$ there is $i_\sigma \in [f_T(n), f_T(n+1))$ such that $\{\tau(i_\sigma) : \tau \in T|_\sigma,\ |\tau| = i_\sigma + 1\} = F(i_\sigma)$. We will need the following notation and lemmas from [Pet93]: For an increasing function $h \in \omega^\omega$ define

$$J^+(h) = \{X \subseteq \omega : limsup_n |[h(n), h(n+1)) \cap X| = \omega\}$$

Lemma 3.5.10. *(folklore, [Pet93]) If $\mathcal{F} \subseteq \omega^\omega$ is unbounded, then for every $X \in [\omega]^\omega$ there is $h \in \mathcal{F}$ such that $X \in J^+(h)$.*

For each $T \in \mathbb{IE}_F$ using the lemma 3.5.10 pick $h_T \in \omega^\omega$ such that $rng(f_T) \in J^+(h_T)$. It follows that the set $X_T =: \{n \in \omega : |[h_T(n), h_T(n+1)) \cap rng(f_T)| \geq 2\}$ is infinite. Then using the lemma 3.5.10 again, pick $g_T \in \omega^\omega$ such that $X_T \in J^+(g_T)$. For $h, g \in \omega^\omega$ denote then

$$E(h, g) = \{T \in \mathbb{IE}_F : h_T = h,\ g_T = g\}.$$

Clearly $\mathbb{IE}_F = \bigcup_{h,g\in\omega^\omega} E(h, g)$. The following lemma is crucial for the proof

Lemma 3.5.11. *(Simon, Balcar, [Pet93]) For every strictly increasing function $g \in \omega^\omega$ there is an almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ with the properties that, first, every $A \in \mathcal{A}$ intersects every interval $[g(n), g(n+1))$ in at most one point and, second, for every $X \in J^+(g)$ the set $\{A \in \mathcal{A} : A \subseteq X\}$ is of size continuum.*

For each $g \in \mathcal{F}$ let $\mathcal{A}_g$ be the almost disjoint family from the lemma 3.5.11 for the function g. For $X \in J^+(g)$ let $\mathcal{A}_g(X) = \{A \in \mathcal{A}_g : A \subseteq X\}$. Recall the following result

Lemma 3.5.12. *(Sierpiński) For every collection of sets $\{A_\alpha : \alpha < \mathfrak{c}\}$ such that $|A_\alpha| = \mathfrak{c}$ for each $\alpha < \mathfrak{c}$ there are is a collection $\{B_\alpha : \alpha < \mathfrak{c}\}$ such that $B_\alpha \subseteq A_\alpha$, $|B_\alpha| = \mathfrak{c}$ and $B_\alpha \cap B_\beta$ for all $\alpha, \beta < \mathfrak{c}$, $\alpha \neq \beta$.*

The family $\{\mathcal{A}_g(X) : X \in J^+(g)\}$ clearly satisfy the assumption of the above lemma so by applying the lemma we obtain a family $\{\mathcal{B}_g(X) : X \in J^+(g)\}$ such that for every $X, Y \in J^+(g)$ we have

1. $\mathcal{B}_g(X) \subseteq \mathcal{A}_g(X)$,
2. $|\mathcal{B}_g(X)| = \mathfrak{c}$,
3. $\mathcal{B}_g(X) \cap \mathcal{B}_g(Y) = \emptyset$ for $X \neq Y$.

For every sets $C \in [\omega]^\omega$ and $T \in \mathbb{IE}_F$ we define a tree $T[C]$ as follows:
Intuitively the tree $T[C]$ on the intervals $[h_T(g_T(n)), h_T(g_T(n+1)))$ where $n \in C$ retains all the nodes of T and on the intervals $[h_T(g_T(n)), h_T(g_T(n+1)))$ where $n \notin C$ it always chooses the left-most branch of T. Formally, $T[C]$ is constructed inductively as an increasing union of finite trees $F_n \subseteq T$ such that $|\sigma_n| = g_T(f_T(n))$ for every terminal node $\sigma \in F_n$. Let $F_0 = \{\emptyset\}$ and assume that F_n has been built. Then, if $n \in C$, let F_{n+1} be the set of all $\tau \in T$ satisfying $|\tau| \leq g_T(f_T(n+1))$, such that τ is either contained or extends some terminal of F_{n+1} i.e. $F_{n+1} = F_n \cup \bigcup_{\sigma \in term(F_n)} \{\tau \in T|_\sigma : |\tau| \leq g_T(f_T(n+1))\}$. If on the other hand $n \notin C$, we extend terminal nodes of F_n only by leftmost branches of T i.e. F_{n+1} is the tree generated by the set $\{y|_{g_T(f_T(n+1))} : y \text{ is the left-most branch of } T|_\sigma \text{ where } \sigma \in term(F_n)\}$. This finishes the induction step and the construction of $T[C]$. Note that $T[C] \in \mathbb{IE}_F$ because of $rng(f_T) \in J^+(h_T)$.
We are now ready to define the antichains $\{\mathcal{A}_{h,g} : h, g \in \mathcal{F}\}$ witnessing for the $\mathfrak{b}$-nowhere distributivity of the tree type $\mathbb{IE}_F$. For $h, g \in \mathcal{F}$ define $\mathcal{A}_{h,g} = \{T[A] : T \in E(h,g), A \in \mathcal{B}_g(X_T)\}$. By the last claim we have $\mathcal{A}_{h,g} \subseteq \mathbb{IE}_F$. To see that each $\mathcal{A}_{h,g}$ forms in fact an antichain in $\mathbb{IE}_F$ we will prove the following

Claim 3.5.13. $\forall T, S \in E(h,g), T \neq S \forall A \in \mathcal{B}_g(X_T) \forall B \in \mathcal{B}_g(X_S)$ $[T[A]] \cap [S[B]]$ *is finite.*

Proof. Let $N \in \omega$ be such that $A \cap B \subseteq N$. Assume towards contradiction that $[T[A]] \cap [S[B]]$ is infinite. Then the tree $T[A] \cap S[B]$ must have infinitely many nodes that splits in $T[A] \cap S[B]$ into at least two successors. Then, as $Seq(F)$ is finitely splitting tree, we are able to pick a sequence $< \sigma_n : n \in \omega >$ of these splitnodes such that $|\sigma_n| < |\sigma_{n+1}|$ for every $n \in \omega$. Let $n \in \omega$ be such that $|\sigma_n| > g(h(N))$ and that $|\sigma_n| \in [g(h(m)), g(h(m+1)))$ for some $m \in \omega$. Now, if σ_n splits in $T[A]$, by the construction of $T[A]$, it must be that $m \in A$. Similarly, we have $m \in B$ because of the $S[B]$. This implies $m \in (A \cap B) \setminus N$ which contradicts the choice of N. □

To finish the proof of the theorem, notice that the antichains $\{\mathcal{A}_{h,g} : h, g \in \mathcal{F}\}$ clearly witness the $\mathfrak{b}$-nowhere distributivity because if $T \in \mathbb{IE}_F$, then $T \in E(h,g)$ for some $h, g \in \mathcal{F}$ and then every $T[C]$ for $C \in \mathcal{B}_g(X_T)$ is compatible with T (is even a subtree of T) while there are continuum many $C \in \mathcal{B}_g(X_T)$. □

As a corollary we get the following result

Corollary 3.5.14. *If* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ *then* $\mathrm{add}(\mathcal{I}(\mathbb{IE}_F)), \mathrm{add}(\mathcal{I}(\mathbb{S}_F)) \leq \mathfrak{b}$.

Proof. Directly follows from the Theorem 3.5.9, the Proposition 3.2.5 and the Proposition 3.5.8. □

We apply these inequalities to get the following consistency result.

Theorem 3.5.15. *Both inequalities* $\mathrm{add}(\mathcal{I}(\mathbb{S}_F)) < \mathrm{cov}(\mathcal{I}(\mathbb{S}_F))$ *and* $\mathrm{add}(\mathcal{I}(\mathbb{IE}_F)) < \mathrm{cov}(\mathcal{I}(\mathbb{IE}_F))$ *are consistent with ZFC.*

Proof. Consider the countable support iteration $\{\mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\}$ of length ω_2 such that for each $\mathbb{T} \in \{\mathbb{S}_F, \mathbb{IE}_F, \mathbb{FM}\}$ the set $\{\alpha \in \omega_2 : \Vdash_\alpha \text{"}\mathbb{Q}_\alpha = \mathbb{T}\text{"}\}$ is a stationary subset of ω_2. We will show that if $\mathbf{V}$ is a model of CH, then in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ the desired inequalities hold. As each of the tree types has axiom A, all cardinal numbers are preserved and it is a standard argument that $\mathfrak{c} = \omega_2$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ (see [Bau10]).
First, we show that the covering numbers are equal ω_2 in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. We need the following lemma.

Lemma 3.5.16. *Let* $\mathcal{A} \subseteq \mathbb{T}$ *be an maximal antichain in* $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ *and* $\mathbb{T} \in \{\mathbb{S}_F, \mathbb{IE}_F\}$. *Then there is a club* $C \subseteq \{\alpha < \omega_2 : cf(\alpha) = \omega_1\}$ *such that for every* $\alpha \in C$ *we have that* $\mathbf{V}^{\mathbb{P}_\alpha} \vDash \text{"}\mathcal{A} \cap \mathbf{V}^{\mathbb{P}_\alpha}$ *is a maximal antichain in* $\mathbb{T}$"*.*

Proof. In $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ we define the function $w : \mathbb{T} \to \omega_2$ as $w(T) = \min\{\alpha \in \omega_2 : \exists S \in \mathcal{A} \cap \mathbf{V}^{\mathbb{P}_\alpha} S||T\}$. The set over which the minimum is taken, is non-empty because, first, $\mathcal{A}$ is maximal in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ and for every $S \in \mathbb{T}$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ there is $\alpha \in \omega_2$ such that $S \in \mathbf{V}^{\mathbb{P}_\alpha}$ (see [Bau10]). Define $f : \omega_2 \to \omega_2$ as follows: for $\alpha \in \omega_2$ let

$$f(\alpha) = \sup\{w(T) : T \in \mathbf{V}^{\mathbb{P}_\alpha}\}$$

Note that $f(\alpha) \in \omega_2$ because CH holds in $\mathbf{V}^{\mathbb{P}_\alpha}$ for every $\alpha \in \omega_2$. Let $C \subseteq \omega_2$ be the set of closure points of f i.e. $C = \{\alpha \in \omega_2 : cf(\alpha) = \omega_1$ and $f(\gamma) < \alpha$ for every $\gamma < \alpha\}$. It is well-known that such set is club in ω_2. We check that for every $\alpha \in C$ we have that $\mathbf{V}^{\mathbb{P}_\alpha} \vDash$ "$\mathcal{A} \cap \mathbf{V}^{\mathbb{P}_\alpha}$ is a maximal antichain in $\mathbb{T}$". Let $\alpha \in C$ and $T \in \mathbb{T} \cap \mathbf{V}^{\mathbb{P}_\alpha}$. As T can be thought of as a real (a subset of $Seq(F)$, if $\mathbb{T}$ lives on ΠF) and $cf(\alpha) = \omega_1$, there is $\beta < \alpha$ such that $T \in \mathbf{V}^{\mathbb{P}_\beta}$. Then in $\mathbf{V}^{\mathbb{P}_{w(T)}}$ there is an $S \in \mathcal{A}$ compatible with T. But $w(T) \leq f(\beta) \leq \alpha$. It follows that in $\mathbf{V}^{\mathbb{P}_\alpha}$ we can find $S \in \mathcal{A}$ compatible with T. □

We continue the proof of the theorem. In $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, let $\{X_\gamma : \gamma < \omega_1\} \subseteq \mathcal{I}(\mathbb{T})$ and $\mathbb{T} \in \{\mathbb{S}_F, \mathbb{IE}_F\}$. We will show that $\bigcup_\gamma X_\gamma \neq 2^\omega$. By the Proposition 2.1.3, for every $\gamma < \omega_1$ we may choose a maximal antichain $\mathcal{A}_\gamma \subseteq \mathbb{T}$ such that $X_\gamma \subseteq \mathcal{X}(\mathcal{A}_\gamma)$. For every $\gamma < \omega_1$ let C_γ be a club given by the Lemma 3.5.16. Let $C =: \bigcap_{\gamma<\omega_1} C_\gamma$ which is a club in ω_2. Let $\alpha \in C$ be such that $\Vdash_\alpha$ "$\mathbb{Q}_\alpha = \mathbb{T}$". In $\mathbb{V}^{\mathbb{P}_\alpha}$, for every $\gamma \in \omega_1$ we have that $\mathcal{A}_\gamma \cap \mathbf{V}^{\mathbb{P}_\alpha}$ is a maximal antichain in $\mathbb{T}$. It follows that in $\mathbb{V}^{\mathbb{P}_{\alpha+1}}$, for every $\gamma \in \omega_1$ there is $T_\gamma \in \mathcal{A}_\gamma \cap \mathbf{V}^{\mathbb{P}_\alpha}$ such that $r_\alpha \in [T_\gamma]$ (where r_α is the generic real given by the $\mathbb{Q}_\alpha$). As $r_\alpha \in [T_\gamma]$ is absolute, it holds in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. It follows that $r_\alpha \notin \bigcup_{\gamma\in\omega_1} \mathcal{X}(\mathcal{A}_\gamma)$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$.

We will now show that $\mathrm{add}(\mathcal{I}(\mathbb{S}_F)) = \mathrm{add}(\mathcal{I}(\mathbb{IE}_F)) = \omega_1$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. First, observe that as Cohen reals were added by $\mathbb{FM}$ (see the Proposition 3.1.3) stationarly often, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. By the Proposition 3.2.5, we then have that $\mathrm{is}(\mathbb{S}_F) = \mathrm{is}(\mathbb{IE}_F) = \mathfrak{c}$. By the Proposition 3.5.8 and Theorem 3.5.9 we get $\max\{\mathrm{add}(\mathcal{I}(\mathbb{S}_F)), \mathrm{add}(\mathcal{I}(\mathbb{IE}_F))\} \leq \mathfrak{b}$. The proof will be finished by showing that $\mathrm{non}(\mathcal{M}) = \omega_1$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. Notice that each of the forcing notions we use for the iteration can be expressed as Borel $\setminus \mathcal{I}$ where $\mathcal{I}$ is a σ-ideal generated by closed sets. We will use Hurewicz schemas and Solecki's dichotomy [Sol94] to show that $\mathrm{non}(\mathcal{M}) = \omega_1$ in the final model. According to the Theorem 6.1.13, Lemma 6.1.8 and Lemma 6.3.17 from Bartoszyński's and Judah's book [BJ95], if we will prove the following claim, then in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, $\mathrm{non}(\mathcal{M}) = \omega_1$ holds. Recall that $\mathbb{P}_\mathcal{I} = \mathrm{Borel} \setminus \mathcal{I}$.

Claim 3.5.17. *Suppose that a σ-ideal $\mathcal{I}$ is generated by closed sets and $\mathbf{M}$ be an elementary substructure of some $H(\kappa)$, containing* Borel$\setminus\mathcal{I}$. *Let $B \in \mathbb{P}_\mathcal{I} \cap \mathbf{M}$ and let c be a Cohen real over $\mathbf{M}$. Then there exists $B' \subseteq B$, $B' \in \mathbb{P}_\mathcal{I}$ such that the following two conditions holds:*

- *$B' \Vdash$ "c is a Cohen over $\mathbf{M}[\dot{G}]$"*
- *$B' \Vdash$ "$\dot{G} \cap D \cap \mathbf{M} \neq \emptyset$"*

Proof. In the proof we will constantly use the Solecki's dichotomy 2.2.49. Recall the result 2.2.4 of [BJ95] saying that any meager subset of 2^ω is covered by a set of form

$$\{x \in 2^\omega : \forall^\infty_n\ x|_{[f(n),f(n+1))} \neq y|_{[f(n),f(n+1))}\}$$

for some $y \in 2^\omega$ and some increasing $f \in \omega^\omega$. Enumerate in $\{(\dot{y}_N, \dot{f}_N) : 0 < N < \omega\}$ all $\mathbb{P}_\mathcal{I}$-names for such parameters for the meager sets in $\mathbf{M}^{\mathbb{P}_\mathcal{I}}$. Enumerate also as $\{D_n : n \in \omega\}$, all dense subsetes of Borel $\setminus \mathcal{I}$ which are in $\mathbf{M}$. We inductively construct a sequence $\{\mathcal{U}^N : N < \omega\}$ of Hurewicz schemas $\mathcal{U}^N = \{U^N_\sigma : \sigma \in \omega^{<\omega}\} \in \mathbf{M}$ and a sequence of functions $\{g_N : N \in \omega\}$ such that for every $N \in \omega$ the following conditions holds:

$(1)_N$ $\{U^N_\sigma : \sigma \in \omega^{\leq N}\} = \{U^{N+1}_\sigma : \sigma \in \omega^{\leq N}\}$,

$(2)_N$ $g_N : [\mathcal{U}^N] \to 2^\omega \times {}^4(\omega^\omega \uparrow)$ i continuous and such that $[\mathcal{U}^N] \Vdash$ "$g_N(\dot{r}_{gen}) = (\dot{y}_N, \dot{f}_N)$",

$(3)_N$ $[\mathcal{U}^N] \Vdash$ "$\forall_{j\leq N}\ \exists F_j \subseteq \omega, |F_j| = N\ \forall i \in F_j\ c|_{[\dot{f}_j(i),\dot{f}_j(i+1))} = \dot{y}_j|_{[\dot{f}_j(i),\dot{f}_j(i+1))}$",

$(4)_N$ $[\mathcal{U}^N] \Vdash$ "$\dot{G} \cap D_N \cap \mathbf{M} \neq \emptyset$"

When the construction is over, we define $\mathcal{U}^\infty$ as $\bigcup_{N\in\omega}\{U^N_\sigma : \sigma \in \omega^{\leq N}\}$. By the conditions $(1)_N$, $N \in \omega$, $\mathcal{U}^\infty$ is in fact a Hurewicz schema. By the conditions $(3)_N$, $N \in \omega$, we have that

$$[\mathcal{U}^\infty] \Vdash \text{"}\forall j \in \omega\ \exists^\infty_i\ c|_{[\dot{f}_j(i),\dot{f}_j(i+1))} = \dot{y}_j|_{[\dot{f}_j(i),\dot{f}_j(i+1))}\text{"}$$

This means that $[\mathcal{U}^\infty] \Vdash$ "c is a Cohen over $\mathbf{M}[\dot{G}]$" and the claim follows. We now deal with the inductive construction. Start with a Hurewicz schema $\mathcal{U}^{-1} \in \mathbf{M}$ such that $[\mathcal{U}^{-1}] \subseteq B$. Assume that $\mathcal{U}^N$ has been built. First, for every $\sigma \in \omega^{<\omega}$ and $n \in \omega$ use continuous reading of names of $\mathbb{P}_\mathcal{I}$ (Theorem 1.2.6) to find a Hurewicz schema $\mathcal{V}^{(\sigma,n)} \in \mathbf{M}$, $\mathcal{V}^{(\sigma,n)} = \{V^{(\sigma,n)}_\rho : \rho \in \omega^{<\omega}\}$ and a continuous function $h^{(\sigma,n)} : [\mathcal{V}^{(\sigma,n)}] \to 2^\omega \times (\omega^\omega \uparrow)$ such that $[\mathcal{V}^{(\sigma,n)}] \subseteq U^N \cap [\mathcal{U}^N]$, $V^{(\sigma,n)}_\emptyset \subseteq U^N_{\sigma^\frown n}$ and $[\mathcal{V}^{(\sigma,n)}] \Vdash$ "$h^{(\sigma,n)}(\dot{r}_{gen}) = (\dot{y}_{N+1}, \dot{f}_{N+1})$". Let also $x^{(\sigma,n)} \in [\mathcal{V}^{(\sigma,n)}] \cap \mathbf{M}$. Define $g_{N+1} = \bigcup_{\sigma\in\omega^N}\bigcup_{n\in\omega} h^{(\sigma,n)}$. By strengthening each $\mathcal{V}^{(\sigma,n)} \in \mathbf{M}$ if necessary we may assume that $[\mathcal{V}^{(\sigma,n)}] \in D_{N+1}$.

Fix $\sigma \in \omega^N$ and $n \in \omega$ for a moment. By induction of length N we construct a sequence $\rho_{-1} \subseteq \rho_0 \subseteq ... \subseteq \rho_N$ in $\omega^{<\omega}$. Start with $\emptyset = \rho_{-1}$. Assume that ρ_{i-1} has been constructed for some $0 \leq i < N$. Let $g_i(x^{(\sigma,n)}) = (y'_i, f'_i)$. As c is a Cohen real over $\mathbf{M}$ and $(y'_i, f'_i) \in \mathbf{M}$ there are $l^i_0 < l^i_1 < ... < l^i_N$ such that for every $j \leq N$ we have

$$c|_{[f'_i(l^i_j),f'_i(l^i_j+1)} = y'_i|_{[f'_i(l^i_j),f'_i(l^i_j+1)}$$

By the continuity of g_i, it is possible to find $\rho_i \in \omega^{<\omega}$ such that $\rho_{i-1} \subseteq \rho_i$, $x^{(\sigma,n)} \in V^{(\sigma,n)}_{\rho_i}$ and $g_i[V^{(\sigma,n)}_{\rho_i}] \subseteq [y'_i|_{f'_i(l^i_N+1)}] \times [f'_i(l^i_N + 1)]$. When the short induction is done, the ρ^N is constructed. Denote this ρ_N as $\rho^{(\sigma,n)}$. Define now a Hurewicz schema $\mathcal{U}^{N+1} = \{U^{N+1}_\sigma : \sigma \in \omega^{<\omega}\}$ as follows: $U^{N+1}_\sigma = U^N_\sigma$ for $\sigma \in \omega^{\leq N}$ and $U^{N+1}_{\sigma^\frown n^\frown \tau} = V^{(\sigma,n)}_{\rho^{(\sigma,n)}{}^\frown \tau}$ for $\sigma \in \omega^N$, $n \in \omega$ and $\tau \in \omega^{<\omega}$. Cleary, $\mathcal{U}^{N+1} \in \mathbf{M}$. This finishes the construction of $\mathcal{U}^{N+1}$ and the induction process. □

□

Proposition 3.5.18. *Let $\mathbb{T}$ be a combinatorial type with continuous reading of names. Then, if $\mathbb{T}$ adds Cohen reals, the inequality* $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \mathrm{add}(\mathcal{M})$ *holds.*

Proof. On the one hand, by the Proposition 3.2.2 and 3.2.11 we get $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \mathrm{is}(\mathbb{T}) \leq \mathfrak{b}$. On the other, by Theorem 3.1.1 we have $\mathrm{add}(\mathcal{I}(\mathbb{T})) \leq \mathrm{cov}(\mathcal{I}(\mathbb{T})) \leq \mathrm{cov}(\mathcal{M})$. □

As a corollary we get the following consistency result.

Proposition 3.5.19. *If $\mathbb{M}_\mathcal{J}$ adds Cohen reals, then the inequality* $\mathrm{add}(\mathcal{I}(\mathbb{M}_\mathcal{J})) < \mathrm{cov}(\mathcal{I}(\mathbb{M}_\mathcal{J}))$ *is consistent. The same is true for the tree type $\mathbb{FM}$.*

Proof. The inequality holds in the model obtained by countable support iteration of length ω_2 over a model of CH with $\mathbb{M}_\mathcal{J}$ standing as each iterand. By Proposition 3.5.18 we have that $\mathrm{add}(\mathcal{I}(\mathbb{M}_\mathcal{J})) \leq \mathrm{add}(\mathcal{M})$. On the other hand the same argument as in the Theorem 3.5.15 shows that $\mathrm{non}(\mathcal{M}) = \omega_1$ and $\mathrm{cov}(\mathcal{I}(\mathbb{M}_\mathcal{J})) = \omega_2$ in the final model. □

We will finish this section by discussion about nowhere distributivity of $\mathbb{L}_\mathcal{J}$'s and $\mathbb{M}_\mathcal{J}$. For the next result, we will need the notion of base tree and the distributivity number $\mathfrak{h}_\mathcal{J}$.

[4] $\omega^\omega \uparrow$ denotes the set of all strictly increasing functions in ω^ω

Definition 3.5.20. *For an order $\mathbb{P}$, let $\mathfrak{h}_{\mathbb{P}}$ be the distributivity number of $\mathbb{P}$ i.e. the smallest number of open dense subsets of $\mathbb{P}$ with non-dense intersection. If $\mathcal{J}$ is an ideal on ω, then $\mathfrak{h}_{\mathcal{J}}$ is the distributivity of the pre-order $(\mathcal{P}(\omega) \setminus \mathcal{J}, \subseteq_{\mathcal{J}})$.*

The following result will be important to us.

Theorem 3.5.21. *(Balcar, Doucha, Hrušák; Theorem 2.1 in [BDMH12]) Let $\mathcal{J}$ be an ideal on ω such that:*

- *$\mathcal{J}$ is homogeneous,*
- *for every $A \in \mathcal{J}^+$ there are $A_0, A_1 \in \mathbb{P}$ such that $A_0, A_1 \subseteq A$ and $A_0 \cap A_1 \in \mathcal{J}$,*
- *for every $\mathcal{J}$-decreasing sequence $\{A_n : n \in \omega\} \subseteq \mathcal{J}^+$, there is $A \in \mathcal{J}^+$ such that $A \subseteq_{\mathcal{J}} A_n$ for every $n \in \omega$*

Then there is a base tree for $\mathcal{J}$ i.e. there is a collection $\{\mathcal{B}_\alpha : \alpha < \mathfrak{h}_{\mathcal{J}}\}$ such that for every $\alpha < \mathfrak{h}_{\mathcal{J}}$ the following holds:

- *$\mathcal{B}_\alpha \subseteq \mathcal{J}^+$ is such that for $A, B \in \mathcal{B}_\alpha$, if $A \neq B$, then $A \cap B \in \mathcal{J}$,*
- *for every $A \in \mathcal{J}^+$ there is $B \in \mathcal{B}_\alpha$ such that $A \cap B \in \mathcal{J}^+$,*
- *$\mathcal{B}_\alpha$ refines $\mathcal{B}_\beta$ for $\beta < \alpha$ i.e. for every $A \in \mathcal{B}_\alpha$ there is $B \in \mathcal{B}_\beta$ such that $A \subseteq_J B$,*
- *for every $A \in \mathcal{B}_\alpha$ the set $\{B \in \mathcal{B}_{\alpha+1} : B \subseteq_{\mathcal{J}} A\}$ is of cardinality $\mathfrak{c}$,*
- *$\bigcup_{\alpha<\mathfrak{h}_{\mathcal{J}}} \mathcal{B}_\alpha$ is dense in $\mathcal{J}^+$ i.e. for every $A \in \mathcal{J}^+$ there is $\alpha < \mathfrak{h}_{\mathcal{J}}$ and $B \in \mathcal{B}_\alpha$ such that $B \subseteq_{\mathcal{J}} A$*

Corollary 3.5.22. *If $\mathcal{J}$ is a homogeneous F_σ-ideal, then $\mathcal{P}(\omega) \setminus \mathcal{J}$ has a base tree.*

Proof. By the proposition 3.5.2 for every $\mathcal{J}$-decreasing sequence $\{A_n : n \in \omega\} \subseteq \mathcal{J}^+$, there is $A \in \mathcal{J}^+$ such that $A \subseteq_{\mathcal{J}} A_n$ for every $n \in \omega$. Clearly, for any $A \in \mathcal{J}^+$ there are $A_0, A_1 \in \mathbb{P}$ such that $A_0, A_1 \subseteq A$ and $A_0 \cap A_1 \in \mathcal{J}$, □

For the next theorem, the only thing that we need from an ideal $\mathcal{J}$, is that it has a base tree and that for every $\mathcal{J}$-positive set $A \subseteq \omega$ there is a collection $\{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{J}^+$ of $\mathcal{J}$-almost disjoint subsets of A i.e. $A_\alpha \cap A_\beta \in \mathcal{J}$ for all $\alpha \neq \beta$. The latter is true for meager ideals by the Proposition 4.5.1.

