Politechnika Wrocławska

FIELD OF SCIENCE: NATURAL SCIENCE

DISCIPLINE OF SCIENCE: MATHEMATICS

DOCTORAL DISSERTATION

Pricing time-capped American options

Mr. Paweł Stępniak, MSc., Eng.

Supervisor: Prof. dr hab. Zbigniew Bogusław Palmowski

Keywords: optimal stopping, American options, Lévy processes

WROCŁAW SEPTEMBER 2025

Summary

This thesis considers the problem of pricing (both analytically and numerically) of American options with stochastic constraints imposed on the time to maturity. Fundamentally, the problem can be formulated as an optimal stopping problem of the following form:

$$\sup_{\tau \in \mathcal{T}, \tau \leq T} \mathbb{E}[e^{-r\tau \wedge \theta} G(S_{\tau \wedge \theta})]$$

where τ is the optimal stopping time, T is the positive (possibly infinite) time to maturity, S_t is a stochastic process describing the underlying asset price, G is the payout function and θ is the random time at which the contract is terminated. This time is referred to as a time-cap and may or may not be associated with the underlying asset performance.

The motivation for this thesis is to investigate a new and broad class of financial instruments that could potentially be introduced to the market. The development of capped options began in the early 1990s in Chicago, when the Chicago Board Options Exchange launched into the market capped European options on the S&P 100 and S&P 500. These contracts, which were immediately exercised when the underlying index exceeded a predetermined level (the cap), became a practical example of how payoff-limiting features could reduce the seller's risk while retaining investment appeal. Since then, many variations of capped options have been studied, but almost all of them apply the cap to the payoff function or the underlying asset price.

In this work, the approach is different: the cap is placed on the time to maturity. Each chosen time cap defines a new type of instrument, allowing for the study of a wide range of problems. Among the contracts considered are those where the time cap is given by the first exit time of the underlying asset from a specified interval, the last exit time from such an interval, and the first time when the drawdown — the ratio between the current price and the historical maximum — exceeds a certain threshold. The last exit time is not a stopping time, which would make this contract difficult to implement in practice; nevertheless, it is included here as an interesting analytical exercise that fits the overall scope of the thesis.

The price dynamics of the underlying asset is modeled using a spectrally negative geometric Lévy process, which allows for sudden downward jumps while retaining analytical tractability. The analytical part of the work employs the guess-and-verify method: an optimal stopping rule is proposed based on qualitative arguments, and then its validity is established using a verification theorem together with the Hamilton–Jacobi–Bellman system. In the numerical part, a modified version of the Least Squares Monte Carlo method, specifically adapted for the time-capped framework, is applied to illustrate and support

the theoretical results. This thesis is organized as follows.
Chapter 1 provides a theoretical introduction to financial markets and options, as well as an overview of the probability space, Lévy processes, scale functions, and the key parameters used throughout the remainder of the thesis.
Chapter 2 focuses on the valuation of a perpetual American put option with a random maturity date given by the first exit time of the underlying asset price from a pre-specified set. While similar instruments have been studied in the past, the introduction of downward jumps in our setting allows for a Poisson-type drop below the lower boundary. In such a case, the contract is early exercised by the cap event, rather than terminated, which means the holder may still obtain a non-zero payoff. In the proposed solution, we conjecture that the optimal stopping time is the first moment when the asset price falls below a certain level, and then we determine the option price under this assumption. This conjecture is subsequently verified later in the chapter. Finally, in the last part of this chapter, we present a numerical analysis.
Chapter 3 examines options stopped by a drawdown event, defined as the first time when the ratio between the underlying asset price and its historical maximum exceeds a predetermined threshold. The central statement of this chapter is the proof that the optimal stopping time is the first moment when the asset price falls below the value of a certain function of its past historical maximum. We start from the analysis of the special case of the Black-Scholes market when there are no jumps in the asset price. Then, later, we analyze the case with additional exponential downward jumps in the asset price. In both scenarios, we show that the optimal strategy is either to wait until the price falls below a fixed barrier or until the option is stopped by the drawdown event. As in the previous case, the latter does not terminate the option but forces its early exercise, potentially resulting in a positive payoff for the holder.
Chapter 4 considers an option stopped at the moment when the underlying asset price crosses above a given threshold for the last time. Since this moment is not a stopping time, the contract would be difficult to implement in practice. However, it is very interesting from the mathematical point of view and produces intriguing analysis. As before, we use the guess-and-verify method proposing the optimal strategy and then verify it and derive the corresponding option price. This chapter is based on [28].
Chapter 5 presents the description and proof of convergence of a modified Least Squares Monte Carlo method, designed to price American options whose maturity is constrained by an arbitrary stopping time. The chapter concludes with numerical results for several selected instruments of this type. This chapter is based on [29].

Contents

Chapter 1

Introduction

American options are one of the most popular instruments traded on the financial markets. They are a type of derivative security which gives their holders a right, but not obligation, to buy or sell a given asset. For this reason, they are an excellent hedging tool. On the other hand, the risks associated with standard contracts of this kind expose sellers to potentially significant losses. In case of the call options, the possible loss is unlimited. Additionally, investors writing these contracts must have sufficient liquid assets to meet margin requirements. For these reasons, many modifications of the standard American options are considered. They are supposed to reduce the risk and consecutively lower the price of the contract.

One instrument gaining more interest as a method of minimizing buyer's expense is the so-called *capped option*. It can be automatically terminated before the prespecified time to maturity. In most examples, these instruments have the constraint on the payoff. It means that if a certain level of buyer's profit is exceeded, then the contract is exercised, and thus the loss of a seller is limited.

This thesis introduces a new approach to capped American options. Here, the constraint is set on the possible exercise time instead of the holder's profit. This means that the contract can be exercised at any time up to a prespecified random event. In general, such an event may be dependent or independent of the underlying asset price.

1.1 Financial markets and European options

Financial market is a market which allows people to trade variety of financial instruments, commodities and other assets. In addition to goods such as gold or silver, one of the most basic financial instruments is a *share* of a company. It is the ownership of a given part of this firm. Each shareholder is therefore entitled to participate in the distribution of the company's profit. Shareholders receive the money in the form of *dividends* and for many investors it is one of the two main reasons of holding the stock. The other possible profit comes from growth in the value of shares they have.

Every participant of the financial markets must take into account the risk of losing money. The value of stock may decrease over time and thus result in a loss for the shareholder. The risk also concerns exchange rates of currencies, prices of commodities (e.g. precious metals or oil) or indices, i.e., market indicators usually made up from a weighted sum of a given collection of shares. It is generally assumed that *bonds* are one of the few riskless securities. Holders of such an instrument should know *a priori* how much they will earn, and the interest should be accumulated at a *risk free rate*.

One of the possible ways to hedge against risks in financial markets is to use derivative financial instruments (or derivatives). These contracts depend on other more fundamental assets, such as shares or indices. The price and *payoff* (i.e., the amount of money investors gain or lose during the execution of a contract) of a derivative are usually functions of the *underlying* asset performance (e.g. share price or exchange rate of a currency pair). In addition to hedging, derivatives can be used for speculation purposes or as a proxy for the hard or impossible to trade instruments (such as indices).

One of the most commonly used types of derivatives are the so–called *European options*. These instruments provide their holders with the right, but not obligation, to trade a given asset in the future at a predetermined price. On the other hand, the seller of the option has to trade the asset if the buyer demands it. Contracts of this type have been in use for more than 2000 years. The works of Aristotle tell the story of the philosopher Thales, who successfully predicted an extraordinarily fertile olive season. Confident of his predictions, he paid for the right to use the olive press exclusively during the olive harvest. When the abundance of these fruits increased the demand for presses, he rented them at a higher price and thus earned some money (see [8]).

We can distinguish two basic types of options: *call* options and *put* options. The European call options give their holders the right to buy a given asset for a prespecified amount of money (*strike price*) at a prespecified time (*maturity date*), whereas the put European options give the right to sell the asset. The buyer of a call option should exercise his/her contract if the market price of the underlying asset at the maturity date is greater than the strike price. When the strike price is greater than the current price of the underlying, exercising the option would result in a loss. Thus, the option holder's profit at the maturity date can be described by the payoff function given by the formula:

$$h(S_T) = \max(S_T - K, 0), \tag{1.1}$$

where K is the strike price and S_T is the price of the underlying at the maturity date T. Similarly, the put option should not be exercised if $K < S_T$ and therefore the payoff function for this contract is given by the formula:

$$h(S_T) = \max(K - S_T, 0). \tag{1.2}$$

The value of the European options at expiration is often presented in the so-called *payoff diagrams*. These charts allow to easily visualize the payoffs of derivatives based on the value of the underlying (as in Figure 1.1).

Both the formulas (1.1) and (1.2) and the Figure 1.1 indicate that the holder of the option cannot lose money on this contract. The payoff never drops below zero. This means that the options are completely riskless for their buyers. The sellers bear all the risk and for this reason they need to be awarded with a *premium*, i.e., the amount paid for the contract

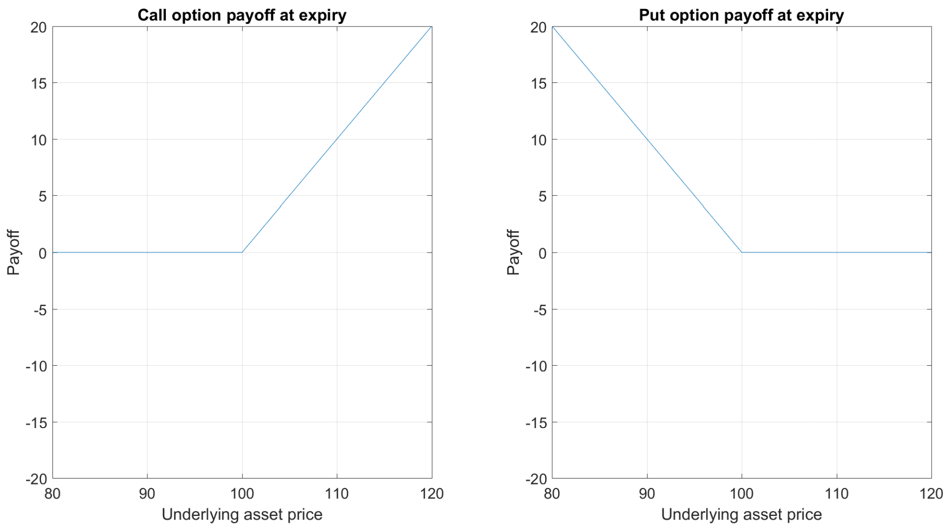

Figure 1.1: Payoff diagrams of the European options. The initial price of the options was not included.

by the holder.
The *fair price* of European options can be found analytically. In 1969 Fischer Black and Myron Scholes proposed a model describing the dynamics of this price (see [3]). It contained many assumptions about financial markets:

- the risk–free rate is known and constant and all participants of the market can lend or borrow any amount of money at this rate,
- there is no *arbitrage* possibility, i.e. nobody can earn money without the risk other than by lending money at the risk–free rate,
- any amount, even fractional, of the underlying assets can be bought or sold; *short selling* is possible,
- there are no transaction costs on the underlying assets — the market is *frictionless*,
- the dynamics of the underlying asset price can by described by the *geometric Brownian motion*.

The authors of the model spent a couple of years justifying it and in 1973 they published the so–called *Black–Scholes equation* describing the dynamics of prices of European call and put options (see e.g. [2], [3], [37]):

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0,$$

where V is the price of the option, t is the time, S is the price of the underlying asset at time t, σ is the *volatility* of the underlying and r is the risk–free rate. This equation, along with the proper boundary conditions, is sufficient to find the closed–form formulas for the prices of European options. The price of the call option is given by the following formula (see e.g. [2]):

$$V(S,T) = S\mathcal{N}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - K\mathrm{e}^{-rT}\mathcal{N}\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

where S is the initial asset price and $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution. Similarly, for a put option, the price is given by the formula:

$$V(S,T) = K\mathrm{e}^{-rT}\mathcal{N}\left(-\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) - S\mathcal{N}\left(-\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

Black-Scholes model allows to price other types of options as well, for example *binary options*. However, not for every option price the analytical solution has been found. One of the examples of such contracts is the *American option*.

1.2 American options

American options are one of the most popular derivative financial instruments (see e.g. [32] or [37]). They can be found in all significant financial markets, such as commodity, equity, currency exchange, energy, credit and insurance markets (see [23]). They differ from their European counterparts in the possibility of exercising them at any time from signing the contract up to the maturity date T or at this date. This feature makes them more interesting for research analysis and more difficult to price.

It is not clear when exactly the history of the American options began. However, in the late 17th century John Houghton published *A Collection for the Improvement of Husbandry* where he described the then English financial market (see [32]). It was probably the first decent description of securities trading in London. Houghton described a contract that allowed its holder to buy a share on any day before a prespecified maturity date. Today, such a contract would be considered an American option.

American options give their holders more rights and more possibilities to earn money than European options. The ability to exercise at any time is certainly valuable for both hedging and speculation purposes. For this reason, American options cannot be cheaper than their European counterparts written on the same underlying with the same strike price and maturity date. In fact, only if a no–dividend case is considered, the prices of the European and American call options should be equal (see e.g. [2]). This is due to the fact that the underlying asset price is assumed to increase over time and therefore call options are not rationally exercised early.

The main issue with pricing other kinds of these contracts is deciding when is the right time to exercise. Despite extensive research on the subject, there is no explicit solution for this problem. The *optimal exercise boundary*, i.e., the set of underlying asset prices over time for which the early exercise should occur, is not known *a priori* for the American puts and must be found as a part of the solution to the problem of pricing the option. The only American put option for which the analytical formula for the price is known is

the so-called perpetual option, where the time to maturity is equal to infinity (see [37]).
Solving the problem of finding the optimal exercise boundary is significant from both a theoretical and a practical point of view. On the theoretical side, this is the example of the *optimal stopping problem*, which is often considered by mathematicians working with optimization theory or stochastic analysis. On the practical side, it may not only be useful for pricing derivatives but also for estimating the optimal time to, for example, sell a house or build a power plant.

For these reasons, many attempts have been made to find a numerical solution to this problem. Among the most popular methods of pricing American options, one can distinguish: bi– or trinomial trees, finite–difference and finite–element schemes or Monte Carlo methods (see [23]).

There are many ways of valuing these derivatives and they all share one common feature: the underlying asset price process is assumed to be time–discrete, as it is not possible to simulate a time–continuous geometric Brownian motion using computers. This leads to a situation where, instead of the American option, the so–called *Bermuda option* is priced. Bermuda option is a contract that can be exercised only at some specified times before expiration (see [37]). However, such simplification may be justified, as when the time between the possible moments of exercising the option approaches zero, the price of the Bermuda option should approach the price of the American option (see [23, 34]). There is no consensus on which of these methods is the best. However, Monte Carlo methods are sometimes considered as last resort methods because the result they give is a random variable and they can be slow. Nevertheless, they may turn out to be very useful in the valuation of more sophisticated derivatives.

1.3 Capped American options

Standard, or *vanilla*, American options along with standard European options are considered basic contracts of this type, unlike their *exotic* counterparts, such as barrier options (see [36]). One of the possible modifications of standard American or European options is adding a **cap** on their payoff function. One of the simplest examples of such contracts are capped European options on the S&P 100 and S&P 500 indices introduced in 1991 by the Chicago Board of Options Exchange (see [5]). They were immediately exercised when the closing index value exceeded a certain threshold (cap). Such instruments are characterized by the limited risk of the seller and thus can be an interesting alternative to the vanilla options for many investors. One of the advantages of these contracts is that for both European and American version the analytical formulas for the price exist for a zero- or low–dividend case (see [5]).

So far, different versions of the capped options have been examined. For example, Boyle and Turnbull [4] analyzed long-term foreign currency contracts and their insensitivity to the volatility parameter. Ott [27] has been working with capped American lookback options. Broadie and Detemple [5] focused on pricing capped American calls with constant and growing caps. Deng and Peng [9] used capped options to improve the Least Squares Monte Carlo and binomial tree methods.

However, little has been done for the options where the cap is not set on the payoff but on the possible exercise time. One of the first such examples is the paper of Egloff, Farkas and Leippold [10], who used numerical methods to price American options with stochastic stopping time constraints, where the exercise of the option is restricted to specific conditions being met. These conditions, defined in terms of the states of a Markov process, are linked to a stochastic performance condition. As the authors noted, such performance-based constraints play an important role in structuring new investment products and designing executive stock option plans with exercise conditions tied, for example, to outperforming a reference index. To some extent, the motivation for considering a time-capped option is very similar: this thesis introduces a class of options that remain valid until they are either terminated by the buyer or triggered by an event described above, whichever occurs first. However, the difference in pricing is crucial. In [10] the buyer can only exercise the option when a prespecified condition, associated with the performance of the underlying asset, is satisfied. Therefore, it was possible to transform the constrained pricing problem into an unconstrained optimal stopping problem that corresponds to a generalized barrier option pricing and a stochastic Cauchy-Dirichlet problem.

Several authors have also explored the concept of time-capping in American options. In [35], the random cap is a first hitting time of a fixed barrier by the underlying asset price. In [27], the fixed time cap is examined for Russian and American look-back options. Finally, there are other related works. For example, [39, 38] investigate Russian options that terminate when the stock price hits its running maximum for the last time, as well as American options that terminate when the stock price reaches a prespecified level for the last time.

Time-cap is closely related to cancellable options, which are terminated early when a specific event occurs. In fact, the price of the stopped option consists of the value of the cancellable option and the discounted payoff (under the risk-neutral measure) at the event time, provided the event happens before maturity. Typically, this event is described as the first or last time when the underlying asset price reaches a specific threshold; see, e.g., [14, 13] and references therein. Similar early termination features can be found in game options (like, e.g., Israeli options) where the seller has the right to terminate the contract early, subject to a fixed penalty paid to the buyer; see the seminal paper [17] and subsequent works such as [24, 40, 41].

This thesis presents a novel approach to the time–capped instruments. The considered American options are immediately exercised after a prespecified event occurs. Since they are exercised and not terminated, the holders of such options can still receive a non-zero payoff. We formulate closed-form price formulas for three types in this class of instruments. We consider contracts where the time cap is the first or the last time when the underlying asset exits some interval and the options terminated by drawdown, i.e., the first time when the ratio between the current asset price and its historical maximum exceeds a certain level. We consider a Lévy market where downward jumps of stock price may occur. We provide analytical formulas for perpetual American time-capped put options. We do not consider the call options here, as they are less interesting when working with spectrally negative Lévy processes. The reason for this is that the downward jumps cannot move the process directly into the stopping region. Additionally, the calculations for the call options would be very similar, so they would not contribute much to the thesis, while at the same time significantly increase its length.

Additionally, we introduce a modified LSMC method which can handle all time-capped American options with finite maturity and the time constraint defined as any stopping time adapted to the natural filtration generated by the stock price process. Thus, we allow a variety of new instruments to be introduced on the OTC financial markets. To the best of our knowledge, the results presented here have not been analyzed or described before.

This thesis is organized as follows. In Section 1.4, we introduce the functions, symbols, and notations used throughout the rest of the dissertation. In Chapter 2 we find the price of the American put option capped by the first time when the underlying asset price exits a predefined interval. In Chapter 3 we deal with the option capped by a drawdown event, i.e., the first moment when the asset price drops below its historical maximum by a fixed (relative) threshold. We first handle the simpler case in Section 3.1, where we consider the Black-Scholes market. Then, in Section 3.2 we provide the price of such an instrument when downward jumps can occur. Next, in Chapter 4 we consider an option stopped when the underlying asset price exceeds a given threshold for the last time. Finally, in Chapter 5 we propose a robust modified Least Squares Monte Carlo method capable of numerically finding the prices of a wider class of the capped American options. We also provide numerical results for the options considered earlier in this thesis.

1.4 Preliminaries

Consider a Lévy market where the asset price follows the process

$$S_t = e^{X_t},$$

with $\{X_t\}_{t\in[0,\infty)}$ being a spectrally negative Lévy process and $s = S_0$ representing the initial asset price. Specifically, we define

$$X_t = x + \mu t + \sigma B_t - \sum_{k=1}^{N_t} U_k, \tag{1.3}$$

where $x = X_0 = \log s$ and $\sigma \geq 0$. In (1.3) B_t is a Brownian motion, N_t is a homogeneous Poisson process with intensity λ and $\{U_k\}_{\{k\in\mathbb{N}\}}$ is a sequence of independent identically distributed exponential random variables with mean $\rho^{-1} > 0$. We assume that B_t, N_t and $\{U_k\}_{\{k\in\mathbb{N}\}}$ are mutually independent and that all considered processes live in a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathbb{Q})$ with natural filtration $\{\mathcal{F}_t\}_{\{t\geq 0\}}$ of X_t satisfying the usual conditions. We allow $\lambda = 0$, which corresponds to the standard Black-Scholes model. For simplicity, we assume that no dividends are paid to the holders of the underlying asset.

The motivation for the choice of the model is two-fold. First, empirical evidence indicates that asset returns are typically characterized by asymmetry and heavy-tailed behavior, with downward jumps being more frequent and more severe than upward ones. Such features cannot be captured by the Gaussian framework, but they are naturally accommodated by spectrally negative Lévy processes, which allow for a diffusion component together with negative jumps of arbitrary size. Second, from a mathematical perspective, spectrally negative Lévy processes constitute a tractable class within the family of Lévy processes. In particular, the fluctuation theory for this class is well developed and provides access to a range of explicit identities and analytical tools that are useful in financial modeling. Hence, this modeling framework simultaneously preserves realistic asset price dynamics and analytical tractability.

The Laplace exponent of the process X_t is defined as

$$\Psi(\theta) = \frac{1}{t} \log \mathbb{E}_x^{\mathbb{Q}} e^{\theta X_t}. \tag{1.4}$$

The subscript x of the expected value here denotes that the process X_t starts from x, that is, $X_0 = x$. When λ is non-zero, the Laplace exponent is given by

$$\Psi(\theta) = \mu\theta + \frac{\sigma^2\theta^2}{2} - \frac{\lambda\theta}{\theta + \rho}.$$

We assume that $\mathbb{Q}$ is a risk-neutral measure and $e^{-rt}S_t$ is a $\mathbb{Q}$-local martingale. It leads to condition

$$\Psi(1) = r. \tag{1.5}$$

To ensure this, we take

$$\mu = r - \frac{\sigma^2}{2} + \frac{\lambda}{1 + \rho}.$$

As pointed out in Cont and Tankov [7], the introduction of jumps in asset price modeling leads to market incompleteness, since the randomness introduced by jumps cannot be hedged using the underlying asset alone. In contrast to the classical Black-Scholes setting, where the market is complete and the risk-neutral measure is uniquely determined, in jump models, there exists an infinite family of equivalent martingale measures (EMMs). Consequently, the no-arbitrage principle alone is insufficient to determine a unique option price, and an additional modeling choice must be made to select a particular risk-neutral measure.

In our setting, the martingale condition $\Psi(1) = r$ must hold under the risk-neutral measure. However, this condition alone does not uniquely specify the measure because there is more than one way to achieve it. In general, a change of measure in a jump-diffusion framework involves not only modifying the drift, but also altering the jump intensity λ and/or the distribution of jump sizes (e.g., the parameter ρ). Each such choice defines a different EMM and hence leads to different option prices.

To address this non-uniqueness, we adopt a specific modeling convention for selecting the risk-neutral measure. Following the approach introduced by Merton [25], we adjust only the drift term μ to enforce the martingale condition, while keeping the jump parameters λ and ρ, as well as the diffusion volatility σ, unchanged. This approach deliberately avoids altering the structure or statistical properties of the jump component under the risk-neutral measure. In doing so, we preserve the real-world jump behavior, interpret

the jumps as a non-hedgeable source of risk, and isolate their contribution to derivative pricing purely through their effect on the drift adjustment.
This convention effectively mimics the pricing logic of complete markets, even though the underlying model is incomplete. It allows for analytical tractability, facilitates comparison with the classical Black-Scholes framework, and is widely used in practice for its simplicity and economic interpretability. Alternative risk-neutral measures—obtained, for example, by tilting the jump size distribution or altering the jump intensity—are also valid and may be better suited in contexts where jump risk premia are significant or empirically observable. A comprehensive discussion of such alternatives and their implications for pricing and hedging can be found in Chapter 10 of Cont and Tankov [7].
For $r \geq 0$ we define a continuous function $W^{(r)} : [0, \infty) \to [0, \infty)$, called r**-scale function**, such that

$$\int_0^\infty e^{-\beta x} W^{(r)}(x) \, \mathrm{d}x = \frac{1}{\Psi(\beta) - r} \quad \text{for } \beta > \Phi(r), \tag{1.6}$$

where Φ is the right-inverse function of Ψ, i.e.,

$$\Phi(r) = \sup\{y \geq 0 : \ \Psi(y) = q\}.$$

With the first scale function we associate the second one given by

$$Z^{(r)}(x) = 1 + r \int_0^x W^{(r)}(y) \, \mathrm{d}y. \tag{1.7}$$

The scale functions play a key role in applications involving spectrally negative Lévy processes, especially in problems related to exit times, ruin probabilities, and optimal stopping. Although they are defined via the Laplace transform, they also have a clear probabilistic interpretation and useful analytical properties.
First, $W^{(r)}(x)$ is zero for negative x, continuous, and strictly increasing for $x \geq 0$. When the process has unbounded variation — such as when a Brownian component is present — $W^{(r)}(0) = 0$. For processes of bounded variation, it starts from a strictly positive value at zero. As the discount rate r increases, the function gets squeezed down, and in the limit $r \to \infty$, it tends to zero pointwise.
From a practical perspective, the scale function gives us a direct way to compute quantities like the probability of exiting an interval before ruin, or the expected discounted time of such events. For example, the ratio $W^{(r)}(x)/W^{(r)}(a)$ tells us the probability (under exponential discounting) that the process will reach a level $a > x$ before falling below zero. This makes scale functions particularly useful in risk theory, where they can represent survival probabilities, and in finance, where they appear in the pricing of barrier-type and American-style options.
Scale functions also provide building blocks for constructing value functions in a variety of control problems, such as dividend optimization or capital injection strategies. In such contexts, their increasing and convex behavior (in many cases) aligns well with economic intuition: the further away the process is from a critical boundary (like ruin), the better the expected outcomes. For reference, see e.g. [21] or [7].

Lemma 1. *The scale function for the process X defined by (1.3) is given by*

$$W^{(r)}(x) = \sum_{i=1}^{3} C_i e^{\gamma_i x}, \tag{1.8}$$

where:

$$\gamma_1 = 1, \qquad \gamma_{2/3} = \frac{-1}{2(\rho\sigma^2 + \sigma^2)} \left(2\lambda + 2r + \rho^2\sigma^2 + \rho\sigma^2 + 2r\rho \pm 2\sqrt{\omega}\right), \tag{1.9}$$

and

$$\omega = \lambda^2 + \lambda(\rho+1)(2r + \rho\sigma^2) + (\rho+1)^2 \left(r - \frac{1}{2}\rho\sigma^2\right)^2.$$

Furthermore,

$$C_1 = \frac{2(\gamma_1 + \rho)}{\sigma^2(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad C_2 = \frac{2(\gamma_2 + \rho)}{\sigma^2(\gamma_2 - \gamma_1)(\gamma_2 - \gamma_3)}, \tag{1.10}$$

$$C_3 = \frac{2(\gamma_3 + \rho)}{\sigma^2(\gamma_3 - \gamma_1)(\gamma_3 - \gamma_2)}. \tag{1.11}$$

Proof. From (1.4) and (1.6) it follows that the scale function is of the form (1.8) where γ_i $(i = 1, 2, 3)$ solve $\Psi(\gamma) = r$ (see also e.g. [15, 42]). From (1.5) we can conclude that $\gamma_1 = 1$. Solving the remaining square equation we derive (1.9). Observing that (1.6) is equivalent to

$$\frac{\rho + \theta}{\frac{\sigma^2}{2}(\theta - \gamma_1)(\theta - \gamma_2)(\theta - \gamma_3)} = \sum_{i=1}^{3} \frac{C_i}{\theta - \gamma_i}$$

gives (1.10) and (1.11). □

Additionally

$$Z^{(r)}(x) = 1 + r\int_0^x W^{(r)}(y)\,\mathrm{d}y = \sum_{i=1}^{3} \frac{rC_i}{\gamma_i} e^{\gamma_i x}. \tag{1.12}$$

Note that $\sum_{i=1}^{3} \frac{rC_i}{\gamma_i} = 1$. Now, let us introduce a change of measure that will simplify several calculations later on. This is achieved by changing the **numéraire** from the risk-free asset e^{-rt} to the underlying asset $S_t = e^{X_t}$, which induces a new probability measure $\mathbb{P}$.

In arbitrage pricing theory, a numéraire is any strictly positive, tradable asset that serves as the unit in which all other asset prices are expressed. While asset prices are typically quoted in units of a fixed currency (e.g., USD or EUR), this viewpoint can be generalized: by choosing a different tradable asset as the reference unit, we can re-express all prices relative to that asset. This idea forms the basis of the change of numéraire technique. The change of numéraire is always accompanied by a change of probability measure, defined via a Radon-Nikodym derivative that reflects the relative growth of the new and old numeraires. In practice, this technique provides a powerful and flexible tool to reframe pricing problems under a measure where the relevant quantities become easier to compute or interpret.

Importantly, this new measure $\mathbb{P}$ is not to be interpreted as the historical or real-world measure. Rather, it is an auxiliary measure, introduced solely to facilitate analytical computations. The transformation is defined via the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\bigg|_{\mathcal{F}_t} = \frac{e^{X_t - rt}}{e^x}.$$

Moreover, from [21, eq. (8.6)] we know that $(X, \mathbb{P})$ is also a spectrally negative Lévy process and its Laplace exponent is given by

$$\Psi^{\mathbb{P}}(x) = \Psi(x+1) - \Psi(1) = \Psi(x+1) - r. \tag{1.13}$$

Let us now define $W^{(0)}$ scale function for $(X, \mathbb{P})$. We will denote it by $W^{\mathbb{P}}$:

$$\int_0^\infty e^{-\beta x} W^{\mathbb{P}}(x)\,\mathrm{d}x = \frac{1}{\Psi^{\mathbb{P}}(\beta)} \quad \text{for } \beta > 0.$$

Since (1.13) holds, we can write:

$$\int_0^\infty e^{-\beta x} W^{\mathbb{P}}(x)\,\mathrm{d}x = \frac{1}{\Psi(\beta+1) - r} = \int_0^\infty e^{-x} e^{-\beta x} W^{(r)}(x)\,\mathrm{d}x.$$

This gives us:

$$W^{\mathbb{P}}(x) = e^{-x} W^{(r)}(x). \tag{1.14}$$

Additionally:

$$Z^{\mathbb{P}}(x) = 1 + 0 \int_0^x W^{\mathbb{P}}(y)\,\mathrm{d}y \equiv 1. \tag{1.15}$$

From (1.14) it is clear that the coefficients of the scale function on $\mathbb{P}$ must satisfy the following equalities:

$$W^{\mathbb{P}}(x) = \sum_{i=1}^{3} \tilde{C}_i e^{\tilde{\gamma}_i x} \iff \forall_{i \in \{1,2,3\}} \tilde{C}_i = C_i \text{ and } \tilde{\gamma}_i = \gamma_i - 1$$

Additionally, by looking closer to the Laplace exponent, we get

$$\begin{aligned}
\Psi^{\mathbb{P}}(\theta) &= \Psi(\theta+1) - \Psi(1) = \mu(\theta+1) + \frac{\sigma^2}{2}(\theta+1)^2 - \frac{\lambda(\theta+1)}{\theta+1+\rho} - r \\
&= \frac{\sigma^2}{2}\theta^2 + \theta(\mu+\sigma^2) - \frac{\lambda(\theta+1)}{\theta+1+\rho} + \frac{\sigma^2}{2} + \mu - r = \frac{\sigma^2}{2}\theta^2 + \theta(\mu+\sigma^2) - \frac{\lambda(\theta+1)}{\theta+1+\rho} + \frac{\lambda}{1+\rho} \\
&= \frac{\sigma^2}{2}\theta^2 + \theta(\mu+\sigma^2) - \frac{\lambda\rho\theta}{(\theta+1+\rho)(\rho+1)} = \frac{\sigma^2}{2}\theta^2 + \theta(\mu+\sigma^2) - \frac{\lambda\rho}{\rho+1}\frac{\theta}{\theta+(\rho+1)}.
\end{aligned}$$

Therefore, for $(X, \mathbb{P})$ we have:

$$\tilde{\mu} = \mu + \sigma^2,$$
$$\tilde{\lambda} = \frac{\lambda\rho}{\rho + 1},$$
$$\tilde{\rho} = \rho + 1.$$

Now, since we allow for the case with $\lambda = 0$, we should also recalculate the scale functions for such a scenario. Observe that with $\lambda = 0$, we get a simpler Laplace exponent

$$\Psi(\theta) = \left(r - \frac{\sigma^2}{2}\right)\theta + \frac{\sigma^2\theta^2}{2}.$$

Furthermore, the scale function of X_t is given by

$$W^{(r)}(x) = C_1 e^x + C_2 e^{\gamma x},$$

where

$$\gamma = -\frac{2r}{\sigma^2}$$

and

$$C_1 = \frac{1}{r + \frac{\sigma^2}{2}}, \qquad C_2 = \frac{-1}{r + \frac{\sigma^2}{2}}.$$

Chapter 2

First passage cap

The main objective of this section is to derive a closed-form formula for the price of the perpetual American put option capped by the first exit time. Such instrument resembles a double barrier American option, previously studied by Gapeev in 2006, see [12]. However, a key distinction is that if the underlying asset's price falls below the lower barrier due to a Poissonian jump, the option is exercised immediately rather than canceled. Consequently, the option holder still receives a non-zero payoff.

Both sides benefit from this modification of the classical American payout: it mitigates the risk for the option issuer by terminating the contract before the asset price falls too low, while also providing the buyer with a hedge against losses. Unlike traditional cancellation features, which might leave investors with nothing if the stock price drops too much, this mechanism ensures a non-zero payout. The pricing model is constructed using the guess-and-verify technique and relies on the Hamilton-Jacobi-Bellman framework. We provide explicit formulas for the optimal stopping boundary and the option value, validated through numerical analysis.

2.1 Main result

Let K be the strike price, L a lower and H an upper barrier of the process X_t, such that the option is in force as long as $L < X_t < H$. We assume $H > \log(K) > L$. We introduce the exit time η as:

$$\eta = \inf\{t \geq 0 \ : \ X_t \notin (L, H)\}.$$

Our goal is to identify the value function defined via the following optimization problem:

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}_{e^x}[e^{-r\tau\wedge\eta}(K - S_{\tau\wedge\eta})^+], \tag{2.1}$$

where $\mathcal{T}$ is the family of stopping times with respect to the natural filtration $\mathcal{F}_t$ of X_t. The subscript e^x of the expected value denotes here that the process S_t starts from e^x. Thus, the process X_t starts from x. Here, $p \wedge q = \min(p, q)$ and $p \vee q = \max(p, q)$. Furthermore, we want to identify the so-called optimal stopping rule $\tau^* \in \mathcal{T}$ for which the supremum in (2.1) is attained.

To find the fair value of the option, we need to identify the optimal stopping time.

Let us define a stopping time τ^* as

$$\tau^* = \inf\{t \geq 0 \ : \ X_t \leq a^*\}, \tag{2.2}$$

where

$$a^* = a \vee L \tag{2.3}$$

and a solves the following equation:

$$Z^{(r)}(H - a) = \frac{e^H}{K}. \tag{2.4}$$

We will later show that this is the optimal stopping time.

Theorem 1. *The value function defined in (2.1) is given by*

$$V(x) = KZ^{(r)}(x - a^*) - e^x + \frac{W^{(r)}(x - a^*)}{W^{(r)}(H - a^*)} \left(e^H - KZ^{(r)}(H - a^*)\right) \quad \text{for } x \in (a^*, \ H). \tag{2.5}$$

Moreover, τ^* defined in (2.2) is the optimal stopping time

Remark 1. *We define the stopping region as the region for the asset price when the option should be exercised, either optimally by the buyer or automatically by the cap feature. That is, $\tau^* = \inf\{t \geq 0 : X_t \in D\}$. From this we can conclude that the solution to (2.1) on D is given by $V(x) = (K - e^x)^+$. The continuation region is defined as the completion of the stopping region. From Theorem 1 we have:*

$$C = (a^*, \ H), \quad D = C^{\complement}.$$

Let $\mathcal{L}$ be the extended generator of the process X_t, i.e.,

$$\mathcal{L}f(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x) + \lambda\rho \int_0^\infty \left(f(x - y) - f(x)\right) e^{-\rho y} \, \mathrm{d}y.$$

Lemma 2. *Suppose that function $\hat{V} \in \mathcal{C}^2(\mathbb{R} \setminus \{a^*\})$. Assume that $\hat{V}$ satisfies the following HJB system of equations:*

$$(\mathcal{L}\hat{V} - r\hat{V})(x) = 0 \quad \text{for } H > x > a^*, \tag{2.6}$$
$$(\mathcal{L}\hat{V} - r\hat{V})(x) \leq 0 \quad \text{for } x \leq a^*, \tag{2.7}$$

$$\hat{V}(x) = (K - e^x)^+ \quad \text{for } x \leq a^*, \tag{2.8}$$
$$\hat{V}(x) > (K - e^x)^+ \quad \text{for } H > x > a^*, \tag{2.9}$$

$$\hat{V}(a^*) = (K - e^{a^*})^+, \tag{2.10}$$
$$\hat{V}'(a^*) = \frac{\mathrm{d}}{\mathrm{d}x}(K - e^x)^+\big|_{x=a^*}. \tag{2.11}$$

Then $\hat{V}(x) \geq V(x)$ for every $L < x < H$.

Proof. Assumed smoothness of $\hat{V}$ along with equation (2.11) allow us to use Theorem 3.1 from [31] and write

$$\begin{aligned} e^{-rt}\hat{V}(X_t) = \hat{V}(x) &+ \int_0^t e^{-ru}\left(\mathcal{L}\hat{V} - r\hat{V}\right)(X_u)\,\mathrm{d}u \\ &+ \int_0^t e^{-ru}\hat{V}'(X_u)\sigma\,\mathrm{d}B_u + \sum_{u\le t} e^{-ru}\left[\hat{V}(X_u) - \hat{V}(X_u-)\right] \\ &- \lambda\int_0^t e^{-ru}\int_0^\infty \left(\hat{V}(X_u - y) - \hat{V}(X_u)\right)\rho e^{-\rho y}\,\mathrm{d}y\,\mathrm{d}u. \end{aligned}$$

Let us now define L_t as

$$\begin{aligned} L_t = &\int_0^t e^{-ru}\hat{V}'(X_u)\sigma\,\mathrm{d}B_u + \sum_{u\le t} e^{-ru}\left[\hat{V}(X_u) - \hat{V}(X_u-)\right] \\ &- \lambda\int_0^t e^{-ru}\int_0^\infty \left(\hat{V}(X_u - y) - \hat{V}(X_u)\right)\rho e^{-\rho y}\,\mathrm{d}y\,\mathrm{d}u. \end{aligned}$$

Observe that the double integral in the second line is the compensator of the sum in the first line (see [20, Thm. 3.4]). Moreover, from [16, eq. (4.34)] the integral over B_u is a local martingale. Thus, L_t is a local martingale with a zero mean. Now, with (2.6) and (2.7) we get

$$e^{-rt}\hat{V}(X_t) \le \hat{V}(x) + L_t, \quad t \ge 0.$$

Then, by applying the expected value to both sides of the inequality we get

$$\mathbb{E}_x^{\mathbb{Q}}[e^{-rt}\hat{V}(X_t)] \le \hat{V}(x).$$

Hence, $e^{-rt}\hat{V}(X_t)$ is a supermartingale and $e^{-rt}\hat{V}(x)$ is a superharmonic function that dominates the payout. Now, from [30, (2.2.80), p. 49] we additionally know that $\hat{V}$ is also lower semi-continuous. It allows us to use [30, Thm. 2.7, p. 40] and claim that it is the optimal solution to the considered stopping problem. Hence $\hat{V}(x) \ge V(x)$. □

Now, let us define the first time of hitting the upper barrier H by θ

$$\theta = \inf\{t \ge 0 \;:\; X_t = H\}.$$

Here, we want to introduce a candidate solution for the function $\hat{V}(x)$ as

$$\hat{V}(x) = \begin{cases} 0, \quad x > H, \\ \mathbb{E}_{e^x}^{\mathbb{Q}}[e^{-r\tau^*\wedge\eta}(K - S_{\tau^*\wedge\eta})^+], \quad x \in C, \\ K - e^x, \quad x < a^*, \end{cases} \tag{2.12}$$

for τ^* given in (2.2) and C defined in Remark 1.

Remark 2. *If function $\hat{V}$ that satisfies the HJB system from Lemma 2 exists, then $\hat{V}(x) \ge V(x)$. On the other hand, V is the supremum over all stopping times, so if we choose $\hat{V}$ as in (2.12), that is, the expected value from (2.1) but for a specifically chosen stopping* *time, then clearly $\hat{V}(x) \leq V(x)$. Therefore, if we show that for $\hat{V}$ the HJB system holds, then $\hat{V}(x) = V(x)$.*

Going further, for $x \in C$

$$\begin{aligned}
\hat{V}(x) &= \mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*\wedge\eta}\left(K - e^{X_{\tau^*\wedge\eta}}\right)^+\right] = \mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*\wedge\theta}\left(K - e^{X_{\tau^*\wedge\theta}}\right)^+\right] \\
&= \mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\theta}\left(K - e^{H}\right)^+ \mathbb{I}_{\{\theta<\tau^*\}}\right] + \mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*}\left(K - e^{X_{\tau^*}}\right)\mathbb{I}_{\{\tau^*<\theta\}}\right] \\
&= \mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*}\left(K - e^{X_{\tau^*}}\right)\mathbb{I}_{\{\tau^*<\theta\}}\right] \\
&= K\mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*}\mathbb{I}_{\{\tau^*<\theta\}}\right] - \mathbb{E}_x^{\mathbb{Q}}\left[e^{X_{\tau^*}-r\tau^*}\mathbb{I}_{\{\tau^*<\theta\}}\right],
\end{aligned}$$

where $\mathbb{I}_{\{\omega\}}$ is the indicator function of event ω. The first expected value standing by K is a known two-sided exit problem described e.g. in [21]. Following equation (8.12) from [21] we can write

$$K\mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau^*}\mathbb{I}_{\{\tau^*<\theta\}}\right] = K\left[Z^{(r)}(x-a^*) - Z^{(r)}(H-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(H-a^*)}\right].$$

The second expected value is easy to calculate after changing the numéraire to the underlying asset. Observe that after switching to the $\mathbb{P}$ measure, we get

$$\begin{aligned}
\mathbb{E}_x^{\mathbb{Q}}\left[e^{X_{\tau^*}-r\tau^*}\mathbb{I}_{\{\tau^*<\theta\}}\right] &= e^x\mathbb{E}_x^{\mathbb{P}}\left[\mathbb{I}_{\{\tau^*<\theta\}}\right] = e^x\left[Z^{\mathbb{Q}}(x-a^*) - Z^{\mathbb{Q}}(H-a^*)\frac{W^{\mathbb{Q}}(x-a^*)}{W^{\mathbb{Q}}(H-a^*)}\right] \\
&= e^x\left(1 - e^{H-x}\frac{W^{(r)}(x-a^*)}{W^{(r)}(H-a^*)}\right) = e^x - e^H\frac{W^{(r)}(x-a^*)}{W^{(r)}(H-a^*)}.
\end{aligned}$$

Finally,

$$\hat{V}(x) = KZ^{(r)}(x-a^*) - e^x + \frac{W^{(r)}(x-a^*)}{W^{(r)}(H-a^*)}\left(e^H - KZ^{(r)}(H-a^*)\right). \tag{2.13}$$

Now, let us find such a that the function $\hat{V}(x)$ satisfies equations (2.10) and (2.11). Observe that if $a < L$, then $\hat{V}(x) = K - e^x$ in the neighborhood of a and therefore both equations hold. We will check what happens when $a > L$. Firstly, let us write

$$\hat{V}(a) = KZ^{(r)}(0) - e^a + \frac{W^{(r)}(0)}{W^{(r)}(H-a)}\left(e^H - KZ^{(r)}(H-a)\right) = K - e^a.$$

This immediately proves (2.10). Moving forward to equation (2.11), we write

$$\begin{aligned}
\hat{V}'(x) &= KZ'^{(r)}(x-a) - e^x + \frac{W'^{(r)}(x-a)}{W^{(r)}(H-a)}\left(e^H - KZ^{(r)}(H-a)\right) \\
&= rKW^{(r)}(x-a) - e^x + \frac{W'^{(r)}(x-a)}{W^{(r)}(H-a)}\left(e^H - KZ^{(r)}(H-a)\right).
\end{aligned}$$

Since

$$W'^{(r)}(0) = \sum_{i=1}^{3} C_i \gamma_i = \frac{2}{\sigma^2}$$

(see [19, Lem. 3.2]), then

$$\hat{V}'(a) = -e^a + \frac{2}{\sigma^2 W^{(r)}(H-a)} \left(e^H - KZ^{(r)}(H-a) \right).$$

From (2.10) we know that we search for such a that $\hat{V}'(a) = -e^a$. In other words, we need to ensure that

$$e^H - KZ^{(r)}(H-a) = 0 \tag{2.14}$$

which is equivalent to (2.4). □

Now, we need to verify that the candidate solution from equation (2.5) satisfies the system from Lemma 2. We can immediately state that equations (2.10) and (2.11) hold from our choice of a. Further, equation (2.8) holds directly from the choice of the candidate solution in (2.12).

Now, we can easily check that for the scale functions $W^{(r)}(x)$ and $Z^{(r)}(x)$ defined in (1.8) and (1.12), we have:

$$\mathcal{L}e^x = re^x, \quad \mathcal{L}W^{(r)}(x) - rW^{(r)}(x) = 0 \quad \text{and} \quad \mathcal{L}Z^{(r)}(x) - rZ^{(r)}(x) = 0. \tag{2.15}$$

In the continuation region, the candidate function $\hat{V}$ is a combination of functions $W(x)$, $Z(x)$ and e^x and therefore equation (2.6) holds. Observe that in the stopping region candidate function $\hat{V}(x)$ is either equal to 0 or to $K - e^x$. In both cases $\mathcal{L}\hat{V}(x) = 0$. Therefore, $(\mathcal{L}\hat{V} - r\hat{V})(x)$ is either equal to 0 or $-rK$, which proves (2.7). To prove (2.9) observe that since condition (2.14) holds, we can rewrite (2.13) as:

$$\hat{V}(x) = KZ^{(r)}(x-a^*) - e^x + \frac{W^{(r)}(x-L)}{W^{(r)}(H-L)} \left(e^H - KZ^{(r)}(H-L) \right) \mathbb{I}_{\{a<L\}}. \tag{2.16}$$

From (1.7) it is clear that $Z^{(r)}(x)$ is a strictly increasing function (since $W^{(r)}(x)$ is positive) and that $Z^{(r)}(x) > 1$ for $x > 0$. If $a < L$ then

$$e^H - KZ^{(r)}(H-L) > e^H - KZ^{(r)}(H-a) = 0.$$

Therefore

$$\frac{W^{(r)}(x-L)}{W^{(r)}(H-L)} \left(e^H - KZ^{(r)}(H-L) \right) \mathbb{I}_{\{a<L\}} \geq 0.$$

Further,

$$KZ^{(r)}(x-a^*) - e^x > K - e^x$$

which proves (2.9). Since all conditions from Lemma 2 are satisfied, the candidate $\hat{V}(x)$ is the optimal solution and Theorem 1 is proved. □

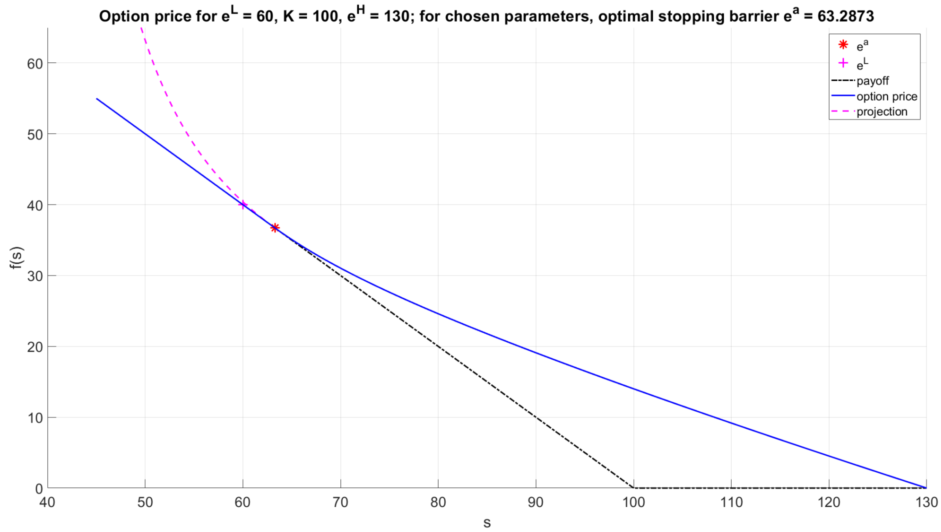

Figure 2.1: Smooth paste of the option price V and the option payoff when $L < a$. The term "Projection" mentioned in the legend refers to the form of function $\hat{V}$ defined on the continuation region, applied to the stopping region.

2.2 Numerical analysis

In this section, we present charts comparing the option price $V(x)$, with the payoff function $(K - e^x)^+$. From the form of function given in equation (2.16), it is evident that the option's value is significantly influenced by whether the optimal threshold, a, is greater or smaller than the lower barrier L. For our analysis, we choose the following set of option parameters and market values: $r = 0.05,\ \sigma = 0.2,\ \rho = 1,\ \lambda = 0.2,\ K = 100,\ e^H = 130$. From (2.4) we get that the optimal stopping boundary is approximately 63. We will consider two scenarios: one where $L < a$ and the other where $L > a$.

In the first case, where $L < a$, the option should be exercised by the buyer before termination at the lower barrier—unless a jump occurs that takes the process from $X_{t-} > a$ to $X_t < L$. In this scenario, we observe smooth pasting of the function $KZ^{(r)}(x - a) - e^x$ and the payoff function $(K - e^x)^+$. The results for $L = 60$ are presented in Figure 2.1.

In contrast, when $L > a$ option price pastes to the payoff function at $S = e^L$, but not in a smooth manner. However, condition (2.11) still holds, as in the neighborhood of a, the function $V(x)$ is equal to $K - e^x$. The interaction between payoff and price for $L = 70$ is shown in Figure 2.2.

Both figures show that even though boundary L does not appear explicitly in the value function, it still significantly affects the option price. The buyer should know the relationship between barriers a and L to make the right decision when to exercise.

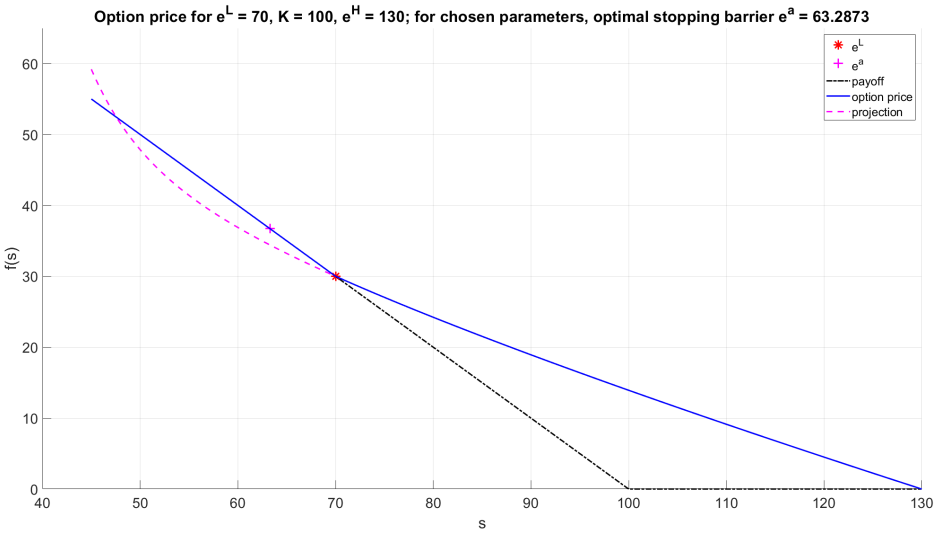

Figure 2.2: Smooth paste of the option price V and the option payoff when $L > a$. The term "Projection" mentioned in the legend refers to the form of function $\hat{V}$ defined on the continuation region, applied to the stopping region.

Chapter 3

Drawdown cap

In this section, our study focuses on a non-deterministic time cap, where the termination of the option is triggered by the asset price experiencing a drawdown that exceeds a fixed threshold. Here, drawdown refers to the relative drop in the asset price compared to its historical maximum. We derive a closed-form formula for the price of the American put option under this random maturity condition and determine the corresponding optimal exercise strategy.

Let us define the running maximum of the asset price as

$$\overline{S}_t = e^{\overline{x}} \vee \sup_{0 \le u \le t} S_u,$$

where $e^{\overline{x}}$ is the historical maximum of the underlying asset price prior to the beginning of the contract. For the fixed threshold $c > 0$ let

$$\tau_D = \inf \left\{ t \ge 0 : \frac{\overline{S}_t}{S_t} \ge e^c \right\}$$

be the first time when the (relative) drawdown is greater than e^c. In the main result of this section we identify the closed-form formula for the price of the American put option with the random maturity determined by a drawdown event given by

$$V(x, \overline{x}) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}_{e^x, e^{\overline{x}}} \left[e^{-r\tau \wedge \tau_D} (K - S_{\tau \wedge \tau_D})^+ \right], \tag{3.1}$$

for a family of stopping times $\mathcal{T}$ and fixed strike price $K > 0$. Here, the subscript denotes that the process S_t starts from e^x and $\overline{S}_t$ from $e^{\overline{x}}$. We restrict the domain of V to $\mathcal{D} := \{(x, \overline{x}) \in \mathbb{R}^2 : x \le \overline{x}\}$.

The first crucial step is to prove that the optimal stopping rule for (3.1) is the first downward crossing epoch of some boundary depending on the running supremum.

Proposition 1. *The optimal stopping time τ^* is of the following form:*

$$\tau^* = \inf\{t \ge 0 : X_t \le a_0(\overline{X_t})\}$$

for some function a_0.

Proof. Let
$$D = \{(x,\overline{x}) \in \mathbb{R}^2 : V(x,\overline{x}) = (K-e^x)^+\}.$$
Note that
$$Z_t = (X_t, \overline{X}_t) \tag{3.2}$$
is a Feller process. By [30, Thm. 2.7, p. 40 and (2.2.80), p. 49] it follows that
$$\tau^* = \inf\{t \geq 0 : X_t \in D\},$$
that is, D is a stopping region, where the option should be exercised immediately. Assume there exist $(x,\overline{x}) \in D$. We additionally assume that $x < \log K$, otherwise the immediate payout is null. Let τ_y and τ_x be the optimal stopping rule for starting point $(y,\overline{x})$ and $(x,\overline{x})$ respectively. Observe that if $y < x$, then, for a chosen c, we have $\tau_D(y,\overline{x}) \leq \tau_D(x,\overline{x})$, where $\tau_D(x,\overline{x})$ denotes τ_D for the starting point $(x,\overline{x})$. Further, we have:

$$\begin{aligned}
&V(y,\overline{x}) - V(x,\overline{x}) \\
&\quad = \mathbb{E}^{\mathbb{Q}}_{y,\overline{x}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{X_{\tau_y\wedge\tau_D(y,\overline{x})}})^+] - \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\tau_x\wedge\tau_D(x,\overline{x})}(K - e^{X_{\tau_x\wedge\tau_D(x,\overline{x})}})^+] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}_{y,\overline{x}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{X_{\tau_y\wedge\tau_D(y,\overline{x})}})^+] - \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{X_{\tau_y\wedge\tau_D(y,\overline{x})}})^+] \\
&\quad = \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{y+X_{\tau_y\wedge\tau_D(y,\overline{x})}})^+] - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{x+X_{\tau_y\wedge\tau_D(y,\overline{x})}})^+] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{y+X_{\tau_y\wedge\tau_D(y,\overline{x})}})] - \mathbb{E}^{\mathbb{Q}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})}(K - e^{x+X_{\tau_y\wedge\tau_D(y,\overline{x})}})] \\
&\quad = (e^x - e^y)\mathbb{E}^{\mathbb{Q}}[e^{-r\tau_y\wedge\tau_D(y,\overline{x})+X_{\tau_y\wedge\tau_D(y,\overline{x})}}] = e^x - e^y.
\end{aligned}$$

Therefore we get
$$V(y,\overline{x}) - V(x,\overline{x}) \leq e^x - e^y = (K-e^y) - (K-e^x)$$
and further
$$V(y,\overline{x}) \leq (K-e^y) \leq (K-e^y)^+.$$
On the other hand the payoff function of the option cannot be higher than its value function, therefore
$$V(y,\overline{x}) \geq (K-e^y)^+.$$
This gives $(y,\overline{x}) \in D$. This leads to the conclusion that for a certain $\overline{x}$ the optimal stopping region can be achieved by the pair $(X_t, \overline{X_t})$ when X_t drops down to some value $a_0(\overline{X_t})$ before it reaches its past maximum.

The question remains if the optimal stopping region can also be achieved from below, when both X_t and $\overline{X_t}$ hit some level for the first time. We will show that this is not possible. Indeed, assume a contrario that there exists such threshold b that there exist x and $\overline{x} < b$ such that $(x,\overline{x}) \notin D$ and $(b,b) \in D$. Let us take a positive ε such that $b - \varepsilon > \overline{x}$. Let θ be the first time when the process X_t hits the level $b-\varepsilon$ from below, i.e.,
$$\theta = \inf\{t > 0 : X_t = b - \varepsilon\}$$
and let τ_b be the first time when the process X_t hits the level b from below. Then $\theta < \tau_b$

and $e^{X_\theta} < e^{X_{\tau_b}}$. Therefore

$$\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\theta\wedge\tau_D(x,\overline{x})}(K - e^{X_{\theta\wedge\tau_D(x,\overline{x})}})^+] \geq \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\tau_b\wedge\tau_D(x,\overline{x})}(K - e^{X_{\tau_b\wedge\tau_D(x,\overline{x})}})^+]. \tag{3.3}$$

But we assumed that τ_b is the optimal stopping rule and therefore it should dominate over the expected value taken with a different stopping rule. Inequality (3.3) leads then to the contradiction, which completes the proof.

□

3.1 Black-Scholes model

Let us first consider a scenario where we set $\lambda = 0$, thus simplifying the market to the standard Black-Scholes setup.

3.1.1 Main result

The main theorems for the Black-Scholes setup in this section are as follows.

Theorem 2. *The optimal stopping barrier is the first downward asset price time*

$$\tau^* = \inf\{t \geq 0 : X_t \leq a^*\}, \tag{3.4}$$

where

$$a^* = \log\left(K\left(\frac{\gamma\,(e^{\gamma c} - e^c)}{(1-\gamma)e^c}\right)^{\frac{W^{(r)}(c)}{W^{(r)\prime}(c)}}\right). \tag{3.5}$$

The option should be stopped either at $X_t = a^*$ or $X_t = \overline{X_t} - c$, whichever occurs first.

Theorem 3. *The value function $V(x, \overline{x})$ is given by*

$$V(x,\overline{x}) = \begin{cases} (K - e^x)^+ & \textit{for } x < a^* \wedge \overline{x} - c \\ V_1(x,\overline{x}) + V_2(x,\overline{x})\Big(V_3(\overline{x}) + V_4(\overline{x})V_5\Big) & \textit{for } x > \overline{x} - c,\ \overline{x} < a^* + c, \\ V_6(x,\overline{x}) + V_7(x,\overline{x})V_8(\overline{x}) & \textit{for } x > \overline{x} - c,\ a^* + c < \overline{x} < \log(K) + c, \\ 0 & \textit{for } \overline{x} > \log(K) + c, \end{cases}$$

where

$$V_1(x,\overline{x}) = \left(K - e^{a^*}\right)\left(Z^{(r)}(x-a^*) - Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right),$$

$$V_2(x,\overline{x}) = \frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)},$$

$$V_3(\overline{x}) = \left(K - e^{a^*}\right)\left(Z^{(r)}(\overline{x}-a^*) - Z^{(r)}(c)\frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}\right),$$

$$V_4(\overline{x}) = \frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)},$$

$$V_5 = \frac{\sigma^2}{2}\left[W^{(r)\prime}(c) - \frac{W^{(r)}(c)W^{(r)\prime\prime}(c)}{W^{(r)\prime}(c)}\right]\left(K\left(1 - \frac{e^{-\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}(\log(K)-a^*)}}{1-\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}}\right) + \frac{\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}e^{a^*}}{1-\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}}\right),$$

$$V_6(x,\overline{x}) = \left(K - e^{\overline{x}-c}\right)\left[Z^{(r)}(x+c-\overline{x}) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(x+c-\overline{x})\right],$$

$$V_7(x,\overline{x}) = \frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)},$$

$$V_8(\overline{x}) = \frac{\sigma^2}{2}\left[W^{(r)\prime}(c) - \frac{W^{(r)}(c)W^{(r)\prime\prime}(c)}{W^{(r)\prime}(c)}\right]K - e^{\overline{x}},$$

$$+ Ke^{\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}(\overline{x}-\log(K)-c)}\left(e^c - \frac{\sigma^2}{2}\left[W^{(r)\prime}(c) - \frac{W^{(r)}(c)W^{(r)\prime\prime}(c)}{W^{(r)\prime}(c)}\right]\right).$$

By $\mathcal{L}$ we denote an infinitesimal generator of the Markov process Z_t defined in (3.2), which equals

$$\mathcal{L}f(x,\overline{x}) = \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}f(z) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}f(z) \quad \text{for} \quad 0 < x < \overline{x}$$

and the domain of this (full) generator includes the functions $f \in \mathcal{C}_0^2(\mathbb{R})$ such that

$$\frac{\partial}{\partial \overline{x}}f(x,\overline{x}) = 0 \quad \text{for} \quad x = \overline{x}. \tag{3.6}$$

In the next step, we prove the following verification lemma.

Lemma 3. *Let $\hat{V}(x,\overline{x}) \in \mathcal{C}_0^2(\mathbb{R})$ be such that* (3.6) *holds. Assume that for some function b,*

$$(\mathcal{L}\hat{V} - r\hat{V})(x,\overline{x}) = 0 \quad \textit{for } x > b(\overline{x}), \tag{3.7}$$

$$(\mathcal{L}\hat{V} - r\hat{V})(x,\overline{x}) \leq 0 \quad \textit{for } x \leq b(\overline{x}), \tag{3.8}$$

$$\hat{V}(x,\overline{x}) = (K - e^x)^+ \quad \textit{for } x \leq b(\overline{x}), \tag{3.9}$$

$$\hat{V}(x,\overline{x}) > (K - e^x)^+ \quad \textit{for } x > b(\overline{x}), \tag{3.10}$$

$$\hat{V}(x,\overline{x})\big|_{x=b(\overline{x})} = (K - e^{b(\overline{x})})^+, \tag{3.11}$$

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=b(\overline{x})} = \frac{\partial}{\partial x}(K - e^{x})^+\big|_{x=b(\overline{x})} \quad \text{if } b(\overline{x}) < \overline{x} - c. \tag{3.12}$$

Then $\hat{V}(x,\overline{x}) \geq V(x,\overline{x})$.

Remark 3. Conditions (3.11) and (3.12) are the so-called smooth paste conditions of the value function. Note that condition (3.11) is mainly required to write (3.12) which is used in the proof of Lemma 3.

Proof. Due to the assumed smoothness of $\hat{V}$, the smooth paste conditions (3.11) - (3.12) and an appropriate version Itô's theorem (see [11, p. 208]) we have

$$\begin{aligned} e^{-rt}\hat{V}(X_t,\overline{X}_t) &= \hat{V}(x,\overline{x}) + \sigma\int_0^t e^{-ru}\frac{\partial}{\partial x}\hat{V}(X_u,\overline{X}_u)\,\mathrm{d}B_u + \int_0^t e^{-ru}\frac{\partial}{\partial \overline{x}}\hat{V}(X_u,\overline{X}_u)\,\mathrm{d}\overline{X}_u \\ &+ \int_0^t e^{-ru}\Big(\mathcal{L}\hat{V}(X_u,\overline{X}_u) - r\hat{V}(X_u,\overline{X}_u)\Big)\,\mathrm{d}u \\ &+ \frac{1}{2}\int_0^t \left(\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=b(\overline{x})} - \frac{\partial}{\partial x}(K-e^x)^+\big|_{x=b(\overline{x})}\right)\mathrm{d}L(s), \end{aligned}$$

where L is a local time of the process $X - b(\overline{X})$ at 0. Now, requirement (3.6) guarantees that the integral over $\mathrm{d}\overline{X}_u$ is zero, since $\overline{X}_t$ can only change when $\overline{X}_t = X_t$. Similarly, the smooth-paste conditions make sure that the integral over local time also vanishes. Finally, relations (3.7) and (3.8) lead to the conclusion that the integral over dx, u is non-positive. If we take the expectation of both sides, we get the following result

$$\begin{aligned} e^{-rt}\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\hat{V}(X_t,\overline{X}_t) &= \hat{V}(x,\overline{x}) + \sigma\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\int_0^t e^{-ru}\frac{\partial}{\partial x}\hat{V}(X_u,\overline{X}_u)\,\mathrm{d}B_u \\ &+ \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\int_0^t e^{-ru}\Big(\mathcal{L}\hat{V}(X_u,\overline{X}_u) - r\hat{V}(X_u,\overline{X}_u)\Big)\,\mathrm{d}u \leq \hat{V}(x,\overline{x}) \end{aligned}$$

since the integral over Brownian motion is a zero-mean $\mathbb{Q}$-local martingale. Hence, by the assumptions made, the process $e^{-rt\wedge\tau_D}\hat{V}(X_{t\wedge\tau_D},\overline{X}_{t\wedge\tau_D}) = e^{-rt\wedge\tau_D}\hat{V}(Z_{t\wedge\tau_D})$ is a supermartingale and $\hat{V}(x,\overline{x})$ is a superharmonic function that dominates the payout. Now, from [30, (2.2.80), p. 49] we additionally know that $\hat{V}$ is also lower semi-continuous. It allows us to use [30, Thm. 2.7, p. 40] and claim that it is the optimal solution to the considered stopping problem. Hence $\hat{V}(x,\overline{x}) \geq V(x,\overline{x})$. □

Remark 4. In $\hat{V}$ we chose a specific stopping time, as the first moment when the process X_t reaches the stopping region. On the other hand, the form of V defined in (3.1) ensures that an optimal strategy is chosen. As a result $\hat{V}(x,\overline{x}) \leq V(x,\overline{x})$. Together with Lemma 3, we finally get $\hat{V}(x,\overline{x}) = V(x,\overline{x})$. If we manage to find such functions $\hat{V}$ and $b(\overline{x})$ that satisfy the HJB system, then we know that we found the optimal stopping rule and the fair value of the option. Moreover, these results are unique.

By Proposition 1 and Lemma 3 it follows now that it is sufficient to show that for τ^* defined in (3.4) the value function

$$\hat{V}(x,\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\tau^*\wedge\tau_D}(K - S_{\tau^*\wedge\tau_D})^+]$$

satisfies all the assumptions of Lemma 3 with

- $b(\overline{x}) = a^*$ where a^* is defined in (3.5) when $\overline{x} < a^* + c$;
- $b(\overline{x}) = \overline{x} - c$ when $a^* + c < \overline{x} < \log(K) + c$.