Theorem 3.5.23. *$\mathbb{L}_{\mathcal{J}}$ and $\mathbb{M}_{\mathcal{J}}$ are $\mathfrak{h}_{\mathcal{J}}$-nowhere-distributive, for meager ideal $\mathcal{J}$ with a base tree.*

Proof. We will prove the theorem only for $\mathcal{J}$-Laver tree as the argument for $\mathcal{J}$-Miller trees is almost identical. Let $\{\mathcal{B}_\alpha : \alpha < \mathfrak{h}\}$ be a base tree in $\mathcal{P}(\omega) \setminus \mathcal{J}$. For $\alpha < \mathfrak{h}$ define

$$\mathbb{L}^\alpha_{\mathcal{J}} = \{T \in \mathbb{L}_{\mathcal{J}} : \forall \sigma \in T \text{ above the stem } \exists B \in \mathcal{B}_\alpha \; B \subseteq_{\mathcal{J}} succ_T(\sigma)\}$$

Claim 3.5.24. *For every $T \in \mathbb{L}_{\mathcal{J}}$ there is $\alpha < \mathfrak{h}_{\mathcal{J}}$ such that $T \in \mathbb{L}^\alpha_{\mathcal{J}}$.*

Proof. Let $T \in \mathbb{L}_{\mathcal{J}}$. For every $\sigma \in T$ above stem there is $\alpha_\sigma \in \mathfrak{h}_{\mathcal{J}}$ such that $B \subseteq_{\mathcal{J}} succ_T(\sigma)$ for some $B \in \mathcal{B}_\alpha$. As $\mathfrak{h}_{\mathcal{J}}$ is a regular cardinal, there is $\alpha < \kappa$ above all α_σ's. Clearly $T \in \mathbb{L}^\alpha_{\mathcal{J}}$. □

We continue the proof of the theorem. For every $\alpha < \mathfrak{h}$ we construct a function $\Phi_\alpha : \mathbb{L}^\alpha_{\mathcal{J}} \to \mathbb{L}_{\mathcal{J}}$ such that for $T, S \in \mathbb{L}^\alpha_{\mathcal{J}}$ the following two conditions are satisfied:

1. $\Phi_\alpha(T) \leq T$,
2. $T \neq S$ implies that $[\Phi_\alpha(T)] \cap [\Phi_\alpha(S)] \in \mathcal{I}_{\mathcal{B}}(\mathbb{L}_{\mathcal{J}})$.

Notice that once the construction of Φ_α's is done the rest of the proof follows easily: we decompose each $\Phi_\alpha(T)$ into $\mathfrak{c}$-many disjoint $\mathbb{L}_\mathcal{J}$-trees i.e. $[\Phi_\alpha(T)] = \bigcup_{\gamma<\mathfrak{c}}[R^\alpha_\gamma(T)]$. For every $\alpha < \mathfrak{h}$ we define $\mathcal{A}_\alpha = \{R^\alpha_\gamma(T) : \gamma < \mathfrak{c} \text{ and } T \in \mathbb{L}^\alpha_\mathcal{J}\}$. Because of the condition 2., each $\mathcal{A}_\alpha$ forms an antichain in $\mathbb{L}_\alpha$, which clearly is of cardinality $\mathfrak{c}$. These antichains form a witness for $\mathfrak{h}$-nowhere-distributivity because for every $T \in \mathbb{L}_\mathcal{J}$ there is $\alpha < \mathfrak{h}$ such that $T \in \mathbb{L}^\mathcal{J}_\alpha$ and then $\{R^\alpha_\gamma(T) : \gamma < \mathfrak{c}\} \subseteq \{S \in \mathcal{A}_\alpha : S||T\}$.

Construction of Φ_α:
For every $T \in \mathbb{L}^\alpha_\mathcal{J}$ and every $\sigma \in T$ above the stem, choose a set $B(T,\sigma) \in \mathcal{B}_\alpha$ such that $B(T,\sigma) \subseteq_\mathcal{J} succ_T(\sigma)$. For each $B \in \mathcal{B}_\alpha$ use 4.5.1 to choose an $\mathcal{J}$-almost-disjoint family $\mathcal{D}(B)$ of size $\mathfrak{c}$ of $\mathcal{J}$-positive subsets of the set B. As $|\mathbb{L}^\alpha_\mathcal{J}| = \mathfrak{c}$ we can enumerate $\mathcal{D}(B) = \{D^B_T : T \in \mathbb{L}^\alpha_\mathcal{J}\}$. Define now inductively a tree $R = \Phi_\alpha(T) \subseteq T$ such that $stem(R) = stem(T)$ and for every $\sigma \in S$ above the stem we have $succ_R(\sigma) = D^{B(T,\sigma)_T}$. Clearly $R \in \mathbb{L}_\mathcal{J}$ and $R \leq T$. We will finish the construction of $\Phi_\alpha(T)$ by proving the condition 2. of the properties $\Phi_\alpha(T)$. Let $S \neq T$ be in $\mathbb{L}^\alpha_\mathcal{J}$ and let R denote the tree $\Phi_\alpha(S) \cap \Phi_\alpha(S)$. Without loss of generality, we may assume that $stem(S)$ and $stem(T)$ are comparable, so let $stem$ be the longer one. Notice that if $\sigma \in R$ extends the $stem$ and $\sigma^\frown i \in R$, then $i \in succ_{\Phi_\alpha(S)}(\sigma) \cap succ_{\Phi_\alpha(T)}(\sigma) \subseteq D^{B(S,\sigma)}_S \cap D^{B(T,\sigma)}_T$. It follows that

$$[R] \subseteq \{x \in \omega^\omega : \forall n \geq |stem| : x(n) \in D^{B(S,x|_n)}_S \cap D^{B(T,x|_n)}_T\}$$

so, by the Theorem 2.2.39 (or Theorem 2.2.31 in case of $\mathcal{J}$-Miller trees) the argument will be finished by checking that $D^{B(S,\sigma)}_S \cap D^{B(T,\sigma)}_T \in \mathcal{J}$ for every such σ. Consider two cases: if $B(S,\sigma) \neq B(T,\sigma)$ then they are $\mathcal{J}$-almost disjoint as taken from $\mathcal{B}_\alpha$. But $D^{B(S,x|_n)}_S \cap D^{B(T,x|_n)}_T \subseteq B(S,\sigma) \cap B(T,\sigma)$. If on the other hand $B(S,\sigma) = B(T,\sigma) = B$ then $D^{B(S,x|_n)}_S \cap D^{B(T,x|_n)}_T \in \mathcal{J}$ because $\mathcal{D}(B)$ is $\mathcal{J}$-almost disjoint. □

Corollary 3.5.25. *$\mathbb{M}_\mathcal{J}$ and $\mathbb{L}_\mathcal{J}$ are $\mathfrak{h}_\mathcal{J}$-nowhere distributive for any homogeneous F_σ-ideal $\mathcal{J}$.*

We finish this section with a question regarding two-dimensional tree types. It is easy to see that for any combinatorial tree type $\mathbb{T}$ the inequality $\mathrm{add}(\mathcal{I}(\mathbb{T}^2)) \leq \mathrm{add}(\mathcal{I}(\mathbb{T}))$ holds in ZFC. By the Proposition 2.3.3 we know that $\mathbb{L}^2$ adds Cohen reals and the inequality $\mathrm{add}(\mathcal{I}(\mathbb{L}^2)) \leq \mathrm{cov}(\mathcal{M})$ holds by the Theorem 3.1.1. On the other hand, in [Spi95] Spinas shower that it is possible get the consistency of $\mathrm{cov}(\mathcal{M}) < \mathrm{add}(\mathcal{I}(\mathbb{L}))$ (by constructing an amoeba forcing for $\mathbb{L}$ with the Laver property). It follows that $\mathrm{add}(\mathcal{I}(\mathbb{L}^2)) < \mathrm{add}(\mathcal{I}(\mathbb{L}))$.

Question 3.5.26. *Is it consistent with ZFC that* $\mathrm{add}(\mathcal{I}(\mathbb{S}^2)) < \mathrm{add}(\mathcal{I}(\mathbb{S}))$*?*

3.6 Cofinality numbers

In the paper [BKW17] Brendle, Khomskii and Wohofsky proved the following theorem:

Theorem 3.6.1. *(Brendle, Khomskii and Wohofsky, [BKW17]) Let $\mathbb{T}$ be a combinatorial tree forcing. If $\mathbb{T}$ satisfies one of the following two properties, then the cofinality of the tree ideal $\mathcal{I}(\mathbb{T})$ is strictly greater than $\mathfrak{c}$.*

- Constant or 1-1 property*:*
  For every $T \in \mathbb{T}$ and every continuous $f : [T] \to 2^\omega$ there is a tree $S \leq T, S \in \mathbb{T}$ such that the restriction $f|_{[S]}$ is either constant or 1-1.
- Incompatibility shrinking property*:*
  For every $T \in \mathbb{T}$ and a collection $\{S_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{T}$ of trees which are incompatible with T, there is a $S \in \mathbb{T}$, $S \subseteq T$ such that $[S] \cap [S_\alpha] = \emptyset$ for each $\alpha < \mathfrak{c}$.

The "Constant or 1-1 property" has been studied in the context of the minimality of forcing extensions. For example, in [Sac71] G.E.Sacks proved the minimality of forcing extensions for Sacks trees and A. Miller proved that $\mathbb{M}$ has Constant or 1-1 property (see [Mil84]). For Laver trees this property has been proved by C. Gray, in [BKW17]. The crucial tool for investigating the constant or 1-1 property will be the possibility of reading reals out of the generic real in a continuous way.
The constant or 1-1 property of the classical Sacks forcing is a well-known fact. The similar argument can be used in the case of $\mathbb{S}_F$ and $\mathbb{IE}_F$ trees

Proposition 3.6.2. *The tree types $\mathbb{S}_F$ and $\mathbb{IE}_F$ have the constant or 1-1 property.*

Proof. We will only proof the proposition in case of $\mathbb{IE}_F$-trees as the one for $\mathbb{S}_F$ is almost identical. Let $T \in \mathbb{IE}_F$ and let $f : [T] \to 2^\omega$ be continuous. Assume that f is not constant on any $[S]$ for $S \leq T$. This in particular means that $f[[S]]$ is infinite for every $S \leq T$. Inductively we will build increasing collection of finite trees $\{H_n : n \in \omega\}$ and open sets $\{U_\tau : \tau \in \bigcup_n term(H_n)\}$ such that for each $n \in \omega$ we have:

(1) H_{n+1} is an end-extension of H_n and $H_n \subseteq T$,

(2) $\forall \sigma \in term(H_n)\ \exists N > |\sigma|\ \{\tau(N) : \tau \in H_{n+1}|_\sigma\} = F(N)$,

(3) $\forall \sigma \in term(H_n)\ f[[T|_\sigma]] \subseteq U_\sigma$,

(4) $\forall \sigma, \tau \in term(H_n)$ if $\sigma \neq \tau$, then $U_\sigma \cap U_\tau = \emptyset$.

Let $H_0 = \{\emptyset\}$ and assume that H_n has been constructed. For each $\sigma \in term(H_n)$ choose $x_\sigma \in [T|_\sigma]$ such that for $\sigma \neq \tau$ we have $f(x_\sigma) \neq f(x_\sigma)$. This is possible as each set $f[[T|_\sigma]]$ is infinite. Find then collection $\{U_\sigma : \sigma \in term(H_n)\}$ of pairwise open sets such that $f(x_\sigma) \in U_\sigma$. By the continuity of f for each $\sigma \in term(H_n)$ there is $N_\sigma \in \omega$ such that $f[[T|_{x|_{N_\sigma}}]] \subseteq U_\sigma$. As $T \in \mathbb{IE}_F$ for every σ as above there is $M_\sigma > N_\sigma$ such that $\{\tau(M_\sigma) : \tau \in T|_{x|_{N_\sigma}}\} = F(M_\sigma)$. Let then H_{n+1} be the tree generated by the sets $\{\tau \in T : \exists \sigma \in term(H_n)\ x|_{N_\sigma} \subseteq \tau$ and $|\tau| = M_\sigma + 1\}$.
When the induction is over, let $S = \bigcup_n H_n$. We claim that $S \in \mathbb{IE}_F$ and the restriction $f|_{[S]}$ is 1-1. To see the first notice that if $\sigma \in S$, then $\sigma \in H_n$ for some $n \in \omega$ and the by the condition (2) in front σ there is a phenomenon required by the definition of $\mathbb{IE}_F$-trees. To see that $f|_{[S]}$ is 1-1, let $x, y \in [S]$, $x \neq y$. Let $\sigma \in Seq(F)$ be the longest common part of x and y. Let $n \in \omega$ and $\tau_x, \tau_y \in term(H_n)$ be such that $\tau_x \subseteq x$ and $\tau_y \subseteq y$. By the condition (3) for $z \in \{x, y\}$ we have that $f(z) \in f[[S|_{\tau_z}]] \subseteq f[[T|_{\tau_z}]] \subseteq U_{\tau_z}$, but by the condition (4) we have $U_{\tau_x} \cap U_{\tau_y} = \emptyset$ which shows that $f(x) \neq f(y)$. □

An immediate consequence is this is that $\mathrm{cof}(\mathcal{I}(\mathbb{IE}_F))$ and $\mathrm{cof}(\mathcal{I}(\mathbb{S}_F))$ are strictly above the continuum in ZFC. In case of $\mathbb{FM}$-trees and $\mathbb{M}_\mathcal{J}$-trees the situation is more complicated.

Theorem 3.6.3. *(Sabok and Zapletal, [SZ11]) Suppose that a σ-ideal $\mathcal{I}$ is generated by closed sets. Then $\mathbb{P}_\mathcal{I}$ has the constant or 1-1 property if and only if $\mathbb{P}_\mathcal{I}$ does not add Cohen reals. The same holds for $\mathbb{M}_\mathcal{J}$-trees.*

The forcing $\mathbb{FM}$ falls into this class of forcing notions and adds Cohen reals ([NRa93]), therefore, $\mathbb{FM}$ does not have the constant or the 1-1 property. But, by the results regarding incompatibility shrinking number of $\mathbb{FM}$ we have that under $\mathrm{add}(\mathcal{M}) = \mathfrak{c}$ the inequality $\mathrm{cof}(fm_0) > \mathfrak{c}$. This was known only under the assumption of CH ([BKW17]). However, the following problem remains unknown.

Question 3.6.4. *Is it consistent with ZFC, that $\mathrm{cof}(fm^0) = \mathfrak{c}$?*

The constant or 1-1 property has the following simple impact on random and Cohen reals

Proposition 3.6.5. *If a tree type $\mathbb{T}$ has continuous reading of names and constant or 1-1 property, then it does not add random or Cohen reals.*

Proof. Assume that $T \in \mathbb{T}$ and that $T \Vdash$ "$f(\dot{r}_{gen})$ is a random (Cohen) real". Strengthening T if necessary, we may assume that f is either constant or 1-1 on its domain, and because $f(\dot{r}_{gen})$ is a new real, the former cannot be true. Assume now that $[T] = \bigcup_{\alpha<\mathfrak{c}}[T_\alpha]$ where each $T_\alpha \in \mathbb{T}$ and $[T_\alpha] \cap [T_\beta] = \emptyset$ for $\alpha \neq \beta$. As f is 1-1, $f[[T]]$ is a disjoint union of $f[[T_\alpha]]$'s and, as $\mathcal{N}$ ($\mathcal{M}$) has c.c.c., there must be $\alpha < \mathfrak{c}$ such that $f[[T_\alpha]] \in \mathcal{N}$ ($f[[T_\alpha]] \in \mathcal{M}$). But then T_α forces $f(\dot{r}_{gen})$ to be covered by a null (meager) set. □

Let us recall the following theorem of Sabok and Zapletal which completely characterizes the constant or 1-1 property for $\mathcal{J}$-Miller trees

Theorem 3.6.6. *(Sabok and Zapletal, [SZ11]) Forcing $\mathbb{M}_\mathcal{J}$ adds Cohen reals if and only if $nwd \leq_K \mathcal{J}|_X$ for some $X \notin \mathcal{J}$. Moreover, either $\mathbb{M}_\mathcal{J}$ has the constant or 1-1 property or adds Cohen reals.*

Proposition 3.6.7. *If $\mathbb{M}_\mathcal{J}$ does not have the constant or 1-1 property, then $\mathbb{L}_\mathcal{J}$ does not have it either.*

Proof. Assume $\mathbb{L}_\mathcal{J}$ has the constant or 1-1 property, $T \in \mathbb{M}_\mathcal{J}$ and $f : [T] \to 2^\omega$ is continuous. Let $T' \in \mathbb{L}_\mathcal{J}$ be the unique tree that is constructed from T by "removing the parts in between the splinodes", i.e. $stem(T') = stem(T)$ and if for any tree T, if $s^T : \omega^{<\omega} \to split(T)$ is the unique $\subseteq$-isomorphism which preserves lexicographical order then $succ_{T'}(s^{T'}(stem^\frown\tau)) = succ_T(s^T(stem^\frown\tau))$ for each $\tau \in \omega^{<\omega}$. Next let $f' : [T'] \to 2^\omega$ be a continuous function defined for $x \in [T']$ as $f'(x) = f(\bigcup_k s^T(\tau_k))$ where $\{\tau_k : k \in \omega\} \subseteq \omega^{<\omega}$ is such that $\{x|_n : n \in \omega\} \cap split(T') = \{s^{T'}(\tau_k) : k \in \omega\}$. Now clearly any restriction $S' \in \mathbb{L}_\mathcal{J}$, $S' \leq T'$ with $f'|_{[S']}$ being constant or 1-1 produce $S \in \mathbb{M}_\mathcal{J}$, $S \leq T$ with constant or 1-1 restriction $f|_{[S]}$. □

We will now focus on proving that $\mathbb{L}_\mathcal{J}$ has the constant of 1-1 property for a certain class of not near maximal P^+-ideals. Recall that a filter $\mathcal{F}$ is *near maximal* if there is a finite to one $h : \omega \to \omega$ such that $f(\mathcal{F})$ is an ultrafilter. An ideal $\mathcal{J}$ is *near maximal* if the dual filter $\mathcal{J}^*$ is near maximal. By Talagrand's lemma (see [Tal80] or [BJ95]), all meager ideals are not near maximal, so in particular all analytic ideals are not maximal. Note that, if $\mathcal{J}$ is a P^+-ideal, and h is a finite to one function such that $h(\mathcal{J})$ is a maximal ideal, then $h(\mathcal{J})^*$ has to be a P-point. Therefore, it is consistent that all P^+-ideals are not near maximal. On the other hand, under filter dichotomy, all non-meager ideals are near maximal (see [AB89]). The proof of the main theorem of this section resembles the proof of the analogous theorem for the usual Laver trees, which appears in [BKW17]. Before we prove this, we need a list of several lemmas.

Lemma 3.6.8. *If $\mathcal{J}$ is a P^+-ideal, then for every $T \in \mathbb{L}_\mathcal{J}$ and every continuous $f : [T] \to 2^\omega$ there is $S \leq_0 T$ and $z \in 2^\omega$ such that for every $N \in \omega$, $\{i \in succ_S(stem) : f[[S|_{stem^\frown i}]] \nsubseteq [z|_N]\} \in \mathcal{F}in$.*

Proof. Assume that $T \in \mathbb{L}_\mathcal{J}$ and that $f : [T] \to 2^\omega$ is continuous. Using the pure decision property, we construct a sequence $\{T_n : n \in \omega\} \subseteq \mathbb{L}_\mathcal{J}$ such that $T_0 = T$, $T_{n+1} \leq_0 T_n$ and that T_n decides the value of $f(\dot{r}_{gen})|_n$ to be x_n. Then, as $\mathcal{J}$ is P^+, there is a pseudointersection $A \in \mathcal{J}^+$ of the family $\{succ_{T_n}(stem(T)) : n \in \omega\}$. Let $x = \bigcup_n x_n \in 2^\omega$ and let $S \in \mathbb{L}_\mathcal{J}$ be such that $stem(S) = stem(T)$, $succ_S(stem(S)) = A$ and for $n \in A$, $S|_{stem(S)^\frown n} = T_k|_{stem(S)^\frown n}$ where $k \in \omega$ is the last such that $n \in succ_{T_k}(stem(S))$. □

Lemma 3.6.9. *Let $\mathcal{J}$ be a P^+-ideal and let $\{A_n : n \in \omega\} \subseteq \mathcal{J}^+$, $z \in 2^\omega$ and $f_n : A_n \to 2^\omega$ be such that, for all $n, k \in \omega$, $f_n^{-1}([z|k] \cap f[A_n]) \in \mathcal{J}^+$ and $z \notin f_n[A_n]$. Then there are $B_n \subseteq A_n$ such that $B_n \in \mathcal{J}^+$ and $\bigcup_{n\in\omega} f_n[B_n]$ is discrete.*

Proof. Using the P^+ property, it is possible to restrict to some $A'_n \subseteq A_n$ such that $f_n[A'_n]$ converges to z, and therefore $f_n[A'_n]$ is discrete. Let

$$B_n = A'_n \setminus f_n^{-1}([z|n])$$

then $B_n \subseteq A_n$, $B_n \in \mathcal{J}^+$ and, since $\bigcup_{n\in\omega} f_n[B_n] \setminus f_n^{-1}([z|_n])$ is finite for every $k \in \omega$, then $\bigcup_{n\in\omega} f_n[B_n]$ is discrete. □

Lemma 3.6.10. *Let $\mathcal{J}$ be a P^+-ideal such that it is not near maximal, $A, B \in \mathcal{J}^+$, $z \in 2^\omega$, $f : A \to 2^\omega$ and $g : B \to 2^\omega$ be such that $z \notin f[A] \cup g[B]$ and both $f[A]$ and $g[B]$ converge to z. Then, there are $A' \subseteq A$ and $B' \subseteq B$ such that $f[A'] \cap g[B'] = \emptyset$.*

Proof. Let $F : A \to \omega$ defined by $F(n) = k$ if and only if $f(n)|_k = z|_k$ but $f(n)|_{k+1} \neq z|_{k+1}$. Define $G : B \to \omega$ in a similar way, using g instead of f. Notice that both F and G are finite to one functions, so there must be $A", B" \subseteq \omega$ such that $A_0 = F^{-1}(A"), A_1 = F^{-1}(\omega \setminus A")$, $B_0 = G^{-1}(B"), B_1 = G^{-1}(\omega \setminus B")$ are all in $\mathcal{J}^+$. Then it is easy to show that, either $A_0 \cap B_0$ and $A_1 \cap B_1$ are in $\mathcal{J}^+$, or $A_1 \cap B_0$ and $A_0 \cap B_1$ are in $\mathcal{J}^+$. □

Lemma 3.6.11. *Let $\mathcal{J}$ be a P^+-ideal such that it is not near maximal, $\{B_n : n \in \omega\} \subseteq \mathcal{J}^+$, $z \in 2^\omega$ and $f_n : B_n \to 2^\omega$ such that $B_n \in \mathcal{J}^+$, $f_n[B_n]$ converges to z, then there are $C_n \subseteq B_n$ such that $C_n \in \mathcal{J}^+$, $\bigcup_{n\in\omega} f_n[C_n]$ is discrete, and for each $i, j \in \omega$, $i \neq j$ implies that $f_i[C_i] \cap f_j[C_j] = \emptyset$.*

Proof. By Lemma 3.6.9, we may assume that $\bigcup_{n\in\omega} f_n[B_n]$ is discrete. Recursively, we are going to construct a family of sequences $\{S^i_j : i \leq j, j \in \omega\}$ such that, for each $i \leq j, j \in \omega$,

1. $S^i_j \subseteq B_j$ is $\mathcal{J}^+$,
2. $S^j_j \subseteq S^i_j$,
3. for $m \geq i$, $f_m[S^i_m] \cap f_i[S^i_i]$ is finite.

Assume we were able to carry out the construction of $\langle S^i_j : i \leq j, j \in \omega\rangle$ for all $i < k$, we are going to construct $\langle S^k_j : k \leq j, j \in \omega\rangle$: Let $S'_k = \bigcap_{i<k} S^i_k$ and recursively apply the previous remark to get two families $\langle S'_\ell : k < \ell\rangle$ and $\langle S^k_\ell : k < \ell\rangle$ such that,

- $S'_\ell, S^k_\ell \in \mathcal{J}^+$,
- $S'_{\ell+1} \subseteq S'_\ell \subseteq S'_k$,
- for all $i < k$, $S^k_\ell \subseteq S^i_\ell$,
- $f_k[S'_\ell] \cap f_\ell[S^k_\ell] = \emptyset$.

Finally, let S^k_k be a positive pseudo-intersection of the family $\langle S'_\ell : k < \ell\rangle$. It is routine to check that $\langle S^k_j : k \leq j, j \in \omega\rangle$ satisfies the properties we are looking for.

Finally, let $C_n = S^n_n \setminus f_n^{-1}(\bigcup_{i<n} f_i[S^i_i])$. It is easy to see that the family $\langle C_n : n \in \omega\rangle$ satisfies the desired properties. □

We are now ready to prove the main theorem of this section.