Indeed, in this case $\hat{V}(x,\overline{x}) \geq V(x,\overline{x})$ by Lemma 3 and $\hat{V}(x,\overline{x}) \leq V(x,\overline{x})$ due to the fact that we choose specific stopping rule.
Observe that when $\overline{x} \geq \log(K) + c$ and when $x < a^*$ then the option is immediately exercised. In this case, the assertion of Theorem 2 is trivial and holds true.
When the second case holds, that is, when $a^*+c \leq \overline{x} < \log(K)+c$, then the level a^* cannot be achieved by the process X_t since the drawdown event will happen first. Therefore, we choose $b(\overline{x}) = \overline{x} - c$ in Lemma 3.
We will first find the value function $\hat{V}(x,\overline{x})$. We will do it via some fluctuation identities using the scale functions. Let us recall from equations (2.15) that

$$\mathcal{L}W^{(r)}(x) - rW^{(r)}(x) = 0 \quad \text{and} \quad \mathcal{L}Z^{(r)}(x) - rZ^{(r)}(x) = 0.$$

3.1.2 The case when $\overline{x} < a^* + c$

We start from the case when $\overline{x} < a^* + c$, that is, when $b(\overline{x}) = a^*$ in Lemma 3 with a^* defined in (3.5). Let $\tau = \tau^* \wedge \tau_D$ be the time then the option is stopped either optimally or by the drawdown. Observe that in this case

$$\begin{aligned}
\hat{V}(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)^+\right] = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)^+\mathbb{I}_{\{\tau^-_{a^*\vee\overline{x}-c}<\tau^+_{\overline{x}}\}}\right] \qquad (3.13)\\
&+\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)^+\mathbb{I}_{\{\tau^-_{a^*\vee\overline{x}-c}>\tau^+_{\overline{x}}\}}\right] = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^-_{a^*}}\left(K-e^{a^*}\right)\mathbb{I}_{\{\tau^-_{a^*}<\tau^+_{\overline{x}}\}}\right]\\
&+\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}_{\{\tau^+_{\overline{x}}<\tau^-_{a^*}\}}\right]\mathbb{E}^{\mathbb{Q}}_{\overline{x},\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)^+\right]\\
&=\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^-_{a^*}}\left(K-e^{a^*}\right)\mathbb{I}_{\{\tau^-_{a^*}<\tau^+_{\overline{x}}\}}\right]+\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}_{\{\tau^+_{\overline{x}}<\tau^-_{a^*}\}}\right]\\
&\times\left(\mathbb{E}^{\mathbb{Q}}_{\overline{x},\overline{x}}\left[e^{-r\tau^-_{a^*}}\left(K-e^{a^*}\right)\mathbb{I}_{\{\tau^-_{a^*}<\tau^+_{a^*+c}\}}\right]+\right.\\
&\left.+\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{a^*+c}}\mathbb{I}_{\{\tau^-_{a^*}>\tau^+_{a^*+c}\}}\right]\mathbb{E}^{\mathbb{Q}}_{a^*+c,a^*+c}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}_{\{\tau_D<\tau^+_{\log K+c}\}}\right]\right),\\
&:= V_1(x,\overline{x})+V_2(x,\overline{x})\Big(V_3(\overline{x})+V_4(\overline{x})V_5\Big),
\end{aligned}$$

where

$$\tau^-_x = \inf\{t\geq 0:\ X_t\leq x\} \quad \text{and} \quad \tau^+_x = \inf\{t\geq 0:\ X_t\geq x\}.$$

The condition $\tau_D < \tau^+_{\log K+c}$ in V_5 is necessary to ensure that $K \geq e^{X_{\tau_D}}$ so that $\left(K-e^{X_{\tau_D}}\right)^+ = \left(K-e^{X_{\tau_D}}\right)$. Further in this chapter we will skip one subscript when both processes X_t and $\overline{X_t}$ start from the same point, i.e., we will denote $\mathbb{E}^{\mathbb{Q}}_{\overline{x}}f(X_t,\overline{X_t}) := \mathbb{E}^{\mathbb{Q}}_{\overline{x},\overline{x}}f(X_t,\overline{X_t})$. We will analyze the above components one by one. Using formula [26, eq. (2.4)] we get that

$$\begin{aligned}
V_1(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K-e^{a^*}\right)\mathbb{I}_{\{\tau^-_{a^*}<\tau^+_{\overline{x}}\}}\right]\\
&=\left(K-e^{a^*}\right)\left(Z^{(r)}(x-a^*)-Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right) \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned} V_3(\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau^-_{a^*}}\left(K-e^{a^*}\right)\mathbb{I}_{\{\tau^-_{a^*}<\tau^+_{a^*+c}\}}\right] \\ &= \left(K-e^{a^*}\right)\left(Z^{(r)}(\overline{x}-a^*)-Z^{(r)}(c)\frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}\right). \end{aligned} \tag{3.15}$$

From [26, eq. (2.3)] we also have that

$$V_2(x,\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}_{\{\tau^+_{\overline{x}}<\tau^-_{a^*}\}}\right] = \frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)} \tag{3.16}$$

and

$$V_4(\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{a^*+c}}\mathbb{I}_{\{\tau^-_{a^*}>\tau^+_{a^*+c}\}}\right] = \frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}. \tag{3.17}$$

To find the last component $V_5 = \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}_{\{\tau_D<\tau^+_{\log K+c}\}}\right]$ we introduce the following notations:

$$\eta^{\mathbb{Q}} = \frac{W^{(r)\prime}(d)}{W^{(r)}(d)}, \tag{3.18}$$

$$F_{r,d}(y) = \eta^{\mathbb{Q}}e^{-y\eta^{\mathbb{Q}}}, \quad y\in\mathbb{R}_+,$$

$$\Delta^{\mathbb{Q}} = \frac{\sigma^2}{2}\left[W^{(r)\prime}(d)-\frac{1}{\eta^{\mathbb{Q}}}W^{(r)\prime\prime}(d)\right]. \tag{3.19}$$

Now, by [26, eq. (3.11)] we obtain

$$\begin{aligned} V_5 =& \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}_{\{\tau_D<\tau^+_{\log K+c}\}}\right] = K\mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}_{\{\tau_D<\tau^+_{\log K+c}\}}\right] \\ &- \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D+X_{\tau_D}}\mathbb{I}_{\{\tau_D<\tau^+_{\log K+c}\}}\right] = K\int_{a^*+c}^{\log K+c} F_{r,c}(v-(a^*+c))\Delta^{\mathbb{Q}}\,\mathrm{d}v \\ &- \int_{a^*+c}^{\log K+c} e^{v-c}F_{r,c}(v-(a^*+c))\Delta^{\mathbb{Q}}\,\mathrm{d}v \\ &= \Delta^{\mathbb{Q}}e^{(a^*+c)\eta^{\mathbb{Q}}}\left(K\left(e^{-(a^*+c)\eta^{\mathbb{Q}}}-e^{-(\log K+c)\eta^{\mathbb{Q}}}\right)\right. \\ &\left. - e^{-c}\frac{\eta^{\mathbb{Q}}}{1-\eta^{\mathbb{Q}}}\left(e^{(1-\eta^{\mathbb{Q}})(\log(K)+c)}-e^{(1-\eta^{\mathbb{Q}})(a^*+c)}\right)\right) \\ &= \Delta^{\mathbb{Q}}\left(K\left(1-\frac{e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}}{1-\eta^{\mathbb{Q}}}\right)+\frac{\eta^{\mathbb{Q}}e^{a^*}}{1-\eta^{\mathbb{Q}}}\right). \end{aligned} \tag{3.20}$$

Inserting formulas (3.14)-(3.17) and (3.20) into (3.13) gives the price $\hat{V}(x,\overline{x})$ which is in $\mathcal{C}^2_0$ and it is a linear combination of the scale functions W and Z and therefore by (2.15) condition we have that $\mathcal{L}\hat{V}-r\hat{V}(x,\overline{x})=0$ and conditions (3.7) and (3.8) are satisfied. We recall that to have $\hat{V}(x,\overline{x})$ in the domain of the generator $\mathcal{L}$ of the Markov process Z_t (see (3.6)) we need

$$\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}} = 0$$

which we will show now. Observe that

$$\frac{\partial}{\partial \overline{x}}\hat{V}(x,\overline{x}) =\frac{\partial}{\partial \overline{x}}V_1(x,\overline{x}) + V_5\left(V_4(\overline{x})\frac{\partial}{\partial \overline{x}}V_2(x,\overline{x}) + V_2(x,\overline{x})\frac{\partial}{\partial \overline{x}}V_4(\overline{x})\right)$$
$$+ V_2(x,\overline{x})\frac{\partial}{\partial \overline{x}}V_3(\overline{x}) + V_3(\overline{x})\frac{\partial}{\partial \overline{x}}V_2(x,\overline{x}).$$

Direct verification gives that for all x and $\overline{x}$ we have $V_4(\overline{x})\frac{\partial}{\partial \overline{x}}V_2(x,\overline{x})+V_2(x,\overline{x})\frac{\partial}{\partial \overline{x}}V_4(\overline{x}) = 0$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial \overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}} =& -\left(K-e^{a^*}\right)\frac{Z^{(r)\prime}(\overline{x}-a^*)W^{(r)}(\overline{x}-a^*) - W^{(r)\prime}(\overline{x}-a^*)Z^{(r)}(\overline{x}-a^*)}{W^{(r)}(\overline{x}-a^*)}\\
&- \frac{W^{(r)\prime}(\overline{x}-a^*)}{W^{(r)}(\overline{x}-a^*)}\left(K-e^{a^*}\right)\left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)\right)\\
&+ \left(K-e^{a^*}\right)\left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)\prime}(c)}{W^{(r)}(c)}W^{(r)\prime}(\overline{x}-a^*)\right)\\
=& \frac{K-e^{a^*}}{W^{(r)}(\overline{x}-a^*)}\Big(W^{(r)\prime}(\overline{x}-a^*)Z^{(r)}(\overline{x}-a^*) - Z^{(r)\prime}(\overline{x}-a^*)W^{(r)}(\overline{x}-a^*)\\
&- W^{(r)\prime}(\overline{x}-a^*)Z^{(r)}(\overline{x}-a^*) + \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)W^{(r)\prime}(\overline{x}-a^*)\\
&+Z^{(r)\prime}(\overline{x}-a^*)W^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)W^{(r)\prime}(\overline{x}-a^*)\Big) = 0
\end{aligned}$$

which gives (3.6).

We recall that in the first case $b(\overline{x}) = a^*$. Observe that $W^{(r)}(0) = 0$ and $Z^{(r)}(0) = 1$. Because of that, $V_1(a^*,\overline{x}) = \left(K-e^{a^*}\right)$ and $V_2(a^*,\overline{x}) = 0$. This immediately gives (3.11). We will show that the smooth paste condition (3.12) holds, that is,

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=a^*} = -e^{a^*}.$$

Observe that out of factors V_1 to V_5 only V_1 and V_2 are dependent on x. Therefore

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=a^*} = \frac{\partial}{\partial x}V_1(x,\overline{x})\big|_{x=a^*} + (V_3(\overline{x}) + V_4(\overline{x})V_5)\frac{\partial}{\partial x}V_2(x,\overline{x})\big|_{x=a^*}.$$

Direct computations give

$$\frac{\partial}{\partial x}V_1(x,\overline{x})\big|_{x=a^*} = \left(K-e^{a^*}\right)\left(-(1-\gamma)C_1\frac{Z^{(r)}(\overline{x}-a^*)}{W^{(r)}(\overline{x}-a^*)}\right)$$

and

$$\frac{\partial}{\partial x}V_2(x,\overline{x})\big|_{x=a^*} = \frac{(1-\gamma)C_1}{W(\overline{x}-a^*)}.$$

Further,

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=a^*} = \frac{(1-\gamma)C_1}{W(\overline{x}-a^*)}\Big[-\Big(K-e^{a^*}\Big)Z^{(r)}(\overline{x}-a^*)$$

$$+\Big(K-e^{a^*}\Big)\left(Z^{(r)}(\overline{x}-a^*)-Z^{(r)}(c)\frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}\right)+V_4(\overline{x})V_5\Bigg]$$

$$=\frac{(1-\gamma)C_1}{W(\overline{x}-a^*)}\left[-\Big(K-e^{a^*}\Big)Z^{(r)}(c)\frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}+\frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}V_5\right]$$

$$=\frac{(1-\gamma)C_1}{W(c)}\left[\Delta^{\mathbb{Q}}\left(K\left(1-\frac{e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}}{1-\eta^{\mathbb{Q}}}\right)+\frac{\eta^{\mathbb{Q}}e^{a^*}}{1-\eta^{\mathbb{Q}}}\right)-Z^{(r)}(c)\Big(K-e^{a^*}\Big)\right]$$

$$=\frac{1-\gamma}{e^c-e^{\gamma c}}\left[\frac{\Delta^{\mathbb{Q}}}{1-\eta^{\mathbb{Q}}}\Big(K\Big(1-\eta^{\mathbb{Q}}-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)+\eta^{\mathbb{Q}}e^{a^*}\Big)-\frac{e^{\gamma c}-\gamma e^c}{1-\gamma}\Big(K-e^{a^*}\Big)\right]$$

$$=\frac{1-\gamma}{e^c-e^{\gamma c}}\left[-\Big(K-e^{a^*}\Big)\left(\frac{e^{\gamma c}-\gamma e^c}{1-\gamma}+\frac{\eta^{\mathbb{Q}}\Delta^{\mathbb{Q}}}{1-\eta^{\mathbb{Q}}}\right)+\frac{K\Delta^{\mathbb{Q}}}{1-\eta^{\mathbb{Q}}}\Big(1-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)\right].$$

Observe that

$$\frac{\eta^{\mathbb{Q}}\Delta^{\mathbb{Q}}}{1-\eta^{\mathbb{Q}}}=-e^c.$$

Therefore,

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=a^*}=\frac{1-\gamma}{e^c-e^{\gamma c}}\left[\Big(K-e^{a^*}\Big)\left(e^c-\frac{e^{\gamma c}-\gamma e^c}{1-\gamma}+\right)+\frac{Ke^c}{\eta^{\mathbb{Q}}}\Big(1-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)\right]$$

$$=\frac{1-\gamma}{e^c-e^{\gamma c}}\left[\Big(K-e^{a^*}\Big)\frac{e^c-e^{\gamma c}}{1-\gamma}-Ke^c\frac{e^c-e^{\gamma c}}{e^c-\gamma e^{\gamma c}}\Big(1-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)\right]$$

$$=-e^{a^*}+K\left[1-\frac{(1-\gamma)e^c}{e^c-\gamma e^{\gamma c}}\Big(1-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)\right].$$

Now, a^* given in (3.5) is chosen in such a way that

$$1-\frac{(1-\gamma)e^c}{e^c-\gamma e^{\gamma c}}\Big(1-e^{-\eta^{\mathbb{Q}}(\log(K)-a^*)}\Big)=0$$

which completes the proof of Theorem 2 and the second case of Theorem 3. □

3.1.3 The case of $a^*+c<\overline{x}<\log(K)+c$

We recall that in this case we choose $b(\overline{x})=\overline{x}-c$ in Lemma 3. Furthermore, we have

$$\begin{aligned}\hat{V}(x,\overline{x})&=\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau_D}\Big(K-e^{X_{\tau_D}}\Big)\mathbb{I}_{\{\tau^-_{\overline{x}-c}<\tau^+_{\overline{x}}\}}\right]\\&\quad+\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}_{\{\tau^-_{\overline{x}-c}>\tau^+_{\overline{x}}\}}\right]\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau_D}\Big(K-e^{X_{\tau_D}}\Big)\mathbb{I}_{\{\tau_D<\tau^+_{\log(K)+c}\}}\right]\\&:=V_6(x,\overline{x})+V_7(x,\overline{x})V_8(\overline{x}).\end{aligned}$$

Now, we use [26, equations (2.3), (2.4), (3.11)] to get

$$
\begin{aligned}
V_6(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}_{\{\tau^-_{\overline{x}-c}<\tau^+_{\overline{x}}\}}\right]\\
&= \left(K-e^{\overline{x}-c}\right)\left[Z^{(r)}(x+c-\overline{x})-\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(x+c-\overline{x})\right],\\
V_7(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}_{\{\tau^-_{\overline{x}-c}>\tau^+_{\overline{x}}\}}\right] = \frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)},\\
V_8(\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}_{\{\tau_D<\tau^+_{\log(K)+c}\}}\right]\\
&= \Delta^{\mathbb{Q}}K-e^{\overline{x}}+Ke^{\eta^{\mathbb{Q}}(\overline{x}-\log(K)-c)}\left(e^c-\Delta^{\mathbb{Q}}\right).
\end{aligned}
$$

Observe that in this case $\hat{V}(x,\overline{x})$ is again in $\mathcal{C}^2_0$ and it is a linear combination of the scale functions W and Z and therefore by (2.15) we have that $(\mathcal{L}\hat{V}-r\hat{V})(x,\overline{x})=0$ and conditions (3.7) and (3.8) are satisfied. To have (3.6) satisfied, we need to verify that $\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}}=0$. Note that

$$
\begin{aligned}
&\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}} = \left(\frac{\partial}{\partial\overline{x}}V_6(x,\overline{x})+V_7(x,\overline{x})\frac{\partial}{\partial\overline{x}}V_8(\overline{x})+V_8(\overline{x})\frac{\partial}{\partial\overline{x}}V_7(x,\overline{x})\right)\Big|_{x=\overline{x}}\\
&= \left(K-e^{\overline{x}-c}\right)\left[-Z^{(r)\prime}(c)+\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)\prime}(c)\right]-e^{\overline{x}}+K\eta^{\mathbb{Q}}e^{\eta^{\mathbb{Q}}(\overline{x}-\log(K)-c)}\left(e^c-\Delta^{\mathbb{Q}}\right)\\
&-\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}\left(\Delta^{\mathbb{Q}}K-e^{\overline{x}}+Ke^{\eta^{\mathbb{Q}}(\overline{x}-\log(K)-c)}\left(e^c-\Delta^{\mathbb{Q}}\right)\right).
\end{aligned}
$$

We also have $Z^{(r)\prime}(c)=rW^{(r)}(c)$ and $\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}=\eta^{\mathbb{Q}}$. Therefore

$$
\begin{aligned}
&\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}} = \left(K-e^{\overline{x}-c}\right)\left[-Z^{(r)\prime}(c)+\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)\prime}(c)\right]\\
&-e^{\overline{x}}\left(1-\eta^{\mathbb{Q}}\right)-\eta^{\mathbb{Q}}\Delta^{\mathbb{Q}}K = \left(K-e^{\overline{x}-c}\right)\left[-Z^{(r)\prime}(c)+\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)\prime}(c)\right]\\
&-e^{\overline{x}}\left(1-\eta^{\mathbb{Q}}\right)-\frac{\sigma^2}{2}K\left(\eta^{\mathbb{Q}}W^{(r)\prime}(c)-W^{(r)\prime\prime}(c)\right)\\
&= K\left(\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)\prime}(c)-rW^{(r)}(c)-\frac{\left(\sigma W^{(r')}(c)\right)^2}{2W^{(r)}(c)}+\frac{\sigma^2}{2}W^{(r)\prime\prime}(c)\right)\\
&+e^{\overline{x}}\left(rW^{(r)}(c)e^{-c}-\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)\prime}(c)e^{-c}-1+\frac{W^{(r)\prime}(c)}{W^{(r)}(c)}\right).
\end{aligned}
\tag{3.21}
$$

We will prove that both brackets are equal to zero. Note that $Z^{(r)}(c)=\frac{1}{1-\gamma}\left(e^{\gamma}c-\gamma e^{c}\right)$

and observe that:

$$
\begin{aligned}
&\frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)'}(c) - rW^{(r)}(c) - \frac{\left(\sigma W^{(r')}(c)\right)^2}{2W^{(r)}(c)} + \frac{\sigma^2}{2}W^{(r)''}(c) \qquad (3.22)\\
&= \frac{1}{1-\gamma}\left(e^{\gamma c}-\gamma e^{c}\right)\frac{e^c-\gamma e^{\gamma c}}{e^c-e^{\gamma c}} - \frac{r}{r+\frac{\sigma^2}{2}}\left(e^c-e^{\gamma c}\right) - \frac{\frac{\sigma^2}{2}}{r+\frac{\sigma^2}{2}}\left(\frac{(e^c-\gamma e^{\gamma c})^2}{e^c-e^{\gamma c}}\right)\\
&= \frac{1}{1-\gamma}\left(e^{\gamma c}-\gamma e^{c}\right)\frac{e^c-\gamma e^{\gamma c}}{e^c-e^{\gamma c}} + \frac{\gamma}{1-\gamma}\left(e^c-e^{\gamma c}\right) - \frac{1}{1-\gamma}\left(\frac{(e^c-\gamma e^{\gamma c})^2}{e^c-e^{\gamma c}}\right)\\
&= -\frac{1}{r+\frac{\sigma^2}{2}}(e^c-e^{\gamma c}) + \frac{(e^c-e^{\gamma c})^{-1}}{1-\gamma}\\
&\times\left[e^{c(1+\gamma^2)} - \gamma(e^{2c}+e^{2\gamma c}) - e^{2c} - \gamma^2 e^{2\gamma c} + 2\gamma e^{c(1+\gamma)} + e^{2c} + \gamma^2 e^{2\gamma c} - (1+\gamma^2)e^{c(1+\gamma)}\right]\\
&= \frac{\gamma(e^c-e^{\gamma c})^{-1}}{1-\gamma}\left(e^{2c}+e^{2\gamma c}-2e^{c(1+\gamma)}\right) - \frac{1}{r+\frac{\sigma^2}{2}}(e^c-e^{\gamma c})\\
&= (e^c-e^{\gamma c})\frac{1}{1-\frac{1}{\gamma}} - \frac{1}{1-\frac{1}{\gamma}}(e^c-e^{\gamma c}) = 0.
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
&rW^{(r)}(c)e^{-c} - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)'}(c)e^{-c} - 1 + \frac{W^{(r)'}(c)}{W^{(r)}(c)} = \qquad (3.23)\\
&= \frac{1}{W^{(r)}(c)}\left[r(W^{(r)}(c))^2e^{-c} - Z^{(r)}(c)W^{(r)'}(c)e^{-c} - W^{(r)}(c) + W^{(r)'}(c)\right]\\
&= \frac{C_1e^{-c}}{W^{(r)}(c)}\left[rC_1(e^c-e^{\gamma c})^2 - \frac{1}{1-\gamma}(e^{\gamma c}-\gamma e^c)(e^c-\gamma e^{\gamma c}) - e^c(e^c-e^{\gamma c}-e^c+\gamma e^{\gamma c})\right]\\
&= \frac{C_1e^{-c}}{W^{(r)}(c)}\left[\frac{1}{1-\gamma}\left(\gamma e^{2c}+\gamma e^{2\gamma c} - e^{c(1+\gamma)} - \gamma(e^{2c}+e^{2\gamma c}-2e^{c(1+\gamma)})\right) + (1-\gamma)e^{c(1+\gamma)}\right] = 0.
\end{aligned}
$$

Combining (3.21), (3.22) and (3.23) we get that $\frac{\partial}{\partial \overline{x}}\hat{V}(x,\overline{x})\big|_{x=\overline{x}} = 0$. This completes the proof. □

3.1.4 Numerical analysis

In this section, we explore several properties of the options capped by drawdown. First, Figure 3.1 illustrates the smooth-paste condition defined by equations (3.11) and (3.12). The parameters are selected such that $\overline{x} < a^*+c$, which allows us to observe the occurrence of a drawdown event.

Next, Figure 3.2 demonstrates how the option price depends on the initial values $X_0 = x$ and $\overline{X}_0 = \overline{x}$. Notably, when $\overline{x}$ exceeds $\log(K) + c$, the option becomes worthless unless $x < \log(K)$. This is because, at such a level of $\overline{x}$, it is no longer possible for the stock price process to reach the strike before the option is terminated by the drawdown. Of course, one can hypothetically consider a pair $(x, \overline{x})$ such that the difference between them is greater than c. In that case, the option is immediately exercised and its price is equal to the immediate payoff. For $\overline{x} < \log(K) + c$, the plot shows a smooth pasting of the price function to the payoff function.

Finally, we perform a sensitivity analysis of both the stopping barrier and the option price with respect to the volatility σ and the risk-free rate r. In Figure 3.3, we analyze the optimal barrier e^{a^*} of the underlying asset price process. It is evident that the barrier increases with higher r and lower σ, indicating that an increase in the drift parameter of X_t leads to an upward shift in the barrier.

Similarly, Figure 3.4 presents the sensitivities of the option price. In contrast to the previous chart, we can see that the function increases with higher interest rate and lower volatility. This behavior is intuitive: greater σ leads to higher uncertainty for the seller, which has an effect on the risk premium. On the other hand, the decrease of r translates to discount factors closer to 1 and in consequence: to higher present value of future payouts.

The relation between Figures 3.3 and 3.4 is also reasonable. The highest option price coincides with the lowest optimal stopping barrier. This is logical, since reaching a lower asset price can lead to a higher payout than if the contract were exercised earlier.

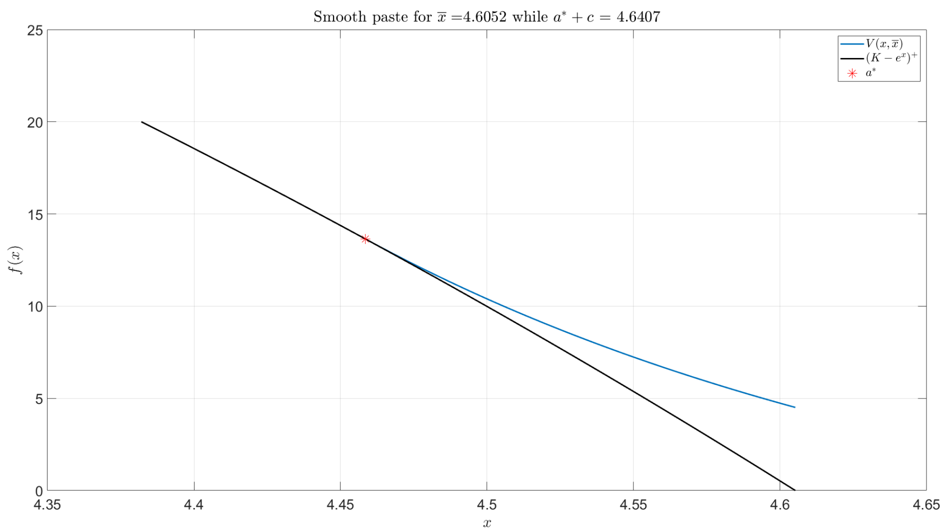

Figure 3.1: Smooth paste of the option price V and the option payoff. Parameters of the model: $r = 0.1,\ \sigma = 0.2,\ e^c = 1.2,\ e^{\overline{x}} = K = 100.$

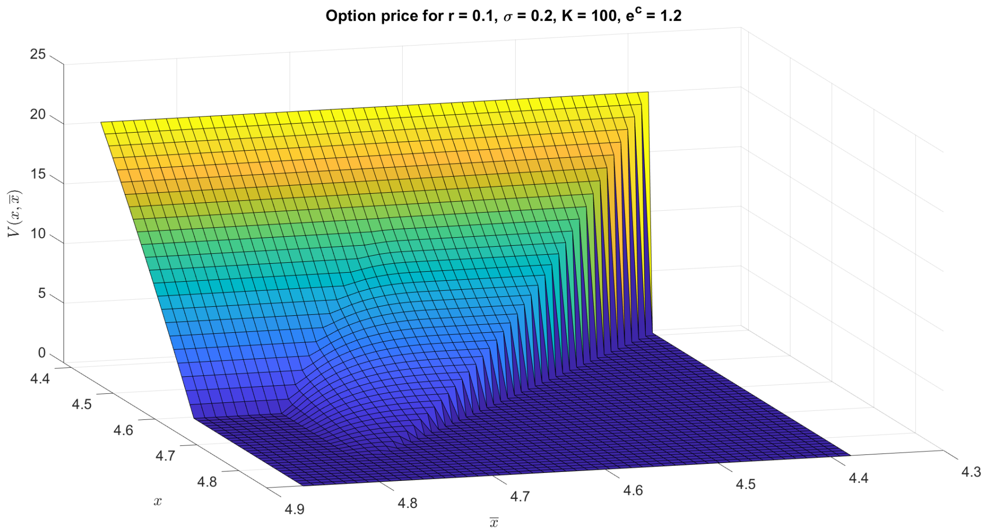

Figure 3.2: Option price depending on x and $\overline{x}$. Note that the function is not defined for $\overline{x} < x$.

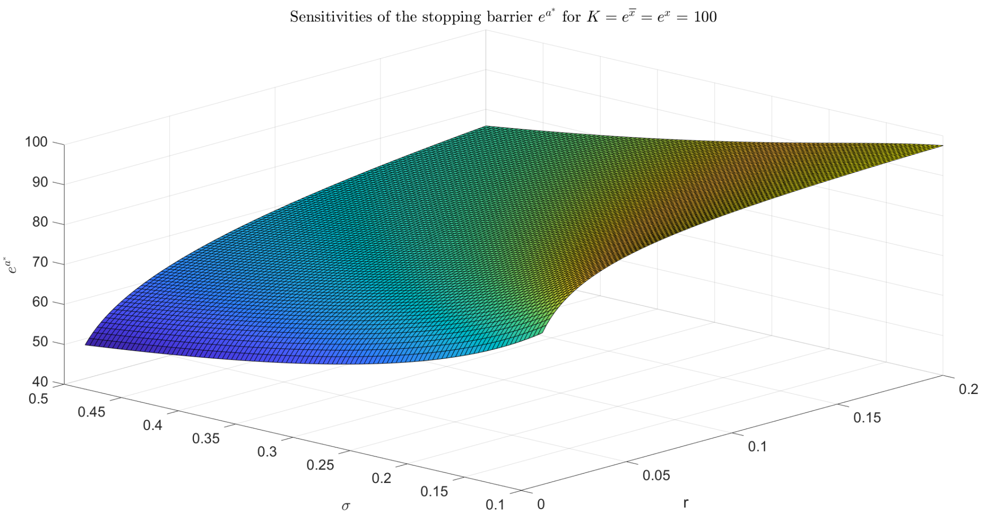

Figure 3.3: Stopping barrier e^{a^*} of the underlying asset price process S_t depending on the risk-free rate r and the volatility σ.

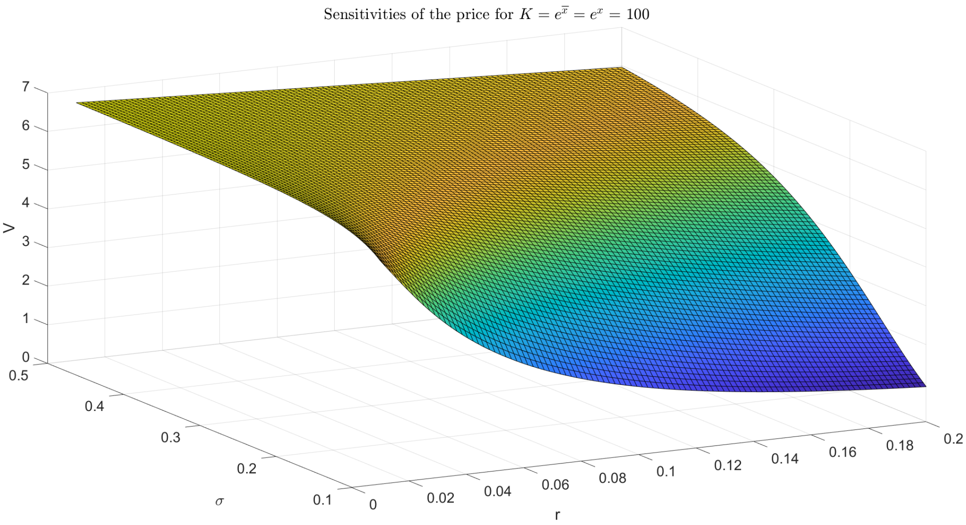

Figure 3.4: Sensitivities of the option price depending on the risk-free rate r and the volatility σ.

3.2 Jump-diffusion model

In this section, we no longer assume that $\lambda = 0$, thus allowing for the downward jumps. In the main theorem of this part, we postulate the formula for the option price. First, we start with claiming the form of the optimal stopping rule:

Theorem 4. *The optimal stopping barrier is the first downward asset price time*

$$\tau^* = \inf\{t \geq 0 : X_t \leq a^*\},$$

where a^* is the unique solution of equation (3.48)*. Moreover, we have $V(x, \overline{x}) = \hat{V}(x, \overline{x})$ for the function $\hat{V}(x, \overline{x})$ identified in Theorem 5.*

Remark 5. By Proposition 1 we know that the optimal stopping rule τ^* is of the one-sided form. In the next step, we postulate that the optimal stopping boundary is even more specific, namely, that

- $b(\overline{x}) = a^*$ for some optimal a^* when $\overline{x} < a^* + c$;
- $b(\overline{x}) = \overline{x} - c$ when $a^* + c < \overline{x} < \log(K) + c$.

We will calculate the value function

$$\hat{V}(x, \overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}[e^{-r\tau^* \wedge \tau_D}(K - S_{\tau^* \wedge \tau_D})^+]$$

for this postulated stopping rule τ^* and we will show that all the assumptions of Lemma 3 are satisfied for $\hat{V}$. Hence in this case $\hat{V}(x, \overline{x}) \geq V(x, \overline{x})$ by Lemma 3 and $\hat{V}(x, \overline{x}) \leq V(x, \overline{x})$ due to the fact that we choose specific stopping rule. Thus, $\hat{V}(x, \overline{x}) = V(x, \overline{x})$ is a true value function and τ^* is the optimal stopping rule.

Similarly as in Remark 4, from general stopping theory applied to the Markov process $(S_{t\wedge\tau_D}, \overline{S}_{t\wedge\tau_D})$ (see [30, Thm. 2.7, p. 40 and (2.2.80), p. 49]) it follows that the stopping region is the set when the value function meets the payout function and hence it is unique which is due to the existence of the value function $V(x,\overline{x})$. In other words, our stopping region is unique as well.

3.2.1 Main result

Theorem 5. *The following holds.*
For $x > a^*$ and $\overline{x} - x < c$:

(i) If $\overline{x} < a^* + c$ then we have

$$\hat{V}(x,\overline{x}) = V_1(x,\overline{x}) + V_2(x,\overline{x})(V_3(\overline{x}) + V_4(\overline{x})(V_5 + V_6V_7))$$

, where V_1, V_2, V_3, V_4, V_5, V_6, V_7 are given in (3.26), (3.24), (3.27), (3.25), (3.31), (3.34), (3.36), respectively.

(ii) If $a^* + c < \overline{x} < \log(K) + c$ then we have

$$\hat{V}(x,\overline{x}) = V_{10}(x,\overline{x}) + V_{11}(x,\overline{x})(V_{12}(\overline{x}) + V_{13}(\overline{x})V_7)$$

, where V_{10}, V_{11}, V_{12}, V_{13} are given in (3.37), (3.38), (3.40), (3.41), respectively.

(iii) If $\overline{x} > \log(K) + c$ then we have

$$\hat{V}(x,\overline{x}) = V_{14}(x,\overline{x}) + V_{15}(x,\overline{x})V_{16}(\overline{x})$$

, where V_{14}, V_{15}, V_{16} are given in (3.44), (3.45), (3.46), respectively.