Theorem 3.6.12. *If $\mathcal{J}$ is a P^+-ideal and $\mathcal{J}$ is not near maximal, then $\mathbb{L}_\mathcal{J}$ has the constant or 1-1 property.*

Proof. Assume that $T \in \mathbb{L}_{\mathcal{J}}$ and $f : [T] \to 2^\omega$ is continuous. Using a standard fusion argument and Lemma 3.6.8, we can get a tree $S \leq_0 T$ and a collection $\{z_\sigma : \sigma \in split(S)\} \subseteq 2^\omega$ such that, for each $\sigma \in split(S)$ we have

$$\forall N \in \omega \ \{i \in succ_S(\sigma) : f[[S|_{\sigma^\frown i}]] \nsubseteq [z_\sigma|_N]\} \in \mathcal{F}in. \tag{3.5}$$

This in particular means that the sequence $\{z_{\sigma^\frown n} : n \in \omega\}$ converges to z_σ. To decide whether the restriction we are looking for is constant or 1-1 we will use a rank argument. Define the function $rank : split(S) \to \omega_1$ as:

$$rank(\sigma) = \begin{cases} 0 & \text{if } \{n \in \omega : z_{\sigma^\frown n} \neq z_\sigma\} \in \mathcal{J}^+, \\ \alpha & \text{if } \alpha \text{ is the smallest such that } \{n \in \omega : rank(\sigma^\frown n) < \alpha\} \in \mathcal{J}^+. \end{cases}$$

Now we consider two cases:

Case 1: For some $\sigma \in split(S)$ the rank is not defined. This implies that $\{n \in \omega : rank(\sigma^\frown n)$ is defined $\} \in \mathcal{I}$. Using this, together with a standard fusion argument, we trim S to get a $R \leq_0 S_\sigma$ such that for all $\tau \in split(R)$ we have $z_\tau = z_\sigma$. We conclude that the restriction $f|_{[R]}$ is constant with value z_σ.

Case 2: The rank is defined for all $\sigma \in split(S)$. We will inductively construct an increasing sequence $\{F_n : n \in \omega\}$ of well-founded subtrees of S and disjoint clopen sets $\{V_{\sigma|_{|\sigma|-1}} : \sigma \in term(F_n)\}$ such that for all $\sigma \in \omega^{<\omega}$ and $n \in \omega$ we have:

A. if $\sigma \in F_n$ is not a terminal node of F_n then $succ_{F_n}(\sigma) \in \mathcal{J}^+$,

B. if $\sigma \in F_{n+1} \setminus F_n$ then σ extends a terminal node of F_n,

C. for each $\sigma \in term(F_n)$ we have $f[[S_\sigma]] \subseteq V_{\sigma|_{|\sigma|-1}}$.

Assume for a moment that such a sequence has been constructed and let $R = \bigcup_n F_n$. Clearly $R \leq S$, and the condition A implies $R \in \mathbb{L}_{\mathcal{J}}$. We claim that the restriction $f|_{[R]}$ is 1-1. Let $x, y \in [R]$, and $x \neq y$. Then there are $\sigma, \tau \in \omega^{<\omega}$, $\sigma \neq \tau$ and $n \in \omega$ such that $\sigma \subseteq x, \tau \subseteq y$ and σ, τ are terminal nodes of F_n. The condition C guarantees that $f(x) \in V_{\sigma|_{|\sigma|-1}}$ and $f(y) \in V_{\tau|_{|\tau|-1}}$ so $f(x) \neq f(y)$.

To finish the proof, we will show how to carry out the inductive construction: Let $F_0 = \{stem(S)\}$ and assume that F_n has been built. Fix a terminal node σ of F_n. As the rank of σ is defined, there is a well-founded tree $G_\sigma \subseteq \{\tau \in \omega^{<\omega} : \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$ such that all terminal nodes of G_σ have rank 0 and all non-terminal nodes of G_σ splits into $\mathcal{J}$-positive sets. We then apply Lemma 3.6.11 to the family $\{A_\tau : \tau \in term(G_\sigma)\} \subseteq \mathcal{J}^+$, $A_\tau = \{n \in \omega : z_{\tau^\frown n} \neq z_\tau\}$, and the functions $f_\tau : A_\tau \to 2^\omega$ defined as $f_\tau(n) = z_{\tau^\frown n}$. We obtain $B_\tau \subseteq A_\tau$, $B_\tau \in \mathcal{J}^+$ such that $D = \bigcup\{f_\tau[B_\tau] : \tau \in term(G_\sigma)\}$ is a discrete subset of 2^ω and for $\tau \neq \tau'$, we have that $f_\tau[B_\tau] \cap f_{\tau'}[B_{\tau'}] = \emptyset$. Let then $\{U_z : z \in D\}$ be a collection of disjoint clopen sets such that $z \in U_z$ for each $z \in D$. By the condition (3.5), we can remove $\mathcal{J}$-many points from each B_τ to obtain $\mathcal{J}$-positive sets $C_\tau \subseteq B_\tau$ such that for every $m \in C_\tau$ we have $f[[S|_{\tau^\frown m}]] \subseteq U_{z_{\tau^\frown m}}$. Let H_σ be the tree generated by the set $\{\tau^\frown m : \tau \in term(G_\sigma)$ and $m \in C_\tau\}$. Finally, we unfix σ and let F_{n+1} be the union of all H_σ's where σ is a terminal node of F_n. This finishes the construction. □

As a corollary, we get the following result

Corollary 3.6.13. *If $\mathcal{J}$ is a P^+-ideal, then* $\mathrm{cof}(\mathcal{I}(\mathbb{L}_{\mathcal{J}})) > \mathfrak{c}$.

The above theorem cannot characterize constant or 1-1 property for $\mathbb{L}_{\mathcal{J}}$-trees. An example of non P-Borel ideal $\mathcal{J}$ for which the $\mathbb{L}_{\mathcal{J}}$-trees have constant or 1-1 property is the following. It has been obtained by Osvaldo Guzmán and Francisco Santiago during the work on this tesis [GS24]

Theorem 3.6.14. *(Guzmán and Santiago, [GS24])* $\mathbb{L}_{\mathcal{F}in\times\mathcal{F}in}$ *has the constant or 1-1 property.*

I do not know if there is a characterization for $\mathbb{L}_{\mathcal{J}}$-tree of having the constant or 1-1 property for $\mathcal{J}$-Laver trees. In particular we do not know the answer to the following question.

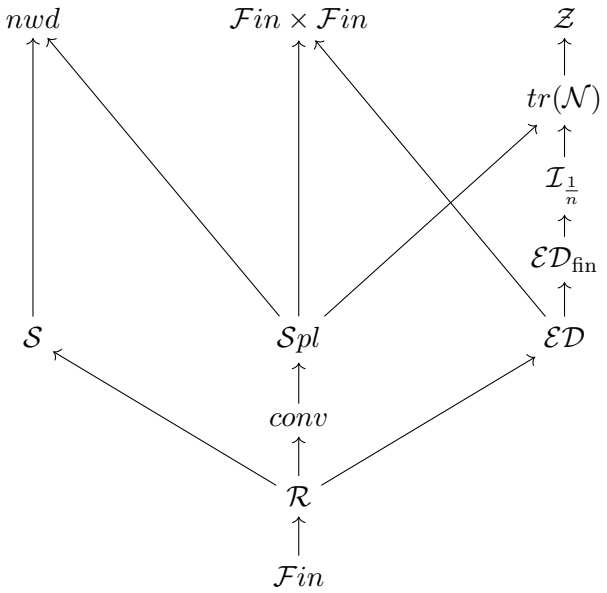
Question 3.6.15. *Does* $\mathbb{L}_{\mathcal{Z}}$ *has the constant or 1-1 property?*

Chapter 4

Cardinal invariants of ideals on ω

4.1 Catalogue of ideals on ω

Here we gather all the ideals with their cardinal characteristics, which are relevant to this work. Let us list all ideals that are of particular importance for us in this thesis. For each one, we will discuss the basic properties we are going to need, cardinal invariants and their ω-versions. For more information about these, the reader may consult [Hru11].
The following diagram shows the complete picture of the mentioned ideals in the Katetov order. The arrows point toward the ideal which is Katetov stronger.



4.1.1 F_σ-ideals

A function $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is a lower semi-continuous submeasure (l.s.c.s.m. for short) if the following conditions hold:

- $\phi(\emptyset) = 0$, $\phi(\{n\}) < \infty$ for every $n \in \omega$,
- $\phi(X) \leq \phi(Y)$ for $X \subseteq Y$,
- $\phi(X \cup Y) \leq \phi(X) + \phi(Y)$ for all $X, Y \subseteq \omega$,
- $\phi(X) = \sup\{\phi(X \cap n) : n \in \omega\}$ for each $X \subseteq \omega$.

There is a natural way to associate an ideals to such ϕ. Define

$$Fin(\phi) = \{X \subseteq \omega : \phi(X) < \infty\}$$

It is easy to see that this is an F_σ and that such ideal is a P^+-ideal. The following is a well-known characterization of F_σ-ideals.

Theorem 4.1.1. *(Mazur,[Maz91]) An ideal $\mathcal{J}$ is F_σ if and only if $\mathcal{J} = Fin(\phi)$ for some l.s.c.s.m. ϕ.*

We start the list of ideals important to us with the following.

Summable ideals

Given a function $f : \omega \to [0, +\infty)$ such that $\Sigma_{n\in\omega} f(n) = \infty$ we define a summable ideal corresponding to f as follows:

$$\mathcal{I}_f = \{A \subseteq \omega : \Sigma_{n\in A} f(n) < \infty\}$$

Clearly $\mathcal{I}_f$ is an F_σ-ideal with the associated submeasure defined as $\phi(A) = \Sigma_{n\in A} f(n)$. It follows that $\mathcal{I}_f$ is a P^+-ideal. It is also a P-ideal. $\mathcal{I}_f$ is tall if and only if $lim_n f(n) = 0$. In this thesis, we will only consider the classical summable ideal $\mathcal{I}_{\frac{1}{n}}$ defined as $\mathcal{I}_f$ where $f(n) = \frac{1}{n+1}$.

The eventually different ideal

Let $\mathcal{ED}$ be an ideal on $\omega \times \omega$ consisting of these sets A such that for some N for all $n > N$ the set $\{m : (n, m) \in A\}$ has at most N elements. Let $\mathcal{ED}_{fin}$ be the restriction $\mathcal{ED}|_\Delta$ where $\Delta = \{(n, m) \in \omega \times \omega : m \leq n\}$. For $F \in \omega^\omega$ one could consider an ideal $\mathcal{ED}_F$ defined as restriction of $\mathcal{ED}$ to the set of form $\{(n, m) \in \omega \times \omega : m \leq F(m)\}$. It is well-known, that such $\mathcal{ED}_F$ is Katetov equivalent with $\mathcal{ED}_{fin}$. It follows that the two have the same covering and uniformity numbers.
Both $\mathcal{ED}$ and $\mathcal{ED}_{fin}$ are F_σ-ideals: for an $A \subseteq \Delta$, the submeasure $\phi(A)$ measures how large, in the sense of cardinality, a vertical sections of A are. It follows that both $\mathcal{ED}$ and $\mathcal{ED}_{fin}$ are P^+-ideals but neither $\mathcal{ED}$ nor $\mathcal{ED}_{fin}$ is a P-ideal. The cardinal invariants have the following values:

Theorem 4.1.2. *(Hrušák, Meza-Alcántara, Minami, [HMAM10]) The following equalities hold:*

- $\text{add}^*(\mathcal{ED}) = \text{non}^*(\mathcal{ED}) = \omega$,
- $\text{cov}^*(\mathcal{ED}) = \text{non}(\mathcal{M})$,
- $\text{cof}^*(\mathcal{ED}) = \mathfrak{c}$.

Theorem 4.1.3. *(Hrušák, Meza-Alcántara, Minami, [HMAM10]) The following equalities hold:*

- $\text{add}^*(\mathcal{ED}_{fin}) = \omega$,
- $\text{cov}(\mathcal{M}) \leq \text{non}^*(\mathcal{ED}_{fin}) \leq \mathfrak{r}$,
- $\mathfrak{s} \leq \text{cov}^*(\mathcal{ED}_{fin}) \leq \text{non}(\mathcal{M})$,
- $\min\{\mathfrak{d}, \text{non}^*(\mathcal{ED}_{fin})\} = \text{cov}(\mathcal{M})$,
- $\max\{\mathfrak{b}, \text{cov}^*(\mathcal{ED}_{fin})\} = \text{non}(\mathcal{M})$,
- $\text{cof}^*(\mathcal{ED}_{fin}) = \mathfrak{c}$.

The last point of the above theorem shows that cov^* and non^* of the $\mathcal{ED}_{fin}$ ideal in a sense completes Cichońs diagram by making the $\text{cov}(\mathcal{M})$ and $\text{non}(\mathcal{M})$ the minimum and maximum of the corresponding invariants. We also have

Theorem 4.1.4. *(Hrušák, Meza-Alcántara, Minami, [HMAM10]) The* $\mathrm{cov}^*(\mathcal{ED}_{fin}) < \mathrm{add}(\mathcal{M})$ *holds in the Hechlers model.* $\mathrm{cof}(\mathcal{M}) < \mathrm{non}^*(\mathcal{ED}_{fin})$ *holds in the dual Hechlers model.*

In case of the ω-versions of the above we have the following result.

Proposition 4.1.5. *The following holds:*

- $\mathrm{add}^*_\omega(\mathcal{ED}) = \mathrm{add}^*_\omega(\mathcal{ED}_{fin}) = \omega_1$,
- $\mathrm{cof}^*_\omega(\mathcal{ED}) = \mathrm{cof}^*_\omega(\mathcal{ED}_{fin}) = \mathfrak{c}$.

Proof. We will focus only for the $\mathcal{ED}$ ideal as the arguments for the $\mathcal{ED}_{fin}$ are identical. For $N \in \omega$ and $F \in [\omega^\omega]^N$ denote by $[N, F]$ the set $\{(n, m) : n < N \text{ or } m \in \{f(n) : f \in F\}\}$. Clearly, any $A \in \mathcal{ED}$ can be covered by the set of this form. To prove the first equality fix an eventually different family $\{e_\alpha : \alpha < \omega_1\}$. We claim that $\{e_\alpha : \alpha < \omega_1\}$ is a witness for $add^*_\omega(\mathcal{ED})$. To show this, assume that we are given $\{A_n : n \in \omega\} \subseteq \mathcal{ED}$ where $A_n = [N_n, F_n]$ for some $N_n \in \omega$ and $F_n \in [\omega^\omega]^{<\omega}$. Observe the following

Claim 4.1.6. *For each* $F \in [\omega^\omega]^{<\omega}$ *the set* H_F *defined as* $\{\alpha \in \omega_1 : e_\alpha$ *is almost contained in* $\bigcup F\}$ *is of cardinality at most* $|F|$.

Proof. Towards contradiction assume that there are $|F| + 1$ elements of the set H_F. Let $N \in \omega$ be large enough so that any two members of H_F take different values above the N. But then we have $|F| + 1 = |\{e(N + 1) : e \in F\}| \leq |F|$, a contradiction. □

We continue the proof of the proposition. Because of the claim we may choose $\alpha_0 \in \omega_1 \setminus \bigcup_n H_{F_n}$. But then for each $n \in \omega$ the set $e_{\alpha_0} \setminus A_n$ is infinite. This finishes the proof of the first equality.
To show the second equality, assume that $\kappa < \mathfrak{c}$ and that $\{B_\beta : \beta < \kappa\}$ is a basis of $\mathcal{ED}$. Assume that each B_β is of form $[N_\beta, F_\beta]$ for some $N_\beta \in \omega$ and $F_\beta \in [\omega^\omega]^{<\omega}$. Let $\{e_\alpha : \alpha < \mathfrak{c}\}$ be an eventually different family. By the claim above we may choose $\alpha_0 \in \mathfrak{c} \setminus \bigcup_{\beta<\kappa} H_{F_\beta}$. Let $\overline{A} \in [\mathcal{ED}]^\omega$ be defined as a constant sequence with the value e_{α_0}. But then (every) element of $\overline{A}$ is not almost covered by any B_β which shows that B_β's did not form a basis in the first place. □

Proposition 4.1.7. $\mathrm{non}^*_\omega(\mathcal{ED}) = \mathrm{cov}(\mathcal{M})$

Proof. As $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}^*(\mathcal{ED}) \leq \mathrm{non}^*_\omega(\mathcal{ED})$, it is enough to prove $\mathrm{non}^*_\omega(\mathcal{ED}) \leq \mathrm{cov}(\mathcal{M})$. Let $\kappa < \mathrm{non}^*_\omega(\mathcal{ED})$ and let $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$. We will find $h \in \omega^\omega$ that is infinitely equal to all f_α's. Let $(I_n)_{n\in\omega}$ be such an interval partition that $|I_n| = n+1$. For every $n \in \omega$ fix a correspondence between the functions $\sigma : I_n \to \omega$ and elements of ω. For every $\alpha < \kappa$ let $g_\alpha \in \omega^\omega$ be such that $g_\alpha(n)$ encodes $f_\alpha|_{I_n}$. As $\kappa < \mathrm{non}^*_\omega(\mathcal{ED})$, there is a countable collection of $\mathcal{ED}$-sets $\{X_n : n \in \omega\}$ such that every g_α has an infinite intersection with some X_n. Every set X_n can be covered by a set of form $(N_n \times \omega) \cup \bigcup F_n$ where $N_n \in \omega$ and $F_n \in [\omega]^{<\omega}$. Note that $\mathcal{H} = \bigcup_n F_n$ is such a subset of ω^ω that every g_α infinitely intersects some $h \in \mathcal{H}$. Enumerate $\mathcal{H}$ as $\{h_n : n \in \omega\}$. We will now construct a function $h \in \omega^\omega$ that infinitely intersects every f_α. Fix $n \in \omega$. We will define $h|_{I_n}$. Let i be the k´th element of I_n. Let $\sigma : I_n \to \omega$ be the function that corresponds to the $h_k(n)$. Define then $h(i) = \sigma(i)$. We will finish the proof by showing that $|h \cap f_\alpha| = \omega$ for every $\alpha \in \kappa$. By the property of $\mathcal{H}$, there is $h_n \in \mathcal{H}$ and $A \in [\omega]^\omega$ such that for every $m \in A$, $h_n(m) = g_\alpha(m)$. But then for every $m \in A \setminus n$ there is $i \in I_m$ such that $h(i) = f_\alpha(i)$ meaning that h is infinitely equal to every f_α. □

Question 4.1.8. *Is* $\mathrm{non}^*(\mathcal{ED}_{fin}) = \mathrm{non}^*_\omega(\mathcal{ED}_{fin})$*?*

The Solecki's ideal

Let $\mathcal{S}$ be the Solecki's ideal, an ideal on $\Omega =: \{C \subseteq 2^\omega : C \text{ is a clopen set and } \lambda(C) = 1/2\}$ generated by sets of the form $[F] =: \{C \in \Omega : C \cap F \neq \emptyset\}$ where $F \in [2^\omega]^{<\omega}$. The $\mathcal{S}$ ideal is a tall F_σ-ideal: for an $A \subseteq \Omega$, the submeasure $\phi(A)$ measures how many elements of 2^ω do we need to intersects every element of A. It follows that $\mathcal{S}$ is a P^+-ideal. It is not a P-ideal. The following proposition is known:

Theorem 4.1.9. *(Hrušák, Meza-Alcántara, Minami, [HMAM10]) The following equalities hold:*

- $\mathrm{add}^*(\mathcal{S}) = \mathrm{non}^*(\mathcal{S}) = \omega$,
- $\mathrm{cov}^*(\mathcal{S}) = \mathrm{non}(\mathcal{N})$,
- $\mathrm{cof}^*(\mathcal{S}) = \mathfrak{c}$.

In case of ω-versions of the invariants we have the following result. The equality $\mathrm{add}^*_\omega(\mathcal{S}) = \omega_1$ is known and appears in the work of Filipów and Kwela [FK23].

Proposition 4.1.10. *The following holds:*

- $\mathrm{add}^*_\omega(\mathcal{S}) = \omega_1$,
- $\mathrm{cof}^*_\omega(\mathcal{S}) = \mathfrak{c}$.

Proof. The prove the first equality let $\{x_\alpha : \alpha < \omega_1\} \subseteq 2^\omega$ consists of distinct points. We claim that the collection $\{[\{x_\alpha\}] : \alpha < \omega_1\}$ is a witness for $\mathrm{add}^*_\omega(\mathcal{S})$. To show this assume that we are given $\{A_n : n \in \omega\} \subseteq \mathcal{S}$ where $A_n = [F_n]$ and $F_n \in [2^\omega]^{<\omega}$. Enumerate $\bigcup_n F_n$ as $\{y_n : n \in \omega\}$ and pick $x_\alpha \notin \{y_n : n \in \omega\}$. Clearly, for each $n \in \omega$, there are infinitely many clopen sets C of measure $1/2$ which contain x_α but not y_n.
The prove the second part let $\kappa < \mathfrak{c}$ and assume that the collection $\{\overline{A}_\alpha : \alpha < \kappa\}$ forms a witness for $\mathrm{cof}^*_\omega(\mathcal{S})$. Assume that $\overline{A}_\alpha = \{A^\alpha_n : n \in \omega\} \in \mathcal{S}^\omega$ where $A^\alpha_n = [F^\alpha_n]$ for some $F^\alpha_n \in [2^\omega]^{<\omega}$. Pick $x \in 2^\omega \setminus \bigcup_{\alpha,n} F^\alpha_n$ and define $\overline{B} \in \mathcal{S}^\omega$ to be a constant sequence with the value $[\{x\}]$. The proof is finished because for every $n \in \omega$ and every $\alpha \in \kappa$ there are infinitely many clopens $C \in \Omega$ such that $x \in C$ but $C \cap F^\alpha_n = \emptyset$. □

An important property of the ideal $\mathcal{S}$, which we going to use later on, is the following. We will say that an ideal $\mathcal{J}$ has the Fubini property if for every Borel subset B of $\omega \times 2^\omega$ and every $\epsilon > 0$, if $\{n \in \omega : \mu(B_n) > \epsilon\} \in \mathcal{J}^+$, then $\mu^*(\{x \in 2^\omega : (B)^x \in \mathcal{J}^+\}) \geq \epsilon$. We have the following result.

Theorem 4.1.11. *(Solecki, [Sol00]; Meza-Alcantara [MA09]) For any ideal $\mathcal{J}$ the following are equivalent:*

- *there is $Y \in \mathcal{J}^+$ such that $\mathcal{S} \leq_K \mathcal{J}|_Y$*
- *$\mathcal{J}$ does not have the Fubini property, i.e. there is a Borel set $B \subseteq \omega \times 2^\omega$ and $\epsilon > 0$ such that $\{n \in \omega : \lambda(B_n) > \epsilon\} \in \mathcal{J}^+$ and $\lambda^*(\{x \in 2^\omega : B^x \in \mathcal{J}^+\}) < \epsilon$*

Van der Waerden ideal

Let $\mathcal{W}$ be an ideal on ω consisting of these $A \subseteq \omega$ which does not contain an arithmetical progression of arbitrary length. The ideal $\mathcal{W}$ is tall and is not a P-ideal. The $\mathcal{W}$ ideal is an F_σ-ideal: for $A \subseteq \omega$, the submeasure $\phi(A)$ measures how long arithmetical sequence does A contain. It follows that $\mathcal{W}$ is a P^+-ideal. Van der Waerden's well-known theorem says that $\mathcal{W}$ is a proper ideal. Szemerédi's famous theorem says that $\mathcal{W} \subseteq \mathcal{Z}$. The stronger statement $\mathcal{W} \subseteq \mathcal{I}_{\frac{1}{n}}$ is a still open Erdos prize problem.

Theorem 4.1.12. *(Brendle, Flaskova; [Fla10]) The following equalities hold:*

- $\mathrm{add}^*(\mathcal{W}) = \omega$,
- $\mathrm{non}^*(\mathcal{ED}_{fin}) \leq \mathrm{non}^*(\mathcal{W}) \leq \min\{\mathfrak{r}, \mathrm{non}^*(\mathcal{Z})\}$,
- $\max\{\mathfrak{s}, \mathrm{cov}^*(\mathcal{Z})\} \leq \mathrm{cov}^*(\mathcal{W}) \leq \mathrm{cov}^*(\mathcal{ED}_{fin})$,
- $\mathrm{cof}^*(\mathcal{W}) = \mathfrak{c}$.

Random graph ideal

Let (ω, E), $E \subseteq [\omega]^2$ be the random graph, i.e. up to isomorphism the only graph with the following property: for every pair $A, B \subseteq \omega$ of non-empty, finite, disjoint sets there is an $n \in \omega \setminus (A \cup B)$ such that $\{\{n, a\} : a \in A\} \subseteq E$ and $\{\{n, b\} : b \in b\} \cap E = \emptyset$. The random graph ideal $\mathcal{R}$ is the ideal generated by homogeneous subsets (cliques and anticliques) of the random graph. $\mathcal{R}$ is tall. It is an F_σ-ideal: for $A \subseteq E$, the submeasure $\phi(A)$ measures what is the minimum amount of cliques and anticliques in E needed to cover A. It follows that $\mathcal{R}$ is a P^+-ideal. It is not a P-ideal. The following is known.

Theorem 4.1.13. *(Meza-Alcantara, [MA09]) The following equalities hold:*

- $\mathrm{add}^*(\mathcal{R}) = \mathrm{non}^*(\mathcal{R}) = \omega$,
- $\mathrm{cov}^*(\mathcal{R}) = \mathrm{cof}^*(\mathcal{R}) = \mathfrak{c}$.

Proposition 4.1.14. *The following equalities hold:*

- $\mathrm{add}^*_\omega(\mathcal{R}) = \mathrm{non}^*_\omega(\mathcal{R}) = \omega_1$,
- $\mathrm{cof}^*_\omega(\mathcal{R}) = \mathfrak{c}$.

Proof. The equality $\mathrm{non}^*_\omega(\mathcal{R}) = \omega_1$ follows from $\mathcal{R} \leq_K conv$ and the fact that $\mathrm{non}^*_\omega(conv) = \omega_1$ (Proposition 4.1.33) and that Katetov reduction preserves non^*_ω (Proposition 1.2.12). The $\mathrm{cof}^*_\omega(\mathcal{R}) = \mathfrak{c}$ follows from $\mathrm{cof}^*(\mathcal{R}) = \mathfrak{c}$. □

4.1.2 analytic P-ideals

The following is a well-known characterization of analytic P-ideal using lower semi continuous submeasures.

Theorem 4.1.15. *(Solecki,[Sol99]) Every analytic P-ideal is of form*

$$Exh(\phi) = \{X \subseteq \omega : lim_n \phi(X \setminus n) = 0\}$$

for some lower semi continuous submeasure ϕ.