For $x \leq a^*$ or $\overline{x} - x \geq c$:

(iv)

$$\hat{V}(x,\overline{x}) = (K - e^x)^+ .$$

Proof. Assume first that $\overline{x} < a^* + c$. Then

$$\begin{aligned}\hat{V}(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau\wedge\tau_D}\left(K - e^{X_{\tau\wedge\tau_D}}\right)^+\right] = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K - e^{X_\tau}\right)\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] \\ &+ \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}\{\tau > \tau^+_{\overline{x}}\}\right]\left(\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau}\left(K - e^{X_\tau}\right)\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] + \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau^+_{a^*+c}}\mathbb{I}\{\tau > \tau^+_{a^*+c}\}\right]\right. \\ &\times\left(\mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\left(K - e^{X_{\tau_D}}\right)\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right] + \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau^+_{\log(K)+c}}\mathbb{I}\{\tau_D > \tau^+_{\log(K)+c}\}\right]\right. \\ &\times\left.\left.\mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\left(K - e^{X_{\tau_D}}\right)^+\right]\right)\right) = V_1(x,\overline{x}) + V_2(x,\overline{x})(V_3(\overline{x}) + V_4(\overline{x})(V_5 + V_6V_7)).\end{aligned}$$

From [26, eq. (2.3)] we have that

$$V_2(x,\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}\{\tau^+_{\overline{x}} < \tau\}\right] = \frac{W^{(r)}(x - a^*)}{W^{(r)}(\overline{x} - a^*)} \tag{3.24}$$

and

$$V_4(\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau^+_{a^*+c}}\mathbb{I}\{\tau^+_{a^*+c} < \tau\}\right] = \frac{W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}. \tag{3.25}$$

Now, from [26, eq. (2.4)], we get

$$\begin{aligned}
V_1(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] = K\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] \\
&- \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau+X_\tau}\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] = K\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau}\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] - e^x\mathbb{E}^{\mathbb{P}}_{x,\overline{x}}\left[\mathbb{I}\{\tau < \tau^+_{\overline{x}}\}\right] \\
&= K\left(Z^{(r)}(x-a^*) - Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right) \\
&- e^x\left(Z^{\mathbb{P}}(x-a^*) - Z^{\mathbb{P}}(\overline{x}-a^*)\frac{W^{\mathbb{P}}(x-a^*)}{W^{\mathbb{P}}(\overline{x}-a^*)}\right) \\
&= K\left(Z^{(r)}(x-a^*) - Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right) - \left(e^x - \frac{e^xW^{\mathbb{P}}(x-a^*)}{W^{\mathbb{P}}(\overline{x}-a^*)}\right) \\
&= K\left(Z^{(r)}(x-a^*) - Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right) - \left(e^x - \frac{e^{\overline{x}}W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right)
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
V_3(\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau}\left(K-e^{X_\tau}\right)\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] = K\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau}\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] \\
&- \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau+X_\tau}\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] = K\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau}\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] - e^x\mathbb{E}^{\mathbb{P}}_{\overline{x}}\left[\mathbb{I}\{\tau < \tau^+_{a^*+c}\}\right] \\
&= K\left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)\right) - e^{\overline{x}}\left(Z^{\mathbb{P}}(\overline{x}-a^*) - \frac{Z^{\mathbb{P}}(c)}{W^{\mathbb{P}}(c)}W^{\mathbb{P}}(\overline{x}-a^*)\right) \\
&= K\left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)\right) - \left(e^{\overline{x}} - \frac{e^{\overline{x}}W^{\mathbb{P}}(\overline{x}-a^*)}{W^{\mathbb{P}}(c)}\right) \\
&= K\left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(\overline{x}-a^*)\right) - \left(e^{\overline{x}} - \frac{e^{a^*+c}W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)}\right).
\end{aligned} \tag{3.27}$$

Moving on to V_5, observe that

$$\begin{aligned}
V_5 &= \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right] \\
&= K\mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right] - e^{a^*+c}\mathbb{E}^{\mathbb{P}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right].
\end{aligned}$$

Let us introduce the following notations to deal with last terms

$$\begin{aligned}
R(r,\mathrm{d}y) &= \left[\frac{1}{\eta^{\mathbb{Q}}}W^{(r)\prime}(y)\,\mathrm{d}y - W^{(r)}(y)\,\mathrm{d}y\right], \\
\overline{X}_t &= \sup_{0\le u\le t} X_u \ \vee\ \overline{x}, \\
\underline{X}_t &= \inf_{0\le u\le t} X_u, \\
D_t &= \overline{X}_t - X_t.
\end{aligned}$$

By

$$\Lambda(y-c-\mathrm{d}h)=\lambda\rho e^{\rho(y-c-h)}\,\mathrm{d}h,\ h\in(0,\infty)$$

we denote the Lévy measure of the Lévy proc:ess X_t. Now, let us define the following two events:

$$A_o=\{\underline{X}_{\tau_D}\geq u;\ \overline{X}_{\tau_D}\in\mathrm{d}v;D_{\tau_D-}\in\mathrm{d}y;D_{\tau_D}-c\in\mathrm{d}h\},$$
$$A_c=\{\underline{X}_{\tau_D}\geq u;\ \overline{X}_{\tau_D}\in\mathrm{d}v;D_{\tau_D-}=c\}.$$

The first one is associated with drawdown exceeding the threshold with a Poissonian jump, the latter is related to the hitting the threshold by creeping. From [26, eq. (3.10, 3.11)] we have

$$\mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau_D}\mathbb{I}_{A_o}\right]=\frac{W^{(r)}((x-u)\wedge c)}{W^{(r)}(c)}F^{\mathbb{Q}}(v-(x\vee(u+c)))\,\mathrm{d}vR(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h).$$

$$\mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau_D}\mathbb{I}_{A_c}\right]=\frac{W^{(r)}((x-u)\wedge c)}{W^{(r)}(c)}F^{\mathbb{Q}}(v-(x\vee(u+c)))\Delta^{\mathbb{Q}}.$$

We can represent events A_o and A_c as follows

$$A_o^5=\{\overline{X}_{\tau_D}\in\mathrm{d}v,\ v\in[a^*+c,\log(K)+c);D_{\tau_D-}\in\mathrm{d}y,\ y\in[0,c)\,;$$
$$D_{\tau_D}-c\in\mathrm{d}h,\ h\in(0,\infty)\}.$$

$$A_c^5=\{\underline{X}_{\tau_D}\geq a^*;\ \overline{X}_{\tau_D}\in\mathrm{d}v,\ v\in[a^*+c,\log(K)+c);\ D_{\tau_D}=c\}.$$

For the first event, A_o^5, the first condition $\underline{X}_{\tau_D}\geq u$ disappears as the jump sizes are unbounded, allowing $\underline{X}_{\tau_D}$ to be arbitrarily small. Additionally, observe that no matter if we take $u=-\infty$ or $u=a^*$, we get $(a^*+c-u)\wedge c=c$ and $a^*+c\vee(u+c)=a^*+c$. As a result, we get $\frac{W^{(r)}((x-u)\wedge a^*)}{W^{(r)}(a^*)}=1$. This gives

$$\mathbb{E}_{a^*+c}^{\mathbb{Q}}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D<\tau_{\log(K)+c}^+\}\right]=\int\limits_{a^*+c}^{\log(K)+c}F^{\mathbb{Q}}(v-(a^*+c))\,\mathrm{d}v\left[\Delta^{\mathbb{Q}}+\int\limits_0^\infty\int\limits_0^c R(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h)\right].$$

The first integral equals

$$\int\limits_{a^*+c}^{\log(K)+c}F^{\mathbb{Q}}(v-(a^*+c))\,\mathrm{d}v=\int\limits_{a^*+c}^{\log(K)+c}\eta^{\mathbb{Q}}e^{-(v-(a^*+c))\eta^{\mathbb{Q}}}\,\mathrm{d}v$$
$$=e^{(a^*+c)\eta^{\mathbb{Q}}}\left(e^{-(a^*+c)\eta^{\mathbb{Q}}}-e^{-(\log(K)+c)\eta^{\mathbb{Q}}}\right)=1-e^{(a^*-\log(K))\eta^{\mathbb{Q}}}$$

and the double integral from the square bracket is

$$\int_0^\infty \int_0^c R(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h) = \int_0^c \int_0^\infty \left[\eta^{\mathbb{Q}-1} W^{(r)'}(y) - W^{(r)}(y)\right] \lambda\rho e^{\rho(y-d-h)}\,\mathrm{d}h\,\mathrm{d}y$$

$$= \lambda e^{-\rho c} \int_0^\infty \rho e^{-\rho h}\,\mathrm{d}h \int_0^c e^{\rho y}\left[\eta^{\mathbb{Q}-1} W^{(r)'}(y) - W^{(r)}(y)\right]\mathrm{d}y$$

$$= \lambda e^{-\rho c} \int_0^\infty \rho e^{-\rho h}\,\mathrm{d}h \int_0^c e^{\rho y}\left[\eta^{\mathbb{Q}-1}\sum_{i=1}^3 C_i\gamma_i e^{\gamma_i x} - \sum_{i=1}^3 C_i e^{\gamma_i x}\right]\mathrm{d}y$$

$$= \lambda e^{-\rho c}\sum_{i=1}^3 \frac{C_i}{\gamma_i+\rho}\left(\frac{\gamma_i}{\eta^{\mathbb{Q}}}-1\right)\left(e^{c(\gamma_i+\rho)}-1\right) := \Gamma_{\mathbb{Q}}. \tag{3.28}$$

Observe that $\sum_{i=1}^3 \frac{C_i}{\gamma_i+\rho} = 0$ since $C_i = \frac{2}{\sigma^2}\frac{\gamma_i+\rho}{(\gamma_i-\gamma_j)(\gamma_i-\gamma_k)}$ for $i,j,k \in \{1,2,3\},\ i\neq j, i\neq k, j\neq k$. Additionally,

$$\lambda e^{-\rho c}\sum_{i=1}^3 \frac{C_i}{\gamma_i+\rho}\frac{\gamma_i}{\eta^{\mathbb{Q}}} = \lambda e^{-\rho c}\frac{1}{\eta^{\mathbb{Q}}}\sum_{i=1}^3 \frac{C_i\gamma_i}{\gamma_i+\rho} = 0 \tag{3.29}$$

because

$$\sum_{i=1}^3 \frac{C_i\gamma_i}{\gamma_i+\rho} = \frac{2}{\sigma^2}\sum_{i=1}^3 \frac{\gamma_i+\rho}{(\gamma_i-\gamma_j)(\gamma_i-\gamma_k)}\frac{\gamma_i}{\gamma_i+\rho}$$
$$= \frac{2}{\sigma^2}\left[\frac{\gamma_1}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{\gamma_2}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)}\right]$$
$$= \frac{2}{\sigma^2(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)(\gamma_2-\gamma_3)}[\gamma_1(\gamma_2-\gamma_3)-\gamma_2(\gamma_1-\gamma_3)+\gamma_3(\gamma_1-\gamma_2)] = 0.$$

Therefore, we can simplify $\Gamma_{\mathbb{Q}}$ in the following way

$$\Gamma_{\mathbb{Q}} = \lambda\sum_{i=1}^3 \frac{C_i}{\gamma_i+\rho}\left(\frac{\gamma_i}{\eta^{\mathbb{Q}}}-1\right)e^{\gamma_i c}. \tag{3.30}$$

To handle the expected value on $\mathbb{P}$ measure, we need to introduce the similar notations as for $\mathbb{Q}$ measure:

$$\eta^{\mathbb{P}} = \frac{W^{\mathbb{P}'}(c)}{W^{\mathbb{P}}(c)} = \frac{e^{-c}\left(W^{(r)'}(c) - W^{(r)}(c)\right)}{e^{-c}W^{(r)}(c)} = \eta^{\mathbb{Q}} - 1,$$

$$F^{\mathbb{P}}(y) = \eta^{\mathbb{P}} e^{-y\eta^{\mathbb{P}}}, \quad y \in \mathbb{R}_+$$

and

$$\begin{aligned}
\Delta^{\mathbb{P}} &= \frac{\sigma^2}{2}\left[W^{\mathbb{P}\prime}(c) - \eta^{\mathbb{P}-1}W^{\mathbb{P}\prime\prime}(c)\right] \\
&= \frac{\sigma^2}{2}e^{-c}\left(W^{(r)\prime}(c) - W^{(r)}(c) - \frac{W^{(r)}(c)\left(W^{(r)\prime\prime}(c) - 2W^{(r)\prime}(c) + W^{(r)}(c)\right)}{W^{(r)\prime}(c) - W^{(r)}(c)}\right) \\
&= \frac{\sigma^2}{2}e^{-c}\left(W^{(r)\prime}(c) - \frac{W^{(r)}(c)\left(W^{(r)\prime\prime}(c) - W^{(r)\prime}(c)\right)}{W^{(r)\prime}(c) - W^{(r)}(c)}\right) \\
&= \frac{\sigma^2}{2}e^{-c}\left(W^{(r)\prime}(c) - \frac{W^{(r)\prime\prime}(c) - W^{(r)\prime}(c)}{\eta^{\mathbb{Q}} - 1}\right) = \frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\frac{\sigma^2}{2}e^{-c}\left(W^{(r)\prime}(c) - \frac{W^{(r)\prime\prime}(c)}{\eta^{\mathbb{Q}}}\right) \\
&= \frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}e^{-c}\Delta^{\mathbb{Q}}.
\end{aligned}$$

Finally, let

$$\begin{aligned}
\Gamma_{\mathbb{P}} &= \tilde{\lambda}e^{-\tilde{\rho}c}\sum_{i=1}^{3}\frac{\tilde{C}_i}{\tilde{\gamma}_i + \tilde{\rho}}\left(\frac{\tilde{\gamma}_i}{\eta^{\mathbb{P}}} - 1\right)\left(e^{c(\tilde{\gamma}_i + \tilde{\rho})} - 1\right) \\
&= \lambda\frac{\rho}{\rho + 1}e^{-c}e^{-\rho c}\sum_{i=1}^{3}\frac{C_i}{\gamma_i + \rho}\left(\frac{\gamma_i - 1}{\eta^{\mathbb{Q}} - 1} - 1\right)\left(e^{c(\gamma_i + \rho)} - 1\right) \\
&= \frac{\rho e^{-c}}{\rho + 1}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\lambda e^{-\rho c}\sum_{i=1}^{3}\frac{C_i}{\gamma_i + \rho}\left(\frac{\gamma_i - 1}{\eta^{\mathbb{Q}}} - \frac{\eta^{\mathbb{Q}} - 1}{\eta^{\mathbb{Q}}}\right)\left(e^{c(\gamma_i + \rho)} - 1\right) \\
&= \frac{\rho e^{-c}}{\rho + 1}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\lambda e^{-\rho c}\sum_{i=1}^{3}\frac{C_i}{\gamma_i + \rho}\left(\frac{\gamma_i -}{\eta^{\mathbb{Q}}} - 1\right)\left(e^{c(\gamma_i + \rho)} - 1\right) = \frac{\rho e^{-c}}{\rho + 1}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\Gamma_{\mathbb{Q}}.
\end{aligned}$$

By combining everything together, we get

$$\begin{aligned}
V_5 &= K\mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right] - e^{a^*+c}\mathbb{E}^{\mathbb{P}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right] \\
&= K\left[\left(1 - e^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\right)\right] - e^{a^*+c}\left[\left(1 - e^{(a^* - \log(K))\eta^{\mathbb{P}}}\right)\left(\Delta^{\mathbb{P}} + \Gamma_{\mathbb{P}}\right)\right] \\
&= K\left[\left(1 - e^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\right)\right] \\
&- e^{a^*+c}\left[\left(1 - e^{(a^* - \log(K))\eta^{\mathbb{Q}} + \log(K) - a^*}\right)\left(\Delta^{\mathbb{P}} + \Gamma_{\mathbb{P}}\right)\right] \\
&= \left[\left(K - Ke^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\right)\right] - \left[\left(e^{a^*+c} - Ke^{c}e^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{P}} + \Gamma_{\mathbb{P}}\right)\right] \\
&= \left[\left(K - Ke^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\right)\right] \\
&- \left[\left(e^{a^*} - Ke^{(a^* - \log(K))\eta^{\mathbb{Q}}}\right)\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\left(\Delta^{\mathbb{Q}} + \frac{\rho}{\rho + 1}\Gamma_{\mathbb{Q}}\right)\right] \\
&= Ke^{(a^* - \log(K))\eta^{\mathbb{Q}}}\frac{\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\frac{\rho + 1 - \eta^{\mathbb{Q}}}{\rho + 1}}{\eta^{\mathbb{Q}} - 1} - e^{a^*}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}} - 1}\left(\Delta^{\mathbb{Q}} + \frac{\rho}{\rho + 1}\Gamma_{\mathbb{Q}}\right) + K\left(\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\right).
\end{aligned} \tag{3.31}$$

Now, let us consider an event, when the first drawdown of the process X_t starting from

$a^* + c$ occurs after the process hits level $\log(K) + c$. We can then split the time until drawdown into two sub-intervals: from 0 to $\tau^+_{\log(K)+c}$ and from $\tau^+_{\log(K)+c}$ to τ_D. As time before reaching $\log(K) + c$ is independent from time to first drawdown starting from $\log(K) + c$, we can write the following relation:

$$
\begin{aligned}
V_6 &= \mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau^+_{\log(K)+c}}\mathbb{I}\{\tau_D > \tau^+_{\log(K)+c}\}\right] \\
&= \frac{\mathbb{E}^{\mathbb{Q}}_{a^*+c}\left[e^{-r\tau_D}\mathbb{I}\{\sup_{t\in(\tau^+_{a^*+c},\tau_D)} X_t \geq \log(K) + c\}\right]}{\mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\right]} := \frac{V_9}{V_8}.
\end{aligned} \tag{3.32}
$$

Let us first handle the denominator. Again, we want to specify events A_o and A_c for V_8:

$$
\begin{aligned}
A^8_o = \{&\overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in [\log(K) + c, \infty); D_{\tau_D -} \in \mathrm{d}y,\ y \in [0, c)\,; \\
&D_{\tau_D} - c \in \mathrm{d}h,\ h \in (0, \infty)\}.
\end{aligned}
$$

$$
A^8_c = \{\underline{X}_{\tau_D} \geq \log(K);\ \overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in [\log(K) + c, \infty);\ D_{\tau_D} = c\}.
$$

With these, we get the formula for V_8 as follows

$$
V_8 = \int\limits_{\log(K)+c}^{\infty} F^{\mathbb{Q}}(v - (\log(K) + c))\,\mathrm{d}v \left[\Delta^{\mathbb{Q}} + \int\limits_0^{\infty}\int\limits_0^{c} R(r, \mathrm{d}y)\Lambda(y - c - \mathrm{d}h)\right] = \Delta^{\mathbb{Q}} + \Gamma_Q
$$

as the first integral is equal to 1 and the double integral is the same as one derived for in V_5 in (3.28). On the other hand, from formula (3.3) from [26] we have:

$$
V_8 = Z^{(r)}(c) - r\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)}
$$

and therefore

$$
\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}} = Z^{(r)}(c) - r\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)}. \tag{3.33}
$$

Similarly, using the same argument for $\mathbb{P}$ counterparts and applying the equality (1.15), we get

$$
\Delta^{\mathbb{P}} + \Gamma_{\mathbb{P}} = Z^{\mathbb{P}}(c) - 0 \cdot \frac{W^{\mathbb{P}}(c)^2}{W^{\mathbb{P}\prime}(c)} = 1.
$$

Now, observe that for V_9, event A_o is the same as for V_8:

$$
\begin{aligned}
A^9_o = A^8_o = \{&\overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in [\log(K) + c, \infty); D_{\tau_D -} \in \mathrm{d}y,\ y \in [0, c)\,; \\
&D_{\tau_D} - c \in \mathrm{d}h,\ h \in (0, \infty)\}.
\end{aligned}
$$

and only the lower bound of $\underline{X}_{\tau_D}$ changes for A_c in the following way:

$$
A^9_c = \{\underline{X}_{\tau_D} \geq a^*;\ \overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in [\log(K) + c, \infty);\ D_{\tau_D} = c\}.
$$

Similarly to V_8, we get

$$V_9 = \int\limits_{\log(K)+c}^{\infty} F^{\mathbb{Q}}(v-(a^*+c))\,\mathrm{d}v \left[\Delta^{\mathbb{Q}} + \int\limits_0^{\infty}\int\limits_0^{c} R(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h)\right] = e^{\eta^{\mathbb{Q}}(a^*-\log(K))}(\Delta^{\mathbb{Q}}+\Gamma_Q).$$

Finally, by (3.32)

$$V_6 = \frac{V_9}{V_8} = e^{\eta^{\mathbb{Q}}(a^*-\log(K))}. \tag{3.34}$$

For the last component of V, i.e., V_7, we only need the event A_o. It is impossible to get a non-zero payout from the option for the stock price starting from Ke^c, if the drawdown does not occur by a Poissonian jump. Hence in this case

$$\begin{aligned} A_o^7 = \{\overline{X}_{\tau_D} \in \mathrm{d}v,\ v\in(\log(K)+c,\infty); D_{\tau_D-}\in\mathrm{d}y,\ y\in[0,c)\,; \\ D_{\tau_D}-c\in\mathrm{d}h,\ h\in(v-c-\log(K),\infty)\}. \end{aligned} \tag{3.35}$$

and then

$$V_7 = \mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\right] = K\mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\mathbb{I}\{A_o^7\}\right] - Ke^c\mathbb{E}^{\mathbb{P}}_{\log(K)+c}\left[\mathbb{I}\{A_o^7\}\right].$$

Furthermore, note that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\mathbb{I}\{A_o^7\}\right] &= \int\limits_{\log(K)+c}^{\infty}\int\limits_{v-c-\log(K)}^{\infty}\int\limits_0^c F^{\mathbb{Q}}(v-(\log(K)+c))R(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h)\,\mathrm{d}v \\
&= \int\limits_{\log(K)+c}^{\infty}\int\limits_{v-c-\log(K)}^{\infty}\int\limits_0^c \eta^{\mathbb{Q}}e^{-(v-(\log(K)+c))\eta^{\mathbb{Q}}}\left[\frac{W^{(r)\prime}(y)}{\eta^{\mathbb{Q}}} - W^{(r)}(y)\right]\lambda\rho e^{\rho(y-c-h)}\,\mathrm{d}y\,\mathrm{d}h\,\mathrm{d}v \\
&= \int\limits_{\log(K)+c}^{\infty} \eta^{\mathbb{Q}}e^{-(v-(\log(K)+c))\eta^{\mathbb{Q}}} \int\limits_{v-c-\log(K)}^{\infty} \rho e^{-\rho h}\,\mathrm{d}h\,\mathrm{d}v \int\limits_0^c \lambda e^{\rho(y-c)}\left[\frac{W^{(r)\prime}(y)}{\eta^{\mathbb{Q}}} - W^{(r)}(y)\right]\mathrm{d}y \\
&= \Gamma_Q \int\limits_{\log(K)+c}^{\infty} \eta^{\mathbb{Q}}e^{-(v-(\log(K)+c))\eta^{\mathbb{Q}}}e^{\rho(\log(K)+c-v)}\,\mathrm{d}v \\
&= \Gamma_Q e^{\left(\rho+\eta^{\mathbb{Q}}\right)(\log(K)+c)}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}\left[-e^{-v\left(\eta^{\mathbb{Q}}+\rho\right)}\right]_{\log(K)+c}^{\infty} \\
&= \frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}e^{\left(\rho+\eta^{\mathbb{Q}}\right)(\log(K)+c)-\left(\rho+\eta^{\mathbb{Q}}\right)(\log(K)+c)} = \frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}.
\end{aligned}$$

Similarly:

$$\mathbb{E}^{\mathbb{P}}_{\log(K)+c}\left[e^{-r\tau_D}\mathbb{I}\{A_o^7\}\right] = \frac{\eta^{\mathbb{P}}\Gamma_{\mathbb{P}}}{\eta^{\mathbb{P}}+\tilde{\rho}} = \frac{\left(\eta^{\mathbb{Q}}-1\right)\frac{\rho e^{-c}}{\rho+1}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1+\rho+1} = \frac{\rho}{\rho+1}\frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}e^{-c}.$$

Finally:

$$V_7 = K\left(\frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho} - \frac{\rho}{\rho+1}\frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}\right) = \frac{K\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\left(\eta^{\mathbb{Q}}+\rho\right)(\rho+1)}. \tag{3.36}$$

This completes the derivation of value function V for $\overline{x} < a^* + c$.

Let us now consider the case where $a^* + c \le \overline{x} < \log(K) + c$. In this case we have

$$\begin{aligned}\hat{V}(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau\wedge\tau_D}\left(K - e^{X_{\tau\wedge\tau_D}}\right)^+\right] = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^-_{\overline{x}-c}}\left(K - e^{X_{\tau^-_{\overline{x}-c}}}\right)\mathbb{I}\{\tau^-_{\overline{x}-c} < \tau^+_{\overline{x}}\}\right] \\ &+ \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}\{\tau^-_{\overline{x}-c} > \tau^+_{\overline{x}}\}\right]\left(\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau_D}\left(K - e^{X_{\tau_D}}\right)\mathbb{I}\{\tau_D < \tau^+_{\log(K)+c}\}\right]\right. \\ &\left.+\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau^+_{\log(K)+c}}\mathbb{I}\{\tau_{\log(K+c)^+<\tau_D}\}\right]\mathbb{E}^{\mathbb{Q}}_{\log(K)+c}\left[e^{-r\tau_D}\left(K - e^{X_{\tau_D}}\right)^+\right]\right) \\ &= V_{10}(x,\overline{x}) + V_{11}(x,\overline{x})(V_{12}(\overline{x}) + V_{13}(\overline{x})V_7).\end{aligned}$$

Observe that V_7 appears in both cases when $\overline{x}$ is smaller or greater than $a^* + c$.

Calculations of V_{10} are similar to calculations of V_3:

$$\begin{aligned}V_{10}(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^-_{\overline{x}-c}}\left(K - e^{X_{\tau^-_{\overline{x}-c}}}\right)\mathbb{I}\{\tau^-_{\overline{x}-c} < \tau^+_{\overline{x}}\}\right] = K\mathbb{E}^{\mathbb{Q}}_{x}\left[e^{-r\tau^-_{\overline{x}-c}}\mathbb{I}\{\tau^-_{\overline{x}-c} < \tau^+_{\overline{x}}\}\right] \\ &- e^{x}\mathbb{E}^{\mathbb{P}}_{x}\left[\mathbb{I}\{\tau^-_{\overline{x}-c} < \tau^+_{\overline{x}}\}\right] = K\left(Z^{(r)}(x+c-\overline{x}) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(x+c-\overline{x})\right) \\ &- e^{\overline{x}}\left(Z^{\mathbb{P}}(x+c-\overline{x}) - \frac{Z^{\mathbb{P}}(c)}{W^{\mathbb{P}}(c)}W^{\mathbb{P}}(x+c-\overline{x})\right) \\ &= K\left(Z^{(r)}(x+c-\overline{x}) - \frac{Z^{(r)}(c)}{W^{(r)}(c)}W^{(r)}(x+c-\overline{x})\right) - \left(e^{x} - \frac{e^{\overline{x}}W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)}\right).\end{aligned} \tag{3.37}$$

Now, similarly to V_2 and V_4, we have:

$$V_{11}(x,\overline{x}) = \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}\{\tau^-_{\overline{x}-c} > \tau^+_{\overline{x}}\}\right] = \frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)}. \tag{3.38}$$

The term V_{12} can be identified in the similar way as it was done for V_5. We introduce

$$\begin{aligned}A^{12}_o = \{&\overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in (\overline{x}, \log(K)+c); D_{\tau_D-} \in \mathrm{d}y,\ y \in [0,c)\,; \\ &D_{\tau_D} - c \in \mathrm{d}h,\ h \in (0,\infty)\}\end{aligned}$$

and

$$A^{12}_c = \{\underline{X}_{\tau_D} \ge \overline{x} - c;\ \overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in (\overline{x}, \log(K)+c);\ D_{\tau_D} = c\}.$$

Then

$$
\begin{aligned}
V_{12}(\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)\mathbb{I}\{\tau_D<\tau^+_{\log(K)+c}\}\right]\\
&= K\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D<\tau^+_{\log(K)+c}\}\right]-e^{\overline{x}}\mathbb{E}^{\mathbb{P}}_{\overline{x}}\left[\mathbb{I}\{\tau_D<\tau^+_{\log(K)+c}\}\right]\\
&= K\int\limits_{\overline{x}}^{\log(K)+c} F^{\mathbb{Q}}(v-\overline{x})\,\mathrm{d}v\left(\Delta^{\mathbb{Q}}+\Gamma_Q\right)-e^{\overline{x}}\int\limits_{\overline{x}}^{\log(K)+c} F^{\mathbb{P}}(v-\overline{x})\,\mathrm{d}v\left(\Delta^{\mathbb{P}}+\Gamma_P\right)\\
&= K\left(1-e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\right)-e^{\overline{x}}\left[\left(1-e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{P}}}\right)\left(\Delta^{\mathbb{P}}+\Gamma_{\mathbb{P}}\right)\right]\\
&= K\left(1-e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\right)\\
&- e^{\overline{x}}\left[\left(1-e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}Ke^{c-\overline{x}}\right)\frac{\eta^{\mathbb{Q}}e^{-c}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_{\mathbb{Q}}\right)\right]\\
&= K\left(1-e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\right)\\
&- \frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left[\left(e^{\overline{x}-c}-Ke^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_{\mathbb{Q}}\right)\right]\\
&= K\left(1+\frac{e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}}{\eta^{\mathbb{Q}}-1}\right)\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\right) \qquad (3.39)\\
&- \frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left[e^{\overline{x}-c}\Delta^{\mathbb{Q}}+\frac{\Gamma_Q}{\rho+1}\left(\rho e^{\overline{x}-c}+Ke^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\right]. \qquad (3.40)
\end{aligned}
$$

Following the analysis of V_6 changing the starting point of X_t from $\overline{x}$ to a^*+c we can write

$$
V_{13}=\mathbb{E}^{\mathbb{Q}}_{\overline{x}}\left[e^{-r\tau^+_{\log(K)+c}}\mathbb{I}\{\tau_{\log(K+c)^+<\tau_D}\}\right]=e^{-\eta^{\mathbb{Q}}(\log(K)+c-\overline{x})}. \tag{3.41}
$$

Now let us consider the case when $\overline{x}\geq\log(K)+c$. Then

$$
\begin{aligned}
\hat{V}(x,\overline{x}) &= \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau\wedge\tau_D}\left(K-e^{X_{\tau\wedge\tau_D}}\right)^+\right]=\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\mathbb{I}\{\tau_D<\tau^+_{\overline{x}}\}\right] \qquad (3.42)\\
&+ \mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau^+_{\overline{x}}}\mathbb{I}\{\tau^+_{\overline{x}}<\tau^-_{\overline{x}+c}\}\right]\mathbb{E}_{\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\right]=V_{14}(x,\overline{x})+V_{15}(x,\overline{x})V_{16}(\overline{x}).
\end{aligned}
$$

Observe, that in this case

$$
\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau_D}\mathbb{I}\{\tau_D<\tau^+_{\overline{x}}\}\mathbb{I}\{K>e^{X_{\tau_D}}\}\right]=\mathbb{E}^{\mathbb{Q}}_{x}\left[e^{-r\tau^-_{\overline{x}-c}}\mathbb{I}\{\tau^-_{\overline{x}-c}<\tau^+_{\overline{x}}\}\mathbb{I}\{X_{\tau^-_{\overline{x}-c}}<\log(K)\}\right].
$$

To calculate this expected value, we will use the following Gerber-Shiu measure

$$
K^{(r)}(a^*,x,\mathrm{d}y,\mathrm{d}z)=\mathbb{E}_x\left[e^{-r\tau_0^-};\ -X_{\tau_0^-}\in\mathrm{d}y;\ X_{\tau_0^- -}\in\mathrm{d}z;\tau_0^-<\tau^+_{a^*}\right]
$$

for $x,z\in[0,a^*]$ and $y\geq 0$. By [20, Thm. 5.5] we have

$$
K^{(r)}(a^*,x,\mathrm{d}y,\mathrm{d}z)=\frac{W^{(r)}(x)W^{(r)}(a^*-z)-W^{(r)}(a^*)W^{(r)}(x-z)}{W^{(r)}(a^*)}\Lambda(z+\mathrm{d}y)\,\mathrm{d}z.
$$

Above, the Gerber-Shiu measure is associated with the first downward crossing of 0. The proof of [20, Thm. 5.5] is given for the classical Cramer-Lundberg risk process, but it remains true without any changes for our Lévy process X_t given in (1.3). We are, on the other hand, interested in the first time when the process starting from x hits the level $\overline{x}-c$. Using the stationarity and independence of increments of Lévy processes, we can shift our starting point from x to $x+c-\overline{x}$ and thus we can search for the first time then X_t becomes negative. Additionally, to make the condition $\{X_{\tau^-_{\overline{x}-c}} < \log(K)\}$ true, we allow the undershoot y of our shifted process to take values from set $(\overline{x}-\log(K)-c,\ \infty)$. Thus, we get that

$$\mathbb{E}^{\mathbb{Q}}_x\left[e^{-r\tau^-_{\overline{x}-c}}\mathbb{I}\{\tau^-_{\overline{x}-c}<\tau^+_{\overline{x}}\}\mathbb{I}\{X_{\tau^-_{\overline{x}-c}}<\log(K)\}\right]$$
$$=\int\limits_{(\overline{x}-\log(K)-c}^{\infty}\int\limits_{0}^{c}\lambda\rho e^{-\rho(y+z)}\frac{W^{(r)}(x+c-\overline{x})W^{(r)}(c-z)-W^{(r)}(c)W^{(r)}(x+c-\overline{x}-z)}{W^{(r)}(c)}\,\mathrm{d}z\,\mathrm{d}y.$$

The calculation of this double integral can be separated into three smaller single integrals. Firstly,

$$\int\limits_{(\overline{x}-\log(K)-c}^{\infty}\rho e^{-\rho y}\,\mathrm{d}y=e^{\rho(\log(K)+c-\overline{x})}. \tag{3.43}$$

Next,

$$\int\limits_0^c e^{-\rho z}W^{(r)}(c-z)\,\mathrm{d}z=\sum_{i=1}^3 C_i\int\limits_0^c e^{\gamma_i c-z(\gamma_i+\rho)}\,\mathrm{d}z=\sum_{i=1}^3 C_ie^{\gamma_i c}\int\limits_0^c e^{-z(\gamma_i+\rho)}\,\mathrm{d}z$$
$$=\sum_{i=1}^3 C_ie^{\gamma_i c}\frac{1-e^{-c(\gamma_i+\rho)}}{\gamma_i+\rho}.$$

Analogically, we have

$$\int\limits_0^c e^{-\rho z}W^{(r)}(x+c-\overline{x}-z)\,\mathrm{d}z=\int\limits_0^{x+c-\overline{x}} e^{-\rho z}W^{(r)}(x+c-\overline{x}-z)\,\mathrm{d}z$$
$$=\sum_{i=1}^3 C_ie^{\gamma_i c}\frac{1-e^{-(x+c-\overline{x})(\gamma_i+\rho)}}{\gamma_i+\rho}$$

because $W^{(r)}(x)=0$ for $x<0$. Combining everything together, we get

$$\mathbb{E}^{\mathbb{Q}}_x\left[e^{-r\tau^-_{\overline{x}-c}}\mathbb{I}\{\tau^-_{\overline{x}-c}<\tau^+_{\overline{x}}\}\mathbb{I}\{X_{\tau^-_{\overline{x}-c}}<\log(K)\}\right]=\lambda e^{\rho(\log(K)+c-\overline{x})}$$
$$\times\sum_{i=1}^3 C_ie^{\gamma_i c}\left[\frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)}\frac{1-e^{-c(\gamma_i+\rho)}}{\gamma_i+\rho}-e^{\gamma_i(x-\overline{x})}\frac{1-e^{-(x+c-\overline{x})(\gamma_i+\rho)}}{\gamma_i+\rho}\right].$$

In order to calculate $\mathbb{E}^{\mathbb{Q}}_{x,\overline{x}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\mathbb{I}\{\tau_D<\tau^+_{\overline{x}}\}\right]$ we also need to consider the following equality

$$\begin{aligned}
&\mathbb{E}_x^{\mathbb{Q}}\left[e^{X_{\tau_{\overline{x}-c}^-}-r\tau_{\overline{x}-c}^-}\mathbb{I}\{\tau_{\overline{x}-c}^- < \tau_{\overline{x}}^+\}\mathbb{I}\{X_{\tau_{\overline{x}-c}^-} < \log(K)\}\right] = e^x\mathbb{E}_x^{\mathbb{P}}\left[\mathbb{I}\{\tau_{\overline{x}-c}^- < \tau_{\overline{x}}^+\}\mathbb{I}\{X_{\tau_{\overline{x}-c}^-} < \log(K)\}\right] \\
&= e^x\tilde{\lambda}e^{\tilde{\rho}(\log(K)+c-\overline{x})}\sum_{i=1}^{3}\tilde{C}_i e^{\tilde{\gamma}_i c}\left[\frac{W^{\mathbb{P}}(x+c-\overline{x})}{W^{\mathbb{P}}(c)}\frac{1-e^{-c(\tilde{\gamma}_i+\tilde{\rho})}}{\tilde{\gamma}_i+\tilde{\rho}} - e^{\tilde{\gamma}_i(x-\overline{x})}\frac{1-e^{-(x+c-\overline{x})(\tilde{\gamma}_i+\tilde{\rho})}}{\tilde{\gamma}_i+\tilde{\rho}}\right] \\
&= \frac{K\rho}{\rho+1}\lambda e^{\rho(\log(K)+c-\overline{x})}\sum_{i=1}^{3}C_i e^{\gamma_i c}\left[\frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)}\frac{1-e^{-c(\gamma_i+\rho)}}{\gamma_i+\rho} - e^{\gamma_i(x-\overline{x})}\frac{1-e^{-(x+c-\overline{x})(\gamma_i+\rho)}}{\gamma_i+\rho}\right].
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
V_{14}(x,\overline{x}) &= \mathbb{E}_{x,\overline{x}}^{\mathbb{Q}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\mathbb{I}\{\tau_D < \tau_{\overline{x}}^+\}\right] \\
&= K\mathbb{E}_x^{\mathbb{Q}}\left[e^{-r\tau_{\overline{x}-c}^-}\mathbb{I}\{\tau_{\overline{x}-c}^- < \tau_{\overline{x}}^+\}\mathbb{I}\{X_{\tau_{\overline{x}-c}^-} < \log(K)\}\right] \\
&- e^x\mathbb{E}_x^{\mathbb{P}}\left[\mathbb{I}\{\tau_{\overline{x}-c}^- < \tau_{\overline{x}}^+\}\mathbb{I}\{X_{\tau_{\overline{x}-c}^-} < \log(K)\}\right] = \frac{K}{\rho+1}\lambda e^{\rho(\log(K)+c-\overline{x})} \\
&\times\sum_{i=1}^{3}C_i e^{\gamma_i c}\left[\frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)}\frac{1-e^{-c(\gamma_i+\rho)}}{\gamma_i+\rho} - e^{\gamma_i(x-\overline{x})}\frac{1-e^{-(x+c-\overline{x})(\gamma_i+\rho)}}{\gamma_i+\rho}\right].
\end{aligned} \tag{3.44}$$

Moving on to the penultimate expected value in (3.42), observe that

$$V_{15}(x,\overline{x}) = \mathbb{E}_{x,\overline{x}}^{\mathbb{Q}}\left[e^{-r\tau_{\overline{x}}^+}\mathbb{I}\{\tau_{\overline{x}}^+ < \tau_{\overline{x}+c}^-\}\right] = \frac{W^{(r)}(x+c-\overline{x})}{W^{(r)}(c)} \tag{3.45}$$

as in (3.24) and (3.25).

Now, similarly to (3.35), we consider the event

$$\begin{aligned}
A_o^{16} = \{\overline{X}_{\tau_D} \in \mathrm{d}v,\ v \in (\overline{x},\infty); D_{\tau_D-} \in \mathrm{d}y,\ y \in [0,c)\,; \\
D_{\tau_D} - c \in \mathrm{d}h,\ h \in (v-c-\log(K),\infty)\}.
\end{aligned}$$

Thus, we can split the last expected value into

$$V_{16}(\overline{x}) = \mathbb{E}_{\overline{x}}^{\mathbb{Q}}\left[e^{-r\tau_D}\left(K-e^{X_{\tau_D}}\right)^+\right] = K\mathbb{E}_{\overline{x}}^{\mathbb{Q}}\left[e^{-r\tau_D}\mathbb{I}\{A_o^{16}\}\right] - e^{\overline{x}}\mathbb{E}_{\overline{x}}^{\mathbb{P}}\left[\mathbb{I}\{A_o^{16}\}\right].$$

We start by calculating the expected value under $\mathbb{Q}$

$$\mathbb{E}_{\overline{x}}^{\mathbb{Q}}\left[e^{-r\tau_D}\mathbb{I}\{A_o^{16}\}\right] = \int\limits_{\overline{x}}^{\infty}\int\limits_{v-c-\log(K)}^{\infty}\int\limits_{0}^{c} F^{\mathbb{Q}}(v-(\log(K)+c))R(r,\mathrm{d}y)\Lambda(y-c-\mathrm{d}h)\,\mathrm{d}v$$

$$= \int\limits_{\overline{x}}^{\infty}\int\limits_{v-c-\log(K)}^{\infty}\int\limits_{0}^{c}\eta^{\mathbb{Q}}e^{-(v-\overline{x})\eta^{\mathbb{Q}}}\left[\frac{W^{(r)'}(y)}{\eta^{\mathbb{Q}}} - W^{(r)}(y)\right]\lambda\rho e^{\rho(y-c-h)}\,\mathrm{d}y\,\mathrm{d}h\,\mathrm{d}v$$

$$= \int\limits_{\overline{x}}^{\infty}\eta^{\mathbb{Q}}e^{-(v-\overline{x})\eta^{\mathbb{Q}}}\int\limits_{v-c-\log(K)}^{\infty}\rho e^{-\rho h}\,\mathrm{d}h\,\mathrm{d}v\int\limits_{0}^{c}\lambda e^{\rho(y-c)}\left[\frac{W^{(r)'}(y)}{\eta^{\mathbb{Q}}} - W^{(r)}(y)\right]\mathrm{d}y$$

$$= \Gamma_Q\int\limits_{\overline{x}}^{\infty}\eta^{\mathbb{Q}}e^{-(v-\overline{x})\eta^{\mathbb{Q}}}e^{\rho(\log(K)+c-v)}\,\mathrm{d}v = \Gamma_Q e^{\rho(\log(K)+c)+\eta^{\mathbb{Q}}\overline{x}}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}\left[-e^{-v\left(\eta^{\mathbb{Q}}+\rho\right)}\right]_{\overline{x}}^{\infty}$$

$$= \frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}e^{\rho(\log(K)+c-\overline{x})}.$$

Redoing the calculation on the $\mathbb{P}$ measure and combining the results together, we get

$$V_{16}(\overline{x}) = \frac{K}{\rho+1}\frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}+\rho}e^{\rho(\log(K)+c-\overline{x})}. \tag{3.46}$$

□

3.2.2 Optimal stopping threshold a^*

To find the optimal level a we choose a^* satisfying condition (3.12), that is

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x})\big|_{x=a^*} = -e^{a^*}. \tag{3.47}$$

Theorem 6. *There exists a unique optimal level a^* satisfying* (3.47) *and it solves the following equation*

$$e^{a^*+c} - rK\frac{W^{(r)}(c)^2}{W^{(r)'}(c)} - \frac{\eta^{\mathbb{Q}}e^{a^*}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right) + \frac{Ke^{\eta^{\mathbb{Q}}(a^*-\log(K))}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right) = 0, \tag{3.48}$$

where $\eta^{\mathbb{Q}}$, $\Delta^{\mathbb{Q}}$, $\Gamma_{\mathbb{Q}}$ are given in (3.18), (3.19), (3.30), respectively.

Proof. We start the proof from observing that

$$\frac{\partial}{\partial x}\hat{V}(x,\overline{x}) = \frac{\partial}{\partial x}V_1(x,\overline{x}) + (V_3(\overline{x}) + V_4(\overline{x})(V_5 + V_6V_7))\frac{\partial}{\partial x}V_2(x,\overline{x}).$$

Furthermore,

$$\frac{\partial}{\partial x}V_1(x,\overline{x}) = K\left(rW^{(r)}(x-a^*) - Z^{(r)}(\overline{x}-a^*)\frac{W^{(r)'}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right) - \left(e^{x} - \frac{e^{\overline{x}}W^{(r)'}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}\right)$$

and

$$\frac{\partial}{\partial x}V_2(x,\overline{x}) = \frac{W^{(r)\prime}(x-a^*)}{W^{(r)}(\overline{x}-a^*)}.$$

Note that $W^{(r)}(0)=0$, $W^{(r)\prime}(0)=\frac{2}{\sigma^2}$, $Z^{(r)}(0)=1$. Now, by setting $x=a^*$, we get

$$\frac{\partial}{\partial x}V_1(x,\overline{x})\big|_{x=a^*} = \frac{-2KZ^{(r)}(\overline{x}-a^*)}{\sigma^2 W^{(r)}(\overline{x}-a^*)} - e^{a^*} + \frac{2e^{\overline{x}}}{\sigma^2 W^{(r)}(\overline{x}-a^*)}$$

and that

$$\frac{\partial}{\partial x}V_2(x,\overline{x})\big|_{x=a^*} = \frac{2}{\sigma^2 W^{(r)}(\overline{x}-a^*)}.$$

Combining everything together we have

$$\begin{aligned}\frac{\partial}{\partial x}V(x,\overline{x})\big|_{x=a^*} &= \frac{-2KZ^{(r)}(\overline{x}-a^*)}{\sigma^2 W^{(r)}(\overline{x}-a^*)} - e^{a^*} + \frac{2e^{\overline{x}}}{\sigma^2 W^{(r)}(\overline{x}-a^*)} + \frac{2KZ^{(r)}(\overline{x}-a^*)}{\sigma^2 W^{(r)}(\overline{x}-a^*)} \\ &- \frac{2KZ^{(r)}(c)}{\sigma^2 W^{(r)}(c)} - \frac{2e^{\overline{x}}}{\sigma^2 W^{(r)}(\overline{x}-a^*)} + \frac{2e^{a^*+c}}{\sigma^2 W^{(r)}(c)} + \frac{2}{\sigma^2 W^{(r)}(c)}(V_5+V_6V_7) \\ &= -e^{a^*} + \frac{2e^{a^*+c}}{\sigma^2 W^{(r)}(c)}\left(e^{a^*+c} - KZ^{(r)}(c) + V_5 + V_6V_7\right).\end{aligned}$$

In order to fulfill condition (3.47) (and (3.12)) we search for a^* satisfying

$$e^{a^*+c} - KZ^{(r)}(c) + V_5 + V_6V_7 = 0. \tag{3.49}$$

Now, using (3.33), we can rewrite the left-hand side of above equation as follows:

$$\begin{aligned}&e^{a^*+c} - KZ^{(r)}(c) + V_5 + V_6V_7 = e^{a^*+c} - KZ^{(r)}(c) + Ke^{(a^*-\log(K))\eta^{\mathbb{Q}}}\frac{\Delta^{\mathbb{Q}} + \Gamma_{\mathbb{Q}}\frac{\rho+1-\eta^{\mathbb{Q}}}{\rho+1}}{\eta^{\mathbb{Q}}-1} \\ &- e^{a^*}\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho}{\rho+1}\Gamma_{\mathbb{Q}}\right) + K\left(Z^{(r)}(c) - r\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)}\right) \\ &+ \frac{K\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{(\eta^{\mathbb{Q}}+\rho)(\rho+1)}e^{\eta^{\mathbb{Q}}(a^*-\log(K))} = a^{a^*+c} - rK\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)} - \frac{\eta^{\mathbb{Q}}e^{a^*}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right) \\ &+ Ke^{\eta^{\mathbb{Q}}(a^*-\log(K))}\left(\frac{\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1} - \frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\frac{\Gamma_{\mathbb{Q}}}{\rho+1} + \frac{\Gamma_{\mathbb{Q}}\eta^{\mathbb{Q}}}{(\eta^{\mathbb{Q}}+\rho)(\rho+1)}\right) \\ &= e^{a^*+c} - rK\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)} - \frac{\eta^{\mathbb{Q}}e^{a^*}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right) + \frac{Ke^{\eta^{\mathbb{Q}}(a^*-\log(K))}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right).\end{aligned}$$

This gives the equation (3.48). Using the intermediate value theorem we can prove that the solution of equation (3.49) always exists. Indeed, observe that by taking some a instead of a^* above and then taking $a \to -\infty$, the last expression above becomes $rK\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)}$, which is smaller than 0. On the other hand, with $a = \log(K)$, we get

$$e^{a+c} - KZ^{(r)}(c) + V_5 + V_6V_7\big|_{a=\log(K)} = Ke^c - rK\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)} - \frac{K\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right)$$
$$+ \frac{K}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}} + \frac{\rho\Gamma_{\mathbb{Q}}}{\rho+1}\right) = K\left[e^c - r\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)} - \Delta^{\mathbb{Q}}\right.$$
$$\left.+ \Gamma_{\mathbb{Q}}\left(\frac{\rho}{(\eta^{\mathbb{Q}}-1)(\eta^{\mathbb{Q}}+\rho)} - \frac{\rho\eta^{\mathbb{Q}}}{(\eta^{\mathbb{Q}}-1)(\rho+1)}\right)\right] = K\left[e^c - r\frac{W^{(r)}(c)^2}{W^{(r)\prime}(c)} - \Delta^{\mathbb{Q}} - \Gamma_{\mathbb{Q}}\right.$$
$$\left.+ \frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{(\eta^{\mathbb{Q}}+\rho)(\rho+1)}\right] = K\left[e^c - Z^{(r)}(c) + \frac{\eta^{\mathbb{Q}}\Gamma_{\mathbb{Q}}}{(\eta^{\mathbb{Q}}+\rho)(\rho+1)}\right].$$

Note that $Z^{(r)}(0) = 1$ for the function $Z^{(r)}(x)$ defined in (1.7) and hence $\sum_{i=1}^{3}\frac{rC_i}{\gamma_1} = 1$. The function $Z^{(r)}(x)$ is then a weighted average of three exponential functions with $C_2, C_3, \gamma_2, \gamma_3 < 0$. Therefore, using fact that $\gamma_1 = 1$ (see (1.9)), we have $e^c - Z^{(r)}(c) > 0$ and hence above expression is strictly positive for $a^* = \log(K)$. Thus indeed solution a^* always exists. □

3.2.3 Proof of the optimal stopping boundary form

In this section, we will prove Theorem 4. According to Remark 5, it is sufficient to show that $\hat{V}(x,\overline{x})$ is in the domain of the infinitesimal generator $\mathcal{L}$, that is, that $\hat{V}(x,\overline{x}) \in \mathcal{C}_0^2(\mathbb{R})$ and that the boundary condition (3.6) is satisfied, that is, that

$$\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x}) = 0 \quad \text{for} \quad x = \overline{x}.$$

Furthermore, we have to verify that all the conditions given in the verification Lemma 3 are satisfied.

The fact that $\hat{V}(x,\overline{x}) \in \mathcal{C}^2(\mathbb{R})$ follows from Theorem 5 and from the form of the scale functions $W^{(r)}(x)$ and $Z^{(r)}(x)$ given in (1.8) and (1.7). The fact that $\hat{V}(x,\overline{x})$ disappear as $x, \overline{x}$ tend to infinity follows directly from the definition of $\hat{V}(x,\overline{x})$. We will now verify that the condition (3.6) holds true.

We start from the case $\overline{x} < a^* + c$ for which we have

$$\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x}) = \frac{\partial}{\partial\overline{x}}V_1(x,\overline{x}) + V_3(\overline{x})\frac{\partial}{\partial\overline{x}}V_2(x,\overline{x}) + V_4(\overline{x})(V_5+V_6V_7)\frac{\partial}{\partial\overline{x}}V_2(x,\overline{x})$$
$$+ V_2(x,\overline{x})\frac{\partial}{\partial\overline{x}}V_3(\overline{x}) + V_2(x,\overline{x})(V_5+V_6V_7)\frac{\partial}{\partial\overline{x}}V_4(\overline{x}).$$

Note that

$$\frac{\partial}{\partial\overline{x}}V_2(x,\overline{x}) = \frac{-W^{(r)\prime}(\overline{x}-a^*)W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)^2}$$

and that

$$\frac{\partial}{\partial\overline{x}}V_4(\overline{x}) = \frac{-W^{(r)\prime}(\overline{x}-a^*)}{W^{(r)}(c)}.$$

Therefore

$$V_4(\overline{x})\frac{\partial}{\partial\overline{x}}V_2(x,\overline{x}) + V_2(x,\overline{x})\frac{\partial}{\partial\overline{x}}V_4(\overline{x}) = 0.$$

Additionally, we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{x}} V_1(x,\overline{x}) &= K W^{(r)}(x-a^*) \frac{Z^{(r)\prime}(\overline{x}-a^*)W^{(r)}(\overline{x}-a^*) - W^{(r)\prime}(\overline{x}-a^*)Z^{(r)}(\overline{x}-a^*)}{W^{(r)}(\overline{x}-a^*)^2} \\
&+ e^{\overline{x}} \frac{W^{(r)}(x-a^*)W^{(r)}(\overline{x}-a^*) - W^{(r)\prime}(\overline{x}-a^*)W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)^2} \\
&= \frac{1}{W^{(r)}(\overline{x}-a^*)^2} \left[W^{(r)}(\overline{x}-a^*)W^{(r)}(x-a^*) \left(e^{\overline{x}} - rKW^{(r)}(\overline{x}-a^*) \right) \right. \\
&\left. + W^{(r)}(x-a^*)W^{(r)\prime}(\overline{x}-a^*) \left(KZ^{(r)}(\overline{x}-a^*) - e^{\overline{x}} \right) \right]
\end{aligned}$$

and that

$$\frac{\partial}{\partial \overline{x}} V_3(\overline{x}) = rKW^{(r)}(\overline{x}-a^*) - K\frac{Z^{(r)}(c)}{W^{(r)}(c)} W^{(r)\prime}(\overline{x}-a^*) - e^{\overline{x}} + \frac{a^{a^*+c}}{W^{(r)}(c)} W^{(r)\prime}(\overline{x}-a^*).$$

Finally, putting all terms together,

$$\begin{aligned}
\frac{\partial}{\partial \overline{x}} \hat{V}(x,\overline{x}) &= \frac{\partial}{\partial \overline{x}} V_1(x,\overline{x}) + V_3(\overline{x}) \frac{\partial}{\partial \overline{x}} V_2(x,\overline{x}) + V_2(x,\overline{x}) \frac{\partial}{\partial \overline{x}} V_3(\overline{x}) = \frac{-W^{(r)\prime}(\overline{x}-a^*)W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)^2} \\
&\times \left[K \left(Z^{(r)}(\overline{x}-a^*) - \frac{Z^{(r)}(c)}{W^{(r)}(c)} W^{(r)}(\overline{x}-a^*) \right) - e^{\overline{x}} + \frac{a^{a^*+c}W^{(r)}(\overline{x}-a^*)}{W^{(r)}(c)} \right] \\
&+ \frac{W^{(r)}(x-a^*)}{W^{(r)}(\overline{x}-a^*)} \left[rKW^{(r)}(\overline{x}-a^*) - K\frac{Z^{(r)}(c)}{W^{(r)}(c)} W^{(r)\prime}(\overline{x}-a^*) - e^{\overline{x}} + \frac{a^{a^*+c}}{W^{(r)}(c)} W^{(r)\prime}(\overline{x}-a^*) \right] \\
&+ \frac{1}{W^{(r)}(\overline{x}-a^*)^2} \left[W^{(r)}(\overline{x}-a^*)W^{(r)}(x-a^*) \left(e^{\overline{x}} - rKW^{(r)}(\overline{x}-a^*) \right) \right. \\
&\left. + W^{(r)}(x-a^*)W^{(r)\prime}(\overline{x}-a^*) \left(KZ^{(r)}(\overline{x}-a^*) - e^{\overline{x}} \right) \right].
\end{aligned}$$

When we set $x = \overline{x}$, all terms cancel out and the condition (3.6) is fulfilled.

We consider now the case of $a^* + c < \overline{x} < \log(K) + c$. In this case, we have

$$\begin{aligned}
\frac{\partial}{\partial \overline{x}} \hat{V}(x,\overline{x}) &= \frac{\partial}{\partial \overline{x}} V_{10}(x,\overline{x}) + (V_{12}(\overline{x}) + V_{13}(\overline{x})V_7) \frac{\partial}{\partial \overline{x}} V_{11}(x,\overline{x}) \\
&+ V_{11}(x,\overline{x}) \left(\frac{\partial}{\partial \overline{x}} V_{12}(\overline{x}) + \frac{\partial}{\partial \overline{x}} V_{13}(\overline{x}) V_7 \right).
\end{aligned}$$

Further,

$$\frac{\partial}{\partial \overline{x}} V_{10}(x,\overline{x})\big|_{x=\overline{x}} = -rKW^{(r)}(c) + \eta^{\mathbb{Q}} \left(KZ^{(r)}(c) - e^{\overline{x}} \right) + e^{\overline{x}}$$

and

$$\frac{\partial}{\partial \overline{x}} V_{11}(x,\overline{x})\big|_{x=\overline{x}} = -\frac{W^{(r)\prime}(c)}{W^{(r)}(c)} = -\eta^{\mathbb{Q}}, \quad V_{11}(\overline{x},\overline{x}) = 1.$$

Moreover, we have

$$\frac{\partial}{\partial \overline{x}} V_{13}(\overline{x}) = \eta^{\mathbb{Q}} e^{-\eta^{\mathbb{Q}}(\log(K)+c-\overline{x})} = \eta^{\mathbb{Q}} V_{13}(\overline{x})$$

and

$$V_7V_{13}(\overline{x})\frac{\partial}{\partial\overline{x}}V_{11}(x,\overline{x})\big|_{x=\overline{x}}-V_7V_{11}(\overline{x},\overline{x})\frac{\partial}{\partial\overline{x}}V_{13}(\overline{x})=0.$$

Furthermore,

$$\begin{aligned}\frac{\partial}{\partial\overline{x}}V_{12}(\overline{x})&=K\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\right)e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\\&\quad-\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left[e^{\overline{x}-c}\Delta^{\mathbb{Q}}+\frac{\Gamma_Q}{\rho+1}\left(\rho e^{\overline{x}-c}+K\eta^{\mathbb{Q}}e^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\right)\right]\\&=\frac{\eta^{\mathbb{Q}}}{\eta^{\mathbb{Q}}-1}\left[Ke^{-(\log(K)+c-\overline{x})\eta^{\mathbb{Q}}}\left(\Delta^{\mathbb{Q}}+\Gamma_{\mathbb{Q}}\left(1-\frac{\eta^{\mathbb{Q}}}{\rho+1}\right)\right)-e^{\overline{x}-c}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_{\mathbb{Q}}\right)\right].\end{aligned}$$

This gives

$$\begin{aligned}\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})&=\frac{\partial}{\partial\overline{x}}V_{10}(x,\overline{x})\big|_{x=\overline{x}}+V_{12}(\overline{x})\frac{\partial}{\partial\overline{x}}V_{11}(x,\overline{x})\big|_{x=\overline{x}}+V_{11}(\overline{x},\overline{x})\frac{\partial}{\partial\overline{x}}V_{12}(\overline{x})\\&=-rKW^{(r)}(c)+e^{\overline{x}}+\eta^{\mathbb{Q}}\left[KZ^{(r)}(c)-e^{\overline{x}}-K\left(\Delta^{\mathbb{Q}}+\Gamma_Q\right)+e^{\overline{x}-c}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_Q\right)\right]\\&=-rKW^{(r)}(c)+e^{\overline{x}}+\eta^{\mathbb{Q}}\left[K\left(Z^{(r)}(c)-Z^{(r)}(c)+\frac{rW^{(r)}(c)}{\eta^{\mathbb{Q}}}\right)-e^{\overline{x}}+e^{\overline{x}-c}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_Q\right)\right]\\&=e^{\overline{x}}\left(1-\eta^{\mathbb{Q}}\right)+\eta^{\mathbb{Q}}e^{\overline{x}-c}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_Q\right)=e^{\overline{x}}\left[1-\eta^{\mathbb{Q}}+\eta^{\mathbb{Q}}e^{-c}\left(\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\Gamma_Q\right)\right]\\&=e^{\overline{x}}\left[1-\eta^{\mathbb{Q}}+(\eta^{\mathbb{Q}}-1)\left(\frac{\eta^{\mathbb{Q}}e^{-c}}{\eta^{\mathbb{Q}}-1}\Delta^{\mathbb{Q}}+\frac{\rho}{\rho+1}\frac{\eta^{\mathbb{Q}}e^{-c}}{\eta^{\mathbb{Q}}-1}\Gamma_Q\right)\right]=e^{\overline{x}}\left[1-\eta^{\mathbb{Q}}+(\eta^{\mathbb{Q}}-1)\left(\Delta^{\mathbb{P}}+\Gamma_{\mathbb{P}}\right)\right]\\&=e^{\overline{x}}\left[1-\eta^{\mathbb{Q}}+(\eta^{\mathbb{Q}}-1)\left(\Delta^{\mathbb{P}}+\Gamma_{\mathbb{P}}\right)\right]=e^{\overline{x}}(1-\eta^{\mathbb{Q}}+\eta^{\mathbb{Q}}-1)=0.\end{aligned}$$

Thus, condition (3.6) is satisfied.

We now analyze the remaining case of $\log(K)+c<\overline{x}$. In this case, we have

$$\frac{\partial}{\partial\overline{x}}\hat{V}(x,\overline{x})=\frac{\partial}{\partial\overline{x}}V_{14}(x,\overline{x})+V_{15}(x,\overline{x})\frac{\partial}{\partial\overline{x}}V_{16}(\overline{x})+V_{16}(\overline{x})\frac{\partial}{\partial\overline{x}}V_{15}(x,\overline{x}).$$

Further,

$$\begin{aligned}\frac{\partial}{\partial\overline{x}}V_{14}(x,\overline{x})&=-\rho V_{14}(x,\overline{x})+\frac{K}{\rho+1}\lambda e^{\rho(\log(K)+c-\overline{x})}\\&\times\sum_{i=1}^{3}C_ie^{\gamma_ic}\left[\frac{-W^{(r)\prime}(x+c-\overline{x})}{W^{(r)}(c)}\frac{1-e^{-c(\gamma_i+\rho)}}{\gamma_i+\rho}-\frac{-\gamma_ie^{\gamma_i(x-\overline{x})}-\rho e^{-\gamma_ic-\rho(x+c)+\rho\overline{x}}}{\gamma_i+\rho}\right]\end{aligned}$$

Using (3.29) we get

$$\begin{aligned}\frac{\partial}{\partial\overline{x}}V_{14}(x,\overline{x})\big|_{x=\overline{x}}&=\frac{K\lambda e^{\rho(\log(K)+c-\overline{x})}}{\rho+1}\sum_{i=1}^{3}\frac{C_ie^{\gamma_ic}}{\gamma_i+\rho}\left[-\eta^{\mathbb{Q}}+\gamma_i+(\rho+\eta^{\mathbb{Q}})e^{-c(\gamma_i+\rho)}\right]\\&=\frac{K\lambda e^{\rho(\log(K)+c-\overline{x})}}{\rho+1}\left[\eta^{\mathbb{Q}}\sum_{i=1}^{3}\frac{C_ie^{\gamma_ic}}{\gamma_i+\rho}\left(\frac{\gamma_i}{\eta^{\mathbb{Q}}}-1\right)+\sum_{i=1}^{3}\frac{C_i(\rho+\eta^{\mathbb{Q}})e^{-c\rho}}{\gamma_i+\rho}\right]\\&=\frac{K\eta^{\mathbb{Q}}e^{\rho(\log(K)+c-\overline{x})}}{\rho+1}\Gamma_{\mathbb{Q}}=(\eta^{\mathbb{Q}}+\rho)V_{16}(\overline{x}).\end{aligned}$$

Additionally, we have

$$\frac{\partial}{\partial \overline{x}} V_{15}(x, \overline{x}) = \frac{-W^{(r)'}(x + c - \overline{x})}{W^{(r)}(c)}, \quad \frac{\partial}{\partial \overline{x}} V_{15}(x, \overline{x})\big|_{x=\overline{x}} = -\eta^{\mathbb{Q}}$$

and

$$\frac{\partial}{\partial \overline{x}} V_{16}(\overline{x}) = -\rho V_{16}(\overline{x}).$$

This gives

$$\frac{\partial}{\partial \overline{x}} \hat{V}(x, \overline{x})\big|_{x=\overline{x}} = (\eta^{\mathbb{Q}} + \rho) V_{16}(\overline{x}) - \eta^{\mathbb{Q}} V_{16}(\overline{x}) - \rho V_{16}(\overline{x}) = 0$$

which completes the proof of condition (3.6).

In the final step, we will verify that all the conditions of the verification Lemma 2 are satisfied.

First, observe that the scale functions $W^{(r)}(x)$ and $Z^{(r)}(x)$ defined as in (1.8) and (1.7) fulfill condition (3.7). Additionally, e^x also satisfies this condition, since $\gamma_1 = 1$. Now, if $x > a^*$, then for all three cases specified in Theorem 5, function $V_{a^*}(x, \overline{x})$ is just a combination of $W^{(r)}(x)$, $Z^{(r)}(x)$ and e^x. Thus, condition (3.7) holds.

Now, from the fourth case specified in Theorem 5 we can see that $(\mathcal{L}V_{a^*} - rV_{a^*})(x, \overline{x}) = -rK$ when $x \leq a^*$. Therefore, condition (3.8) is satisfied as well. The form of function $V_{a^*}(x, \overline{x})$ defined in Theorem 5 makes equation (3.9) directly to be satisfied for $x \leq a^*$.

Let us first handle the smooth paste conditions before moving on to the fourth condition of the HJB system. Note that equality (3.12) is true due to our choice of a^* solving (3.49). Furthermore, note that $V_1(a^*, \overline{x}) = K - e^{a^*}$ and $V_2(a^*, \overline{x}) = 0$, which implies that condition (3.11) is satisfied.

Finally, to prove (3.10), we need to ensure that $V_{a^*}(x, \overline{x})$ dominates over the gain function. Proposition 1 implies that the stopping region is reached by X_t process from above. On the other hand, from conditions (3.11) and (3.12) we know that $\tau_{a^*}^-$ is the first time when the candidate for the value function equals its payoff. Let us consider the pair $(x, \overline{x}) = (a^* + c, a^* + c)$. Note that in this case, we have $V_1(a^* + c, a^* + c) = 0$, $V_2(a^* + c, a^* + c) = 1$, $V_3(a^* + c) = 0$ and $V_4(a^* + c) = 1$. Therefore, from (3.49) we get that $V_{a^*}(a^* + c, a^* + c) = KZ^{(r)}(c) - e^{a^* + c}$. On the other hand, the immediate payout if equal to $K - e^{a^* + c}$. By (1.7) it immediately follows that $Z(x) > 1$ for all $x > 0$ which makes condition (3.10) satisfied.

□

3.3 Numerical analysis

In this section, we analyze several properties of the options capped by drawdown. In Figure 3.5, we present the smooth-pasting of the payoff and value functions, defined by Equations (3.11) and (3.12). The term "Projection", mentioned in the legend, refers to the form of the function V, defined on the continuation region, applied to the stopping region. The chart confirms that the smooth-paste condition indeed holds.

Next, Figure 3.6 illustrates how the option price depends on the initial values $X_0 = x$ and $\overline{X}_0 = \overline{x}$. Half of the chart is set to zero due to domain limitation $x \leq \overline{x}$. The plot reveals an interesting behavior of the function, explained below, when both x and $\overline{x}$ are sufficiently high. Figure 3.7 shows the same function, zoomed into the lower-right corner. The first notable feature is the non-differentiability of the function at $\overline{x} = \log(K) + c$, when $x \in (\log(K), \log(K) + c)$. The explanation is straightforward: if $\overline{x} < \log(K) + c$, a non-zero payout can still be achieved via diffusion, whereas for $\overline{x} > \log(K) + c$, it can only result from a Poissonian jump. Consequently, the structure of the value function changes, leading to non-differentiability.