It is not difficult to see that $Exh(\phi)$ is in fact an $F_{\sigma\delta}$-ideal. It is well-known, that for a l.s.c.s.m. ϕ, the ideal $Exh(\phi)$ is tall if and only if $lim_n \phi(\{n\}) = 0$. Our first example of analytic P-ideal is the already defined summable ideal $\mathcal{I}_{\frac{1}{n}}$. Regarding the additivities and cofinalities of analytic P-ideals, we have the following theorem.

Theorem 4.1.16. *(Todorcevic [Tod96]) For any tall analytic P-ideal $\mathcal{J}$ the Tukey reductions $(\emptyset \times \mathcal{F}in, \subseteq^*, \emptyset \times \mathcal{F}in) \leq_T (\mathcal{J}, \subseteq^*, \mathcal{J}) \leq_T (\mathcal{N}, \subseteq, \mathcal{N})$ take place. In particular the following inequalities hold:*

- $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}^*(\mathcal{J}) \leq \mathfrak{b}$,
- $\mathfrak{d} \leq \mathrm{cof}^*(\mathcal{J}) \leq \mathrm{cof}(\mathcal{N})$.

Regarding the covering and uniformity numbers, the following fact is well-known.

Theorem 4.1.17. *(folklore, [MA09]) For any tall analytic P-ideal $\mathcal{J}$ we have $\mathcal{ED}_{fin} \leq_K \mathcal{J}$. In particular the following inequalities hold:*

- $\mathrm{cov}^*(\mathcal{J}) \leq \mathrm{non}(\mathcal{M})$,
- $\mathrm{cov}(\mathcal{M}) \leq \mathrm{non}^*(\mathcal{J})$.

We will be interested in the following two examples of analytic P-ideals.

Trace of null ideal and density zero ideal

The trace of null ideal $tr(\mathcal{N})$ is an ideal on $2^{<\omega}$ consisting of those sets $A \subseteq 2^{<\omega}$ such that the set $G(A) := \{x \in 2^\omega : \exists_n^\infty x|_n \in A\}$ is of Lebesgue measure zero. The density zero ideal $\mathcal{Z}$ consists of these $A \subseteq \omega$ such that $lim_{n\to\infty} \frac{|A\cap n|}{n} = 0$. Both of these ideals are tall and both can be expressed in the form of $Exh(\phi)$ for certain l.s.c.s.m. ϕ: for the density zero ideal, if $A \subseteq \omega$ then $\phi(A) =: sup\{\frac{|A\cap[2^n,2^{n+1})|}{2^n} : n \in \omega\}$. For the trace of null ideal, if $A \subseteq 2^{<\omega}$, $\phi(A) = \Sigma\{\frac{1}{2^{|\sigma|}} : \sigma \in A$ is $\subseteq$-minimal in $A\}$. It is known that the density zero ideal is Katetov equivalent with the ideal of subsets $A \subseteq 2^{<\omega}$ such that $lim_{n\to\infty} \frac{|A\cap 2^n|}{2^n} = 0$ ([HHH07], Theorem 3.4).
Recall that for P-ideal $\mathcal{J}$, we have $\mathrm{add}^*_\omega(\mathcal{J}) = \mathrm{add}^*(\mathcal{J})$ $\mathrm{non}^*_\omega(\mathcal{J}) = \mathrm{non}^*(\mathcal{J})$ and $\mathrm{cof}^*_\omega(\mathcal{J}) = \mathrm{cof}^*(\mathcal{J})$. In the case of our examples $\mathcal{I}_{\frac{1}{n}}$, $tr(\mathcal{N})$ and $\mathcal{Z}$ the the following is known.

Theorem 4.1.18. *(Fremlin, [Fre04])* $\mathrm{add}^*(\mathcal{Z}) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}^*(\mathcal{Z}) = \mathrm{cof}(\mathcal{N})$ *hold.*

Theorem 4.1.19. *(Hernández-Hernández and Hrušák, [HHH07])* $\mathrm{add}^*(tr(\mathcal{N})) = \mathrm{add}^*(\mathcal{I}_{\frac{1}{n}}) = \mathrm{add}(\mathcal{N})$,

Theorem 4.1.20. *(Hrušák and Zapleta, [HZ04]) The following hold:*

- $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}^*(tr(\mathcal{N})) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$,
- $\min\{\mathfrak{b}, \mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}^*(\mathcal{Z}) \leq \min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\}$.

Theorem 4.1.21. *(Raghavan and Shelah, [RS17]) The* $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{d}$ *and* $\mathfrak{b} \leq \mathrm{non}^*(\mathcal{Z})$ *hold.*

In the section regarding the splitting ideal we will show that $\mathrm{cov}^*(\mathcal{Z})$ (and in fact $\mathrm{cov}^*(tr(\mathcal{N}))$) is below the reaping number $\mathfrak{r}$. This gives more support to the following open problem:

Question 4.1.22. *(Hernández-Hernández and Hrušák, [HHH07]) Is* $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{b}$ *and* $\mathfrak{d} \leq \mathrm{non}^*(\mathcal{Z})$ *provable in ZFC?*

4.1.3 Other examples

The nowhere dense ideal

Usually the ideal *nwd* is defined as $\{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A}$ is nowgere dense subset of $[0,1]\}$. We prefer to work with the following, different presentation of *nwd* ideal. By the Lemma 81 of [Guz] the *nwd* ideal is Katetov equivalent with the ideal on $2^{<\omega}$ described below. Because of this, we will overload the symbol *nwd* for the second ideal. Let *nwd* be the ideal on $2^{<\omega}$ consisting of these sets A that are nowhere dense in $2^{<\omega}$ i.e. for any $\sigma \in 2^{<\omega}$ there is $\tau \in 2^{<\omega}$, $\sigma \subseteq \tau$ with $\tau \notin A$. Clearly, any $A \in nwd$ can be covered by the set of form of a tree $T \subseteq 2^{<\omega}$ with $[T]$ being nowhere dense set in 2^ω. The ideal *nwd* is not a P-ideal and not a P^+-ideal. It compexity is $F_{\sigma\delta}$. The following proposition is known:

Theorem 4.1.23. *(Balcar, Hernández-Hernández and Hrušák, [BHHH04]) The following equalities hold:*

- $\mathrm{add}^*(nwd) = \mathrm{non}^*(nwd) = \omega$,

– $\mathrm{cov}^*(nwd) = \mathrm{cov}(\mathcal{M})$,

– $\mathrm{cof}^*(nwd) = \mathrm{cof}(\mathcal{M})$.

Later we will need the following characterization regarding the Katetov relation to *nwd*.

Lemma 4.1.24. *(Hrušák, Meza, [MA09]) The following are equivalent:*

– $\mathcal{J} \leq_K nwd$,

– $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*(\mathcal{J})$.

We will now start calculations of the ω-versions of the cardinal invariants of *nwd*.

Proposition 4.1.25. $\mathfrak{m}_{\sigma-centered} \leq \mathrm{add}^*_\omega(nwd) \leq \mathrm{add}(\mathcal{M})$

Proof. To prove the second inequality simply notice that if $\{N_\alpha : \alpha < \kappa\}$ is a collection of closed nowhere dense subsets of 2^ω such that $\bigcup_\alpha N_\alpha$ is not meager, then the nowhere dense trees $\{T_\alpha : \alpha < \kappa\} \subseteq nwd$, $[T_\alpha] = N_\alpha$ witness for $\mathrm{add}^*_\omega(nwd)$.
To prove the first inequality one assume that $\kappa < \mathfrak{m}_{\sigma-centered}$ and that we are given $\{T_\alpha : \alpha < \kappa\}$ with $T_\alpha \in nwd$. We define a partial order $\mathbb{P}$ consisting of finite partial functions from ω to $nwd \times \omega$. Let $p = \{(A^p_k, i^p_k) : k < n\}$. The order is given by $p \leq q$ if $dom(q) \subseteq dom(p)$ and for each $k \in dom(q)$ we have $i^p_k > i^q_k$, $A^q_k \subseteq A^p_k$ and $A^p_k \cap 2^{i^q_k} = A^p_k \cap 2^{i^q_k}$. First, observe that $\mathbb{P}$ is σ-centered: if p and q have the same domains, and for each $k \in dom(p)$, $i^p_k = i^q_k = i$ and $A^p_k \cap 2^i = A^q_k \cap 2^i$, then p and q are compatible. Second, note that for any $\alpha < \kappa$ the set $D_\alpha = \{p \in \mathbb{P} : A^p_k = T_\alpha \text{ for some } k \in dom(p)\}$ is dense in $\mathbb{P}$. If $G \subseteq \mathbb{P}$ is generic filter over the family $\{D_\alpha : \alpha < \kappa\}$ then, the sequence $\{B_n : n \in \omega\} \subseteq nwd$ defined as $B_n = \bigcup\{A^p_n \cap i^p_n : p \in G\}$ is such that for each α there is $n \in \omega$ with $T_\alpha \subseteq B_n$, which finishes the proof. □

Proposition 4.1.26. $\mathfrak{b} \leq \mathrm{non}^*_\omega(nwd) \leq \mathrm{non}(\mathcal{M})$.

Proof. We will first show that $\mathrm{non}^*_\omega(nwd) \leq \mathrm{non}(\mathcal{M})$. Let $X \subseteq 2^\omega$ be a witness for $\mathrm{non}(\mathcal{M})$. For $x \in X$ let $Y_x = \{x|_n : n \in \omega\} \in [2^{<\omega}]^\omega$. Clearly Y_x's, $x \in X$ witness for $\mathrm{non}^*_\omega(nwd)$. To see this let $\{T_n : n \in \omega\}$ be a countable collection of nowhere-dense trees. Then there is $x \in X$ such that $x \notin \bigcup_n [T_n]$, but then, Y_x is almost disjoint with all T_n.
We will now show that $\mathfrak{b} \leq \mathrm{non}^*_\omega(nwd)$. Assume that $\kappa < \mathfrak{b}$ and that we are given a collection $\{X_\alpha : \alpha < \kappa\} \subseteq [2^{<\omega}]^\omega$. Shrinking each X_α is necessary we may assume that each X_α converges to a point x_α in the sense that for every $n \in \omega$ almost all elements of X_α extends x_α. We may also assume that each X_α forms either a chain or an antichain in $2^{<\omega}$. Assume that $X = \{\sigma^\alpha_i : i \in \omega\}$ where $|\sigma^\alpha_i| < |\sigma^\alpha_{i+1}|$ for all $i \in \omega$. As $\kappa < \mathrm{non}(\mathcal{M})$, there are nowhere dense sets $N_n \subseteq 2^\omega$ such that $\{x_\alpha : \alpha < \kappa\} \subseteq \bigcup_n N_n$. Assume that T_n's are such trees on $2^{<\omega}$ such that $N_n = [T_n]$. For every $n \in \omega$ and $l \in \omega$ let $S^n_l(T)$ consists of all $\sigma \in 2^l$ such that $\sigma \notin T_n$ but $\sigma|_{|\sigma|-1} \in T_n$. For $n \in \omega$ and $f \in \omega^\omega$ let $R_n(f) = \{\sigma^\frown\tau : \sigma \in S^n_l \text{ and } \tau \in 2^{\leq f(l)} \text{ for } l \in \omega\}$.

Claim 4.1.27. *For any $n \in \omega$ and $f \in \omega^\omega$ the set $R_n(f)$ is a nowhere dense subset of $2^{<\omega}$.*

Proof. Let $\rho \in R_n(f)$. As T_n is nowhere dense tree, there is $\rho' \in S^n_l$ for some $l \in \omega$ such that $\rho \subseteq \rho'$. Then ρ' extended by $(f(l)+1)$-many zeros, does not belong to the set $R_n(f)$. □

We continue the proof of the proposition. Fix $n \in \omega$. Let $A_n = \{\alpha < \kappa : x_\alpha \in N_n\}$. Call $\alpha \in A_n$ not ok if $X_\alpha \cap T_n =^* \emptyset$. Call α ok, it isnt not ok. For every $\alpha \in A_n$ we construct $f^n_\alpha \in \omega^\omega$ as follows: for $i \in \omega$ if $m^\alpha_i \leq |\sigma^\alpha_i|$ is such that $\sigma^\alpha_i|_{m^\alpha_i} \in T_n$ but $\sigma^\alpha_i|_{m^\alpha_i+1} \in T_n$, then let $f^n_\alpha = |\sigma^\alpha_i| - m^\alpha_i + 1$. As $\kappa < \mathfrak{b}$ there is $g^n \in \omega^\omega$ that dominates every f^n_α where $\alpha \in A_n$ is not ok. Unfix $n \in \omega$. We claim that the sets $\{R_n(g^n) : n \in \omega\}$ are such that for every $\alpha < \kappa$ there is $n \in \omega$ such that the intersection $R_n(g^n) \cap X_\alpha$ is infinite. This will finish tre proof. To see this let $\alpha < \kappa$ and let $n \in \omega$ be such that $x_\alpha \in N_\alpha$. If X_α forms a chain we are done. If X_α forms an antichain and is ok then we are also done. If an antichain X_α is not ok. As g^n

dominates f^n_α, for almost all $i \in \omega$ we have that $\sigma^\alpha_i \in \{\sigma^\alpha_i|_{m^\alpha_i} {}^\frown\tau : \tau \in 2^{\leq f^n_\alpha}(i)\} \subseteq \{\sigma^\alpha_i|_{m^\alpha_i} {}^\frown\tau : \tau \in 2^{\leq g^n}(i)\} \subseteq R_n(g^n)$. □

Question 4.1.28. *Is* $\mathrm{add}^*_\omega(nwd) = \mathrm{add}(\mathcal{M})$ *and* $\mathrm{non}^*_\omega(nwd) = \mathrm{non}(\mathcal{M})$*?*

The Fubini products of $\mathcal{F}in$

Let $\mathcal{F}in \times \mathcal{F}in$ be ideal on $\omega \times \omega$ consisting of these sets A such that for almost all $n \in \omega$ the set $\{m : (n, m) \in A\}$ is finite. $\mathcal{F}in \times \mathcal{F}in$ is tall but not a P-ideal and not a P^+-ideal. Its descriptive complexity is $F_{\sigma\delta\sigma}$. The cardinal invariants are as follows.

Proposition 4.1.29. *([MA09]) The following equalities hold:*

- $\mathrm{add}^*(\mathcal{F}in \times \mathcal{F}in) = \mathrm{non}^*(\mathcal{F}in \times \mathcal{F}in) = \omega$,
- $\mathrm{cov}^*(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{b}$,
- $\mathrm{cof}^*(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{d}$.

To calculate the ω-version of these invariants we will need the following lemma

Lemma 4.1.30. *(Roitman, [Roi79] or Lemma 1.3.4. in [BJ95]) The following conditions are equivalent:*

- $\kappa < \mathfrak{d}$,
- *for every* $\mathcal{F} \in [\omega^\omega]^\kappa$ *and* $\mathcal{X} \in [[\omega]^\omega]^\kappa$ *there is* $g \in \omega^\omega$ *such that for every* $f \in \mathcal{F}$ *and* $X \in \mathcal{X}$ *there are infinitely many* $n \in X$*'s with* $f(n) < g(n)$.

The equality $\mathrm{add}^*_\omega(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{b}$ is known and appears in the work of Filipów and Kwela [FK23].

Proposition 4.1.31. *The following equalities hold:*

- $\mathrm{add}^*_\omega(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{b}$,
- $\mathrm{non}^*_\omega(\mathcal{F}in \times \mathcal{F}in) = \mathrm{cof}^*_\omega(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{d}$.

Proof. Notice that an unbounded family witness for $\mathrm{add}^*_\omega(\mathcal{F}in \times \mathcal{F}in)$ and less than $\mathfrak{b}$ sets from $\mathcal{F}in \times \mathcal{F}in$ cannot. It follows that $\mathrm{add}^*_\omega(\mathcal{F}in \times \mathcal{F}in) = \mathfrak{b}$.
We will now show that $\mathfrak{d} \leq \mathrm{non}^*_\omega(\mathcal{F}in \times \mathcal{F}in)$. Assume that $\kappa < \mathfrak{d}$ and that we are given a family $\{X_\alpha : \alpha < \kappa\} \subseteq [\omega \times \omega]^\omega$.Without loss of generality we may assume that each of X_α's is one of the following three forms:

- X_α is contained in a column $\{N\} \times \omega$ for some $N \in \omega$,
- X_α is contained in a row $\omega \times \{N\}$ for some $N \in \omega$,
- X_α is a partial function defined on an infinite set A_α.

In the third case, we extend X_α to a total function $X'_\alpha \in \omega^\omega$. Now, as $\kappa < \mathfrak{d}$, by the Lemma 4.1.30 there is $g \in \omega$ such that for every X_α of the third kind, there are infinitely many n's in A_α with $X'_\alpha(n) = X_\alpha(n) < g(n)$. Define $Z \subseteq \omega \times \omega$ to be the area below g. Also, let $Z^0_N = \{N\} \times \omega$ and $Z^1_N = \omega \times \{N\}$. We want to show that for each $\alpha \in \kappa$ one of the sets Z^i_N, $i \in 2$, $N \in \omega$, or Z, infinitely intersects X_α. Clearly this is true if X_α is contained in a column or in a row. If X_α is of the third form, by the choice of g, we have $|Z \cap X_\alpha| = \omega$. This finishes the proof of $\mathfrak{d} \leq \mathrm{non}^*_\omega(\mathcal{F}in \times \mathcal{F}in)$.
Because $\mathrm{non}^*_\omega(\mathcal{F}in \times \mathcal{F}in) \leq \mathrm{cof}^*_\omega(\mathcal{F}in \times \mathcal{F}in)$, we will finish the proof of the proposition by showing that $\mathrm{cof}^*_\omega(\mathcal{F}in \times \mathcal{F}in) \leq \mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family. For each

$\alpha < \mathfrak{d}$ let $\overline{A}^\alpha = (A^\alpha_n)_{n\in\omega} \in (\mathcal{F}in \times \mathcal{F}in)^\omega$ be such that $A^\alpha_n = \{(a,b) : \text{if } a > n \text{ then } b < f_\alpha(a)\}$. Clearly the collection of $\overline{A}^\alpha$'s witnesses for $\mathrm{cof}^*_\omega(\mathcal{F}in \times \mathcal{F}in)$ because any $X \in \mathcal{F}in \times \mathcal{F}in$ is contained in A^α_n for some $\alpha < \kappa$ and $n \in \omega$. □

Conv

The convergence ideal *conv* is defined as the ideal on $\mathbb{Q}\cap[0,1]$ generated by convergent sequences. Its descriptive complexity is $F_{\sigma\delta\sigma}$. *conv* is tall but not a P-ideal. The cardinal invariants are summarized in the following.

Theorem 4.1.32. *([MA09]) The following equalities hold:*

- $\mathrm{add}^*(conv) = \mathrm{non}^*(conv) = \omega$,
- $\mathrm{cov}^*(conv) = \mathrm{cof}^*(conv) = \mathfrak{c}$.

Proposition 4.1.33. *The following equalities hold:*

- $\mathrm{add}^*_\omega(conv) = \mathrm{non}^*_\omega(conv) = \omega_1$,
- $\mathrm{cof}^*_\omega(conv) = \mathfrak{c}$.

Proof. To prove the first equality, it is enough to show that $\mathrm{non}^*_\omega(conv) \leq \omega_1$. Let $\{x_\alpha : \alpha < \omega_1\} \subseteq [0,1]$, for each $\alpha < \omega_1$ choose $A_\alpha \subseteq \mathbb{Q}$ to be a sequence converging to x_α. We claim that $\{A_\alpha : \alpha \in \omega_1\} \subseteq conv$ witnesses $\mathrm{non}^*_\omega(conv)$. If $\{B_n : n \in \omega\} \subseteq conv$ is given, choose $\alpha \in \omega_1$ such that x_α is not a convergence point of any B_n. Clearly, $A_\alpha \cap B_n$ is finite. To see the second equality notice that by the previous Theorem we have $\mathfrak{c} = \mathrm{cof}^*(conv) \leq \mathrm{cof}^*_\omega(conv)$. □

We will also need the following two facts about the ideal *conv*.

Lemma 4.1.34. *(Meza-Alcántara, [MA09]) The following are equivalent:*

- $conv \leq_K \mathcal{J}$,
- *there is* $\{X_n : n \in \omega\} \subseteq [\omega]^\omega$ *such that for every* $Y \in \mathcal{J}^+$ *there is* $n \in \omega$ *with* $|Y \cap X_n| = |Y \setminus X_n| = \omega$.

Lemma 4.1.35. *(Meza-Alcántara, [MA09]) If* $conv \leq_K \mathcal{J}$ *there* $\mathcal{J}$ *cannot be a* P^+*-ideal. In particular, no* F_σ*-ideal is Katetov above conv*

4.2 The splitting ideal

Definition 4.2.1. *(Sabok and Zapletal, [SZ11]) Let* $\mathcal{S}pl$ *be the splitting ideal, an ideal on* $2^{<\omega}$ *generated by the sets of form* $[A]$ *for* $A \subseteq \omega$. *Here* $[A]$ *consists of these* $\sigma \in 2^{<\omega}$ *such that* $\sigma|_A$ *is constant.*

Proposition 4.2.2. $\mathcal{S}pl$ *is a tall ideal.*

Proof. Let $\{\sigma_n : n \in \omega\}$ be an infinite subset of $2^{<\omega}$. By passing to a subsequence we may assume that $\{\sigma_n : n \in \omega\}$ forms either chain of antichain in the order $(2^{<\omega}, \subseteq)$. If $\{\sigma_n : n \in \omega\}$ is a chain then for $A = \bigcup_n \sigma_n \in 2^\omega \approx [\omega]^\omega$ we have $\{\sigma_n : n \in \omega\} \subseteq [A]$. Assume then that $\{\sigma_n : n \in \omega\}$ is an antichain. By Konigs lemma argument let $y \in 2^\omega$ be an infinite branch in the tree generated by σ_n's. Let $a_n \in \omega$ be defined as $min\{i \in |\sigma_n| : \sigma_n(i) \neq y(i)\}$. The a_n's are well defined as $\{\sigma_n : n \in \omega\}$ is an antichain. Again, by passing to a subsequence we may assume that for each $n \in \omega$ we have that $|\sigma_n| < a_{n+1} < |\sigma_{n+1}|$. Let $A \in [\omega]^\omega$ be such that $|A \cap [|\sigma_n|, a_{n+1})| \leq 1$ for each $n \in \omega$ and $y|_A$ is constant. Clearly then $\{\sigma_n : n \in \omega\} \subseteq [A]$. □

Question 4.2.3. *What do canonical $\mathcal{S}pl$-positive sets look like? Is $\mathcal{S}pl$ homogeneous?*

Proposition 4.2.4. *The $\mathcal{S}pl$ ideal is Katetov below nwd, $tr(\mathcal{N})$ and $\mathcal{F}in \times \mathcal{F}in$.*

Proof. To show $\mathcal{S}pl \leq_K \mathcal{F}in \times \mathcal{F}in$ let for each $n \in \omega$ $x_n \in 2^\omega$ be the binary sequence starting with n-th many 1's followed by 0's only. Let $\pi : \omega \times \omega \to 2^{<\omega}$ be defined as $\pi(n,m) = x_n|_m$. To show that π is a Katetov reduction let $a \in [\omega]^\omega$. Let $a = \{a_n : n \in \omega\}$ and let $f \in \omega^\omega$ be such that $f|_{[a_n, a_{n+1})}$ is constant with value a_{n+1}. To finish the proof note that

$$\pi^{-1}[[a]] \subseteq \{(n,m) \in \omega \times \omega : n \leq a_0 \text{ or } m \leq f(n)\}.$$

To show that $\mathcal{S}pl \leq_K tr(\mathcal{N})$ let π be the identity map. Observe that for each $a \in [\omega]^\omega$ and $n \in \omega$ we have that

$$\frac{|[[a]] \cap 2^n|}{2^n} = \frac{1}{2^{|a \cap n|}}$$

It follows that $[[a]] \in \mathcal{N}$. Finally, to see that the Katetov reduction $\mathcal{S}pl \leq_K nwd$ holds, note that $[a] \in nwd$ for each $a \in [\omega]^\omega$. □

We will now show that $\mathcal{S}pl$ is Katetov above $conv$. For this we will need the following lemma.

Lemma 4.2.5. *If $X \subseteq 2^{<\omega}$ is such that for some $y \in 2^\omega$ it is the case that:*

$$\forall n \in \omega\ \forall i \in 2, \text{ if } i \neq y(n), \text{ then } \{\sigma \in X : y|_n^\frown i \subseteq \sigma\} \text{ is finite}$$

then $X \in \mathcal{S}pl$.

Proof. For $j \in \omega$ let

$$C_j = \{\sigma \in X : \sigma \subseteq y|_j \text{ or } (\sigma|_j = y|_j \text{ and } \sigma|_{j+1} \neq y|_{j+1})\}.$$

By the assumption, each C_j is finite. Let $i \in 2$ be such that $y(n) = i$ for infinitely many n's. Inductively we construct two sequences $a = \{a_n : n \in \omega\}$ and $k = \{k_n : n \in \omega\}$ such that for every $n \in \omega$ we have:

- $y(a_n) = y(k_n) = i$,
- $k_n < a_n < k_{n+1}$,
- $k_{n+1} > \max\{|\sigma| : \sigma \in \bigcup\{C_j : j \in [k_n, a_n)\}\}$,
- $a_{n+1} > \max\{|\sigma| : \sigma \in \bigcup\{C_j : j \in [a_n, k_{n+1})\}\}$.

We claim that $X \subseteq [a] \cup [k]$. Let $\sigma \in X$ and let $j \in \omega$ be such that $\sigma|_j = y|_j$ but $\sigma|_{j+1} \neq y|_{j+1}$. Let $n \in \omega$ be such that $j \in [k_n, a_n)$ or $i \in [a_n, k_{n+1})$. In the first case, we have that $\sigma \in [k]$, and in the second that $\sigma \in [a]$. This finishes the proof. □

Proposition 4.2.6. *$conv \leq_K \mathcal{S}pl$*

Proof. To show this, we will use the Lemma 4.1.34 which says that it is enough to construct countably many infinite subsets of $2^{<\omega}$ which split all $\mathcal{S}pl$-positive sets. For $\sigma \in 2^{<\omega}$ let $X_\sigma = \{\tau \in 2^{<\omega} : \sigma \subseteq 2^{<\omega}\}$. Assume that $X \in \mathcal{S}pl^+$ is not splitted by any X_σ. It is easy to see that there is $y \in 2^\omega$ such that for every $n \in \omega$ and every $i \in 2$, if $i \neq y(n)$, then $X_{y|_n^\frown i} \cap X$ is finite. By the Lemma 4.2.5, X cannot be $\mathcal{S}pl$-positive. A contradiction. □

It follows from the Lemma 4.1.35 that $\mathcal{S}pl$ cannot be P^+-ideal and thus cannot be an F_σ-ideal. We will prove a slightly stronger result using the following result.

Lemma 4.2.7. *(Kwela, [Kwe23] Proposition 4.9) Suppose that $\mathcal{J}$ is such that there is $A \in \mathcal{J}^+$ and a bijection $\pi : A \to \omega$ such that $B \subseteq A$ is in $\mathcal{J}$ if and only if $\pi[B] \in \mathcal{F}in \times \mathcal{F}in$. Then is not a Π^4_0-ideal.*

Proposition 4.2.8. *$\mathcal{S}pl$ is not a P^- and thus not a Π^4_0-ideal.*

Proof. Notice that if $\phi : \omega \times \omega \to$ is the witness for $\mathcal{S}pl \leq_K \mathcal{F}in \times \mathcal{F}in$ constructed in the Proposition 4.2.4, then the set $X = \phi[\omega \times \omega]$ is $\mathcal{S}pl$-positive and is disjoint union of the sets $X_n = \phi[\{n\} \times \omega] \in \mathcal{S}pl$. On the other hand, if $Y \subseteq X$ has a finite intersection with each X_n, then by the Lemma 4.2.5 it must belong to $\mathcal{S}pl$. □

We will now focus on the cardinal invariants of the splitting ideal.

Proposition 4.2.9. $\mathrm{add}^*(\mathcal{S}pl) = \mathrm{non}^*(\mathcal{S}pl) = \omega$. *In particular, $\mathcal{S}pl$ is not a P-ideal.*

Proof. For each $\sigma \in 2^{<\omega}$, let x_σ be the element of 2^ω that starts with σ and continues with zeros only. Let $X_\sigma = \{x_\sigma|_n : n \in \omega\}$. We claim that $\{X_\sigma : \sigma \in 2^{<\omega}\}$ witesses for $\mathrm{non}^*(\mathcal{S}pl)$. To show this, let $A \in [\omega]^\omega$. Let then $\sigma \in 2^{<\omega}$ be such that $A \cap |\sigma|$ consists of two elements and let the values of σ on these two elements are different. Then clearly $X_\sigma \cap [A]$ is finite. □

To calculate the covering number of $\mathcal{S}pl$ we will need the following lemma about certain strengthening of reaping families. Call a family $\mathcal{R} \subseteq [\omega]^\omega$ hereditarily reaping if it is reaping and it is reaping inside of every its element, i.e. for every $R \in \mathcal{R}$ and every $A \in [R]^\omega$ there is $R' \in \mathcal{R}$ such that $R' \subseteq A$ either $R' \subseteq^* A$ or $A \cap R' =^* \emptyset$. The following lemma is known.

Lemma 4.2.10. *There is a hereditarily reaping family of size $\mathfrak{r}$.*

Proof. Fix a reaping family $\mathcal{R}$ of size $\mathfrak{r}$. We will inductively construct an increasing sequence $\{\mathcal{R}_n : n \in \omega\}$ with $|\mathcal{R}_n| = \mathfrak{r}$ such that $\mathcal{R}' := \bigcup_n \mathcal{R}_n$ is hereditarily reaping. Let $\mathcal{R}_0 = \mathcal{R}$ and to construct $\mathcal{R}_{n+1}$ out of $\mathcal{R}_n$ we do as follows: for each $R \in \mathcal{R}_n$ use any bijection between R and ω to place a copy of $\mathcal{R}$ inside of R. Decleare then $\mathcal{R}_{n+1}$ to be $\mathcal{R}_n$ with all such copies for all $R \in \mathcal{R}_n$. Clearly $\mathcal{R}'$ is hereditarily reaping. □

Proposition 4.2.11. *$cov^*(\mathcal{S}pl) = \mathfrak{r}$,*

Proof. To show $\mathfrak{r} \leq \mathrm{cov}^*(\mathcal{S}pl)$ assume that $\{[A_\alpha] : \alpha < \kappa\} \subseteq \mathcal{S}pl$ where $A_\alpha \in [\omega]^\omega$ and $\kappa < \mathfrak{r}$. Let B be such an infinite subset of ω that splits each A_α. Define $y \in 2^\omega$ such that $y(n) = 1$ if and only if $n \in B$. Then note that $Y = \{y|_n : n \in \omega\} \subseteq 2^{<\omega}$ is almost disjoint from each $[A_\alpha]$. To show that $\mathrm{cov}^*(\mathcal{S}pl) \leq \mathfrak{r}$ let $\mathcal{R}$ be a hereditary reaping family of size $\mathfrak{r}$. We may assume that $\mathcal{R}$ is closed under finite modifications. We claim that the family $\{[R] : R \in \mathcal{R}\}$ witnesses for $\mathrm{cov}^*(\mathcal{S}pl)$. Take arbitrary $Y \in [2^{<\omega}]^\omega$. Consider the following two possibilities. If Y contains $\subseteq$-chain $\{\sigma_n : n \in \omega\}$ there must be an $R \in \mathcal{R}$ such that for $y = \bigcup_n \sigma_n \in 2^\omega$ we have $R \cap y^{-1}[1] = \emptyset$ or $R \subseteq y^{-1}[1]$. Clearly in any of the two cases we have $\{\sigma_n : n \in \omega\} \subseteq [R]$. Assume then that Y contains no $\subseteq$-chain. Shrinking Y if necessary we may assume that Y forms an antichain in $(2^{<\omega}, \subseteq)$ and the tree generated by Y contains exactly one infinite branch $y \in 2^\omega$. Let $\{\sigma_n : n \in \omega\}$ be an length increasing enumeration of the set Y and let $a_n \in \omega$ be the smallest $m \in \omega$ such that $\sigma_n|_m \neq y|_m$. Shrinking Y once more we may assume that $a_n < |\sigma_n| < a_{n+1}$ for each $n \in \omega$. Let then I_n be the interval $[a_n, |\sigma_n|)$. Pick now $R_0 \in \mathcal{R}$ be such that $R_0 \subseteq y^{-1}[0]$ or $R_0 \subseteq y^{-1}[1]$. Next, using hereditary reaping family twice we may find $R_1 \in \mathcal{R}$, $R_1 \subset R_0$ such that one of the following three possibilities holds:

$$R_1 \cap \bigcup_n I_n = \emptyset \text{ or } R_1 \subseteq \bigcup_n I_{2n} \text{ or } R_1 \subseteq \bigcup_n I_{2n+1}$$

Note that in the first case we have $\{\sigma_n : n \in \omega\} \subseteq [R_1]$ and in the second and third we have $\{\sigma_{2n+1} : n \in \omega\} \subseteq [R_1]$ or $\{\sigma_{2n} : n \in \omega\} \subseteq [R_1]$ respectively. It follows that $Y \cap [R_1]$ is infinite which finishes the proof. □

Several people thought about the covering number of the density zero ideal. It follows from the position of $\mathcal{Z}$ in the Katekov order that $\mathrm{cov}^*(\mathcal{Z}) \leq \mathrm{non}(\mathcal{M})$ because of $\mathcal{ED}$ ideal. In [HHH07] it is proven that $\mathrm{cov}^*(\mathcal{Z}) \leq max\{\mathrm{cov}(\mathcal{N}), \mathfrak{d}\}$. Also, Raghavan and Shelah in [RS17] proved that $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{d}$ holds. The $\mathrm{cov}^*(\mathcal{Z}) \leq \mathfrak{b}$ remains an open problem. Here, as $\mathcal{Z} \leq_K tr(\mathcal{N})$ we get a new upper boundary of the covering number of density zero ideal

Corollary 4.2.12. $\mathrm{cov}^*(\mathcal{Z}) \leq \mathrm{cov}^*(tr(\mathcal{N})) \leq \mathfrak{r}$

Let us now dualize the inequality $cov^*(tr(\mathcal{N})) \leq \mathfrak{r}$.

Proposition 4.2.13. $\mathfrak{s} \leq non^*(tr(\mathcal{N}))$

Proof. Assume that $\{X_\alpha : \alpha < \kappa\} \subseteq [2^{<\omega}]^\omega$ witnesses for $non^*(tr(\mathcal{N}))$. Again, we may assume that each X_α forms either a chain or an antichain in $(2^{<\omega}, \subseteq)$. Let us deal with the more difficult case, which is when X_α's form antichains. Again, shrinking X_α if necessary, we may assume that the tree generated by X_α has exactly one infinite branch $y_\alpha \in 2^\omega$. Let $X_\alpha = \{\sigma^\alpha_n : n \in \omega\}$ and let $\{a^\alpha_n : n \in \omega\}$ be such that a^α_n is defined as the smallest $m \in \omega$ such that $\sigma^\alpha_n|_m \neq y_\alpha|_m$. Shrinking X_α once more, we may assume that $a^\alpha_n < |\sigma^\alpha_n| < a^\alpha_{n+1}$ for each $n \in \omega$. Define now $y^0_\alpha, y^1_\alpha \in 2^\omega$ as follows: for $i \in 2$ and $k \in \omega$ let

$$y^i_\alpha(k) = \begin{cases} \sigma^\alpha_{2n+i}(k) & \text{if } k \in [a^\alpha_{2n+i}, |\sigma^\alpha_{2n+i}|), \\ y_\alpha(k) & \text{otherwise} \end{cases}$$

We claim that the set $\{y_\alpha : \alpha < \kappa\} \cup \{y^i_\alpha : i \in 2, \alpha < \kappa\}$ forms a splitting family. We may assume that this family is closed under finite modifications. Let $R \in [\omega]^\omega$. Then $[R] \in \mathcal{S}pl \subseteq tr(\mathcal{N})$ so there must be $\alpha < \kappa$ such that $[R] \cap X_\alpha$ is finite. We will show that y_α, y^0_α or y^1_α (treated as subset of ω) splits the R. Assume that y_α does not split R. We only consider the case when $|y_\alpha \cap R| < \omega$ as the case $R \subseteq^* y_\alpha$ is analogous. We may assume that $y_\alpha \cap R = \emptyset$. Then is must be that $|y^0_\alpha \cap R| = |y^1_\alpha \cap R| = \omega$. The proof is finished by noticing that one of the sets $R \setminus y^0_\alpha$, $R \setminus y^1_\alpha$ must be infinite. □

Next, we focus on the cofinality of the splitting ideal

Proposition 4.2.14. $\mathrm{cof}^*(\mathcal{S}pl) = \mathfrak{c}$

Proof. Let $\mathcal{A} \subseteq [\omega]^\omega$ be a maximal almost disjoint family of cardinality $\mathfrak{c}$. Suppose that $\kappa = \mathrm{cof}^*(\mathcal{S}pl) < \mathfrak{c}$ and that $\{A_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ is such that $\{[A_\alpha] : \alpha < \kappa\}$ witnesses for $\mathrm{cof}^*(\mathcal{S}pl)$. Without loss of generality we may assume that each A_α is an almost subset of some $B_\alpha \in \mathcal{A}$. Let then $B \in \mathcal{A} \setminus \{B_\alpha : \alpha < \kappa\}$. We claim that no $[A_\alpha]$ can cover $[B]$. □

Regarding the ω-invariants we have the following

Proposition 4.2.15. $\mathrm{add}^*_\omega(\mathcal{S}pl) = \omega_1$ *and* $\mathrm{cof}^*_\omega(\mathcal{S}pl) = \mathfrak{c}$.

Proof. To show that $\mathrm{add}^*_\omega(\mathcal{S}pl) \leq \omega_1$ let $\{A_\alpha : \alpha < \omega_1\}$ be an almost disjoint family on ω. We claim that the collection $\{[A_\alpha] : \alpha < \omega_1\}$ witnesses $\mathrm{add}^*_\omega(\mathcal{S}pl)$. Suppose then that we are given countably many elements of $\mathcal{S}pl$. Without loss of generality we may assume that then are of form $[B_n]$, where $B_n \subseteq \omega$. Note that $A \subseteq B$ implies that $[B] \subseteq [A]$ and therefore we may shrink B_n's if necessary and assume that each B_n is a subset of some A_{α_n}. Pick $\alpha \in \omega_1 \setminus \{\alpha_n : n \in \omega\}$. For each $n \in \omega$, as A_α and B_n are almost disjoint, we cant have $[A_\alpha] \subseteq^* [B_n]$.
To show that $\mathrm{cof}^*_\omega(\mathcal{S}pl) = \mathfrak{c}$ simply notice, that by proposition 4.2.14 we have that $\mathfrak{c} = \mathrm{cof}^*(\mathcal{S}pl) \leq \mathrm{cof}^*_\omega(\mathcal{S}pl)$. □

4.3 The infinitely equal ideals

In this section we always assume that $F \in \omega^\omega$ is increasing and above 0. For such F we have the following.

Definition 4.3.1. *Let $\mathcal{IE}$ be the infinitely equal ideal, an ideal on $\omega^{<\omega}$ generated by sets of form $E_{f,N}$ for $f \in \omega^\omega$, $N \in \omega$ where $E_{f,N}$ consists of these $\sigma \in \omega^{<\omega}$ such that $\sigma(i) \neq f(i)$ for all $i \in |\sigma| \setminus N$. The F-bounded infinitely equal ideal $\mathcal{IE}_F$ is defined identically except that f and σ's are below the function F i.e. $f \in \Pi F$ and $\sigma \in Seq(F)$.*

The answers to the following basic questions are not known to me

Question 4.3.2. *Are the ideals $\mathcal{IE}$ and $\mathcal{IE}_F$ homogeneous? How does a typical positive sets for $\mathcal{IE}$ or $\mathcal{IE}_F$ look like?*

Clearly, both $\mathcal{IE}$ and $\mathcal{IE}_F$ are analytic. However, I do not know the exact complexity of these ideals.

Question 4.3.3. *Are $\mathcal{IE}$ or $\mathcal{IE}_F$ analytic complete or Borel?*

Regarding the cardinal invariants of these ideals we start with the following

Proposition 4.3.4. *The following holds:*

- $\mathrm{non}^*(\mathcal{IE}) = \mathrm{non}^*(\mathcal{IE}_F) = \omega$,
- $\mathrm{cof}^*(\mathcal{IE}) = \mathrm{cof}^*(\mathcal{IE}_F) = \mathfrak{c}$,

In particular $\mathcal{IE}$ and $\mathcal{IE}_F$ are not P-ideals.

Proof. We prove this for $\mathcal{IE}$ only as the proof for $\mathcal{IE}_F$ is almost identical.
• For each $\sigma \in \omega^{<\omega}$ let x_σ be an element of ω^ω which starts with σ and is continued with zeros only. Let $X_\sigma = \{x_\sigma|_n : n \in \omega\}$. We claim that $\{X_\sigma : \sigma \in \omega^{<\omega}\}$ witnesses for $\mathrm{non}^*(\mathcal{IE})$. To show this let $f \in \omega^\omega$ and $N \in \omega$. Let then $\sigma \in \omega^{<\omega}$ be such that $\sigma(i) = f(i)$ for some $i \in |\sigma| \setminus N$. Then clearly $X_\sigma \cap I_{f,N}$ is finite.
• Suppose that we are give a collection $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$ and $\{N_\alpha : \alpha < \kappa\} \subseteq \omega$ where $\kappa < \mathfrak{c}$. Let $g \in \omega^\omega$ be any function which differs with each f_α on infinitely many coordinates. Notice, that the set $E_{g,0}$ cannot be covered by any E_{f_α,N_α}. □

Next, we estimate the covering numbers. To do so, need the following relation system: for $\{f,g\} \in [\Pi F]^2$ and $h \in \Pi F$ we define $\{f,g\} \neq_2^* h$ if h is eventually different to both f and g.

Proposition 4.3.5. *The following holds:*

- $\mathfrak{d}(\Pi F, \neq^*, \Pi F) \leq \mathrm{cov}^*(\mathcal{IE}_F) \leq \mathfrak{d}([\Pi F]^2, \neq_2^*, \Pi F)$,
- $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*(\mathcal{IE})$.

Proof. To prove the first inequality, let $\kappa < \mathfrak{d}(\Pi F, \neq^*, \Pi F)$ and assume that we are given $\{E_{f_\alpha,N_\alpha} : \alpha < \kappa\} \subseteq \mathcal{IE}_F$ where the functions $\{f_\alpha : \alpha < \kappa\}$ are bounded by F and $\{N_\alpha : \alpha < \kappa\}$ are natural numbers. By the assumption, there is g below F which is infinitely equal to every f_α. As in the previous proposition, the set $\{g|_n : n \in \omega\}$ has finite intersection with every E_{f_α,N_α} for $\alpha < \kappa$. This shows that $\{E_{f_\alpha,N_\alpha} : \alpha < \kappa\}$ cannot witness $\mathrm{cov}^*(\mathcal{IE}_F)$. To prove the second inequality, let $\{f_\alpha : \alpha < \kappa\} \subseteq \Pi F$ be a witness for the $\mathfrak{d}([\Pi F]^2, \neq_2^*, \Pi F)$. We claim that the collection $\{E_{f_\alpha,N} : \alpha < \kappa,\ N \in \omega\} \subseteq \mathcal{IE}_F$ is a witness for the $\mathrm{cov}^*(\mathcal{IE}_F)$. To see this, let $X = \{\sigma_n : n \in \omega\} \in [Seq(F)]^\omega$. Shrinking X is necessary, we may assume that we may assume that $|\sigma_n| < |\sigma_{n+1}|$ for every $n \in \omega$. Consider the following two cases:
First case is that X forms a chain. Define then $x \in \Pi F$ as $\bigcup_n \sigma_n$. Let $\alpha < \kappa$ and $N \in \omega$ be such that f_α is different from x above N. But then $|X \cap E_{f_\alpha,N}| = \omega$. Second case is that X forms

an antichain. By the compactness of ΠF there is $x \in \Pi F$ such that for every $m \in \omega$ almost all σ_n's extend $x|_m$. For $n \in \omega$ define $a_n = \min\{j \in \omega : \sigma_n(j) \neq x(j)\}$. Shrinking X again, we may assume that $a_n < |\sigma_n| < a_{n+1}$ for each $n \in \omega$. Let $y \in \Pi F$ be defined as follows: for $i \in \omega$ let

$$y(i) = \begin{cases} x(i) & \text{if } i \notin \bigcup_n [a_n, |\sigma_n|), \\ \sigma_n(i) & \text{if } i \in [a_n, |\sigma_n|) \text{ for some } n \in \omega. \end{cases}$$

Let $\alpha < \kappa$ and $N \in \omega$ be such that f_α is different from both x and y above $N \in \omega$. But then $|X \cap E_{f_\alpha,N}| = \omega$. This finishes the proof of the second inequality.
To prove that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*(\mathcal{IE})$ let $\{E_{f_\alpha,N_\alpha} : \alpha < \kappa\}$ be a collection of sets from $\mathcal{IE}$ and that $\kappa < \mathrm{cov}(\mathcal{M})$. By Bartoszyński-Miller (see [BJ95]) characterization of $\mathrm{cov}(\mathcal{M})$ there is $f \in \omega^\omega$ which is infinitely equal to every f_α. We claim that the set $\{f_n : n \in \omega\}$ is almost disjoint from each E_{f_α,N_α} which means that $\{E_{f_\alpha,N_\alpha} : \alpha < \kappa\}$ does not wittiness for $\mathrm{cov}^*(\mathcal{IE})$. To see this let $i > N_\alpha$ be such that $f(i) = f_\alpha(i)$. Clearly, then $f_n \notin E_{f_\alpha,N_\alpha}$ for every $n > i$. □

We now focus on the Katetov position of the infinitely equal ideals.

Proposition 4.3.6. *$\mathcal{IE}_F \leq_K \mathcal{IE} \leq_K nwd$*

Proof. We will first show that $\mathcal{IE}_F \leq_K \mathcal{IE}$. Let $\pi : \omega^{<\omega} \to Seq(F)$ be such that $|\pi(\sigma)| = |\sigma|$ and $\pi(\sigma)(i) = \sigma(i)$ $modF(i)$ for every $\sigma \in \omega^{<\omega}$ and $i \in |\sigma|$. This defines a Katetov reduction as $\pi^{-1}[E_{f,N}] \subseteq E_{f,N}$ holds. Notice that $E_{f,N}$ on the left side is a subset of $Seq(F)$ while the right one of $\omega^{<\omega}$.
We will show that $\mathcal{IE} \leq_K nwd$. By the lemma 4.1.24,the Katetov reducibility $\mathcal{J} \leq_K nwd$ is equivalent with $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*(\mathcal{J})$ which is true by the Proposition 4.3.5. □

Question 4.3.7. *Is $\mathcal{IE}$ Katetov equivalent with the nwd? Is $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*(\mathcal{IE})$?*

Next, we investigate the relation between the ideals $\mathcal{IE}_F$ and $\mathcal{S}$.

Proposition 4.3.8. *$\mathcal{IE}_F$ does not have Fubini property for $F \in \omega^\omega$ such that $\Sigma_i \frac{1}{F(i)} < 1$.*

Proof. Let $\epsilon > 0$ be such that $\Sigma_i \frac{1}{F(i)} < \epsilon$ and let $B = \{(\sigma, x) \in Seq(F) \times \Pi F : \forall i \in |\sigma|$ $\sigma(i) \neq x(i)\}$. Clearly, the set is Borel. First, notice that the horizontal section $B^x = \{\sigma : \forall i \in |\sigma|$ $\sigma(i) \neq x(i)\}$ is in $\mathcal{IE}_f$ for every $x \in \Pi F$. It follows that $\mu^*(\{x \in \Pi F : B^x \in \mathbb{IE}_F^+\}) = 0$. On the other hand for every $x \in \Pi F$ we have that $\{\sigma \in Seq(F) : \mu(B_\sigma) > \epsilon\} = \{\sigma \in Seq(F) : \mu(\Pi F \setminus B_\sigma) \leq \epsilon\} = \{\sigma \in Seq(F) : \mu(\Pi F \setminus B_\sigma) \leq \Sigma_{i<|\sigma|} \frac{1}{F(i)} \leq \Sigma_{i\in\omega} \frac{1}{F(i)} < \epsilon\}$. It follows that $\{\sigma \in Seq(F) : \mu(B_\sigma) > \epsilon\} = Seq(F) \in \mathbb{IE}_F^+$. □

Corollary 4.3.9. *There is an $\mathcal{IE}_F$-positive set Y such that $\mathcal{S} \leq_K \mathcal{IE}_F|_Y$.*

Question 4.3.10. *Is $\mathcal{S} \leq_K \mathcal{IE}_F$ true?*

We finish this section with analysis of ω-invariants of infinitely different ideals. We have the following

Proposition 4.3.11. *We have the following:*

- $\mathrm{add}^*_\omega(\mathcal{IE}) = \mathrm{add}^*_\omega(\mathcal{IE}_F) = \omega_1$,
- $\mathrm{cof}^*_\omega(\mathcal{IE}) = \mathrm{cof}^*_\omega(\mathcal{IE}_F) = \mathfrak{c}$.

Proof. We focus on the ideal $\mathcal{IE}$ as the proof for $\mathcal{IE}_F$ is identical. We start with the additivities. Let $\{f_\alpha : \alpha \in \omega_1\} \subseteq \omega^\omega$ be arbitrary collection of different functions. We claim that the collection $\{E_{f_\alpha,0} : \alpha < \omega_1\}$ witnesses for $\mathrm{add}^*_\omega(\mathcal{IE})$. To see this let $\{g_n : n \in \omega\} \subseteq \omega^\omega$ and $\{N_n : n \in \omega\} \subseteq \omega$. Let $\alpha < \omega_1$ be such that f_α is not equal to any g_n, $n \in \omega$ modulo finite modifications. We will finish the proof by showing that for every $n \in \omega$ the set $E_{f_\alpha,0} \setminus E_{g_n,N_m}$

is infinite. Fix $n \in \omega$. Define $h \in \omega^\omega$ such that $h(i) = g_n(i)$ if $f_\alpha(i) \neq g_n(i)$, and if not, let $h(i)$ be any value different than $f_\alpha(i)$ and $g_n(i)$. Then for any $M > N_n$ such that $f_\alpha(M) \neq g_n(M)$ we have that $\{h|_k : k > M\} \subseteq E_{f_\alpha,0} \setminus E_{g_n,N_m}$. This shows that $\mathrm{add}^*_\omega(\mathcal{IE}) = \omega_1$.
The calculation of cofinalities follows from 4.3.4 and the fact that $\mathrm{cof}^*(\mathcal{IE}) \leq \mathrm{cof}^*_\omega(\mathcal{IE})$. □

4.4 More on covering numbers

In this section, we will investigate the +-versions of cov^* numbers. We will also prove more results about the covering numbers. Recall that

- $\mathrm{cov}^*_+(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J}$ and every $X \in \mathcal{J}^+$ infinitely intersects some $F \in \mathcal{F}\}$,
- $\mathrm{non}^*_+(\mathcal{J}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{J}^+$ and every $Y \in \mathcal{J}$ is almost disjoint with some $F \in \mathcal{F}\}$.

The first invariant was investigated in the work [FZ22] of Farkas and Zdomskyy. The second invariant were studied in the work [BF17] of Brendle and Flaskova. We have the following result.

Theorem 4.4.1. *(Farkas and Zdomskyy, [FZ22])*

- $\mathrm{cov}^*_+(\mathcal{ED}_{fin}) = \mathrm{cov}^*(\mathcal{ED}_{fin})$,
- $\mathrm{cov}^*_+(\mathcal{ED}) = \mathrm{cov}^*(\mathcal{ED})$,
- $\mathrm{cov}^*_+(\mathcal{S}) = \mathrm{cov}^*(\mathcal{S})$,
- $\mathrm{cov}^*_+(conv) = \mathfrak{c}$,
- $\mathrm{cov}^*_+(\mathcal{R}) = \mathfrak{c}$,
- $\mathrm{cov}^*_+(\mathcal{F}in \times \mathcal{F}in) = \omega$.

The calculation of the cov^*_+-number comes from [BHHH04] of Balcar, Hernández-Hernández and Hrušák. However, what they did is they analyzed the cov^*_+ invariant of the order $(\mathbb{P}_{dense}, \subseteq_{nwd})$ where $\mathbb{P}_{dense} = \{A \subseteq [0,1] \cap \mathbb{Q} : A$ is dense in $[0,1]\}$ and $A \subseteq_{nwd} B$ means $A \setminus B$ is nowhere dense. Because this is a little difference, we include the proof.