Additionally, we observe that for $\overline{x} > \log(K) + c$, the value function is zero when $x < \overline{x} - c$, which is expected, as the option is out of the money and immediately stopped by the drawdown. However, for $x > \overline{x} - c$, the value function increases with respect to x, which might be surprising, since for $\overline{x} < \log(K) + c$ it behaves in the opposite manner. This can be explained by the fact that the higher the value of x, the longer it takes for the drawdown to occur. As a result, the probability of observing a downward jump that causes x to fall below $\log(K)$ increases.

Next, we perform a sensitivity analysis of both the stopping barrier and the option price with respect to the volatility σ, the risk-free rate r, the jump size parameter ρ, and the jump frequency parameter λ.

In Figure 3.8, we examine the optimal barrier e^{a^*} of the underlying asset price process. It is evident that the barrier increases with higher values of r and lower values of σ, indicating that an increase in the drift parameter of X_t results in an upward shift of the barrier.

Figure 3.9 presents the sensitivities of the option price with respect to the risk-free rate and volatility. It is clear that the highest price is obtained for the lowest values of r and σ. However, it is also evident that for a high value of $r = 0.5$, the value function increases with σ. This suggests that the impact of volatility on the option price depends on the values of the other parameters.

A similar analysis is conducted for ρ and λ. Figure 3.10 shows that the optimal stopping barrier decreases with increasing λ and decreasing ρ. This is intuitive, as a smaller ρ implies a larger mean jump size. Likewise, the option price, presented in Figure 3.11, is highest for the largest value of λ and the smallest value of ρ, which is sensible, since in such a scenario the likelihood of the process X_t dropping significantly below $\log(K)$ increases.

Figures 3.12 and 3.13 present side views of the chart in Figure 3.11. There, we observe that the option price grows exponentially as ρ decreases and linearly as λ increases. This behavior can be explained by the fact that while λ controls the frequency of jumps, ρ exponentially influences the jump sizes, affecting the payout after a jump occurs.

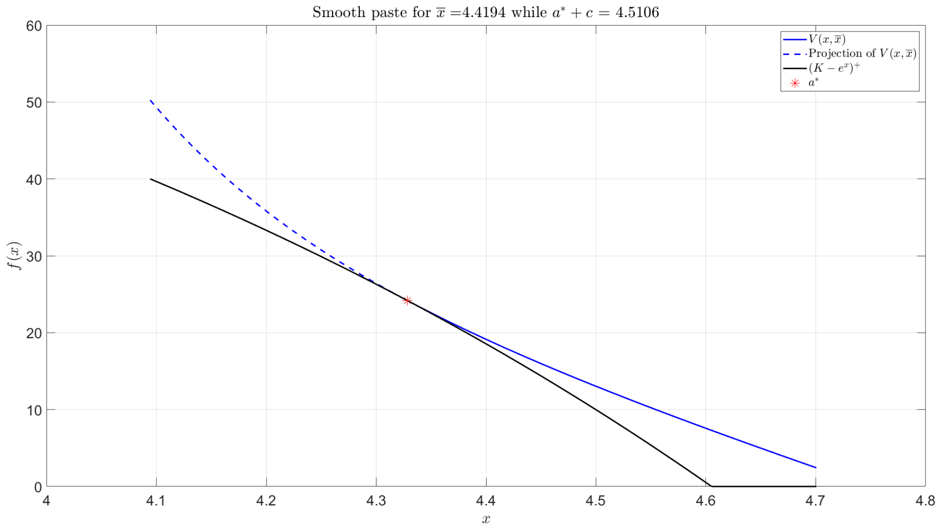

Figure 3.5: Smooth paste of the option price V and the option payoff. Parameters of the model: $r = 0.1,\ \sigma = 0.2,\ e^c = 1.2,\ \rho = 3,\ \lambda = 0.2,\ K = 100.$

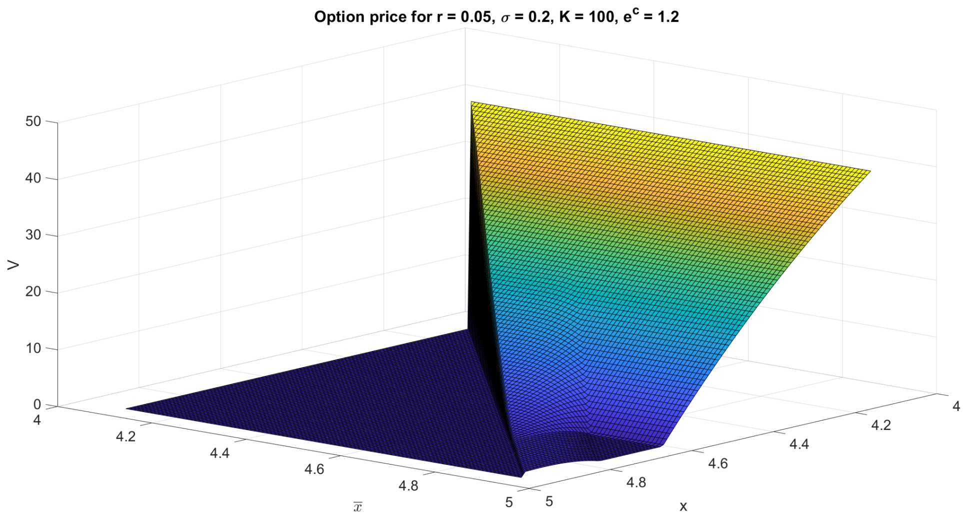

Figure 3.6: Option price depending on x and $\overline{x}$. Note that the function is not defined for $\overline{x} > x$.

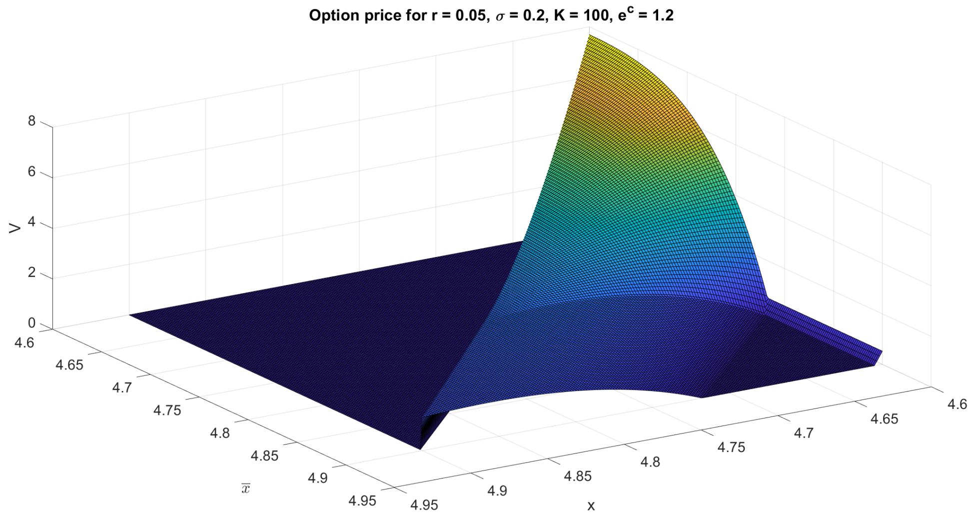

Figure 3.7: Option price depending on x and $\overline{x}$. Here, non-differentiability is clearly visible for $s = \log(K) + c$.

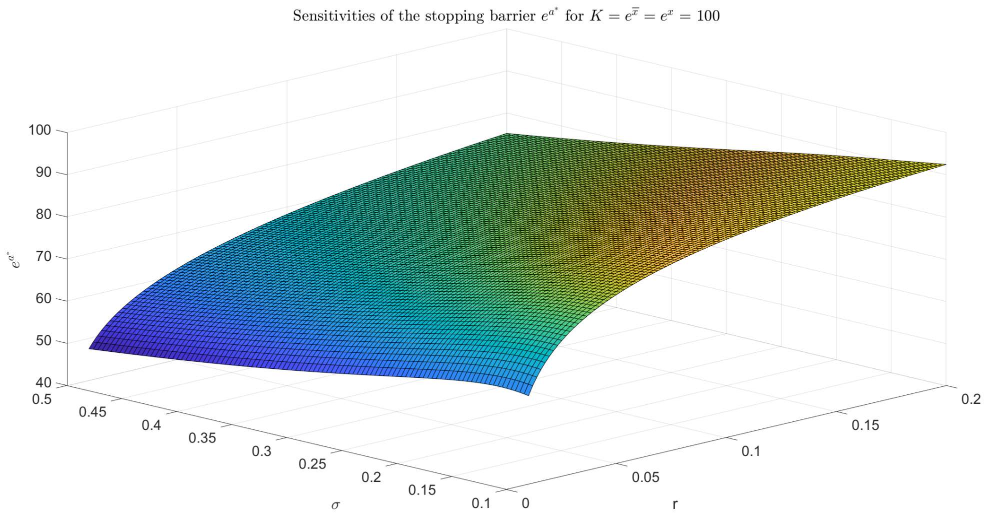

Figure 3.8: Stopping barrier e^{a^*} of the underlying asset price process S_t depending on the risk-free rate r and the volatility σ.

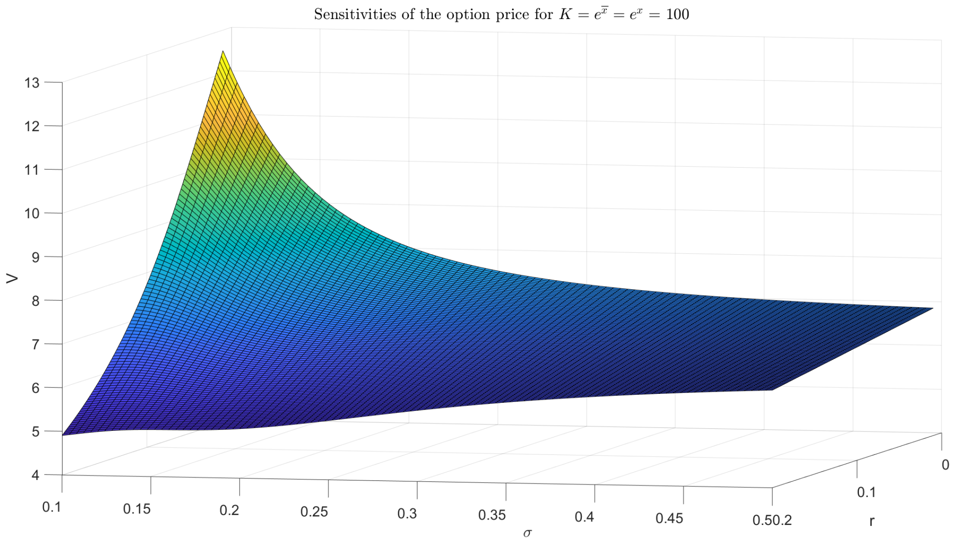

Figure 3.9: Sensitivities of the option price depending on the risk-free rate r and the volatility σ.

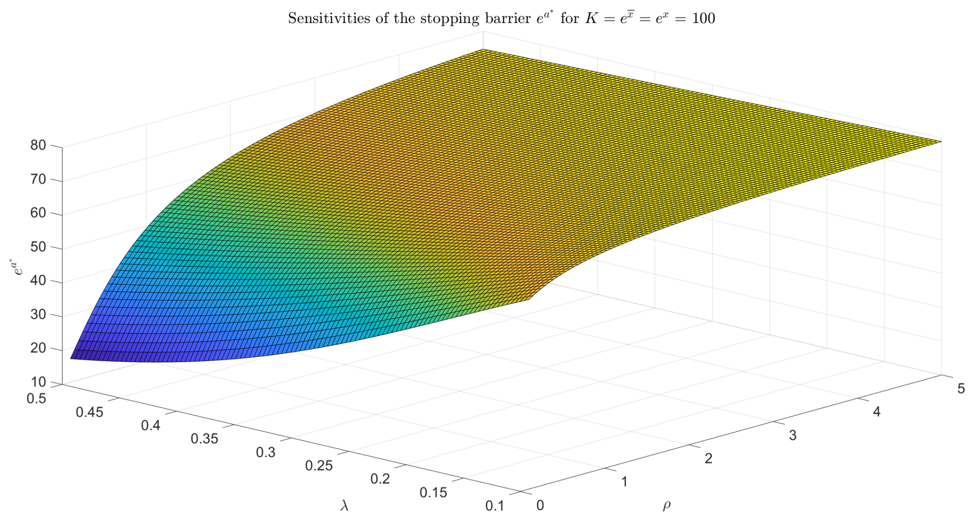

Figure 3.10: Stopping barrier e^{a^*} of the underlying asset price process S_t depending on the jump size ρ and the jump intensity λ.

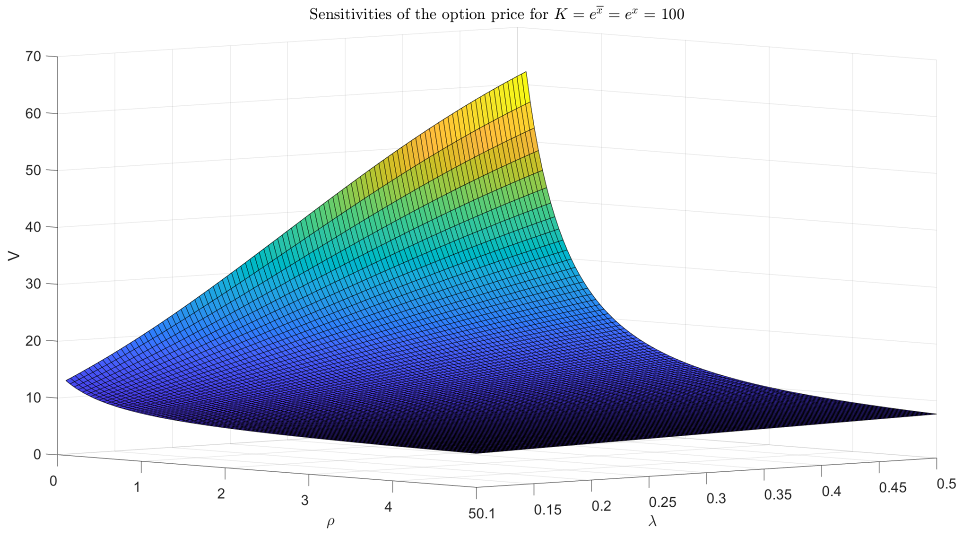

Figure 3.11: Sensitivities of the option price depending on the jump size ρ and the jump intensity λ.

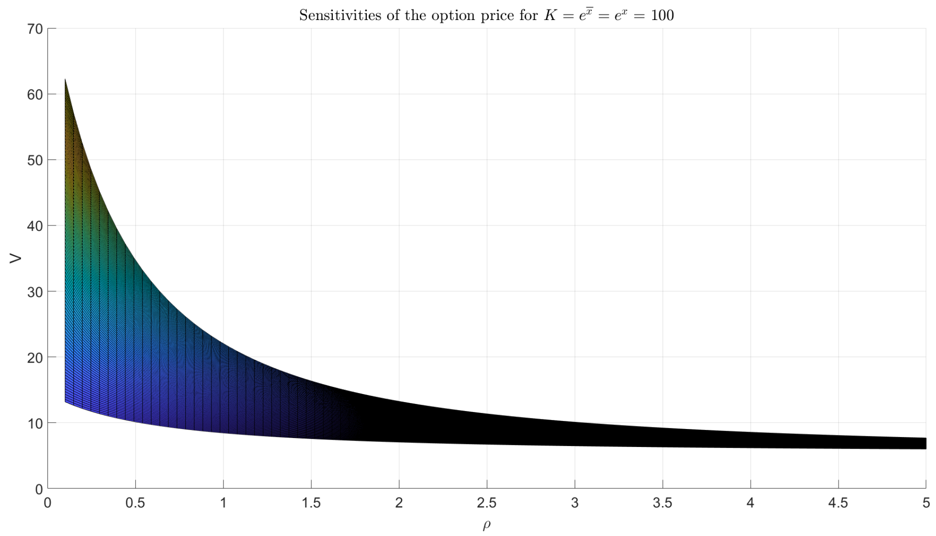

Figure 3.12: Sensitivities of the option price depending on the jump size ρ and the jump intensity λ.

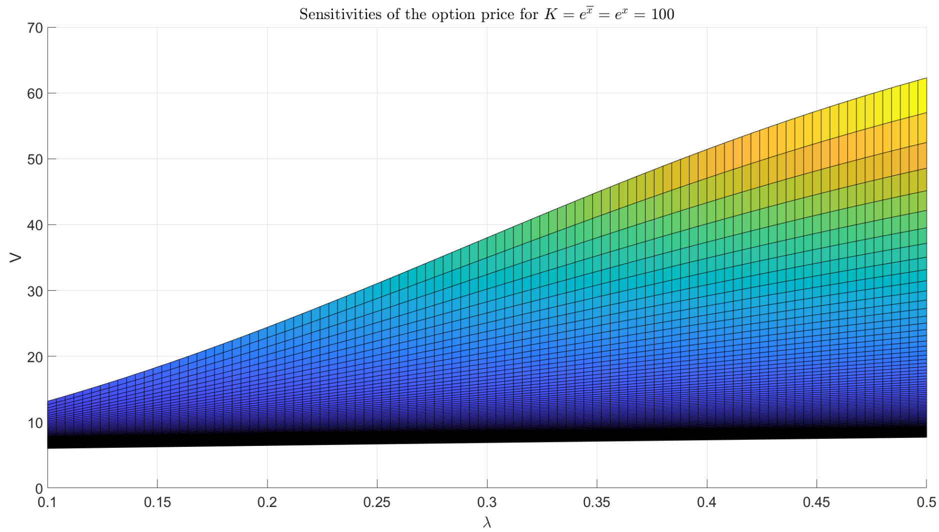

Figure 3.13: Sensitivities of the option price depending on the jump size ρ and the jump intensity λ.

Chapter 4

Last passage cap

The main goal of this section is to find the closed-form formula for the price of the perpetual American put option cancelled at the last passage time of the underlying above some fixed level h. More formally, in this section we find the following value function

$$V(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau}(K - S_\tau)^+; \tau < \theta], \tag{4.1}$$

where

$$\theta = \sup\{t \geq 0 : S_t \geq h\},$$

for some fixed threshold $h > K$. We will show that the optimal exercise time is of the form

$$\tau_{a^*} = \inf\{t \geq 0 : S_t \leq a^*\}. \tag{4.2}$$

The threshold a^* must be lower than the strike price K (and hence of the cancelling threshold h) so that exercising the option can be profitable to the holder.

This work continues the research done by [13] where the authors also evaluate the American-style option (4.1), but they do this by solving an appropriate HJB system of equations. In [13] the underlying asset price was described by geometric Brownian motion, for which the above approach is very natural due to the locality of the diffusive generator of the asset price process S_t. Still, in the context of non-local generators, a 'guess-and-verify' method used in this paper seems to be more efficient. To show the relationship of these approaches, we also provide a connection of our results to the seminal HJB equations.

We will take

$$0 < a^* < K. \tag{4.3}$$

We denote

$$G(s) = (K - s)\left(\left(\frac{h}{s}\right)^{\alpha} \wedge 1\right). \tag{4.4}$$

For some negative α.

4.1 Main result

Theorem 7. *The random time* τ_{a^*} *defined in* (4.2) *is the optimal stopping rule for problem (4.1), where*

$$a^* = \frac{K\left(\frac{\sigma^2}{2}\sum_{i=2}^3 C_i\gamma_i(\gamma_i-1)+\alpha+\frac{\rho}{\rho-\alpha}\sum_{i=2}^3 C_i\gamma_i\left[r\left(\frac{1}{\gamma_i}-1\right)-\frac{\sigma^2}{2}(\gamma_i-1)\right]\right)}{\frac{\sigma^2}{2}\sum_{i=2}^3 C_i\gamma_i(\gamma_i-1)-(1-\alpha)+\frac{\rho}{\rho-\alpha+1}\sum_{i=2}^3 C_i\gamma_i\left[r\left(\frac{1}{\gamma_i}-1\right)-\frac{\sigma^2}{2}(\gamma_i-1)\right]}. \tag{4.5}$$

Additionally, we also give the price of the considered option.

Theorem 8. *The price of the perpetual American cancellable put option defined in (4.1) equals*

$$V(s) = \frac{\sigma^2}{2}\left[W^{(r)\prime}\left(\log\frac{s}{a^*}\right)-W^{(r)}\left(\log\frac{s}{a^*}\right)\right]G(a^*)$$

$$+\left[Z^{(r)}\left(\log\frac{s}{a^*}\right)-\frac{\sigma^2}{2}W^{(r)\prime}\left(\log\frac{s}{a^*}\right)-W^{(r)}\left(\log\frac{s}{a^*}\right)\left(r-\frac{\sigma^2}{2}\right)\right]$$
$$\times\rho\left(\frac{h}{a^*}\right)^{\alpha}\left(\frac{K}{\rho-\alpha}-\frac{a^*}{\rho-\alpha+1}\right).$$

We start by transforming the value function $V(s)$. Let $Z_t = \mathbb{Q}(\theta > t|\mathcal{F}_t)$ be the conditional survival process. Observe that we can alternatively present it as

$$Z_t = \mathbb{Q}(\theta > t|\mathcal{F}_t) = \mathbb{Q}(\sup_{t\le s} S_s > h|\mathcal{F}_t). \tag{4.6}$$

Additionally, let us introduce a parameter α solving

$$\Psi(-\alpha) = 0.$$

The above equation has three solutions and the only one that can possibly be negative, that is,

$$\alpha = \frac{\rho}{2}+\frac{\mu}{\sigma^2}-\sqrt{\left(\frac{\rho}{2}-\frac{\mu}{\sigma^2}\right)^2+\frac{2\lambda}{\sigma^2}}.$$

Observe that $-1<\alpha$. By definition (1.4) $S_t^{-\alpha}$ is a martingale, therefore, by the optimal stopping theorem

$$\mathbb{E}_s^{\mathbb{Q}}\left[S_{\tau_h^+}^{-\alpha}\mathbb{I}_{\{\tau_h^+<\infty\}}\right] = s^{-\alpha}.$$

On the other hand, due to the lack of positive jumps, we have $S_{\tau_h^+}^{-\alpha} = h^{-\alpha}$. Therefore,

$$\mathbb{E}_s^{\mathbb{Q}}\left[S_{\tau_h^+}^{-\alpha}\mathbb{I}_{\{\tau_h^+<\infty\}}\right] = h^{-\alpha}\mathbb{Q}(\tau_h^+<\infty).$$

Then, by (4.6) we get

$$Z_t = \begin{cases} \left(\frac{h}{S_t}\right)^{\alpha} \wedge 1, & \text{if } \alpha < 0, \\ 1, & \text{if } \alpha \geq 0 \end{cases}$$

and

$$\mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau}(K - S_\tau)^+; \tau < \theta] = \mathbb{E}_s^{\mathbb{Q}}\left[e^{-r\tau}(K - S_\tau)^+ \left(\left(\frac{h}{S_\tau}\right)^{\alpha} \wedge 1\right)\right].$$

Hence,

$$V(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau} G(S_\tau)],$$

where the function G is defined in (4.4). The above representation is very convenient from the point of general optimal stopping theory. However, we can still modify the above representation. Observe that by [33, Thm. 31.5, p. 208]

$$\mathcal{A}f(s) = \tilde{\mu} s f'(s) + \frac{1}{2}\sigma^2 s^2 f''(s) + \lambda\rho \int_0^\infty \left(f(se^{-y}) - f(s)\right) e^{-\rho y}\, \mathrm{d}y$$

is the infinitesimal generator of the process S_t acting on $\mathcal{C}_0^2(\mathbb{R})$, where

$$\tilde{\mu} = r + \frac{\lambda}{1+\rho}.$$

Here, it is more convenient to work on the process S_t instead of X_t, therefore the generator $\mathcal{A}$ differs from previously used $\mathcal{L}$. Due to localization procedure $\mathcal{A}$ is extended generator as well acting on $\mathcal{C}^2(\mathbb{R})$. For $s < K$ we denote

$$H(s) = \mathcal{A}G(s) - rG(s) = \left(\frac{h}{s}\right)^{\alpha} (\delta s - rK)\mathbb{I}_{\{s<h\}} - rK\mathbb{I}_{\{s \geq h\}}, \tag{4.7}$$

where

$$\delta = \alpha\sigma^2 - \frac{\lambda}{1+\rho} + \frac{\lambda\rho}{(\rho-\alpha)(1+\rho-\alpha)}.$$

Let us also introduce the local time $l_t^{\log(h)}(X)$ of the process X at the point $\log(h)$ (see e.g., [31]):

$$l_t^{\log(h)}(X) = \mathbb{P} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}_{\{\log(h)-\varepsilon < X_u < \log(h)+\varepsilon\}}\, \mathrm{d}< X >_u .$$

The key representation of $V(s)$ is given in the next lemma.

Lemma 4. *The following holds true:*

$$V(s) = G(s) + V^*(s),$$

where

$$V^*(s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\log s}^{\mathbb{Q}} \left[\int_0^\tau e^{-r\tau} H(e^{X_u})\, \mathrm{d}u \right.$$
$$\left. + \int_0^\tau e^{-r\tau} h \left(G'(h+) - G'(h-)\right) \mathbb{I}_{\{e^{X_u} = h\}}\, \mathrm{d}l_t^{\log(h)}(X)\right].$$

Proof. By using the change-of-variable formula [31, Thm. 3.1] we have

$$\begin{aligned}
e^{-rt}G(S_t) &= e^{-rt}G(e^{X_t}) \\
&= G(e^x) + \int_0^t e^{-ru}G'(e^{X_u})e^{X_u}\,\mathrm{d}X_u^c - \int_0^t re^{-ru}G(e^{X_u})\,\mathrm{d}u \\
&+ \frac{1}{2}\int_0^t e^{-ru}\left[G'(e^{X_u})e^{X_u} + G''(e^{X_u})e^{2X_u}\right]\mathrm{d}< X_u^c > \\
&+ \sum_{u\le t} e^{-ru}\left[G(e^{X_u}) - G(e^{X_u-})\right] \\
&+ \int_0^t e^{-ru}e^{X_u}\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{e^{X_u}=h\}}\,\mathrm{d}l_t^{\log(h)}(X) \\
= G(s) &+ \int_0^t e^{-ru}\left[\mu S_u G'(S_u) - rG(S_u) + \frac{1}{2}\sigma^2 S_u G'(S_u) + \frac{1}{2}\sigma^2 S_u^2 G''(S_u)\right]\mathrm{d}u \\
&+ \int_0^t e^{-ru}S_u G'(S_u)\sigma\,\mathrm{d}B_u + \sum_{u\le t} e^{-ru}\left[G(S_u) - G(S_u-)\right] \\
&+ \int_0^t e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_t^{\log(h)}(X) \\
&= G(s) + \int_0^t e^{-ru}\left(\mathcal{A}G - rG\right)(S_u)\,\mathrm{d}u \\
&+ \int_0^t e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_t^{\log(h)}(X) \\
&+ \int_0^t e^{-ru}S_u G'(S_u)\sigma\,\mathrm{d}B_u + \sum_{u\le t} e^{-ru}\left[G(S_u) - G(S_u-)\right] \\
&- \lambda\int_0^t\int_0^\infty \left(G(S_u e^{-y}) - G(S_u)\right)\rho e^{-\rho y}\,\mathrm{d}y\,\mathrm{d}u, \qquad (4.8)
\end{aligned}$$

where $< X_t^c >$ is the quadratic variation of the continuous part of X_t process. Furthermore, by [16, eq. (4.34), p. 47] and [20, Thm. 3.4, p. 18 and Rem. 3.5, p. 20] the sum of three last increments of (4.8) is a zero-mean martingale. In fact it is a uniformly integrable (UI) martingale. Indeed, this follows from triangle inequality, fact that $\int_0^t e^{-ru}S_u G'(S_u)\sigma\,\mathrm{d}B_u$ is UI martingale and that

$$\mathbb{E}^{\mathbb{Q}}_{\log s}\left[\sup_{t\ge 0}\int_0^t e^{-ru}\left|H(e^{X_u})\right|\mathrm{d}u\right] < \infty \qquad (4.9)$$

and

$$\mathbb{E}^{\mathbb{Q}}_{\log s}\left[\sup_{t\ge 0}\int_0^t e^{-ru}\left|G'(h+) - G'(h-)\right|\mathbb{I}(e^{X_u} = h)\,\mathrm{d}l_u^{\log(h)}(X)\right] < \infty. \qquad (4.10)$$

To show (4.9) observe that from equation (4.7) it follows that for $0 < s < h$ function $H(s)$ is continuous and hence bounded and for $s \ge h$ the function $H(s)$ is constant. To prove (4.10) note that

$$G'(h+) - G'(h-) = -\alpha\frac{h-k}{h}.$$

Furthermore,

$$\mathbb{E}^{\mathbb{Q}}_{\log s}\left[\sup_{t\geq 0}\int_0^t e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]\leq \mathbb{E}^{\mathbb{Q}}_{\log s}\left[\int_0^\infty e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]<+\infty.$$

Indeed, defining the sequence of consecutive downward passage times of a^* by $\tau_1(a^*)=\tau_{a^*}$ and $\tau_{k+1}(a^*)=\inf\{t>\tau_k : S_t\leq a^*\}$ and recalling that our price process S_t is upward skip-free (hence passing upward $\log s$ in a continuous way), from the Markov property of S_t we have we have

$$\mathbb{E}^{\mathbb{Q}}_{\log s}\left[\int_0^\infty e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]\leq \mathbb{E}^{\mathbb{Q}}_{\log s}\left[\int_0^{\tau_{a^*}} e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]\left(1+\sum_{k=1}^\infty \mathbb{E}^{\mathbb{Q}}_{\log s}e^{-r\tau_k(a^*)}\right)$$
$$\leq \mathbb{E}^{\mathbb{Q}}_{\log s}\left[\int_0^{\tau_{a^*}} e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]\left(1+\sum_{k=1}^\infty \left(\mathbb{E}^{\mathbb{Q}}_{\log s}e^{-r\tau_{a^*}}\right)^k\right)<+\infty$$

because

$$\mathbb{E}^{\mathbb{Q}}_{\log s}\left[\int_0^{\tau_a} e^{-ru}\,\mathrm{d}l_u^{\log(h)}(X)\right]\leq \mathbb{E}^{\mathbb{Q}}_{\log s}l_{\tau_a}^{\log(h)}(X)<+\infty$$

by [22, Cor. 3.4].
The proof of the main assertion follows now from Optional Stopping Theorem. □

The next step is the Verification Theorem, which allows us to identify $V^*(s)$.

Theorem 9. *Suppose that function $\hat{V}\in\mathcal{C}^2(\mathbb{R})$ except the point h and the point a^* where it is of class $\mathcal{C}^1(\mathbb{R})$. Assume that $\hat{V}$ satisfies the following HJB system of equations*

$$(\mathcal{A}\hat{V}-r\hat{V})(s)\leq -H(s), \tag{4.11}$$
$$\hat{V}'(h+)-\hat{V}'(h-)=-(G'(h+)-G'(h-)). \tag{4.12}$$

Then $\hat{V}(s)\geq V^*(s)$.

Proof. First, we apply the change-of-variable formula to $e^{-r\tau}V(S_t)$ to get

$$\begin{aligned} e^{-rt}\hat{V}(e^{X_t}) &= \hat{V}(s)+\int_0^t e^{-ru}\left(\mathcal{A}\hat{V}-r\hat{V}\right)(S_u)\,\mathrm{d}u \\ &+\int_0^t e^{-ru}h\left(\hat{V}'(h+)-\hat{V}'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_u^{\log(h)}(X)\\ &+\int_0^t e^{-ru}a^*\left(\hat{V}'(a^*+)-\hat{V}'(a^*-)\right)\mathbb{I}_{\{S_u=a^*\}}\,\mathrm{d}l_u^{\log(a^*)}(X)\\ &+\int_0^t e^{-ru}S_u\hat{V}'(S_u)\sigma\,\mathrm{d}B_u+\sum_{u\leq t}e^{-ru}\left[\hat{V}(S_u)-\hat{V}(S_u-)\right]\\ &-\lambda\int_0^t\int_0^\infty\left(\hat{V}(S_ue^{-y})-\hat{V}(S_u)\right)\rho e^{-\rho y}\,\mathrm{d}y\,\mathrm{d}u. \end{aligned} \tag{4.13}$$

Note that $\int_0^t e^{-ru}a^*\left(\hat{V}'(a^*+)-\hat{V}'(a^*-)\right)\mathbb{I}_{\{S_u=a^*\}}\,\mathrm{d}l_u^{\log(a^*)}(X)=0$ due to assumed smoothness of $\hat{V}$ at a^*. Further, let L_t is the sum of three last increments of (4.13). Note that

L_t is a mean-one local martingale (see [16, eq. (4.34), p. 47]). Using (4.11) and (4.12), we can conclude that

$$\begin{aligned}\hat{V}(s) + L_t &\geq e^{-rt}V(e^{X_t}) + \int_0^t e^{-ru}H(S_u)\,\mathrm{d}u \\ &+ \int_0^t e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_u^{\log(h)}(X) \\ &\geq \int_0^t e^{-ru}H(S_u)\,\mathrm{d}u + \int_0^t e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_u^{\log(h)}(X).\end{aligned}$$

Let $(\kappa_n)_{n\in\mathbb{N}}$ be a localizing sequence for L_t. Using Optional Stopping Theorem, we can write for any stopping time τ:

$$\begin{aligned}&\mathbb{E}_s^{\mathbb{Q}}\left[\int_0^{\tau\wedge\kappa_n} e^{-ru}H(S_u)\,\mathrm{d}u + \int_0^{\tau\wedge\kappa_n} e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_u^{\log(h)}(X)\right] \\ &\quad\leq V(s) + \mathbb{E}_s^{\mathbb{Q}}[L_{\tau\wedge\kappa_n}] = \hat{V}(s).\end{aligned}$$

Now taking the limit with n tending to infinity and applying Lebesgue dominated convergence theorem, we get:

$$\mathbb{E}_s^{\mathbb{Q}}\left[\int_0^{\tau} e^{-ru}H(S_u)\,\mathrm{d}u + \int_0^{\tau} e^{-ru}h\left(G'(h+) - G'(h-)\right)\mathbb{I}_{\{S_u=h\}}\,\mathrm{d}l_u^{\log(h)}(X)\right] \leq V(s)$$

which completes the proof. □

Now the main idea is to find the value function

$$V_{a^*}(s) = \mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau_{a^*}}(K - S_{\tau_{a^*}}); \tau_{a^*} < \theta], \qquad s > a^*,$$

when the exercise time is the first passage downward time τ_a^* of the asset price defined in (4.2). We let

$$\hat{V}(s) = V_{a^*}(s) - G(s) \quad \text{for } s > a \text{ and } 0 \text{ otherwise}$$

for the unique $0 < a^* < K$ (see assumption (4.3)) solving $\hat{V}(s)\big|_{s=a+} = 0$ and $\hat{V}'(s)\big|_{s=a+} = 0$. In the final step, we will show that $\hat{V}(s)$ satisfies all HJB conditions of the Verification Theorem 9 and hence we get the assertion of the main Theorem 8.