Theorem 4.4.2. $\mathrm{cov}^*_+(nwd) = \mathrm{add}(\mathcal{M})$

Proof. To show that $\mathrm{add}(\mathcal{M}) \leq \mathrm{cov}^*_+(nwd)$ assume that $A \in nwd^+$, $\{T_\alpha : \alpha < \kappa\} \subseteq nwd$ and $\kappa < \mathrm{add}(\mathcal{M})$. Let $G = \{x \in 2^\omega : \exists^\infty \sigma \in A\ \sigma \subseteq x\}$. Note that G is a non-meager set. For each σ pick $x_\sigma \in (G \cap [\sigma]) \setminus \bigcup_\alpha [T_\alpha]$. For each $\alpha < \kappa$ define $f_\alpha : A \to \omega$ such that $x_\sigma|_{f_\alpha(\sigma)} \notin T_\alpha$. Let $f : A \to \omega$ be such a function that dominates all of the f_α's. For every $\sigma \in A$ pick $\tau_\sigma \in A$ such that τ_σ extends $x_\sigma|_{f_\sigma}$ and define $B = \{\tau_\sigma : \sigma \in A\}$. Clearly B is as desired.
To show that $\mathrm{cov}^*_+(nwd) \leq \mathrm{add}(\mathcal{M})$ assume that $\mathcal{F}$ is a collection of nowhere dense subsets of the 2^ω and $|\mathcal{F}| < \kappa < \mathrm{cov}^*_+(nwd)$. Let $\mathbf{M}$ be a model of cardinality κ containing $\mathcal{F}$. As $\kappa < \mathrm{cov}^*_+(nwd)$ there is a $A \in nwd^+$ such that $A \cap T =^* \emptyset$ for each $T \in nwd \cap \mathbf{M}$. If $G = \{x \in 2^\omega : \exists^\infty \sigma \in A\ \sigma \subseteq x\}$, then in particular for each nowhere dense $F \subseteq 2^\omega$ coded is $\mathbf{M}$ we have $G \cap F = \emptyset$. But then also $(G + \mathbb{Q}) \cap F = \emptyset$. As $G + \mathbb{Q}$ is comeager and $\mathcal{F} \subseteq \mathbf{M}$ we have $\bigcup \mathcal{F} \in \mathcal{M}$. □

Theorem 4.4.3. $\mathrm{cov}^*_+(\mathcal{W}) = \mathrm{cov}^*(\mathcal{W})$.

Proof. By the results of [KT17], for any $A \in \mathcal{W}^+$ the ideals $\mathcal{W}$ and $\mathcal{W}|_A$ are isomorphic (there is a bijection $b : \omega \to A$ such that $X \subseteq \omega$ is in $\mathcal{W}$ if and only if $b[X]$ is in $\mathcal{W}|_A$). By the Proposition 1.2.12 we have that $\mathrm{cov}^*_+(\mathcal{W}|_A) = \mathrm{cov}^*_+(\mathcal{W})$ and $\mathrm{cov}^*(\mathcal{W}|_A) = \mathrm{cov}^*(\mathcal{W})$ for any $A \in \mathcal{W}^+$. As +-covering is always smaller than usual covering, it will be enough to show that $\mathrm{cov}^*_+(\mathcal{W}|_A) \geq \mathrm{cov}^*(\mathcal{W}|_A)$ for some set $A \in \mathcal{W}^+$.
Call a set $A \in \mathcal{W}^+$ sparse if A is countable disjoint union of sets A_n where for every $n \in \omega$ we have that:

– A_n is a arithmetical progression of length n,

– $2 \max A_n < \min A_{n+1}$.

It is easy to see that any $\mathcal{W}$-positive set contains a set of this form. We need the following claim

Claim 4.4.4. *If A is sparse and $B \subseteq A$ is finite arithmetical progression, then there are at most two n's such that $B \cap A_n \neq \emptyset$.*

Proof. of the claim Let $B = \{b_0 < b_1 < ... < b_l\}$ and let K be the distance between the two consecutive elements of B. Assume towards a contradiction that B intersects three A_n's and let $n_0 < n_1 < n_2$ be the smallest such n's. Let $i \leq l$ be such that b_i is the largest element of A_{n_1}. In particular $b_{i+1} \in A_{n_2}$. Then, as A is sparse, we have that

$$K = b_{i+1} - b_i \geq (\min A_{n_2} - \max A_{n_1}) \geq (\min A_{n_1+1} - \max A_{n_1}) \geq \max A_{n_1}$$

If $j < l$ is such that b_j is the largest element of A_{n_0}, then $K = b_{j+1} - b_j \geq \max A_{n_1}$ which is impossible because both b_j and b_{j+1} are smaller than $\max A_{n_1}$. □

We continue the proof of the theorem. Fix sparse $A \in \mathcal{W}^+$. To show that $\mathrm{cov}^*_+(\mathcal{W}|_A) \geq \mathrm{cov}^*(\mathcal{W}|_A)$, split every A_{n^2} into n-many consecutive sets $\{B^n_i : i < n\}$, each being an arithmetical progression of length n. Define two functions $\phi : [A]^\omega \to (\mathcal{W}|_A)^+$ and $\psi : \mathcal{W}|_A \to \mathcal{W}|_A$ as follows: for $X \in [A]^\omega$ let $\phi(X) = \bigcup_{k \in X} B^{n(k)}_{i(k)}$ where $n(k)$ is the unique $n \in \omega$ such that $k \in A_n$ and $i(k)$ is such that k is the i-th element of A_n. For $Y \in \mathcal{W}|_A$ let $\psi(Y) = \{k \in A : B^{n(k)}_{i(k)} \cap Y \neq \emptyset\}$.

Claim 4.4.5. *$\phi(X) \in (\mathcal{W}|_A)^+$ and $\psi(Y) \in \mathcal{W}|_A$.*

Proof. proof of the claim To show that $\phi(X)$ contains arbitrary long a.p. sets let $N \in \omega$. Let $k \in X$ be such that $k > N$ and let $k \in A_n$ for some $n \in \omega$. Then $B^{n(k)}_{i(k)} \subseteq \phi(X)$ and $B^{n(k)}_{i(k)}$ is an arithmetical progression of length $n > N$. To see the second one, assume that Y contains no arithmetical progression of length greater than N. We will show that the same is true for $\psi(Y)$. Towards a contradiction, suppose that $C \subseteq \psi(Y)$ is an arithmetical progression of length greater than $N + 2$. By the previous claim C is contained in two sets A_n. The same argument as the previous claim shows that C is contained in some $A_{n'}$ except for one element. Let $C' = C \cap A_{n'}$. But then $\{k \in A : B^{n(k)}_{i(k)} \cap C' \neq \emptyset\}$ contains an arithmetical progression of length $|C'| = N + 1$, a contradiction. □

We continue the proof of the proposition. It follows directly from the definition of ϕ and ψ that for $X \in [A]^\omega$ and $F \in \mathcal{W}|_A$, if $\phi(X) \cap F$ is infinite, then $X \cap \psi(F)$ is infinite. So in particular, if $\mathcal{F} \subseteq \mathcal{W}|_A$ is a witness for $\mathrm{cov}^*_+(\mathcal{W}|_A)$, then $\{\psi(F) : F \in \mathcal{F}\}$ is a witness for $\mathrm{cov}^*(\mathcal{W}|_A)$. □

Proposition 4.4.6. $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}^*_+(\mathcal{S}pl) \leq \mathfrak{r}$

Proof. The inequality $\mathrm{cov}^*_+(\mathcal{S}pl) \leq \mathfrak{r}$ follows from $\mathrm{cov}^*_+(\mathcal{S}pl) \leq \mathrm{cov}^*(\mathcal{S}pl) = \mathfrak{r}$.
Assume that $\kappa < \mathrm{cov}(\mathcal{M})$ and assume that we are given a collection $\{A_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$. Clearly, if $T \subseteq 2^\omega$ is a splitting tree (see definition 2.2.7), then $T \notin \mathcal{S}pl$. Use the notion of forcing (equivalent to Cohen forcing) from the Proposition 3.2.7 to obtain a splitting tree T such that $[T] \cap [X_{A_\alpha}] = \emptyset$ for every $\alpha < \kappa$. This is possible as $\kappa < \mathrm{cov}(\mathcal{M})$ and as the mentioned forcing is equivalent to Cohen forcing. □

Question 4.4.7. *Is $\mathrm{cov}^*_+(\mathcal{S}pl)$ equal to $\mathfrak{r}$ in ZFC?*

Next, we will prove a lower bound for +-coverings of F_σ-ideals. We will need the following relational system involving slaloms: Let $F, b \in \omega^\omega$ be increasing, above 0, and $b(n) < F(n)$ for every $n \in \omega$. Whenever we mention F and b we will always assume that they satisfies these requirements. Let $\mathcal{S}lm(F,b)$ be a collection of all b-slaloms on ΠF i.e. all functions $S : \omega \to [\omega]^{<\omega}$ such that $S(n) \subseteq F(n)$ and $|S(n)| \leq b(n)$. For a function $f \in \Pi F$ and $S \in \mathcal{S}lm(F,b)$, the relation $f \subseteq^* S$ means that for almost all $n \in \omega$ we have $f(n) \in S(n)$. We will consider the relational system $\langle \Pi F, \subseteq^*, \mathcal{S}lm(F,b)\rangle$. The invariant $\mathfrak{b}(\Pi F, \subseteq^*, \mathcal{S}lm(F,b))$ appeared in characterization of transitive additivity of $\mathcal{N}$ ([BJ95]). For more informations about relational systems related to slaloms, the reader may consult the survey article of Cardona and Mejia [AA23].
We have the following result.

Theorem 4.4.8. *Suppose that $\mathcal{J}$ is an F_σ-ideal. For every $b \in \omega^\omega$ there is an $F \in \omega^\omega$ such that the following inequalities hold:*

- $\mathfrak{b}(\Pi F, \subseteq^*, Slm(F,b)) \leq \mathrm{cov}^*_+(\mathcal{J})$,
- $\mathrm{non}^*_+(\mathcal{J}) \leq \mathfrak{d}(\Pi F, \subseteq^*, Slm(F,b))$.

Proof. By the Theorem 4.1.1 of Mazur there is a l.s.c.s.m. φ such that $\mathcal{J} = Fin(\varphi)$. Let $(I_n)_{n\in\omega}$ be an interval partition such that $\varphi(I_n) \geq nb(n) + n$ for every $n \in \omega$. This is possible as $\phi(\omega) = \infty$ and by lower semi continuity of ϕ. For every $n \in \omega$ let $F(n)$ be the cardinality of the set $H_n = \{Y \subseteq I_n : \varphi(Y) \leq n\}$. In this way, we will identify the elements of the set H_n with numbers $i < F(n)$. We will construct two morphisms

$$\phi : Slm(F,b) \to \mathcal{J}^+ \text{ and } \psi : \mathcal{J} \to \Pi F.$$

For $S \in Slm(F,b)$ define $\psi(S) = \bigcup_n (I_n \setminus \bigcup_{Z\in S(n)} Z)$. Notice, that the set $\psi(S)$ is $\mathcal{J}$-positive because for every $n \in \omega$ the following holds.

$$\varphi(S) \geq \varphi(I_n \setminus \textstyle\bigcup_{Z\in S(n)} Z) \geq \varphi(I_n) - \varphi(\bigcup_{Z\in S(n)} Z) \geq nb(n) + n - nb(n) = n$$

For $X \in \mathcal{J}$ define $\psi(X)$ as follows: for $n \in \omega$ we define $\psi(X)(n) = X \cap I_n \in H_n$. Notice that as X is of finite submeasure, the value of $\psi(X)(n)$ are really defined for cofinitely many n's. For the remaining, finitely many n's, we put whatever we like. To prove both inequalities, it will be enough to show that if $\psi(X) \subseteq^* S$, then $|X \cap \psi(S)| < \omega$. If for almost every $n \in \omega$ we have $\psi(X)(n) \in S(n)$, meaning that $X \cap I_n \subseteq \bigcup_{Z\in S(n)} Z$, then $X \subseteq^* \bigcup_n \bigcup_{Z\in S(n)} Z$ and thus $X \cap \psi(S) =^* \emptyset$. □

Recall that $\mathrm{cov}(\mathcal{N}) \leq \mathrm{cov}^*(tr(\mathcal{N})) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ and $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}^*(tr(\mathcal{N})) \leq \mathrm{non}(\mathcal{N})$. A similar inequality holds for the $\mathcal{ED}_{fin}$-ideal

Theorem 4.4.9. *The following inequalities hold:*

- $\mathrm{cov}^*(\mathcal{ED}_{fin}) \leq \max\{\mathfrak{d}, \Pi_F \mathfrak{d}(\Pi F, =^\infty, \Pi F)\}$,
- $\min\{\mathfrak{b}, \Pi_F \mathfrak{b}(\Pi F, =^\infty, \Pi F)\} \leq \mathrm{non}^*(\mathcal{ED}_{fin})$.

Proof. We will show the Tukey reduction

$$\langle [\omega]^\omega, \cap^\infty, \mathcal{ED}_{fin}\rangle \leq_T \langle \mathbb{IP}, \leq^*, \mathbb{IP}\rangle \otimes \Pi_F \langle \Pi F, =^\infty, \Pi F\rangle$$

and then the inequalities will follow from the Proposition 1.2.22.
Let $\Delta = \{(a,b) \in \omega \times \omega : a \leq b\}$. For an interval partition $\overline{I} = (I_n)_{n\in\omega}$ let $g_{\overline{I}} \in \omega^\omega$ counts the number of points on Δ with the first coordinate in I_n's i.e. $g_{\overline{I}}(n) = |\{(a,b) \in \Delta : a \in I_n\}|$. Let $\pi : \Delta \to \omega$ denotes the projection onto the first coordinate. We will define two functions:

$$\phi = (\phi_0, \phi_1) : [\Delta]^\omega \to \mathbb{IP} \times (\Pi_F(\Pi F))^{\mathbb{IP}}$$

and

$$\psi : \mathbb{IP} \times \Pi_F(\Pi F) \to \mathcal{ED}_{fin}$$

To define ϕ let $X \in [\Delta]^\omega$. Define $\phi_0(X)$ to be such an interval partition $(J_n)_{n\in\omega}$, that for every $n \in \omega$ we have $J_n \cap \pi[X] \neq \emptyset$. Define then $\phi_1(X) : \mathbb{IP} \to \Pi_F(\Pi F)$ such that for $\overline{I} = (I_n)_{n\in\omega} \in \mathbb{IP}$ and $F \in \omega^\omega$ the $\phi_1(X)(\overline{I})(F)$ is such a function $f \in \Pi F$ that for each $n \in \omega$ we have:

- if $g_{\overline{I}}(n) > F(n)$ or $\pi[X] \cap I_n = \emptyset$, then $f(n) = 0$,
- otherwise $f(n) = i < F(n)$, where i'th element of the set $\{(a, b) \in \Delta : a \in I_n\}$ is in X

Note that the second item makes sense because $g_{\overline{I}}(n) \leq F(n)$ and $\pi[X] \cap I_n \neq \emptyset$.

For $\overline{I} \in \mathbb{IP}$ and $(x_F)_F \in \Pi_F(\Pi F)$ we define $\psi(\overline{I}, (x_F)_F) \in \mathcal{ED}_{fin}$ in the following way: Pick any $F \in \omega^\omega$ with $g_{\overline{I}} \leq F$ and define $\psi(\overline{I}, (x_F)_F)$ to be $\psi'(g_{\overline{I}}, x_F) \in \mathcal{ED}_{fin}$, which is any set with the following two properties (for every $n \in \omega$):

- $|\psi'(g_{\overline{I}}, x_F) \cap \{(a, b) \in \Delta : a \in I_n\}| = 1$,
- the only element of this intersection is the $(x_F(n) \bmod g_{\overline{I}}(n))$'th element of the set $\{(a, b) \in \Delta : a \in I_n\}$.

This in particular means that $\psi'(g_{\overline{I}}, x_F) \in \mathcal{ED}_{fin}$

To finish the proof we have to show that for any $X \in [\Delta]^\omega$, $\overline{I} \in \mathbb{IP}$ and $(x_F)_F \in \Pi_F(\Pi F)$, if $\phi(X) \sqsubseteq_\otimes (\overline{I}, (x_F)_F)$, then the intersection $X \cap \psi(\overline{I}, (x_F)_F)$ is infinite. Assume that $\phi(X) \sqsubseteq_\otimes (\overline{I}, (x_F)_F)$. By the definition of the relation $\sqsubseteq_\otimes$, we have that $\phi_0(X) \leq^* (\overline{I})$ and $\phi_1(X)(\overline{I})(F) =^\infty x_F$. The former means that for almost every $n \in \omega$ we have $\{(a, b) \in \Delta : a \in I_n\} \neq \emptyset$. Then, by the definition of $\psi'(\overline{I}, x_F)$ and the latter we have that $X \cap \psi(\overline{I}, (x_F)_F)$ is infinite which finishes the proof. □

Proposition 4.4.10. *For any increasing $F \in (\omega \setminus 1)^\omega$ the following inequalities hold:*

- $\mathfrak{d}(\Pi F, =^\infty, \Pi F) \leq \mathrm{cov}^*(\mathcal{ED}_{fin})$,
- $\mathrm{non}^*(\mathcal{ED}_{fin}) \leq \mathfrak{b}(\Pi F, =^\infty, \Pi F)$.

Proof. Fix $F \in \omega^\omega$. Recall that $\mathcal{ED}_{fin}$ is Katetov equivalent with $\mathcal{ED}_G$ for any increasing $G \in \omega^\omega$ and thus the two have the same covering and uniformity numbers. Pick an interval partition $(I_n)_{n\in\omega}$ such that $|I_n| = n + 1$ for every $n \in \omega$. Let $G \in \omega^\omega$ be defined such that $G(n) = |\{\rho : I_n \to \omega : \forall j \in I_n\ \rho(j) < F(j)\}|$. In this way we can identify the elements of the set $\{\rho : I_n \to \omega : \forall j \in I_n\ \rho(j) < F(j)\}$ with natural numbers below $G(n)$. We construct two functions

$$\phi : \mathcal{ED}_G \to \Pi F \text{ and } \psi : \Pi F \to [\{(a, b) \in \omega^2 : b < G(a)\}]^\omega$$

To define ϕ let $A \in \mathcal{ED}_G$. Cover A with graphs of functions $\{f_0^A, f_1^A, ..., f_{N_A}^A\} \subseteq \Pi G$. Using the identification above, for every $i \leq N_A$ there is $f_i' \in \Pi F$ such that we identify $f_i'|_{I_m}$ with $f_i^A(m)$ for every $m \in \omega$. Let then $\phi(A) \in \Pi F$ be any function such that for every $n > N_A$ and every $i \leq N_A$ we have that $\phi(A)|_{I_n} \cap f_i'|_{I_n} \neq \emptyset$. For $g \in \Pi F$ define $\psi(g) = \{(n, g|_{I_n}) : n \in \omega\}$, which, by our identification is an element of ΠG.
We claim that for $A \in \mathcal{ED}_G$ and $g \in \Pi F$, if $|A \cap \psi(g)| = \omega$, then $\phi(A) =^\infty g$. This will finish the proof of the two inequalities: if $\mathcal{F} \subseteq \mathcal{ED}_G$ is a witness for the $\mathrm{cov}^*(\mathcal{ED}_G)$, then $\{\phi(F) : F \in \mathcal{F}\}$ is a witness for the $\mathfrak{d}(\Pi F, =^\infty, \Pi F)$, and if $\mathcal{F} \subseteq \Pi F$ is a witness for the $\mathfrak{b}(\Pi F, =^\infty, \Pi F)$, then $\{\psi(f) : f \in \mathcal{F}\}$ is a witness for the $\mathrm{non}^*(\mathcal{ED}_G)$. Now, if the intersection $A \cap \psi(g)$ is infinite then there is $i < N_A$ and $(m_k)_{k\in\omega} \in \omega^\omega$ such that $(m_k, \psi(g)(m_k)) = f_i^A(m_k)$ for every $k \in \omega$. But then for every $k \in \omega$ such that $m_k > N_A$, there is $j \in I_n$ for which the following holds:

$$\phi(A)(j) = f_i'|_{I_{m_k}}(j) = f_i^A(m_k)(j) = \psi(g)(m_k)(j) = g|_{I_{m_k}}(j) = g(j)$$

which means that $\phi(A) =^\infty g$ and finishes the proof. □

Due to the time restrictions for this thesis, we will not investigate inequalities regarding covering numbers of summable-like ideals and numbers of form $\mathfrak{d}(\Pi F, \in^\infty, Slm(F, b))$.

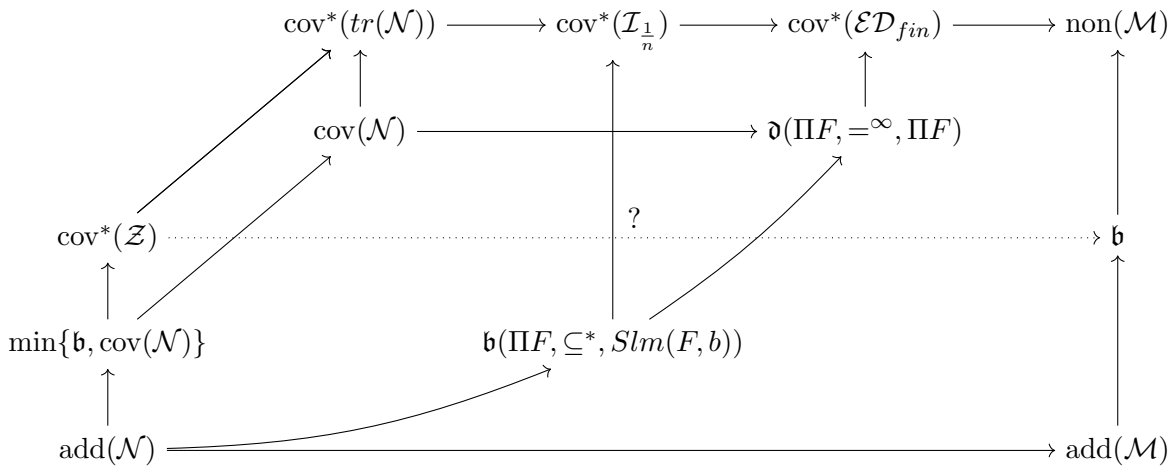
Let $\mathcal{F} \subseteq \mathcal{P}(\omega)$ be a proper filter. Let then $\mathbb{M}a(\mathcal{F})$ be the Mathias-Prikry forcing with filter $\mathcal{F}$ i.e. $\mathbb{M}a(\mathcal{F}) = \{(a, F) : a \in [\omega]^{<\omega},\ F \in \mathcal{F} \text{ and } \max(a) < \min(F)\}$ with the order $(a, F) \leq (b, G)$ if $b \subseteq a$ and $F \subseteq G$. As there are ony countably many choices of the first coordinate, and $\mathcal{F}$ is closed under finite intersections, the forcing $\mathbb{M}a(\mathcal{F})$ is σ-centered for any filter $\mathcal{F}$. The following result says that $\mathbb{M}a(\mathcal{F})$ not only destroys the ideal $\mathcal{F}^*$ but in fact it $+$-destroys it, whenever it is possible.

Theorem 4.4.11. *(Farkas and Zdomskyy, [FZ22]) For any tall Borel ideal $\mathcal{J}$ the following are equivalent:*

- $\mathrm{cov}^*_+(\mathcal{J}) > \omega$,
- *there is a forcing notion which $+$-destroy $\mathcal{J}$,*
- $\mathbb{M}a(\mathcal{J}^*)$ *$+$-destroy $\mathcal{J}$.*

Corollary 4.4.12. $\mathfrak{p} \leq \mathrm{cov}^*_+(\mathcal{J})$ *for any ideal $\mathcal{J}$ with* $\mathrm{cov}^*_+(\mathcal{J}) > \omega$.

Proof. If $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}$ where $\kappa < \mathfrak{p}$, then $\mathbb{M}a(\mathcal{J}^*)$ is σ-centered forcing notion which - by Theorem 4.4.11 - introduces $\mathcal{J}$-positive set which is almost disjoint (modulo finite sets) with each A_α. It follows that $\{A_\alpha : \alpha < \kappa\}$ cannot witness for the $\mathrm{cov}^*_+(\mathcal{J})$. □



Let us now move towards the consistency results regarding the covering numbers of ideals. We know that in case of the ideals $\mathcal{ED}_{fin}$, $\mathcal{I}_{\frac{1}{n}}$ and $tr(\mathcal{N})$, there are two natural lower bounds for the covering numbers cov^*: first, the $+$-covering numbers cov^*_+ and second, the invariants $\mathfrak{d}(\Pi F, =^\infty, \Pi F)$ and $\mathrm{cov}(\mathcal{N})$ (see Theorem 4.4.9). In this context, it is natural to ask, if there is an inequality between the $+$-covering numbers and the other ones. We will show that these two are independent of each other.

Theorem 4.4.13. *Each of the following inequality is consistent with ZFC:*

$$\mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) = \mathrm{cov}^*_+(tr(\mathcal{N})) < \mathrm{cov}(\mathcal{N}) = \mathfrak{d}(\Pi F, =^\infty, \Pi F).$$

Proof. The inequality holds in the random model. On the one hand, in this model we have $\mathrm{cov}(\mathcal{N}) = \mathfrak{c} = \omega_2$ (see [BJ95] or [Bla09]) and the inequality $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{d}(\Pi F, =^\infty, \Pi F)$ holds by the Theorem 4.4.9. On the other hand, by the Theorem 4.4.20, we have that the inequalities $\mathrm{cov}^*_+(tr(\mathcal{N})) \leq \mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) \leq \mathrm{non}(\mathcal{E}) \leq \mathrm{non}(\mathcal{N})$ hold. But it is known that $\mathrm{non}(\mathcal{N}) = \omega_1$ holds in random model (see [BJ95] or [Bla09]). □

We will now show that one can have cov^*_+'es large while keeping the other small.

Theorem 4.4.14. *Each of the following inequality is consistent with ZFC:*

$$\mathrm{cov}(\mathcal{N}) = \mathfrak{d}(\Pi F, =^\infty, \Pi F) < \mathrm{cov}^*_+(tr(\mathcal{N})) = \mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) = \mathrm{cov}^*_+(\mathcal{ED}_{fin})$$

Proof. We will show that the inequality holds after using finite support iteration of length ω_2 of Mathias-Prikry forcing with filter over a model of CH. For $\alpha \leq \omega_2$ denote by $\mathbb{M}a(\mathcal{Z}^*)_{\omega_2}$ the finite support iteration of Mathias-Prikry with filter $\mathcal{Z}^*$. For any forcing notion $\mathbb{P}$ consider the following standard property.

$$(\star)_{\mathbb{P}} : \forall F \in \omega^\omega\ \forall \dot{f} \in \Pi F \text{ } \mathbb{P}\text{-name } \exists \{f_n : n \in \omega\} \subseteq \Pi F\ \forall g \in \Pi F$$
$$\text{if } \forall n \in \omega\ g =^\infty f_n, \text{ then } \mathbb{P} \Vdash \text{"}\dot{f} =^\infty g\text{"}$$

Clearly, this implies that in the $\mathbb{P}$-extension, the ground model elements of ΠF are infinitely equal to any $f \in \Pi F$ from the extension. We would like to show that $(\star)_{\mathbb{M}a(\mathcal{Z}^*)_{\omega_2}}$ holds, as this would mean that $\mathfrak{d}(\Pi F, =^\infty, \Pi F) \leq \omega_1$ in the final model. We will show that $(\star)_{\mathbb{P}}$ holds for σ-centered forcing notions (so in particular for the Mathias-Prikry forcing) and that this property is preserved along finite support iteration. The proof of the theorem will be finished with the following lemma

Lemma 4.4.15. *The following holds:*

- *If $\mathbb{P}$ is σ-centered, then the $(\star)_{\mathbb{P}}$ holds,*
- *if $(\star)_{\mathbb{P}}$ and $\Vdash_{\mathbb{P}}$ "$(\star)_{\dot{\mathbb{Q}}}$", then $(\star)_{\mathbb{P}*\dot{\mathbb{Q}}}$,*
- *if δ is limit ordinal with countable cofinality (say $\{\delta_n : n \in \omega\}$ is cofinal in δ) and for each $n \in \omega$ the $(\star)_{\mathbb{P}_{\delta_n}}$ holds, then $(\star)_{\mathbb{P}_\delta}$ holds.*

Proof. The proof follows the standard preservation argument for finite support iteration. Because of this, we will only sketch the proof. In the proof we will, without mentioning - use the standard fact that if we deal with two step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ and $\dot{f}$ is $\mathbb{P} * \dot{\mathbb{Q}}$-name for a real, then there is a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name $\dot{f}$ such that $\dot{g}$ is equal to the $\dot{g}$ in $\mathbf{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$ (see [Bau10]). We will also use the standard fact that if $\mathbb{P}_\delta$ is f.s.i. and $\gamma < \delta$, then there is a canonical isomorphism between $\mathbb{P}_\delta$ and two-step iteration $\mathbb{P}_\gamma * \dot{\mathbb{P}}_{[\gamma,\delta)}$. For more information the reader may consult Baumgartner's article [Bau10].

• To prove the first item, let $\mathbb{P} = \bigcup_n \mathbb{P}_n$ where each $\mathbb{P}_n$ is centered. Assume that $F \in \Pi F$ and $\dot{f}$ is $\mathbb{P}$-name for an element of ΠF. For $n, k \in \omega$ let

$$A^n_k = \{j < F(k) : \exists p \in \mathbb{P}_n\ p \Vdash \text{"}\dot{f}(k) \neq j\text{"}\}$$

Claim 4.4.16. *For all $n, k \in \omega$ we have $F(k) \setminus A^n_k \neq \emptyset$*

proof of the claim. If not, then for each $i \in F(k)$ there is $p_i \in \mathbb{P}_n$ with $p_i \Vdash \text{"}\dot{f}(k) \neq i\text{"}$. But then any $p \in \mathbb{P}_n$, which is below all p_i's (such p exists by the centered-ness of $\mathbb{P}_n$) cannot decide the $\dot{f}(k)$. □

For each $n \in \omega$ let $f_n \in \Pi F$ be such that $f_n(k) \in F(k) \setminus A^n_k$ for every $k \in \omega$. We will show that the sequence $\{f_n : n \in \omega\}$ is as desired. To see this, let $g \in \Pi F$ be finitely equal to each f_n, and assume that for some $p \in \mathbb{P}$, we have $p \Vdash$ "$g \neq^* \dot{f}$". Strengthening p if necessary, we may assume, that for some $N \in \omega$ we have $p \Vdash$ "$\forall k > N\ g(k) \neq \dot{f}(k)$". Let also $n \in \omega$ be such that $p \in \mathbb{P}_n$. As $f_n =^\infty g$, we may pick $k > N$ such that $f_n(k) = g(k)$. Because $f_n(k) \notin A^n_k$ there must be $p' \leq p$ with $p' \Vdash$ "$\dot{f}(k) = f_n(k) = g(k)$" which contradicts what the p forces.

• To prove the second item, let $(p, \dot{q}) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$ "$\dot{f} \in \Pi F$". As $p \Vdash_{\mathbb{P}}$ "$\dot{q} \Vdash$ "$\dot{f} \in \Pi F$"", we may apply $(\star)_{\dot{\mathbb{Q}}}$ in $\mathbf{V}^{\mathbb{P}}$ to get $\mathbb{P}$-names $\{\dot{g}_n : n \in \omega\}$ for elements of ΠF such that

$$p \Vdash_{\mathbb{P}} \text{"}\forall \dot{g} \in \Pi F \text{ if } (\forall n \in \omega\ \dot{g} =^\infty \dot{g}_n), \text{ then } \dot{q} \Vdash_{\dot{\mathbb{Q}}} \text{"}g =^{\dot{\infty}} \dot{f}\text{""}$$

Then applying $(\star)_{\mathbb{P}}$ to each $\dot{g}_n$ we obtain a collection $\{h^n_m : n, m \in \omega\} \subseteq \Pi F$ such that for every $h \in \Pi F$ and every $n \in \omega$, if $h =^\infty g^n_m$ for every $m \in \omega$, then $p \Vdash_{\mathbb{P}}$ "$h =^\infty \dot{g}^n_m$". But this means that for $h \in \Pi F$, if h is infinitely equal to each h^n_m, $n, m \in \omega$, then $(p.\dot{q}) \Vdash_{\mathbb{P}*\dot{\mathbb{Q}}}$ "$h =^\infty \dot{f}$".

• To prove the last item, let $\{\delta_n : n \in \omega\}$ be an increasing sequence of ordinals with δ as a limit and assume that $(\star)_{\mathbb{P}_{\delta_n}}$ holds for each $n \in \omega$. Let $\dot{f}$ be a $\mathbb{P}_\alpha$-name for an element of ΠF. In each intermediate model $\mathbf{V}^{\mathbb{P}_{\delta_n}}$ we take interpretations of $\dot{f}$ i.e. for every $n \in \omega$ there are $\mathbb{P}_{\delta_n}$-names $\{\dot{q}^n_k : k \in \omega\}$ for elements of $\mathbb{P}_{[\delta_n, \delta)}$ and a sequence $\{\dot{f}_n : n \in \omega\}$ such that $\dot{f}_n$ is $\mathbb{P}_{\delta_n}$-name for an element of ΠF, and that:

$$\Vdash_{\mathbb{P}_{\delta_n}} \text{"}\forall k \in \omega\ \dot{q}^n_{k+1} \leq \dot{q}^n_k \text{ and } \dot{q}^n_k \Vdash_{\mathbb{P}_{[\delta_n,\delta)}} \text{"}\dot{f}|_{k+1} = \dot{f}_n|_{k+1}\text{""}$$

By inductive assumption, there are $\{g^n_i : n, i \in \omega\} \subseteq \Pi F$ such that for every $n \in \omega$ and every $g \in \Pi F$: if for all $i \in \omega\ g =^\infty g^n_i$, then $\Vdash_{\delta_n}$ "$\dot{f}_n =^\infty g$". We claim that:

$$\text{if } g \in \Pi F \text{ is infinitely equal to each } g^n_i \text{ where } n, i \in \omega, \text{ then } \Vdash_{\mathbb{P}_\delta} \text{"}\dot{f} =^\infty g\text{"}.$$

Let $p \in \mathbb{P}_\delta$ and $M \in \omega$. We will find $r \leq p$ such that $r \Vdash$ "$\exists k > M\ g(k) = \dot{f}(k)$". As the support of p is finite, there is $N \in \omega$ such that $p \in \mathbb{P}_{\delta_N}$. In $\mathbf{V}^{\mathbb{P}_{\delta_N}}$ we have that $\exists^\infty k \in \omega\ g(k) = f_N(k)$. Let then $\dot{k}$ be a $\mathbb{P}_{\delta_N}$-name such that $\Vdash_{\mathbb{P}_{\delta_N}}$ "$\dot{k} > M$ and $g(\dot{k}) = \dot{f}_N(\dot{k})$". Then, the condition $r =: (p, \dot{q}^N_{\dot{k}})$ is an element of two-step iteration $\mathbb{P}_{\delta_N} * \dot{\mathbb{P}}_{[\delta_N,\delta)}$ that is isomorphic to $\mathbb{P}_\delta$. But then $r \Vdash$ "$g(\dot{k}) = \dot{f}_N(\dot{k}) = \dot{f}(\dot{k})$ and $\dot{k} > M$". This finishes the proof of the last item. □

□

Recall that both $\mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}})$ and $\mathfrak{d}(\Pi F, =^\infty, \Pi F)$ stand as a lower bound of $\mathrm{cov}^*(\mathcal{ED}_{fin})$. We will show that these inequalities can be strict.

Theorem 4.4.17. *It is consistent with ZFC that* $\mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}}) = \mathfrak{d}(\Pi F, =^\infty, \Pi F) < \mathrm{cov}^*(\mathcal{ED}_{fin})$.

Proof. For $\alpha \leq \omega_2$, denote by $\mathbb{P}_\alpha$ countable support iteration of length α with the forcing $\mathbb{L}_{\mathcal{ED}_{fin}}$ standing as each iterand. Let $\mathbf{V}$ be a model of CH. We want to show that the inequality holds in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. By the Proposition 3.1.8, the tree type $\mathbb{L}_{\mathcal{ED}_{fin}}$ has Laver property. On the other hand, it is well-known, that Laver property is preserved under countable support iteration. It follows that $\mathbb{P}_{\omega_2}$ has Laver property. To show that $\mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}}) = \mathfrak{d}(\Pi F, =^\infty, \Pi F) = \omega_1$ holds in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$ we recall the following two results.

Lemma 4.4.18. *((Hrušák, Meza-Alcántara, Minami, [HMAM10])) If a forcing notion $\mathbb{P}$ has Laver property, then for any $X \in [\omega]^\omega$ in $\mathbf{V}^{\mathbb{P}}$, there is a ground model $Y \in \mathcal{I}_{\frac{1}{n}}$, such that $|X \cap Y| = \omega$.*

Lemma 4.4.19. *(Martinez-Celis and Żuchowski, [MC4]) If a forcing notion $\mathbb{P}$ has Laver property, then for any $\dot{f} \in \Pi F$ in $\mathbf{V}^{\mathbb{P}}$, there is a ground model $g \in \Pi F$, such that $|\dot{f} \cap g| = \omega$.*

As there are only ω_1-many reals in the ground model, $\mathrm{cov}^*(\mathcal{I}_{\frac{1}{n}}) = \mathfrak{d}(\Pi F, =^\infty, \Pi F) = \omega_1$ hold in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. We will show that $\mathrm{cov}^*(\mathcal{ED}_{fin}) = \omega_2 = \mathfrak{c}$ holds in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. Assume that in $\{X_\beta : \beta < \omega_1\}$ is a collection of $\mathcal{ED}_{fin}$-sets in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. As $\mathbb{L}_{\mathcal{J}}$ has axiom A, for every $\beta < \omega_1$ there is $\alpha(\beta) < \omega_2$ such that $X_\beta \in \mathbf{V}^{\mathbb{P}_{\alpha(\beta)}}$. Then there is $\alpha < \omega_2$ such that all X_β's are in $\mathbf{V}^{\mathbb{P}_\alpha}$. By the Proposition 3.1.4, the forcing $\mathbb{L}_{\mathcal{ED}_{fin}}$ adds an infinite set which is almost disjoint from every ground model $X \in \mathcal{ED}_{fin}$. Thus in $\mathbf{V}^{\mathbb{P}_{\alpha+1}}$, there is an infinite Y, almost disjoint from all X_β's, which means that $\{X_\beta : \beta < \omega_1\}$ cannot witness for $\mathrm{cov}^*(\mathcal{ED}_{fin})$. It follows that $\mathrm{cov}^*(\mathcal{ED}_{fin}) = \omega_2$ in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. □

In the article [BF17] of Brendle and Flaskova, the authors showed that $\mathrm{cov}(\mathcal{E}) \leq \mathrm{non}^*_+(\mathcal{Z})$ and $\mathrm{non}(\mathcal{E}) \leq \mathrm{non}^*_+(\mathcal{I}_{\frac{1}{n}})$. As mentioned in the introduction of this thesis, I proved the following result only to realize a couple months later that it is a slight generalization of the arguments from [BF17].

Theorem 4.4.20. *The following inequalities hold:*

- $\mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) \leq \mathrm{cov}(\mathcal{E}) \leq \mathrm{non}^*_+(tr(\mathcal{N}))$,
- $\mathrm{cov}^*_+(tr(\mathcal{N})) \leq \mathrm{non}(\mathcal{E}) \leq \mathrm{non}^*_+(\mathcal{I}_{\frac{1}{n}})$.

Proof. We will first show the inequalities $\mathrm{cov}^*_+(\mathcal{I}_{\frac{1}{n}}) \leq \mathrm{cov}(\mathcal{E})$ and $\mathrm{non}(\mathcal{E}) \leq \mathrm{non}^*_+(\mathcal{I}_{\frac{1}{n}})$ but first we state two observations. First. According to the Theorem 3.4. of the article [HHH07] the ideal $\mathcal{I}_{\frac{1}{n}}$ is Katetov equivalent with the ideal $\{A \subseteq 2^{<\omega} : \Sigma_n \frac{|A \cap 2^n|}{2^n} < \infty\}$. In this proof we will work with this presentation of the summable ideal. Second. For $F \in (\omega \setminus 1)^\omega$, denote by $\mathcal{E}_F$ the σ-ideals generated by closed sets of measure zero in ΠF. In particular, for F constant and equal 2, the σ-ideal $\mathcal{E}_F$ is equal to the standard $\mathcal{E}$. By the Remark 4.1 (3) of the [AA23] we have that $\mathrm{cov}(\mathcal{E}) = \mathrm{cov}(\mathcal{E}_F)$ and $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{E}_F)$ for any F as above. It follows that it does not matter which space of form ΠF we choose for the σ-ideal $\mathcal{E}$ to live on.
We define two reductions: $\phi : \Pi_n 2^n \to \mathcal{I}_{1/n}$ and $\psi : \mathcal{I}^+_{1/n} \to \mathcal{E}$. For a real $x \in \Pi_n 2^n$ we define $\phi(x) = \{x(n) : n \in \omega\}$. Clearly, $\phi(x) \in \mathcal{I}_{1/n}$. For $X \in \mathcal{I}^+_{1/n}$ let $\psi(X) = \{x \in \Pi_n 2^n : \forall^\infty n \; x(n) \notin X\}$. Notice, that because of inequality $ln(1-x) \leq x$ for $x > 0$, for every $N \in \omega$ we have that:

$$ln(\Pi_{n>N}(1 - \tfrac{|X \cap 2^n|}{2^n})) = \Sigma_{n>N} ln(1 - \tfrac{|X \cap 2^n|}{2^n}) \leq -\Sigma_{n>N} \tfrac{|X \cap 2^n|}{2^n}$$

and the last item goes to $-\infty$ as N goes to infinity. It follows that $\psi(X) \in \mathcal{E}$. Now, if $\{x_\alpha : \alpha < \kappa\} \subseteq \Pi_n 2^n$ witnesses for $\mathrm{non}(\mathcal{E})$, then $\{\phi(x_\alpha) : \alpha < \kappa\}$ witnesses for $\mathrm{cov}^*_+(\mathcal{I}_{1/n})$ and, if $\{X_\alpha : \alpha < \kappa\} \subseteq [2^{<\omega}]^\omega$ witnesses for $\mathrm{non}^*_+(\mathcal{I}_{1/n})$, then $\{\psi(X_\alpha) : \alpha < \kappa\}$ witnesses for $\mathrm{cov}(\mathcal{E})$.

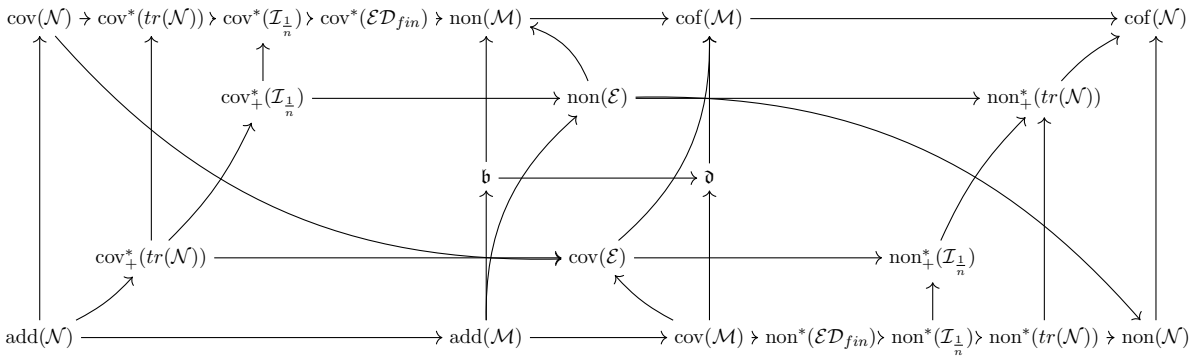
We will now show that $\mathrm{cov}^*_+(tr(\mathcal{N})) \leq \mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{E}) \leq \mathrm{non}^*_+(tr(\mathcal{N}))$. Here we consider the ideal $\mathcal{E}$ on the Cantor space 2^ω. We define two reductions

$$\phi : \mathcal{E} \to tr(\mathcal{N}) \text{ and } \psi : tr(\mathcal{N})^+ \to \mathcal{E}^+$$

For a closed set F of measure zero let $\phi(F) \subseteq 2^{<\omega}$ be such a tree that $[\phi(F)] = F$. Clearly, $\phi(F) \in tr(\mathcal{N})$. For $H \in tr(\mathcal{N})^+$, let $\psi(H) = \{x \in 2^\omega : \exists^\infty_n x|_n \in H\}$. Clearly $\psi(H) \in \mathcal{E}^+$ as a G_δ-set of positive measure. Now, if $\{x_\alpha : \alpha < \kappa\} \subseteq \Pi_n 2^n$ witnesses for $\mathrm{non}(\mathcal{E})$, then $\{\phi(x_\alpha) : \alpha < \kappa\}$ witnesses for $\mathrm{cov}^*_+(\mathcal{I}_{1/n})$ and, if $\{X_\alpha : \alpha < \kappa\} \subseteq [2^{<\omega}]^\omega$ witnesses for $\mathrm{non}^*_+(\mathcal{I}_{1/n})$, then $\{\psi(X_\alpha) : \alpha < \kappa\}$ witnesses for $\mathrm{cov}(\mathcal{E})$.

□

The following diagram summarizes the inequalities between cov^*_+'es and the cardinal invariants of the σ-ideal $\mathcal{E}$.



Notice that both invariants $\mathrm{cov}^*(tr(\mathcal{N}))$ and $\mathrm{cov}(\mathcal{E})$, have $\mathfrak{r}$ and $\max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ as an upper bounds. Also both have $\mathrm{cov}(\mathcal{N})$ as a lower bound. It is impossible to prove in ZFC the inequality $\mathrm{cov}(\mathcal{E}) \leq \mathrm{cov}^*(tr(\mathcal{N}))$. This is because in Cohens model we have $\mathrm{cov}^*(tr(\mathcal{N})) < \mathrm{cov}(\mathcal{E})$ as $\mathrm{cov}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{E})$ and $\mathrm{cov}^*(tr(\mathcal{N})) \leq \mathrm{non}(\mathcal{M})$. This motivates the following question:

Question 4.4.21. *Is* $\mathrm{cov}^*(tr(\mathcal{N})) \leq \mathrm{cov}(\mathcal{E})$ *provable in ZFC?*

4.5 Antichain numbers of $\mathcal{P}(\omega)/\mathcal{J}$'s

Suppose that $\mathcal{J}$ is an ideal on ω. A collection of $\mathcal{J}$-positive sets $\{A_\alpha : \alpha < \kappa\}$ is an $\mathcal{J}$-almost disjoin family is $A_\alpha \cap A_\beta \in \mathcal{J}$ for every $\alpha \neq \beta$. Although, for certain ideal the may be no uncountable such families (for example when $\mathcal{J}$ is maximal), when the ideal $\mathcal{J}$ is meager (and we are interested only in such ideals), uncountable $\mathcal{J}$-almost disjoint families always exists (see the proposition below). Define then $\mathfrak{a}(\mathcal{J})$ is the smallest size of maximal uncountable $\mathcal{J}$-almost disjoint family. In case of the ideal of finite sets, the number $\mathfrak{a}(\mathcal{F}in)$ is simply the classical almost disjoint number $\mathfrak{a}$.
First we will show that for meager $\mathcal{J}$, large $\mathcal{J}$-antichains exist.

Proposition 4.5.1. *(folklore) If* $\mathcal{J}$ *is meager, then there exist an* $\mathcal{J}$*-almost disjoint family of size* $\mathfrak{c}$.

Proof. By Talagrand's Theorem 1.2.14 there is a partition of ω into intervals $\{I_n : n \in \omega\}$ such that if $X \subseteq \omega$ contains infinitely many of these intervals, then X is $\mathcal{J}$-positive. Let $\mathcal{A}' \subseteq \mathcal{P}(\omega)$ be an almost disjoint family of size $\mathfrak{c}$. Then the family $\mathcal{A}$ made of sets of form $\bigcup_{n\in A} I_n$, for $A \in \mathcal{A}'$ as an almost disjoin family of $\mathcal{J}$-positive sets of size $\mathfrak{c}$. □

Let us gather a couple of known results regarding the $\mathcal{J}$-antichain numbers.

Theorem 4.5.2. *The following holds:*

- $\mathfrak{b} \leq \mathfrak{a}(\mathcal{F}in \times \mathcal{F}in) \leq \mathfrak{a}$,
- *it is consistent that* $\mathfrak{d} < \mathfrak{a}(\mathcal{F}in \times \mathcal{F}in)$,
- $\mathfrak{b} \leq \mathfrak{a}(\mathcal{J})$ *holds for all tall analytic P-ideals,*
- $\mathfrak{p} \leq \mathfrak{a}(nwd)$,
- $\mathfrak{a}(nwd) < \mathfrak{b}$ *holds in Lavers model.*

Proof. The first two items are results of Brendle from [Bre]. The third inequality is a result of Farkas and Soukup from [FS09]. The last two inequalities are results of Steprāns from [Ste02]. □

The numbers $\mathfrak{a}(\mathcal{J})$ for fubini products of ideals were investigated by Raghavan and Steprāns in [RS23]. Also, during the work of this thesis, Jonathan Cancino Manriquez showed the following strengthening of the inequality of Steprāns. He presented the result during Wrocław Set Theory seminar in April 2023. It turned out that the two proofs are essentially different.

Theorem 4.5.3. *(Cancino Manriquez, unpublished)* $\mathrm{add}(\mathcal{M}) \leq \mathfrak{a}(nwd)$

To prove our lower bound for the $\mathfrak{a}(\mathcal{J})$ let us introduce the following class of ideals.

Definition 4.5.4. *Suppose that $\mathcal{J}$ is and ideal on ω. We will say that $\mathcal{J}$ is good if for every uncountable, almost disjoint*[1] *$\mathcal{F} \subseteq \mathcal{J}^+$ has a subfamily $\{B_n : n \in \omega\} \subseteq \mathcal{F}$ such that for every $f \in \omega^\omega$ there is a sequence $\{C_n : n \in \omega\} \subseteq \mathcal{J}$, $C_n \subseteq B_n \setminus f(n)$ such that $C := \bigcup_n C_n$ is $\mathcal{J}$-positive.*

We are now ready to prove the main theorem of this section.

Theorem 4.5.5. *If an ideal $\mathcal{J}$ is good, then* $\min\{h\text{-}\mathrm{cov}^*_+(\mathcal{J}), \mathfrak{b}\} \leq \mathfrak{a}(\mathcal{J})$.

Proof. Let $\kappa < \min\{\text{h-cov}^*_+(\mathcal{J}), \mathfrak{b}\}$ and let $\{A_\alpha : \alpha < \kappa\}$ be an $\mathcal{J}$- antichain in $\mathcal{J}^+$. As the inequality $\kappa <\text{h-cov}^*_+(\mathcal{J})$ holds, for every $\alpha < \kappa$ we can find a J-positive set $B_\alpha \subseteq A_\alpha$ such that $B_\alpha \cap A_\beta =^* \emptyset$ for $\beta \neq \alpha$. Suppose that $\{B_n : n < \omega\}$ is as in definition of goodness. Shrinking each B_n if necessary, we may assume that the intersection $B_n \cap \bigcup_{i\leq n} A_i$ is empty for each $n \in \omega$. For every $\alpha \in \kappa \setminus \omega$ define $f_\alpha \in \omega^\omega$ such that $B_n \cap A_\alpha \subseteq f_\alpha(n)$ for each $n \in \omega$. As $\kappa < \mathfrak{b}$, we may choose a function $f \in \omega^\omega$ which dominates all of f_α's. By the property of being good, there is a sequence $\{C_n : n \in \omega\} \subseteq \mathcal{J}$, $C_n \subseteq B_n \setminus f(n)$ such that $C := \bigcup_n C_n$ is $\mathcal{J}$-positive. We claim that $C \cap A_\alpha \in \mathcal{J}$ for every $\alpha < \kappa$ showing that the antichain $\{A_\alpha : \alpha < \kappa\}$ was not maximal. Let $\alpha < \kappa$. If $\alpha = n \in \omega$, then $C \cap A_n \subseteq \bigcup_{i\leq n} C_i$ because for $i > n$ we have $C_i \subseteq B_i$ and $B_i \cap A_n = \emptyset$. If $\alpha \in \kappa \setminus \omega$ let then $N \in \omega$ be such that $f_\alpha(n) < f(n)$ for $n > N$. Then for $n > N$ we have $C_n \cap A_\alpha \subseteq B_n \cap A_\alpha \subseteq f_\alpha(n) \subseteq f(n)$, but $C_n \subseteq (B_n \setminus f(n))$. It follows that $\bigcup_{n>N}(C_n \cap A_\alpha)$ is empty and $C \cap A_\alpha \subseteq \bigcup_{n\leq N} C_n \cap A_\alpha \in \mathcal{J}$. □

Below we will show that most of the ideals we consider in this thesis, are in fact good. However, I cannot precisely identify the class of good ideals. Thus, we have the the following.