We will prove now the following proposition.

Proposition 2. *The value function $V_{a^*}(s)$ equals*

$$\begin{aligned}V_{a^*}(s) = &\frac{\sigma^2}{2}\left[W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\right]G(a^*) \\ &+ \left[Z^{(r)}\left(\log\frac{s}{a^*}\right) - \frac{\sigma^2}{2}W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\left(r - \frac{\sigma^2}{2}\right)\right] \\ &\times \int_0^\infty \rho e^{-\rho y}G\left(a^* e^{-y}\right)\mathrm{d}y. \end{aligned} \tag{4.14}$$

Proof. We start the proof from showing that

$$\mathbb{E}_s^{\mathbb{Q}}\left[e^{-r\tau_{a^*}}; S_{\tau_{a^*}} = a^*\right] = \frac{\sigma^2}{2}\left[W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\right] \tag{4.15}$$

and

$$\begin{aligned} &\mathbb{E}_s^{\mathbb{Q}}\left[e^{-r\tau_{a^*}}; S_{\tau_{a^*}} < a^*\right] \\ &= \left[Z^{(r)}\left(\log\frac{s}{a^*}\right) - \frac{\sigma^2}{2}W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\left(r - \frac{\sigma^2}{2}\right)\right]. \end{aligned} \tag{4.16}$$

Indeed, denoting $\tau_b^- = \inf\{t \geq 0 : X_t < b\}$, from [42] we have

$$\mathbb{E}_y^{\mathbb{Q}}[e^{-r\tau_0^-}] = Z^{(r)}(y) - rW^{(r)}(y) = Z^{(r)}(y) - rW^{(r)}(y)$$

and from [21]:

$$\mathbb{E}_y^{\mathbb{Q}}[e^{-r\tau_0^-}; X_{\tau_0^-} = 0] = \frac{\sigma^2}{2}\left[W^{(r)\prime}(y) - W^{(r)}(y)\right].$$

Now (4.15) and (4.16) follows directly from the fact that $S_t = se^{X_t}$.
In order to find the option price $V(S) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}G(S_\tau)\right]$ we consider two possible scenarios: either the underlying price hits the threshold a^* or it drops below the threshold a^* by a jump. If S_t creeps at a^*, then $G(S_\tau) = G(a^*)$ and therefore

$$\begin{aligned} \mathbb{E}_s^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}G(S_\tau); S_\tau = a\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}; S_\tau = a\right]G(a^*) \\ &= \frac{\sigma^2}{2}\left[W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\right]G(a^*). \end{aligned}$$

In the second scenario $X_{\tau_{a^*}} < \log(a^*)$ and the undershoot $\log(a^*) - X_{\tau_{a^*}}$ has an exponential distribution with parameter ρ due to the lack of memory of this distribution. Additionally, this size of the undershoot is independent of time τ. Observe that in (3.43) the integral over the exponential jump size is independent of the rest of the Gerber-Shiu measure. Here, the situation is analogical. Thus

$$\begin{aligned} &\mathbb{E}_s^{\mathbb{Q}}\left[\mathrm{e}^{-r\tau}G(S_\tau); S_\tau < a^*\right] = \mathbb{E}_s^{\mathbb{Q}}\left[e^{-r\tau}; S_\tau < a^*\right]\mathbb{E}_s^{\mathbb{Q}}\left[G(e^{X_\tau}); S_\tau < a^*\right] \\ &= \mathbb{E}_s^{\mathbb{Q}}\left[e^{-r\tau}; S_\tau < a^*\right]\mathbb{E}_s^{\mathbb{Q}}\left[G(e^{\log(a^*) - U}); S_\tau < a^*\right] \\ &= \left[Z^{(r)}\left(\log\frac{s}{a^*}\right) - \frac{\sigma^2}{2}W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\left(r - \frac{\sigma^2}{2}\right)\right] \\ &\times \int_0^\infty \rho e^{-\rho y}G\left(a^*e^{-y}\right)\mathrm{d}y \\ &= \left[Z^{(r)}\left(\log\frac{s}{a^*}\right) - \frac{\sigma^2}{2}W^{(r)\prime}\left(\log\frac{s}{a^*}\right) - W^{(r)}\left(\log\frac{s}{a^*}\right)\left(r - \frac{\sigma^2}{2}\right)\right] \\ &\times \rho\left(\frac{h}{a^*}\right)^\alpha\left(\frac{K}{\rho - \alpha} - \frac{a^*}{\rho - \alpha + 1}\right). \end{aligned}$$

Using $V_{a^*}(s) = \mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau_{a^*}}G(S_{\tau_{a^*}})]$ and the above identities completes the proof. □

We are now ready to give the proof of the main result of this section.

Proof of Theorem 8. We recall that

$$\hat{V}(s) = \begin{cases} V_{a^*}(s) - G(s), & \text{if } s > a^* \\ 0, & \text{if } 0 < s \le a^* \end{cases}$$

for $V_{a^*}(s)$ defined in (4.14) and a^* solving $V'_{a^*}(s)\big|_{s=a^*+} = 0$. We will show that all equations in HJB system given in Verification Theorem 9.

By [19, Thm.3.10] both scale functions $W^{(r)}$ and $Z^{(r)}$ belong to $\mathcal{C}^2(\mathbb{R})$. Hence by (4.14), $V_{a^*}(s) \in \mathcal{C}^2(\mathbb{R} \setminus \{a^*, h\})$ and of class $\mathcal{C}^1(\mathbb{R})$ at a^* by the choice of a^*. Moreover,

$$\hat{V}'(h+) - \hat{V}'(h-) = V'(h+) - V'(h-) - (G'(h+) - G'(h-)) = -(G'(h+) - G'(h-))$$

and hence (4.12) is satisfied.

Observe that the only candidate for $0 < a^* < K$ which satisfies condition $V'_{a^*}(s)\big|_{s=a^*+} = 0$ is given as a solution of the following equation

$$\frac{1}{a^*}\left(\frac{h}{a^*}\right)^{\alpha}\left[\frac{\sigma^2}{2}(K-a^*)\left[C_2\gamma_2(\gamma_2-1)+C_3\gamma_3(\gamma_3-1)\right](1-\alpha)a^*+\alpha K \right.$$
$$\left. +\rho\left(\frac{K}{\rho-\alpha}+\frac{a^*}{\rho-\alpha+1}\right)\times\sum_{i=2}^{3}C_i\gamma_i\left(r\left(\frac{1}{\gamma_i}-1\right)-\frac{\sigma^2}{2}(\gamma_i-1)\right)\right]=0$$

and hence is given in (4.5). We still have to verify if $0 < a^* < K$. To do so, we rewrite the representation (4.5) of a^* as follows:

$$a^* = K + K\frac{1+\frac{\rho}{(\rho-\alpha)(\rho-\alpha+1)}\sum_{i=2}^{3}C_i\gamma_i\left[r\left(\frac{1}{\gamma_i}-1\right)-\frac{\sigma^2}{2}(\gamma_i-1)\right]}{(\alpha-1)+\sum_{i=2}^{3}C_i\gamma_i\left[\frac{r\rho}{\rho-\alpha+1}\left(\frac{1}{\gamma_i}-1\right)+\frac{\sigma^2}{2}(\gamma_i-1)\left(1-\frac{\rho}{\rho-\alpha+1}\right)\right]}. \tag{4.17}$$

Further,

$$C_2\gamma_2(\gamma_2-1) = \frac{\rho+1}{\omega}(\gamma_2+\rho), \qquad C_3\gamma_3(\gamma_3-1) = -\frac{\rho+1}{\omega}(\gamma_3+\rho) \tag{4.18}$$

and

$$-\gamma_2 < \rho < \gamma_3; \tag{4.19}$$

see e.g. [42]. Therefore, we can see that the right-hand sides of equations in (4.18) are positive, which means that also the left-hand sides are also positive. As both γ_2 and γ_3 are negative, it means that both C_2 and C_3 are also negative. By virtue of the fact that $\sum_{i=2}^{3}C_i\left[r\left(\frac{1}{\gamma_i}-1\right)-\frac{\sigma^2}{2}(\gamma_i-1)\right]=1$ we can see that:

$$C_2\left[r\left(\frac{1}{\gamma_2}-1\right)-\frac{\sigma^2}{2}(\gamma_2-1)\right] = -C_3\left[r\left(\frac{1}{\gamma_3}-1\right)-\frac{\sigma^2}{2}(\gamma_3-1)\right].$$

Now by (4.19)

$$C_2\gamma_2\left[r\left(\frac{1}{\gamma_2}-1\right)-\frac{\sigma^2}{2}(\gamma_2-1)\right] > -C_3\gamma_3\left[r\left(\frac{1}{\gamma_3}-1\right)-\frac{\sigma^2}{2}(\gamma_3-1)\right].$$

which gives

$$\sum_{i=2}^{3} C_i\gamma_i \left[r\left(\frac{1}{\gamma_i}-1\right) - \frac{\sigma^2}{2}(\gamma_i-1)\right] > 0.$$

This leads to the conclusion that the numerator of the rhs of (4.17) is strictly positive. Moreover, since γ_1 and γ_2 are negative and $-1<\alpha<0$, we know that its denominator is negative. This immediately gives $a^* < K$. To show that $a^*>0$, we need to verify that the numerator plus the denominator is smaller than 0, that is, that

$$\begin{aligned}
&1+\frac{\rho}{(\rho-\alpha)(\rho-\alpha+1)}\sum_{i=2}^{3} C_i\gamma_i \left[r\left(\frac{1}{\gamma_i}-1\right) - \frac{\sigma^2}{2}(\gamma_i-1)\right] \\
&+(\alpha-1)+\sum_{i=2}^{3} C_i\gamma_i\left[\frac{r\rho}{\rho-\alpha+1}\left(\frac{1}{\gamma_i}-1\right)+\frac{\sigma^2}{2}(\gamma_i-1)\left(1-\frac{\rho}{\rho-\alpha+1}\right)\right] \\
&=\alpha+\sum_{i=2}^{3} C_i\gamma_i\left[\frac{\rho}{\rho-\alpha}r\left(\frac{1}{\gamma_i}-1\right)+\frac{\sigma^2}{2}(\gamma_i-1)\left(1-\frac{\rho}{\rho-\alpha}\right)\right]<0.
\end{aligned}$$

This follows from the fact that $\alpha, C_2, C_3, \gamma_2, \gamma_3$ are all strictly negative.

Now, note that $e^{-rt\wedge\tau_{a^*}}\hat{V}(S_{t\wedge\tau_{a^*}})$ is a martingale. Indeed, from [1, Rem. 5] we know that $e^{-rt\wedge\tau_{a^*}\wedge\tau_b^+}W^{(r)}(S_{t\wedge\tau_{a^*}\wedge\tau_b^+})$ and $e^{-rt\wedge\tau_{a^*}\wedge\tau_b^+}Z^{(r)}(S_{t\wedge\tau_{a^*}\wedge\tau_b^+})$ are martingales where $\tau_b^+ \inf\{t\geq 0 : S_t \geq b\}$. Furthermore, by (4.15), the process $e^{-rt\wedge\tau_{a^*}\wedge\tau_b^+}W^{(r)\prime}(S_{t\wedge\tau_{a^*}})$ is a martingale as well since $e^{-rt\wedge\tau_{a^*}\wedge\tau_b^+}A(S_{t\wedge\tau_{a^*}})$ is martingale where

$$A(s)=\frac{\sigma^2}{2}\left[W^{(r)\prime}\left(\log\frac{s}{a^*}\right)-W^{(r)}\left(\log\frac{s}{a^*}\right)\right]$$

is the right hand of (4.15). To show this, observe that by the Markov property of S_t we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}_{\log s}[e^{-r\tau_{a^*}}\mathbb{I}_{\{\tau_a<\infty, S_{\tau_{a^*}}=a^*\}}|\mathcal{F}_t] \\
&=\mathbb{I}_{\{\tau_{a^*}>t\}}\mathbb{E}^{\mathbb{Q}}_{\log S_t}[e^{-r\tau_{a^*}}\mathbb{I}_{\{\tau_{a^*}<\infty, S_{\tau_a}=a^*\}}] \\
&+I(\tau_{a^*}\leq t)e^{-r\tau_{a^*}}\mathbb{I}_{\{\tau_{a^*}<\infty, S_{\tau_{a^*}}=a^*\}}=e^{-r\tau_{a^*}\wedge t}A(S_{t\wedge\tau_{a^*}}),
\end{aligned}$$

where we used the fact that $A(s)=0$ for $s<a$ and $A(a)=\frac{\sigma^2}{2}W^{(r)\prime}(0)=1$ because $W^{(r)}(0)=0$ due to the assumption that $\sigma>0$ (see [19, Lem. 3.1 and Lem. 3.2]). Since b appearing in τ_b^+ above is general, hence

$$(\mathcal{A}V_{a^*}-rV_{a^*})(s)=0 \qquad \text{for } s>a^*$$

and thus for $s>a^*$

$$(\mathcal{A}\hat{V}-r\hat{V})(s)=-(\mathcal{A}G-rG)(s)=-H(s)$$

by definition (4.7) of $H(s)$.

Now, for $s<a^*$, $\hat{V}=0$ and $(\mathcal{A}\hat{V}-r\hat{V})(s)=0$. To prove (4.11) one therefore needs to prove that $H(s)=\mathcal{A}G(s)-rG(s)\leq 0$ for $s<a^*$. Using the fact that $a^*<K<h$ we can

write $G(s)$ for $s < a^*$ in the following form:

$$G(s) = (K - s)\left(\frac{h}{s}\right)^{\alpha}.$$

Then

$$\begin{aligned} H(s) &= \tilde{\mu} s G'(s) + \frac{\sigma^2}{2} s^2 G''(s) + \lambda \int_0^{\infty} \left(G(se^{-y}) - G(s)\right) \mathrm{d}F_U(y) - rG(s) \\ &= \left(\frac{h}{s}\right)^{\alpha} \left[s\,(\alpha - 1)\left(r + \frac{\lambda}{1+\rho} - \frac{\sigma^2}{2}\alpha\right) + s\lambda + sr - \frac{s\rho\lambda}{\rho + 1 - \alpha} \right. \\ &\qquad \left. - K(\alpha + 1)\left(r - \frac{\lambda\alpha}{(1+\rho)(\rho - \alpha)} - \frac{\sigma^2}{2}\alpha\right)\right]. \end{aligned}$$

It can be easily seen that the term $K(\alpha+1)\left(r - \frac{\lambda\alpha}{(1+\rho)(\rho-\alpha)} - \frac{\sigma^2}{2}\alpha\right)$ is strictly positive, as $-1 < \alpha < 0$. Additionally, $(\alpha - 1)\left(r + \frac{\lambda}{1+\rho} - \frac{\sigma^2}{2}\alpha\right) + \lambda + r - \frac{\rho\lambda}{\rho+1-\alpha}$ is strictly negative. As both h and s are positive, hence $H(s) < 0$. This completes the proof.

□

4.2 Numerical analysis

4.2.1 Black-Scholes model

As the first case in the numerical analysis, we consider the underlying asset price described by the geometric Brownian motion. We set the intensity λ of the N_t process from equation (1.3) equal to zero and thus X_t becomes the arithmetic Brownian motion with drift parameter $\mu = r - \frac{\sigma^2}{2}$. This example corresponds to the option evaluated in [13]. The scope of the numerical analysis here is to find the optimal exercise level a and the fair price $V(s)$ of the option. The parameters are chosen as follows: the strike price $K = 100$, the threshold $h = 120$ so that $h > K$, the risk-free rate $r = 5\%$ and the volatility of the underlying asset $\sigma^2 = 0.2$. Additionally, the initial price of the underlying asset $s = S_0$ is set to 110. We first start with the calculation of a^*. Using formula (4.5) we obtain $a^* = 50$. This value fits well to the assumption (4.3). Furthermore, when we use formula (43) from [13], we get the same result:

$$a^* = \frac{\eta_2 + \alpha}{\eta_2 + \alpha - 1} K = 100 \frac{-0.5 - 0.5}{-0.5 - 0.5 - 1} = 50.$$

Now, by using Theorem 8, we get the price of the cancellable option $\overline{V}(s) \approx 21.76$. Again, when we use the formula (37) proposed in [13], we get the same result:

$$V(s) = (K - a^*)\left(\frac{s}{a^*}\right)^{\eta_2}\left(\frac{h}{a^*}\right)^{\alpha} = \frac{250}{\sqrt{132}} \approx 21.76.$$

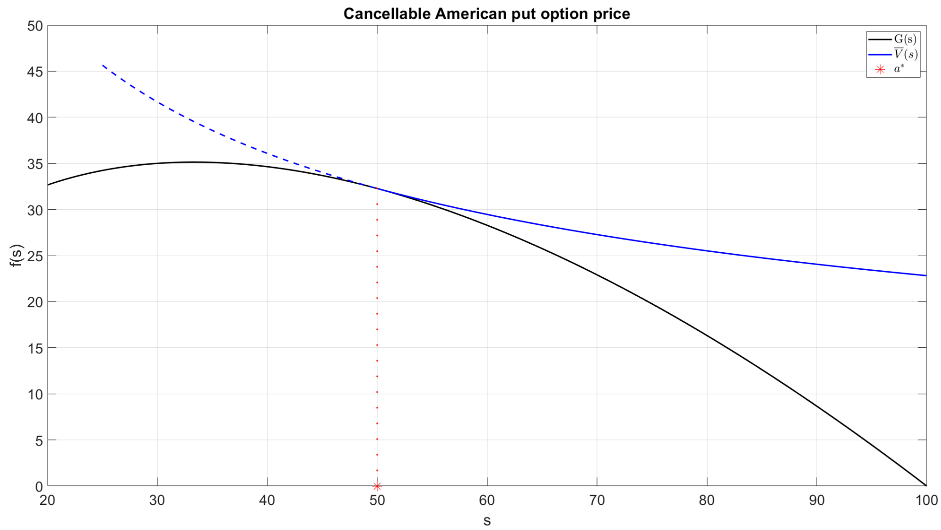

Figure 4.1: Smooth fit of the payoff and the price functions for the geometric Brownian motion with parameters: $\sigma^2 = 0.2$, $r = 0.05$, $K = 100$, $h = 120$.

Now, according to [37, Chap. 9.2], the price of the standard perpetual American option for the no-dividend case is given by:

$$V(s) = Bs^{\alpha^-},$$

where $B = -\frac{1}{\alpha^-}\left(\frac{K}{1-1/\alpha^-}\right)^{1-\alpha^-}$ and $\alpha^- = -\frac{2r}{\sigma^2}$. The price of this option with the parameters chosen as previously in this section is equal to 36.70. This price is significantly higher than the price of the corresponding cancellable option due to the higher risk the issuer of this contract has to deal with.
Finally, in Figure 4.1 we demonstrate the dependence of the cancellable option price on the initial price of the underlying instrument. Note that the price and payoff functions fit smoothly at $s = a^*$.

4.2.2 Jump-diffusion model

Here we perform a similar calculation as in Subsection 4.2.1, but now we set a fixed $\lambda > 0$. We keep the other parameters unchanged, i.e., $h = 120$, $s = S_0 = 110$, $K = 100$, $r = 5\%$, $\sigma^2 = 0.2$ and additionally set $\lambda = 5$ and $\rho = 2$. Again, we start by finding the optimal threshold a^*. With formula (4.5) we obtain $a^* \approx 63.18$. One more time, we use Theorem 8 to find the fair price of the cancellable option and we get $V(s) \approx 18.99$. The price is smaller, although a priori it is hard to expect it uniquely. Indeed, although all the jumps of the underlying are downward the drift is bigger than in the classical B-S model to apply martingale pricing formula.
As the final step, in Figure 4.2 we show the behavior of the payoff and price functions of the cancellable put option, depending on the initial underlying asset price. Again, the smooth fit of the aforementioned functions is clearly visible for $s = a^*$.

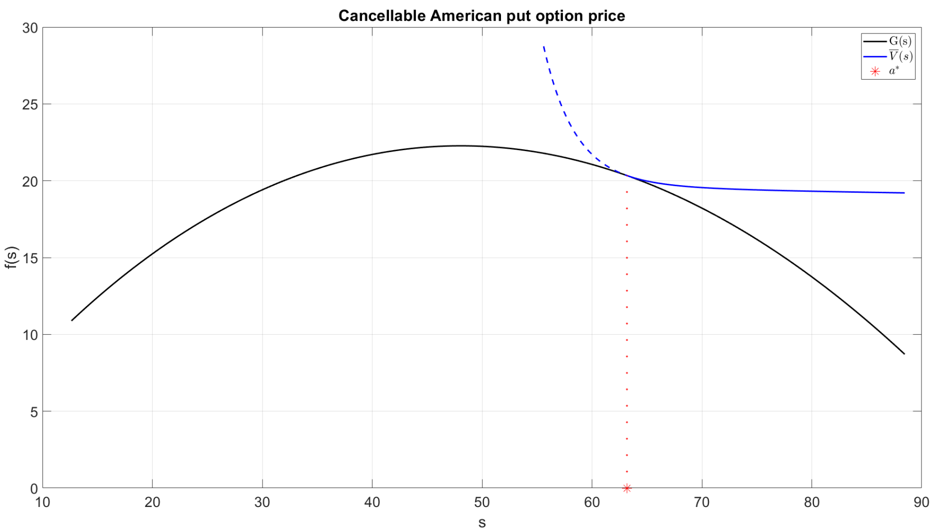

Figure 4.2: Smooth fit of the payoff and the price functions for: $\rho = 2$, $\sigma^2 = 0.2$, $r = 0.05$, $\lambda = 5$, $K = 100$, $h = 120$.

Chapter 5

Modified LSMC method

In this chapter, our main goal is to create a general numerical method of pricing time-capped options. For this purpose, we chose the Least Squares Monte Carlo (LSMC) method proposed by Longstaff and Schwartz in [23]. The main reason for this choice is its robustness to the choice of model. Additionally, it does not fail due to the so-called curse of dimensionality, in contrast to, e.g., finite-difference method or binomial trees. LSMC algorithm is based on an approximate estimate of the expected value of holding the option conditional on the underlying asset price history using a linear combination of the basis polynomials from a chosen orthogonal space. Under the assumption that the holder of the option should exercise it as soon as it is profitable, this procedure allows one to find the stopping time for each of the simulated underlying asset price trajectories. The convergence of this method has been proved by Stentoft [34] and Clement, Lamberton and Protter [6].

In order to price options with the time cap, we introduce a modified LSMC method. It not only allows us to price a new class of instruments, but it is also suitable for options written on the underlying asset described by the geometric Lévy process.

5.1 Market setup

We are now allowing for a more general market setup, as we no longer restrict the jumps to be exponentially distributed. In fact, we do not even have to assume that the considered model has to be spectrally negative.

Let us assume a Lévy market in which the asset price is described by the following process:

$$S_t = e^{X_t},$$

where X_t is a spectrally negative Lévy process and $s = S_0$ is an initial asset price. More specifically, we choose

$$X_t = x + \mu t + \sigma B_t - \sum_{k=1}^{N_t} U_k, \tag{5.1}$$

where $x = X_0 = \log s$ and $\sigma \geq 0$. In (5.1) μ is a fixed drift, B_t is a Brownian motion, N_t is a homogeneous Poisson process with intensity λ and $\{U_k\}_{\{k \in \mathbb{N}\}}$ is a sequence of independent identically distributed random variables with finite second moment. We assume that B_t, N_t and $\{U_k\}_{\{k\in\mathbb{N}\}}$ are mutually independent. We introduce a random variable θ and extend the probability space, where the process X_t is constructed, to have both random objects defined in the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{\{t\geq 0\}}, \mathbb{Q})$. We allow $\lambda = 0$, which leads to the market described by the standard Black-Scholes model. Additionally, for simplicity, we assume that no dividend is paid to the holders of the underlying asset. The main objective of this chapter is to propose a general numerical method of pricing time-capped American options. More formally, our goal is to calculate the following value function:

$$V_s = \sup_{\tau\in\mathcal{T},\tau\leq T} \mathbb{E}_s^{\mathbb{Q}}[e^{-r\tau\wedge\theta}G(S_{\tau\wedge\theta})],$$

where $G(\cdot)$ is the payout function, T is the maturity date, $\mathcal{T}$ is a family of stopping times and θ is the time-cap, i.e., the moment at which an event triggering termination of the option occurs. We assume that θ is a stopping time with respect to $\{\mathcal{F}_t\}_{\{t\geq 0\}}$. The most common examples of the payout function are $G(S) = (K - S)^+$ for the put option and $G(S) = (S - K)^+$ for the call option. Here, we limit our attention to square-integrable payoff functions. Thus, the payoff function belongs to the $L^2(\mathbb{R})$ space. Above, $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation with respect to the measure $\mathbb{Q}$.

Let us define a Laplace exponent of the process X_t as

$$\Psi(z) = \frac{1}{t}\log \mathbb{E}_x^{\mathbb{Q}} e^{zX_t}.$$

For the process X_t defined as in (1.3) we have

$$\Psi(z) = \mu z + \frac{\sigma^2 z^2}{2} + \lambda(\eta(z) - 1),$$

where

$$\eta(z) = \mathbb{E}_x^{\mathbb{Q}} e^{-zU_1}.$$

We are only interested in this distribution of U_1 for which $\eta(1)$ is finite. As $e^{-rt}S_t$ is a local martingale under $\mathbb{Q}$, then

$$\Psi(1) = r.$$

In consequence,

$$\mu = r - \frac{\sigma^2}{2} + \lambda(1 - \eta(1)).$$

5.2 Algorithm modification

American options give their holders the right to exercise them at any time before the maturity time T. To approximate their fair market price using numerical methods, it is necessary to discretize the possible exercise moments, that is, to choose for some L a sequence $\{t_k\}_{k=0}^{L}$ such that $0 = t_0 < t_1 < \ldots < t_L = T$. Then it is assumed that the option can only be exercised at time t_k for any $0 \leq k \leq L$. Such discretization transforms the American option into the so-called Bermuda option, whose price converges to the American option price for $L \to \infty$.

This section is organized as follows: first, we explain the original algorithm, which was designed for the underlying asset described by the geometric Brownian motion. Such assumption is one of the key elements of the Black-Scholes model. We can turn our exponential Lévy process into GBM by setting $\lambda = 0$.
In subsection 5.2.2, we present our modified version of the algorithm, which not only allows to price time-capped options, but is also suitable for the general class of spectrally negative Lévy processes.

5.2.1 Original algorithm

The Least Squares Monte Carlo method proposed by Longstaff and Schwartz is a dynamic programming framework for pricing American options, see [23]. It begins by simulating the N trajectories of the underlying asset price process for some sufficiently large N. Each trajectory is a realization of a geometric Brownian motion. Then, for each trajectory, the payoff at the maturity time is calculated. Then a recursive procedure starts. For each time step t_k, $0 \leq k < L$, and for each trajectory, the expected value of the option payout conditional on the current underlying asset price V_{t_i} is approximated. It allows one to determine whether it is more profitable to hold the option instead of exercising it. Thus, for each trajectory, the optimal stopping time is found. Now, since the payoff function belongs to a Hilbert space, we can present the conditional expectation of the price function mentioned above as a countable linear combination of some basis functions $\{\phi_k(\cdot), k = 1, 2, \ldots\}$ of this space, i.e.,:

$$\mathbb{E}^{\mathbb{Q}}[e^{-r(t_{i+1}-t_i)}V_{t_{i+1}}|\mathcal{F}_{t_i}] = \sum_{k=0}^{\infty} \alpha_k \phi_k(S_{t_i}), \tag{5.2}$$

for some unknown a priori coefficients α_k. The usual choice of the basis assumes that

1. for all $1 \leq j \leq L-1$ the sequence $(\phi_k(S_{t_j}))_{k\geq 0}$ is total in $L^2(\sigma(S_{t_j}))$,
2. for all $0 \leq j \leq L-1$ and $M \geq 1$, if $\sum_{k=0}^{M} \eta_k \phi_k(S_{t_j}) = 0$ almost surely then $\eta_k = 0$ for all $1 \leq k \leq M$.

In [23], the authors choose weighted Laguerre polynomials for the sake of an example. They also recommend Hermite, Legendre, Chebyshev, Gegenbauer or Jacobi polynomials as the possible alternatives. However, other basis functions are also possible, as long as they consistent with the assumptions. Then, the approximation is calculated taking the finite number M of basis functions and finding the estimators α_k^* of the coefficients α_k minimizing the following error in the least squares sense:

$$\left\| \sum_{j=1}^{M} \alpha_j^*(t_i)\phi_j(\boldsymbol{S}_{t_i}) - e^{-r(t_{i+1}-t_i)}\boldsymbol{V}_{t_{i+1}} \right\|, \tag{5.3}$$

where $\boldsymbol{V}_t$ and $\boldsymbol{S}_t$ denote the N-dimensional vectors of option and underlying asset prices at time t consisting of observations from each trajectory.
Observe that in fact this minimization problem is equivalent to the one in linear regression in a matrix form. Therefore, coefficients $\alpha_j^*(t_i)$ can be calculated by solving the following

equation:

$$
\begin{bmatrix} \alpha_1^*(t_i) \\ \alpha_2^*(t_i) \\ \vdots \\ \alpha_M^*(t_i) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{N} \phi_1(S_{t_i}^k)\phi_1(S_{t_i}^k) & \dots & \sum_{k=1}^{N} \phi_1(S_{t_i}^k)\phi_M(S_{t_i}^k) \\ \sum_{k=1}^{N} \phi_2(S_{t_i}^k)\phi_1(S_{t_i}^k) & \dots & \sum_{k=1}^{N} \phi_2(S_{t_i}^k)\phi_M(S_{t_i}^k) \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{N} \phi_M(S_{t_i}^k)\phi_1(S_{t_i}^k) & \dots & \sum_{k=1}^{N} \phi_M(S_{t_i}^k)\phi_M(S_{t_i}^k) \end{bmatrix}^{-1} \begin{bmatrix} \sum_{k=1}^{N} \mathrm{e}^{-r\delta_{t_i}} \phi_1(S_{t_i}^k)V_{t_{i+1}}^k \\ \sum_{k=1}^{N} \mathrm{e}^{-r\delta_{t_i}} \phi_2(S_{t_i}^k)V_{t_{i+1}}^k \\ \vdots \\ \sum_{k=1}^{N} \mathrm{e}^{-r\delta_{t_i}} \phi_M(S_{t_i}^k)V_{t_{i+1}}^k \end{bmatrix},
$$

where $S_{t_i}^k$ is the price of the underlying asset at the moment t_i for k-th trajectory out of N, $V_{t_{i+1}}^k$ is the value of the option at the moment t_{i+1} for k-th trajectory and $\delta_{t_i} = t_{i+1} - t_i$. In more details, the algorithm looks as follows:

1. For a chosen time discretization $0 = t_0 < t_1 < \ldots < t_L = T$ generate N trajectories of the underlying asset price process (geometric Brownian motion).

2. Set the estimated value of the option for each trajectory at the maturity time to $V_T^{*k} = G(S_T^k), k = 1, \ldots, N$, where S_T^k denotes the price of the underlying asset at time T for k-th trajectory.

3. Find coefficients $\alpha_j^*(t_{K-1})$ by minimizing the norm (5.3)

4. For each trajectory update the value function using the formula:

$$
V_{t_{i-1}}^{*k} = \begin{cases} G(S_{t_i}^k) & \text{if } G(S_{t_i}^k) \geq \sum_{j=0}^{M} \alpha_j^*(t_i)\phi_j(S_{t_i}^k) \\ e^{-r(t_{i+1}-t_i)} V_{t_i}^{*k} & \text{otherwise.} \end{cases}
$$

5. repeat steps (3) and (4) until $\boldsymbol{V}_0$ is reached.

6. Calculate the price of the option by taking the average of $\boldsymbol{V}_0$ vector.

5.2.2 Modified approach

We propose a modified LSMC method designed for pricing the time-capped American options, including those analyzed in Chapters 2 and 3, under the assumption of finite maturity. In general, contracts of interest are American-type options that may be exercised up to a fixed time horizon or an earlier random termination time. Our goal is to construct a numerical approach capable of handling such features while maintaining flexibility with respect to the source and nature of the early termination event. In particular, the proposed method applies not only to the specific examples discussed previously but also to a broader family of time-capped American-style derivatives, including those driven by external random mechanisms independent of the underlying asset dynamics.

Our procedure is as follows: we start the algorithm by discretizing the time, i.e., choosing $\{t_k\}_{k=0}^{L}$ such that $0 = t_0 < t_1 < \ldots < t_L = T$. We restrict the values of the stopping rules only to these nest times. Then we simulate N independent trajectories of the underlying asset prices, driven by the exponential Lévy process. Then, for each trajectory we either simulate the random time θ, if it is independent of the underlying asset price, or check if the performance of the underlying led to $\theta < T$ for that trajectory.
In the next step, the recursive procedure starts. The procedure at the terminal time T is unchanged towards the standard method: for each trajectory we set $V_T = G(S_T)$. Then, for each time step t_j, $0 \leq j < T$, the procedure of estimating the expected value V_{t_i} of continuation must take into account the stopping time θ. To do so, we use the counterpart of equation (5.2) and write:

$$\mathbb{E}^{\mathbb{Q}}[e^{-r(t_{i+1}-t_i)}V_{t_{i+1}}\mathbb{I}_{\{t_i<\theta\}}|\mathcal{F}_{t_i}] = \mathbb{I}_{\{t_i<\theta\}}\sum_{k=0}^{\infty}\alpha_k\phi_k(S_{t_i}). \tag{5.4}$$

Note that with the representation of the conditional expected value of continuation given by (5.4) is the same as in (5.2) until time θ is reached. Then, at all time steps from θ to T, the expected value of continuation is equal to zero due to the indicator $\mathbb{I}_{\{t_i\neq\theta\}}$. Due to this, the algorithm ensures that the option can be exercised only up to the moment when it is capped. This method does not change the procedure of estimating coefficients α_j^* and to do that one still needs to minimize the error (5.3).
In summary, the modified procedure looks as follows:

1. For a chosen time discretization $0 = t_0 < t_1 < \ldots < t_L = T$ generate N trajectories of the underlying asset price process (geometric Lévy process).