Question 4.5.6. *Find an example of a meager ideal $\mathcal{J}$ which is not good. Is $\mathcal{S}pl$ any good?*

Proposition 4.5.7. *The ideals nwd and $\mathcal{F}in \times \mathcal{F}in$ are good.*

Proof. First, assume that $\mathcal{F}$ is an uncountable collection of almost disjoint nwd-positive sets. By the definition, for every $F \in \mathcal{F}$ there is $\sigma \in 2^{<\omega}$ such that F is dense below σ, meaning that every τ that extends σ can be extended to a $\rho \in F$. Let $\{B_n : n \in \omega\} \subseteq \mathcal{F}$ be such that all B_n's are dense below the same $\sigma^* \in 2^{<\omega}$. Fix $f \in \omega^\omega$. Inductively construct the sets $\{C_n : n \in \omega\}$ such that for any $n \in \omega$ we have:

$(1)_n$ $C_n \subseteq B_n \setminus f(n)$,

$(2)_n$ $|C_n| = 2^n$,

$(3)_n$ for every $\tau \in 2^{|\sigma^*|+1}$ that extends σ^*, there is unique $\sigma \in C_n$ such that $\tau \subseteq \sigma$,

$(4)_n$ $C_n \cap \bigcup_{i<n} B_i = \emptyset$,

The construction is straightforward. By of the conditions $(2)_n$, each $C_n \in nwd$. By of the conditions $(3)_n$ the set $C = \bigcup_n C_n$ is dense below σ^*, which meant that $C \in nwd^+$. It follows that nwd is good.

[1] in the modulo finite sense

For the case of $\mathcal{F}in \times \mathcal{F}in$, assume that $\mathcal{F}$ is an uncountable collection almost disjoint of $\mathcal{F}in \times \mathcal{F}in$-positive sets. Let $\{B_n : n \in \omega\}$ be any countable sub-collection of $\mathcal{F}$. Given $f \in \omega^\omega$ inductively build an increasing sequence $\{c_n : n \in \omega\}$ of natural numbers such that $C_n := B_n \cap (\{c_n\} \times \omega) \setminus f(n)$ is infinite. Then each $C_n \in \mathcal{F}in \times \mathcal{F}in$ because it is contained in the column $\{c_n\} \times \omega$. We also have that $\mathbb{C} = \bigcup_n C_n \in \mathcal{F}in \times \mathcal{F}in^+$ because $\{c_n : n \in \omega\}$ is increasing. □

In [Ste02] Steprāns proved that $\mathfrak{p} \leq \mathfrak{a}(nwd)$. The following improvement of the result of Steprāns has been independently obtained - in an unpublished note - by Jonathan Cancino Manriquez by a different method.

Corollary 4.5.8. $\mathrm{add}(\mathcal{M}) \leq \mathfrak{a}(nwd)$

Proof. By the Proposition 4.4.2 we have that $\mathrm{cov}^*_+(nwd) = \mathrm{add}(\mathcal{M})$. The inequality follows from the Theorem 4.5.5 and the fact that $\mathrm{add}(\mathcal{M}) = \min \mathfrak{b}, \mathrm{cov}(\mathcal{M})$. □

Proposition 4.5.9. *All F_σ-ideals are good.*

Proof. Let $\mathcal{J}$ be an F_σ-ideal. Let ϕ be a l.s.c.s.m. such that the ideal $\mathcal{J}$ is equal to $Fin(\phi)$. Assume that $\mathcal{F}$ is an uncountable collection of almost disjoint $Fin(\phi)$-positive sets. Let $\{B_n : n \in \omega\} \subseteq \mathcal{F}$ be arbitrary. For every $f \in \omega^\omega$ it is easy to pick finite sets C_n's such that $C_n \subseteq B_n \setminus f(n)$ and $\phi(C_n) > n$. Clearly, then $\phi(C) = \infty$. □

Corollary 4.5.10. *The ideals $\mathcal{S}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{I}_{\frac{1}{n}}$ as well as $\mathcal{ED}$ and $\mathcal{ED}_{fin}$ are good.*

Proposition 4.5.11. *All analytic P-ideals are good.*

Proof. Let ϕ be a l.s.c.s.m. such that the ideal is equal to $Exh(\phi)$. Assume that $\mathcal{F} \subseteq Exh(\phi)^+$ is an uncountable almost disjoint family. Let $\{B_n : n \in \omega\} \subseteq \mathcal{F}$ be such that for some $\epsilon > 0$ we have $limsup_k \phi(B_n \setminus k) > \epsilon$ for each $n \in \omega$. For each $f \in \omega^\omega$ and $n \in \omega$ it is not difficult to pick $C_n \subseteq B_n \setminus f(n)$ with $\phi(C_n) > \epsilon/2$. Clearly, then $lim_k \phi(C \setminus k) = 0$. □

Corollary 4.5.12. *The ideals $\mathcal{Z}$, $tr(\mathcal{N})$ are good.*

Proposition 4.5.13. *conv ideal is good*

Proof. Suppose that we are given a family $\{A_\alpha : \alpha < \omega_1\}$ of almost disjoint *conv*-positive sets.

Claim 4.5.14. *For every A_α there is a collection $\{x^\alpha_{i,j} : i, j \in \omega\} \subseteq A_\alpha$ such that:*

- *for every i and $j_0 \neq j_1$, $x^\alpha_{i,j_0} \neq x^\alpha_{i,j_1}$,*
- *for every i, $\{x^\alpha_{i,j} : j \in \omega\}$ converges to y^α_i,*
- *for all $i_0 \neq i_1$, $y^\alpha_{i_0} \neq y^\alpha_{i_1}$,*
- *$\{y^\alpha_i : i \in \omega\}$ converges to z^α*

proof of the claim. Notice that $\overline{A} \setminus A$ cannot be finite because then $A \in conv$. By compactness of $[0, 1]$ let $\{y_i : i \in \omega\} \subseteq \overline{A} \setminus A$ be an infinite sequence of distinguished points with a limit point z. For each $i \in \omega$ choose a sequence $\{x_{i,j} : j \in \omega\} \subseteq A$ of distinguished point converging to y_i such that $\{x_{i_0,j} : j \in \omega\} \cap \{x_{i_1,j} : j \in \omega\} = \emptyset$ for $i_0 \neq i_1$. □

Let $W = \{\alpha_n : n \in \omega\} \subseteq \omega_1$ be such a countable set that $\{z^{\alpha_n} : n \in \omega\}$ converges to a point w and let $f : \omega \to \mathbb{Q} \cap [0, 1]$. Inductively we construct sets $\{C_n : n \in \omega\}$ such that for each $n \in \omega$ we have:

- there are $i_n \in \omega$ and $N_n \in \omega$ such that $C_n = \{x^{\alpha_n}_{i_n,j} : j > N_n\}$,

– N_n is large enough so that $C_n \cap f[n] = \emptyset$,

– $y^{\alpha_n}_{i_n} \notin \{y^{\alpha_n}_{i_k} : k < n\}$,

– $|y^{\alpha_n}_{i_n} \cap z| < \frac{1}{n}$.

The construction is straightforward. Now, as each C_n is in *conv*, it remains to check that $C = \bigcup_n C_n$ is *conv*-positive. If C would be covered by finitely many convergent sequences, then the limit points of C, it is $\{y^{\alpha_n}_{i_n} : n \in \omega\}$ and z, would be covered by finitely many limit points of the convergent sequences. □

As a result of the Theorem 4.5.5 we obtain the following, known estimate for the antichain numbers

Proposition 4.5.15. $\mathfrak{p} \leq \mathfrak{a}(\mathcal{J})$ *for all good ideals with uncountable* $\mathrm{cov}^*_+(\mathcal{J})$.

Proof. By the Theorem 4.5.5 and the Proposition 4.4.12 we have $\mathfrak{p} \leq \min\{\mathfrak{b}, \mathrm{cov}^*_+(\mathcal{J})\} \leq \mathfrak{a}(\mathcal{J})$ □

In case of the eventually different ideals we have

Corollary 4.5.16. $\min\{\mathfrak{b}, \mathrm{cov}^*(\mathcal{ED}_{fin})\} \leq \mathfrak{a}(\mathcal{ED}_{fin}) \leq \mathfrak{a}$ *and* $\mathfrak{b} \leq \mathfrak{a}(\mathcal{ED}) \leq \mathfrak{a}$ *holds.*

We will finish this section with the following known result.

Theorem 4.5.17. *(Brendle, [Bre09])* $\mathfrak{a}(\mathcal{ED}_{fin}) = \omega_1$ *in Hechlers model.*

It follows that the above corollary cannot be strengthened to the inequality $\mathfrak{b} \leq \mathfrak{a}(\mathcal{ED}_{fin})$.

Bibliography

[AA23] Antonio, Cardona M. ; Alejandro, Mejia D.: Localization and anti-localization cardinals. In: *DOI:10.48550/arXiv.2305.03248* (2023)

[AB89] Andreas Blass, Saharon S.: Near coherence of filters. III. A simplified consistency proof. In: *Notre Dame Journal of Formal Logic* 30 (1989), Nr. 4, S. 530–538. http://dx.doi.org/10.1305/ndjfl/1093635236. – DOI 10.1305/ndjfl/1093635236

[AC23] A. Cardona, Miguel: A friendly iteration forcing that the four cardinal characteristics of E can be pairwise different. In: *Colloquium Mathematicum* (2023), Nr. 173, S. 123–157

[AC24] A. Cardona, Miguel: The cardinal characteristics of the ideal generated by the F measure zero subsets of the reals. In: *preprint: https://arxiv.org/abs/2402.04984* (2024)

[Bau10] Baumgartner, James E.: Iterated Forcing. In: *Cambridge University Press* (2010)

[BDMH12] Balcar, Boguslav ; Doucha, Michal ; M. Hrušák, Michael: Base Tree Property. In: *Order* (2012). http://dx.doi.org/10.1007/s11083-013-9316-2. – DOI 10.1007/s11083–013–9316–2

[BF17] Brendle, Jörg ; Flašková, Jana: Generic existence of ultrafilters on the natural numbers. In: *Fundamenta Mathematicae* 236 (2017), Nr. 3, S. 201–245

[BHHH04] Balcar, Boguslav ; Hernández-Hernández ; Hrušák, Michael: Combinatorics of dense subsets of the rationals. In: *Fundamenta Mathematicae* Bd. 183. 2004, S. 59–80

[BJ95] Bartoszyński, Tomek ; Judah, Haim: *Set theory.* A K Peters, Ltd., Wellesley, MA, 1995. – On the structure of the real line

[BKW17] Brendle, Jörg ; Khomskii, Yurii ; Wohofsky, Wolfgang: Cofinalities of Marczewski-like ideals. In: *Colloq. Math.* 150 (2017), Nr. 2, S. 269–279. – ISSN 0010–1354,1730–6302

[Bla09] Blass, Andreas: *Combinatorial Cardinal Characteristics of the Continuum.* Handbook of Set Theory, 2009. – 395–489 S. – Handbook of Set Theory

[Bre] Brendle, Jörg: Notes on cardinal characteristics of quotients. In: *Unpublished notes*

[Bre99] Brendle, Jörg: Mutual generics and perfect free subsets. In: *Acta Math. Hungar.* 82 (1999), Nr. 1-2, S. 143–161. – ISSN 0236–5294,1588–2632

[Bre09] Brendle, Jörg: Cardinal invariants of analytic quotients. (2009). http://dx.doi.org/https://www.logic.univie.ac.at/2009/esi/pdf/brendle.pdf. – DOI https://www.logic.univie.ac.at/2009/esi/pdf/brendle.pdf

[CMRM24] Cardona, Miguel A. ; Mejía, Diego A. ; Rivera-Madrid, Ismael E.: Uniformity numbers of the null-additive and meager-additive ideals. In: *preprint* (2024). http://dx.doi.org/10.13140/RG.2.2.21087.84646. – DOI 10.13140/RG.2.2.21087.84646

[DR13] Dečo, Michal ; Repický, Miroslav: Strongly dominating sets of reals. In: *Arch. Math. Logic* 52 (2013), Nr. 7-8, S. 827–846. – ISSN 0933–5846,1432–0665

[Eit] Eithsworth, Todd: Private communication.

[FK23] Filipów, Rafał ; Kwela, Adam: Spaces not distinguishing ideal pointwise and -uniform convergence. (2023)

[Fla10] Flaskova, Jana: Ultrafilters on N and van der Waerden ideal. (2010). http://dx.doi.org/https://home.zcu.cz/ blobner/research/KielceSUMTOPO-handout.pdf. – DOI https://home.zcu.cz/ blobner/research/KielceSUMTOPO–handout.pdf

[Fre04] Fremlin, David H.: Measure theory. Set theoretic measure theory. In: *Fundamenta Mathematicae* (2004), Nr. 2, S. 187–200

[FS09] Farkas, Barnabás ; Soukup, Lajos: More on cardinal invariants of analytic P-ideals. In: *Commentationes Mathematicae Universitatis Carolinae* 50 (2009), Nr. 2, S. 281–295

[FZ22] Farkas, Barnabás ; Zdomskyy, Lyubomyr: Ways of Destruction. In: *J. Symbolic Logic* 87 (2022), Nr. 2, S. 938 – 966

[GRSS95] Goldstern, Martin ; Repický, Miroslav ; Shelah, Saharon ; Spinas, Otmar: On tree ideals. In: *Proc. Amer. Math. Soc.* 123 (1995), Nr. 5, S. 1573–1581. – ISSN 0002–9939,1088–6826

[GS24] Guzmán, Osvaldo ; Santiago, Francisco: Laver trees. In: *Private Communication* (2024)

[Guz] Guzmán, Osvaldo: PhD Thesis: P-points, mad families and cardinal invariants. In: *PhD Thesis*

[HHH07] Hernández-Hernández, Fernando ; Hrušák, Michael: Cardinal invariants of analytic P-ideals. In: *Canad. J. Math.* 59 (2007), Nr. 3, S. 575–595. – ISSN 0008–414X,1496–4279

[HMAM10] Hrušák, Michael ; Meza-Alcántara, David ; Minami, Hiroaki: Pair-splitting, pair-reaping and cardinal invariants of F -ideals. In: *Journal of Symbolic Logic* 75 (2010), Nr. 2, S. 661–677

[Hru11] Hrušák, Michael: Combinatorics of filters and ideals. In: *Set theory and its applications* Bd. 533. Amer. Math. Soc., Providence, RI, 2011, S. 29–69

[HS20] Hein, Paul ; Spinas, Otmar: Antichains of perfect and splitting trees. In: *Archive for Mathematical Logic* 59 (2020), S. 367–388

[HZ04] Hrušák, Michael ; Zapletal, Jindřich: Forcing with quotients. (2004)

[Jec02] Jech, Thomas: Set theory, The Third Millennium Edition, 2002. In: *Springer Berlin, Heidelberg* (2002)

[JMS92] Judah, Haim ; Miller, Arnold W. ; Shelah, Saharon: Sacks forcing, Laver forcing, and Martin's axiom. In: *Arch. Math. Logic* 31 (1992), Nr. 3, S. 145–161. – ISSN 0933–5846,1432–0665

[Kec77] Kechris, Alexander S.: On a notion of smallness for subsets of the Baire space. In: *Transactions of the American Mathematical Society* 229 (1977), S. 191–207. – ISSN 0002–9947,1088–6850

[Ken83] Kenneth, Kunen: *Set Theory: An Introduction To Independence Proofs.* ELSEVIER SCIENCE PUBLISHERS B.V., 1983

[KL17] Khomskii, Yurii ; Laguzzi, Giorgio: Full-splitting Miller trees and infinitely often equal reals. In: *Ann. Pure Appl. Logic* 168 (2017), Nr. 8, S. 1491–1506. – ISSN 0168–0072,1873–2461

[KT17] Kwela, Adam ; Tryba, Jacek: Homogeneous ideals on countable sets. In: *Acta Mathematica Hungarica* 151 (2017), S. 139–161

[Kwe23] Kwela, Adam: Unboring ideals. In: *Fundamenta Mathematicae* 261 (2023), Nr. 3, S. 235–272

[KZ] Khomskii, Yurii ; Zuchowski, Tomasz: Eventually Unpredictable Reals. In: *in preparation*

[LMSR23] Laguzzi, Giorgio ; Mildenberger, Heike ; Stuber-Rousselle, Brendan: On splitting trees. In: *Math. Log. Quart.* 69 (2023), Nr. 1, S. 15–30

[MA09] Meza-Alcántara, David: Ideals and filtres on countable sets. In: *Phd thesis* (2009)

[Maz91] Mazur, Krzysztof: F_σ-ideals and (ω_1, ω_1)-gaps in the Boolean algebras $\mathcal{P}(\omega)/I$. In: *Fundamenta Mathematicae* 138 (1991), Nr. 2, S. 103–111. – ISSN 0016–2736

[MC4] Martínez-Celis, Arturo ; Żuchowski, Tomasz: On cardinal invariants related to Rosenthal families and large-scale topology. (2024)

[Mil84] Miller, Arnold W.: Rational perfect set forcing. In: *Axiomatic set theory (Boulder, Colo., 1983)* Bd. 31. Amer. Math. Soc., Providence, RI, 1984. – ISBN 0–8218–5026–1, S. 143–159

[Mil12] Miller, Arnold W.: *Hechler and Laver Trees.* 2012

[MMR4] Mazurkiewicz, Łukasz ; Michalski, Marcin ; Rałowski, Robert ; Żeberski, Szymon: On Algebraic Sums, Trees And Ideals In The Baire Space. In: *Archive for Mathematical Logic* (2024)

[MS16] Malliaris, Maryanthe. ; Shelah, Saharon: Cofinality spectrum theorems in model theory, set theory and general topology. In: *J. Amer. Math. Soc.* 29 (2016), S. 237–297. http://dx.doi.org/10.1090/jams830 MR: 3402699. – DOI 10.1090/jams830 MR: 3402699

[MZ07] Matheron, Étienne ; Zelený, Miroslav: Descriptive Set Theory of Families of Small Sets. In: *Bulletin of Symbolic Logic* 13 (2007), Nr. 4, S. 482–537

[NRa93] Newelski, Ludomir ; Rosł anowski, Andrzej: The ideal determined by the unsymmetric game. In: *Proc. Amer. Math. Soc.* 117 (1993), Nr. 3, S. 823–831. – ISSN 0002–9939,1088–6826

[Pet93] Petr, Simon: Sacks forcing collapses c to b. In: *Commentationes Mathematicae Universitatis Carolinae* 34 (1993), S. 707–710. – ISSN 0010–2628

[Roi79] Roitman, Judith: More paracompact box products. In: *Proceedings of the American Mathematical Society* 34 (1979), S. 171–176

[RS17] Raghavan, Dilip ; Shelah, Saharon: Two inequalities between cardinal invariants. In: *Fund. Math.* 237 (2017), Nr. 2, S. 187–200

[RS23] Raghavan, Dilip ; Steprans, Juris: The almost disjointness invariant for products of ideals. In: *Topology and its Applications* 323 (2023), Nr. 2, S. 795–800

[Sac71] Sacks, Gerald E.: Forcing with perfect closed sets. In: *Axiomatic Set Theory (Proc. Sympos. Pure Math., Vol. XIII, Part I, Univ. California, Los Angeles, Calif., 1967)* Bd. XIII, Part I. 1971, S. 331–355

[Sch22] Schilhan, Jonathan: Some combinatorial properties of splitting trees. 70 (2022), S. 1–12

[She92] Shelah, Saharon: Vive la Diff´erence I: Nonisomorphism of Ultrapowers of Countable Models. In: *Set Theory of the Continuum* (1992), S. 357–405

[Sol94] Solecki, Slawomir: Covering Analytic Sets by Families of Closed Sets. 1994, S. 1022–1031

[Sol99] Solecki, Sławomir: Analytic ideals and their applications. In: *Annals of Pure and Applied Logic* 99 (1999), Nr. 1–3, S. 51–72

[Sol00] Solecki, Sławomir: Filters and sequences. In: *Fundamenta Mathematicae* 163 (2000), Nr. 3, S. 215–228. – ISSN 0016–2736

[Spi95] Spinas, Otmar: Generic trees. In: *J. Symbolic Logic* 60 (1995), Nr. 3, S. 705–726

[Spi97] Spinas, Otmar: Partition numbers. In: *Annals of Pure and Applied Logic* 90 (1997), S. 243–262

[Spi04] Spinas, Otmar: Analytic countably splitting families. In: *J. Symbolic Logic* 69 (2004), S. 101–117

[Spi08] Spinas, Otmar: Perfect set theorems. In: *Fundamenta Mathematicae* 201 (2008), S. 179–195

[Spi16] Spinas, Otmar: Silver trees and Cohen reals. In: *Israel Journal of Mathematics* 211 (2016), S. 473–480

[Spi22] Spinas, Otmar: Tukey reductions of nowhere Ramsey to Silver null sets. In: *Proc. Amer. Math. Soc.* 150 (2022), S. 2715–2727

[SS23] Shelah, Saharon ; Spinas, Otmar: Different cofinalities of tree ideals. In: *Ann. Pure Appl. Logic* 174 (2023), Nr. 8, S. Paper No. 103290, 18

[Ste02] Steprans, Yuris: The Almost Disjointness Cardinal Invariant in the Quotient Algebra of the Rationals Modulo the Nowhere Dense Subsets. In: *Real Analysis Exchange* 27 (2002), Nr. 2, S. 795–800

[SW15] Spinas, Otmar ; Wyszkowski, Marek: Silver Antichains. In: *Journal of Symbolic Logic* 80 (2015), S. 503–519

[SZ11] Sabok, Marcin ; Zapletal, Jindřich: Forcing properties of ideals of closed sets. In: *J. Symbolic Logic* 76 (2011), Nr. 3, S. 1075–1095. – ISSN 0022–4812,1943–5886

[Tal80] Talagrand, Michel: Compacts de fonctions mesurables et filtres non mesurables. In: *Studia Math.* 67 (1980), Nr. 1, 13–43. http://dx.doi.org/10.4064/sm-67-1-13-43. – DOI 10.4064/sm–67–1–13–43. – ISSN 0039–3223,1730–6337

[Tod96] Todorcevi˘c, Stevo: Analytic gaps. In: *Fundamenta Mathematicae* 150 (1996), Nr. 1, 55–66

[Yor03] Yorioka, Teruyuki: Forcings with the countable chain condition and the covering number of the Marczewski ideal. In: *Arch. Math. Logic* 42 (2003), S. 695–710

[Zap04] Zapletal, Jindřich: *Descriptive Set Theory and Definable Forcing.* Memoirs of the American Mathematical Society, 2004

[Zap08] Zapletal, Jindřich: *Cambridge Tracts in Mathematics.* Bd. 174: *Forcing idealized*. Cambridge University Press, Cambridge, 2008. – vi+314 S. – ISBN 978–0–521–87426–7

[Ła96] Łabedzki, Grzegorz: A topology generated by eventually different functions. In: *Acta Universitatis Carolinae. Mathematica et Physica* 37 (1996), Nr. 2, S. 37–53

Index

$\mathcal{J}$-Laver trees
  definition, 29
  dichotomy, 31
  invariants of the Borel part, 31
$\mathcal{J}$-Miller trees
  definition, 26
  dichotomy, 27
  invariants of the Borel part, 27
$\mathcal{J}$-distributivity numbers, 65
$\mathcal{J}$-tower, 58
$\leq_K$, 13

Base tree, 65

Cardinal invariants
  add($\mathcal{I}$), 10
  add$^*(\mathcal{J})$, 12
  $\mathfrak{b}(\mathfrak{R})$, 14
  $\mathfrak{b}_\omega(\mathfrak{R})$, 14
  cof($\mathcal{I}$), 10
  cof$^*(\mathcal{J})$, 12
  cov($\mathcal{I}$), 10
  cov$^*(\mathcal{J})$, 12
  $\mathfrak{d}(\mathfrak{R})$, 14
  $\mathfrak{d}_\omega(\mathfrak{R})$, 14
  non($\mathcal{I}$), 10
  non$^*(\mathcal{J})$, 12
  (un)bounding number, 10
  distributivity number, 11
  dominating number, 10
  groupwise density number, 44
  Martins number, 11
  pseudointersection number, 11
  reaping number, 11
  splitting number, 11
  tower number, 11

F-bounded eventually different trees
  definition, 22
  invariants of the Borel part, 24
F-bounded infinitely equal trees
  definition, 18
  dichotomy, 20
  invariants of the Borel part, 20
F-bounded Sacks trees
  definition, 18
  dichotomy, 20
  invariants of the Borel part, 20
Full-Miller trees
  definition, 24
  dichotomy, 25
  invariants of the Borel part, 25

Hurewicz schema
  definition, 36
  dichotomy, 35

Ideal
  F_σ-ideal, 72
  P-ideal, 12
  $P^+ - ideal$, 12
  $\mathcal{ED}$, 73
  $\mathcal{ED}_{fin}$, 73
  $\mathcal{F}in \times \mathcal{F}in$, 79
  $\mathcal{IE}$, 84
  $\mathcal{IE}_F$, 84
  $\mathcal{R}$, 76
  $\mathcal{S}pl$, 80
  $\mathcal{I}_{\underline{1}}$, 73
  $\mathcal{S}^n$, 75
  $\mathcal{W}$, 75
  $\mathcal{Z}$, 77
  $conv$, 80
  nwd, 77
  $tr(\mathcal{N})$, 77
  is homogeneous, 13
  is Katetov reducible, 13
  analytic P-ideal, 76
  nowhere-maximal, 12
  tall, 12
Infinitely equal ideal, 84
Infinitely equal trees
  definition, 24
  dichotomy, 25
  invariants of the Borel part, 25

Projection of a tree type onto relational system, 54

Relational system

$\mathfrak{b}(\mathfrak{R})$ and $\mathfrak{d}(\mathfrak{R})$, 14
$\langle [\omega]^\omega, reaps, [\omega]^\omega \rangle$, 15
$\langle [\omega]^\omega, splits, [\omega]^\omega \rangle$, 15
$\langle \omega^\omega, =^\infty, \omega^\omega \rangle$ and $\langle \omega^\omega, \neq^*, \omega^\omega \rangle$, 15
$\langle \mathcal{I}, \subseteq, \mathcal{I} \rangle$ and $\langle \Pi F, \in, \mathcal{I} \rangle$, 16
$\langle \mathcal{J}, \subseteq^*, \mathcal{J} \rangle$ and $\langle [\omega]^\omega, \cap^\infty, \mathcal{J} \rangle$, 16
$\langle \mathcal{J}^+, \cap^\infty, \mathcal{J} \rangle$, 16
$\langle \Pi F, =^\infty, \Pi F \rangle$ and $\langle \Pi F, \neq^*, \Pi F \rangle$, 16
$\langle \mathbb{IP}, \leq^*, \mathbb{IP} \rangle$, 15
Composition $\mathcal{R}_0 \otimes \mathcal{R}_1$, 15
definition, 14
Tukey reduction, 14

Splitting ideal, 80
Splitting trees
definition, 20
dichotomy, 21
invariants of the Borel part, 21

Talagrand's Theorem, 13
Trees
$\mathbb{T}$-nwd sets, 17
basis of a tree ideal, 17
Borel part of a tree ideal, 17
combinatorial tree type, 17
tree ideal, 17
Tress
$T, _\sigma 7$
$[T]$, 7
$split(T)$, 7
$succ_T(\sigma)$, 7
end-extension, 7