2. for each trajectory find $t_k = \theta$ provided that $\theta < T$.

3. Set the estimated value of the option for each trajectory at the maturity time to $V_T^{*k} = G(S_T^k), k = 1, \ldots, N$, where S_T^k denotes the price of the underlying asset at time T for k-th trajectory.

4. Find coefficients $\alpha_j^*(t_{K-1})$ by minimizing the norm (5.3).

5. For each trajectory update the value function using the formula:

$$V_{t_{i-1}}^{*k} = \begin{cases} G(S_{t_i}^k) & \text{if } G(S_{t_i}^k) \geq \mathbb{I}_{\{t_i<\theta\}}\sum_{j=0}^{M}\alpha_j^*(t_i)\phi_j(S_{t_i}^k) \\ e^{-r(t_{i+1}-t_i)}V_{t_i}^{*k} & \text{otherwise.} \end{cases}$$

6. repeat steps (3) and (4) until $\boldsymbol{V}_0$ is reached.

7. Calculate the price of the option by taking the average of $\boldsymbol{V}_0$ vector.

5.3 Algorithm convergence

We prove that the modified algorithm converges to a true value. We will use a method similar to that proposed by [6]. We formulate the pricing LSMC procedure from a more general perspective of the optimal stopping with a random time-cap. More precisely, we introduce a capped asset price via

$$\overline{S_t} = S_{t\wedge\theta}$$

and define

$$\overline{Z}_{t_j} = e^{-rt_j} G(\overline{S}_{t_j}).$$

Observe that

$$\sup_{\tau\in\mathcal{T}_L,\tau\leq T} \mathbb{E}^{\mathbb{Q}}\overline{Z}_\tau = \sup_{\tau\in\mathcal{T}_L,\tau\leq T} \mathbb{E}^{\mathbb{Q}}[e^{-r\tau\wedge\theta} G(S_{\tau\wedge\theta})],$$

where $\mathcal{T}_L$ is the set of stopping times with values in moments t_j.

In other words, our pricing problem can re-formulated as finding

$$V = \sup_{\tau\in\mathcal{T}_L,\tau\leq T} \mathbb{E}^{\mathbb{Q}}\overline{Z}_\tau,$$

where $(\overline{Z}_{t_j})_{0\leq j\leq L}$ is a sequence of square-integrable random variables and $\mathcal{T}$ is a set of all possible stopping times. Additionally, we assume in a more general set-up that

$$\overline{Z}_{t_j} = f(t_j, \overline{S}_{t_j})$$

for some Borel function $f(t,s)$ that is non-increasing with respect of t.

We introduce a Snell envelope

$$\overline{U}_{t_j} = \text{ess-}\sup_{\tau\in\mathcal{T}_L,\tau\leq T} \mathbb{E}^{\mathbb{Q}}(\overline{Z}_\tau|\mathcal{F}_{t_j}), \quad j=0,\ldots,L,$$

to formulate the dynamic programming principle

$$\begin{cases} \overline{U}_{t_L} = \overline{Z}_{t_L} \\ \overline{U}_{t_j} = \max(\overline{Z}_{t_j}, \mathbb{E}^{\mathbb{Q}}(\overline{U}_{t_{j+1}}|\mathcal{F}_{t_j})), \quad 0\leq j\leq L-1. \end{cases}$$

Let

$$\overline{\tau}_j = \min\{k\geq j|\overline{U}_{t_k} = \overline{Z}_{t_k}\}.$$

Note that $\overline{U}_{t_j} = \mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_j}|\mathcal{F}_{t_j}]$ and $\mathbb{E}^{\mathbb{Q}}\overline{U}_0 = \sup_{\tau\in\{\overline{\tau}_0,\ldots,\overline{\tau}_L\}} \mathbb{E}^{\mathbb{Q}}\overline{Z}_\tau = \mathbb{E}^{\mathbb{Q}}\overline{Z}_{\overline{\tau}_0}$. Then the above dynamic programming principle can be rewritten as follows

$$\begin{cases} \overline{\tau}_L = t_L \\ \overline{\tau}_j = t_j \mathbb{I}_{\{\overline{Z}_{t_j}\geq\mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{\overline{Z}_{t_j}<\mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}}, \quad 0\leq j\leq L-1. \end{cases}$$

The procedure of identifying the optimal stopping time based on $\overline{\tau}_j$ will be used in our numerical calculations. However, in order to do so, we have to make further adjustments.

To see that some modifications are required, observe that for $t_j\geq\theta$ clearly

$$\mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}] = \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1}, S_\theta)|\mathcal{F}_{t_j}] \leq \mathbb{E}^{\mathbb{Q}}[f(t_j, S_\theta)|\mathcal{F}_{t_j}] = \mathbb{E}^{\mathbb{Q}}[\overline{Z}_{t_j}|\mathcal{F}_{t_j}] = \overline{Z}_{t_j},$$

which means that $\overline{\tau}_j = t_j$. However, if we apply the direct approximation in the numerical algorithm:

$$\mathbb{E}^{\mathbb{Q}}[\overline{Z}_{t_{j+1}}|\mathcal{F}_{t_j}] = \mathbb{E}^{\mathbb{Q}}[e^{-r(t_{j+1}-t_j)}V_{t_{j+1}}|\mathcal{F}_{t_j}] \approx \sum_{k=0}^{M-1} \alpha_k^* \phi_k(\overline{S}_{t_j}),$$

we might get $\mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}] > \overline{Z}_{t_j}$ as the coefficients α_k^* are calculated based on all the trajectories, also those that have not been stopped up to the moment t_j. For this reason, we introduce

$$\begin{cases} \tau_L = t_L \\ \tau_j = t_j \mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \geq \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}} + \tau_{j+1}\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} < \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}\mathbb{I}_{\{t_j<\theta\}}|\mathcal{F}_{t_j}]\}}, \quad 0 \leq j \leq L-1. \end{cases}$$

We start from the first key lemma.

Lemma 5. *For all $0 \leq j \leq L$ we have*

$$\overline{\tau}_j = \tau_j.$$

Proof. If $t_j = t_L = T$ then $\overline{\tau}_j = T = \tau_j$.

Let us consider now the case when $\theta \leq t_j < T$. Then

$$\begin{aligned} \overline{\tau}_j &= t_j \mathbb{I}_{\{\overline{Z}_{t_j} \geq \mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{\overline{Z}_{t_j} < \mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}} \\ &= t_j \mathbb{I}_{\{f(t_j,\overline{S}_j) \geq \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},\overline{S}_{\overline{\tau}_{j+1}})|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{f(t_j,\overline{S}_j) < \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},\overline{S}_{\overline{\tau}_{j+1}})|\mathcal{F}_{t_j}]\}} \\ &= t_j \mathbb{I}_{\{f(t_j,S_\theta) \geq \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},S_\theta)|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{f(t_j,S_\theta) < \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},S_\theta)|\mathcal{F}_{t_j}]\}}. \end{aligned}$$

By the fact that $\overline{\tau}_{j+1} > t_j$ and by monotonicity of the function f with respect of time, we have $f(t_j, S_\theta) \geq \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1}, S_\theta)|\mathcal{F}_{t_j}]$. In consequence, $\overline{\tau}_j = t_j$. Therefore

$$\begin{aligned} \tau_j &= t_j \mathbb{I}_{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \geq \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}} + \tau_{j+1}\mathbb{I}_{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} < \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}\mathbb{I}_{\{t_j<\theta\}}|\mathcal{F}_{t_j}]} \\ &= t_j \mathbb{I}_{\{0 \geq 0\}} + \tau_{j+1}\mathbb{I}_{\{0<0\}} = t_j = \overline{\tau}_j. \end{aligned}$$

We consider now the remaining case of $0 \leq t_j < \theta$. We prove this part by backward mathematical induction. We recall that from previous considerations we know that $\overline{\tau}_j = \tau_j$ for j such that $t_j = \theta$. Assuming that $\overline{\tau}_{j+1} = \tau_{j+1}$, we have

$$\begin{aligned} \overline{\tau}_j &= j\mathbb{I}_{\{\overline{Z}_{t_j} \geq \mathbb{E}^{\mathbb{Q}}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{\overline{Z}_{t_j} < \mathbb{E}[\overline{Z}_{\overline{\tau}_{j+1}}|\mathcal{F}_{t_j}]\}} \\ &= j\mathbb{I}_{\{f(j,\overline{S}_j) \geq \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},\overline{S}_{\overline{\tau}_{j+1}})|\mathcal{F}_{t_j}]\}} + \overline{\tau}_{j+1}\mathbb{I}_{\{f(j,\overline{S}_j) < \mathbb{E}^{\mathbb{Q}}[f(\overline{\tau}_{j+1},\overline{S}_{\overline{\tau}_{j+1}})|\mathcal{F}_{t_j}]\}} \\ &= j\mathbb{I}_{\{f(j,S_{t_j}) \geq \mathbb{E}^{\mathbb{Q}}[f(\tau_{j+1},S_{\tau_{j+1}})|\mathcal{F}_{t_j}]\}} + \tau_{j+1}\mathbb{I}_{\{f(j,S_{t_j}) < \mathbb{E}^{\mathbb{Q}}[f(\tau_{j+1},S_{\tau_{j+1}})|\mathcal{F}_{t_j}]\}} \\ &= j\mathbb{I}_{\{Z_{t_j} \geq \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\}} + \tau_{j+1}\mathbb{I}_{\{Z_{t_j} < \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\}} \\ &= j\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \geq \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}} + \tau_{j+1}\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} < \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}\mathbb{I}_{\{t_j<\theta\}}|\mathcal{F}_{t_j}]\}} = \tau_j. \end{aligned}$$

This completes the proof. □

From Lemma 5 we can conclude that

$$V = \sup_{\tau \in \{\tau_0, \ldots, \tau_L\}} \mathbb{E}^{\mathbb{Q}} Z_\tau,$$

where

$$Z_{t_j} = f(t_j, S_{t_j}).$$

Denoting

$$U_j = \mathbb{E}^{\mathbb{Q}}(Z_{\tau_j} | \mathcal{F}_{t_j}) \tag{5.5}$$

we have

$$\mathbb{E}^{\mathbb{Q}} U_0 = \sup_{\tau \in \{\tau_0, \ldots, \tau_L\}} \mathbb{E}^{\mathbb{Q}} Z_\tau = V.$$

Now, given the approximation $\mathbb{E}^{\mathbb{Q}}[e^{-r(t_{j+1}-t_j)} V_{t_{j+1}} | \mathcal{F}_{t_j}] \approx \sum_{k=0}^{M-1} \alpha_k^* \phi_k(S_{t_j}) = \alpha_j^M \cdot \phi^M(S_{t_j})$ for a chosen value of M and the corresponding vectors $\alpha_j^M = \{\alpha_0^*, \ldots, \alpha_{M-1}^*\}$ (with $\{\alpha_0^*, \ldots, \alpha_{M-1}^*\}$ estimated separately for each j and $\phi^M = \{\phi_0, \ldots, \phi_{M-1}\}$, let us introduce stopping times τ_j^M as:

$$\begin{cases} \tau_L^M = t_L \\ \tau_j^M = t_j \mathbb{I}_{\{Z_{t_j} \mathbb{I}_{\{t_j < \theta\}} \geq [\alpha_j^M \cdot \phi^M(S_{t_j})] \mathbb{I}_{\{t_j < \theta\}}\}} + \tau_{j+1}^M \mathbb{I}_{\{Z_{t_j} \mathbb{I}_{\{t_j < \theta\}} < [\alpha_j^M \cdot \phi^M(S_{t_j})] \mathbb{I}_{\{t_j < \theta\}}\}} \end{cases}$$

for $0 \leq j \leq L-1$. Such stopping times allow us to define an approximation of the value by

$$U_0^M = \max(Z_0, \mathbb{E}^{\mathbb{Q}} Z_{\tau_1^M}).$$

As Z_0 is deterministic, it is sufficient to find $\mathbb{E}^{\mathbb{Q}} Z_{\tau_1^M}$ to be able to obtain the approximation of the option price. For this purpose, Monte Carlo method is used. Let N denote the total number of simulated trajectories. Then, the approximation of the stopping times τ_j for n-th trajectory is obtained using only the first M basis functions ϕ_k. In other words, τ_j is approximated by

$$\begin{cases} \tau_L^{M,N,n} = t_L \\ \tau_j^{M,N,n} = t_j \mathbb{I}_{\{Z_{t_j}^n \mathbb{I}_{\{t_j < \theta\}} \geq [\alpha_j^{M,N} \cdot \phi^M(S_{t_j}^n)] \mathbb{I}_{\{t_j < \theta\}}\}} + \tau_{j+1}^{M,N,n} \mathbb{I}_{\{Z_{t_j}^n \mathbb{I}_{\{t_j < \theta\}} < [\alpha_j^{M,N} \cdot \phi^M(S_{t_j}^n)] \mathbb{I}_{\{t_j < \theta\}}\}}, \end{cases}$$

for $0 \leq j \leq L-1$ and where $S_{t_j}^n$ is the value of n-th trajectory of the underlying asset process, $Z_{t_j}^n = f(t_j, S_{t_j}^n)$ and $\alpha_j^{M,N}$ is a set of first M estimators α_k^* of the coefficients α_k in equation (5.2) evaluated for n-th trajectory at time t_j. In consequence:

$$U_0^{M,N} = \max(Z_0, \frac{1}{N} \sum_{n=1}^{N} Z_{\tau_1^{M,N,n}}^n).$$

is a LSMC estimator of the value V.

It appears that our estimator is consistent in the following sense. In consequence, our algorithm allows to obtain a proper approximation of the capped option price.

Theorem 10. *Assume that for* $0 < j < L$ *we have* $\mathbb{P}(\alpha_j \cdot \phi(S_{t_j}) = Z_{t_j}) = 0$. *Then*

$$U_0^{M,N} \xrightarrow[N\to\infty]{\text{a.s.}} U_0^M \tag{5.6}$$

and

$$\lim_{M\to+\infty} U_0^M = U_0 = V. \tag{5.7}$$

Proof. The convergence (5.6) can be proved in a way similar to Theorem 3.2 of [6]. Indeed, our model fulfills all of the assumptions stated in [6] and in our modified algorithm we do not change the way of approximating α_j^M.

From (5.5) if follows that to prove (5.7) it sufficient to show that for all $0 \le j \le L$ we have:

$$\lim_{M\to\infty} \mathbb{E}^{\mathbb{Q}}[Z_{\tau_j^M}|\mathcal{F}_{t_j}] = \mathbb{E}^{\mathbb{Q}}[Z_{\tau_j}|F_j]. \tag{5.8}$$

We will prove this fact using backward mathematical induction. Equation (5.8) holds for L since $Z_{\tau_L^M} = Z_{\tau_L} = Z_T$. Assume that (5.8) holds for $j+1$. To prove that it holds for j, observe, that

$$Z_{\tau_j^M} = Z_{t_j}\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} + Z_{\tau_{j+1}^M}\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} < [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}}.$$

Furthermore,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[Z_{\tau_j^M} - Z_{\tau_j}|\mathcal{F}_{t_j}] &= \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}^M} - Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} < [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} \\ &+ (Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}])\left(\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} - \mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}}\right).\end{aligned}$$

By induction assumption, $\mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}^M} - Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]$ converges to 0 as M goes to infinity. Let us define B_j^M as the remaining part on the right hand side of above equality, that is,

$$B_j^M = (Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}])\left(\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} - \mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}}\right).$$

We have

$$\begin{aligned}|B_j^M| &= |Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]||\mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} - \mathbb{I}_{\{Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}}| \\ &= |Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]||\mathbb{I}_{\{\mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}} > Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge [\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}}\}} \\ &- \mathbb{I}_{\{[\alpha_j^M\cdot\phi^M(S_{t_j})]\mathbb{I}_{\{t_j<\theta\}} > Z_{t_j}\mathbb{I}_{\{t_j<\theta\}} \ge \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]\mathbb{I}_{\{t_j<\theta\}}\}}| \\ &\le |Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|\mathbb{I}_{\{|Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|\mathbb{I}_{\{t_j<\theta\}} \le |\alpha_j^M\cdot\phi^M(S_{t_j}) - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|\mathbb{I}_{\{t_j<\theta\}}\}} \\ &\le |Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|\mathbb{I}_{\{|Z_{t_j} - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]| \le |\alpha_j^M\cdot\phi^M(S_{t_j}) - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|\}} \\ &\le |\alpha_j^M\cdot\phi^M(S_{t_j}) - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|.\end{aligned}$$

By arguments similar to the one given in the proof of Theorem 3.1 in [6] we can conclude that $|\alpha_j^M\cdot\phi^M(S_{t_j}) - \mathbb{E}^{\mathbb{Q}}[Z_{\tau_{j+1}}|\mathcal{F}_{t_j}]|$ converges to 0 as M tends to infinity. Therefore, B_j^M converges to 0 as M goes to infinity, which completes the proof. □

5.4 Numerical analysis

In a general setup, θ is a stopping time with respect to $\{\mathcal{F}_t\}_{\{t\geq 0\}}$ related to an event observable by the market participants. We allow it to be dependent on and independent of the underlying asset price. However, we will focus particularly on the instruments evaluated analytically previously in this thesis. In addition, we will present the results for options capped by a stopping time not correlated with the underlying asset price.
In all cases, we choose at least 1000 time steps for each trajectory, 20000 trajectories for each price calculation, and the first 6 functions from the Chebyshev basis to approximate the conditional expected value of holding the option. Furthermore, we repeat the calculations for different times to maturity T and perform each calculation 100 times to draw the boxplots of the results. For $T > 5$, we increase the number of time steps to $T \times 200$. We do it to address the problem of decreasing the accuracy of the results for long maturities.

5.4.1 First passage time

We start our analysis with the case of the time cap being the first exit time from an interval, as in Chapter 2. Here, the results are presented for the finite time to maturities, meaning that we are no longer dealing with perpetual options. We would like to compare the numerical results against the analytical formula.
We choose the following set of market parameters: $S_0 = 100,\ K = 100,\ r = 0.05,\ \sigma = 0.2,\ \lambda = 0.2,\ \rho = 1,\ H = \log(130),\ L = \log(60)$. The prices are shown in Figure 5.1. The chart shows option prices with a clear trend: the longer the expiration time, the higher the option price, which is expected. The analytical price of the perpetual option is marked in blue. The expected behavior would be for the prices with finite expiration times to converge toward this perpetual price as the expiration approaches. However, the LSMC method used to obtain the numerical results might have a bias due to the regression-based estimation of the α_j parameters, as discussed in Balazs Kovacs' thesis paper [18]. Additionally, for the chosen parameter set, the options are relatively quickly exercised, with no trajectory lasting more than 24 years before being stopped. The average time-caps are additionally shown in Figure 5.2. In reality, there is a non-zero probability that the option is not stopped through the entire lifespan of the contract. In such case, the payout diminishes through discounting. This leads to a slight overestimation of the price, which could likely be reduced by using more basis polynomials, a different basis, more time steps and more trajectories. Unfortunately, this would require significantly more computational power. Taking into account all of these aspects, the results obtained are reasonable and acceptable.

5.4.2 Drawdown in Black-Scholes model

We start the analysis of drawdown-type cap with the simpler Black-Scholes market. We choose the following set of market parameters: $S_0 = 100,\ \overline{S}_0 = 105,\ K = 100,\ r = 0.1,\ \sigma = 0.2,\ c = \log(1.2)$. The prices are shown in Figure 5.3. Again, there is a clear increasing trend of the prices. As in the previous case, the results compared against the analytical price of perpetual option are slightly overestimated, which is not surprising

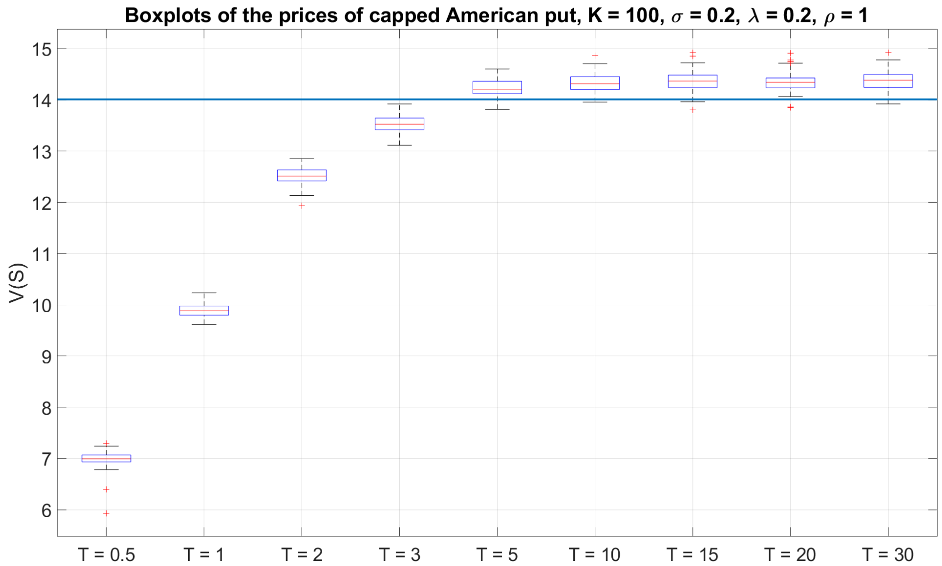

Figure 5.1: Prices of the American put options capped by the first passage time with different maturities compared against the analytically obtained price of perpetual American put option capped by the first passage time.

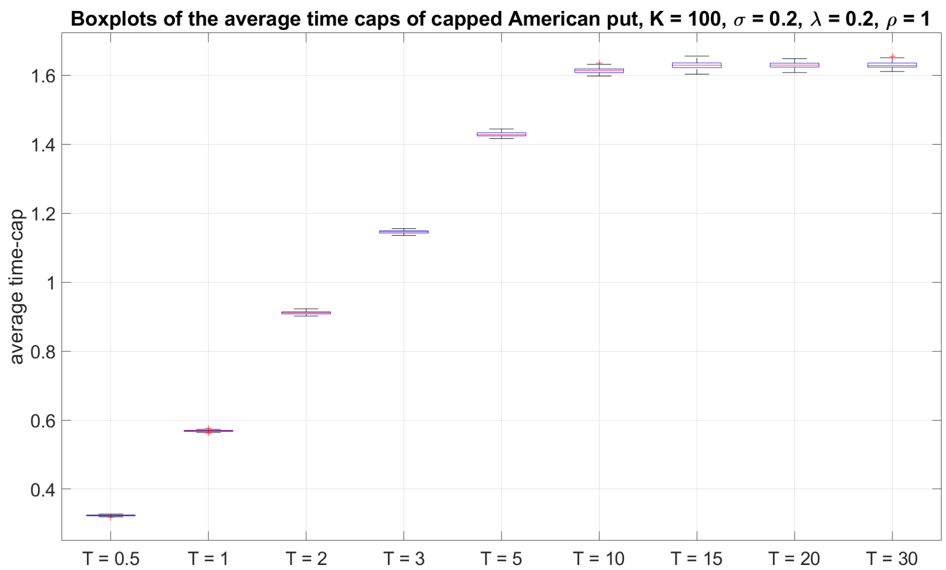

Figure 5.2: Average time-cap of the American put options capped by the first passage time.

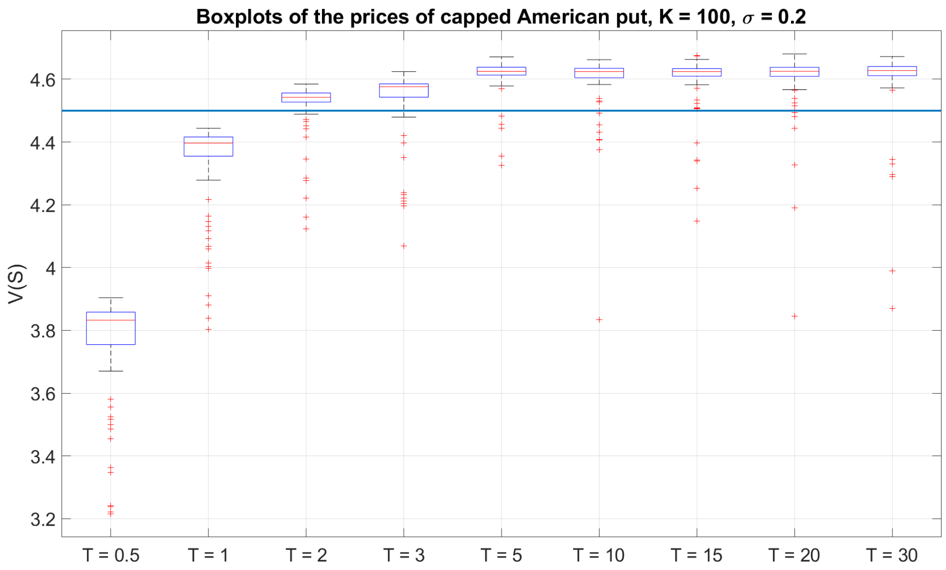

Figure 5.3: Prices of the American put options capped by the drawdown event in Black-Scholes market with different maturities compared against the analytically obtained price of perpetual American put option capped by the drawdown event.

given the circumstances explained before. The average time-caps are shown in Figure 5.4. Again, they are relatively short compared to the maturities, which decreases the accuracy of the results.

5.4.3 Drawdown in jump-diffusion model

After handling the easier Black-Scholes model, we move on to the Lévy market. Here, we choose the following parameters: $S_0 = 100$, $\overline{S}_0 = 105$, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $c = \log(1.2)$, $\lambda = 0.2$, $\rho = 3$. Thus, the drift parameter μ is equal to 0.08, exactly as in Subsection 5.4.2. The prices are shown in Figure 5.5 and behave similarly to both previous examples. Once again, the prices are slightly overestimated compared against the perpetual option price and again the options are relatively quickly exercised, which is proved in Figure 5.6.

5.4.4 Asset-independent caps

Finally, we move on to pricing contracts that were not evaluated previously in this thesis. As mentioned before, the modified LSMC algorithm allows to price options capped by any stopping times, correlated or not with the underlying asset price. Here, as an example, we choose the time cap being an exponential random variable independent of all other stochastic processes. We choose the expected value of this variable to be equal to 10. The other parameters are chosen as follows: $S_0 = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 0.2$, $\rho = 1$. The results are presented in Figure 5.7. Here, the prices remain almost unchanged for $T \geq 15$, which seems sensible given the chosen time-cap. Note that the market parameters chosen here are the same as in Subsection 5.4.1. However, here the

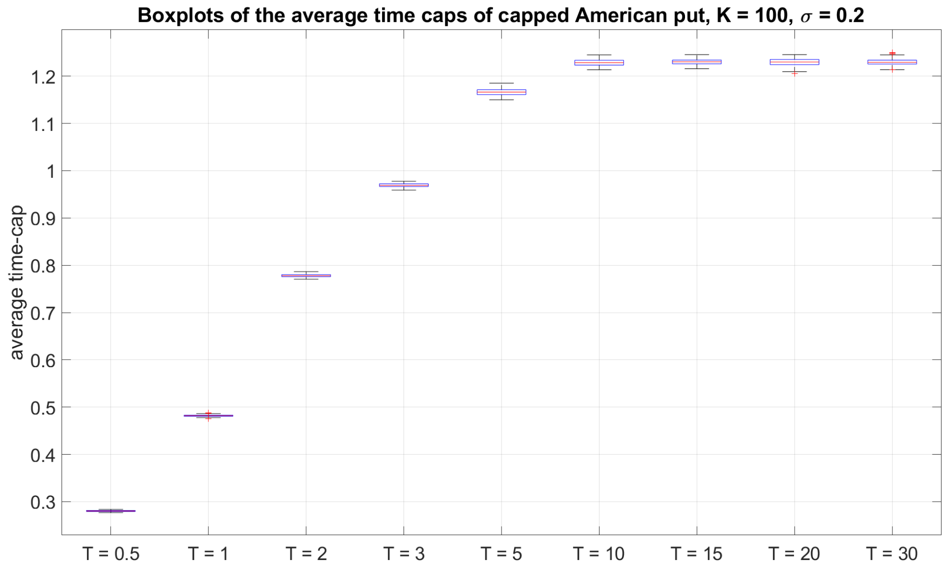

Figure 5.4: Average time-cap of the American put options capped by the drawdown event in Black-Scholes market.

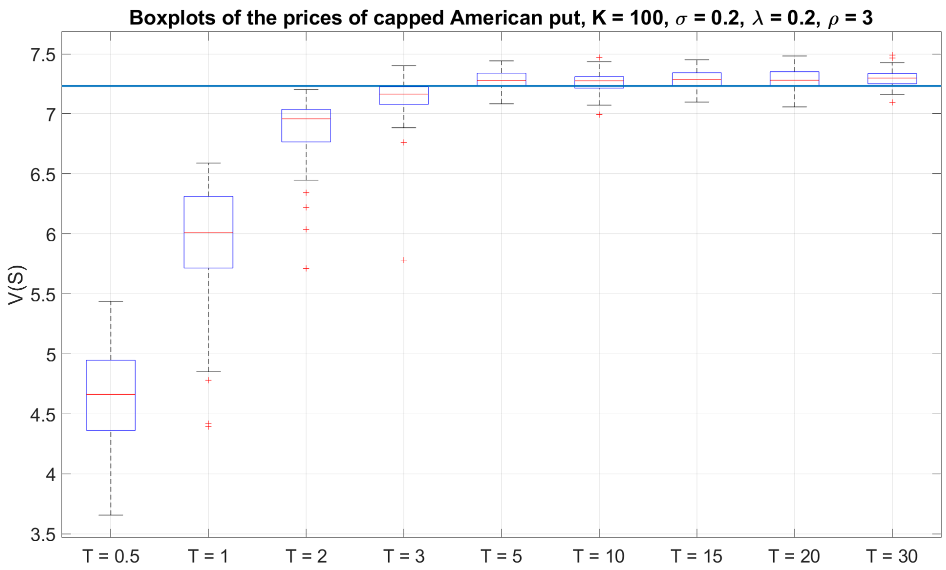

Figure 5.5: Prices of the American put options capped by the drawdown event in Lévy market with different maturities compared against the analytically obtained price of perpetual American put option capped by the drawdown event.

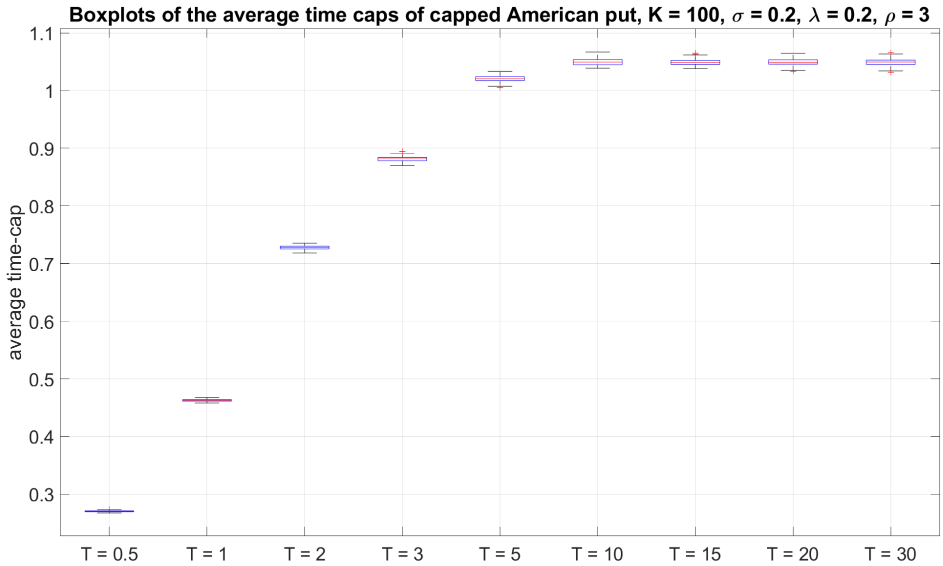

Figure 5.6: Average time-cap of the American put options capped by the drawdown event in Lévy market.

prices are significantly higher. This can be explained by the longer lifespan of the contracts, which is evident from Figure 5.8. The graph of the average time-caps in this case takes a sigmoidal shape, starting from close to 0 for $T = 0.5$ and ending at above 9 for $T = 30$. The lower values for shorter maturities can be justified by the fact that for many trajectories the time-cap event did not occur before the option was terminated.

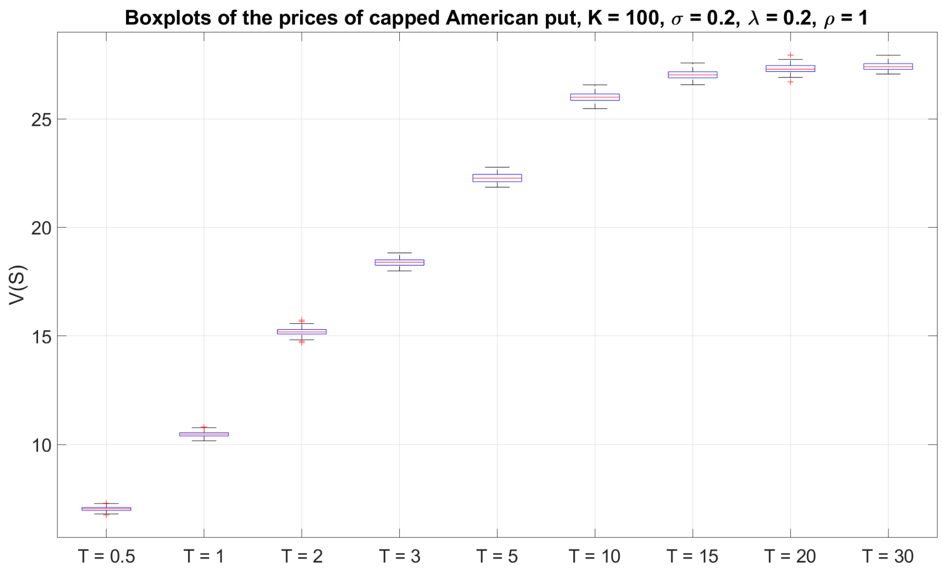

Figure 5.7: Prices of the American put options capped by the exponential cap independent of asset price with different maturities compared.

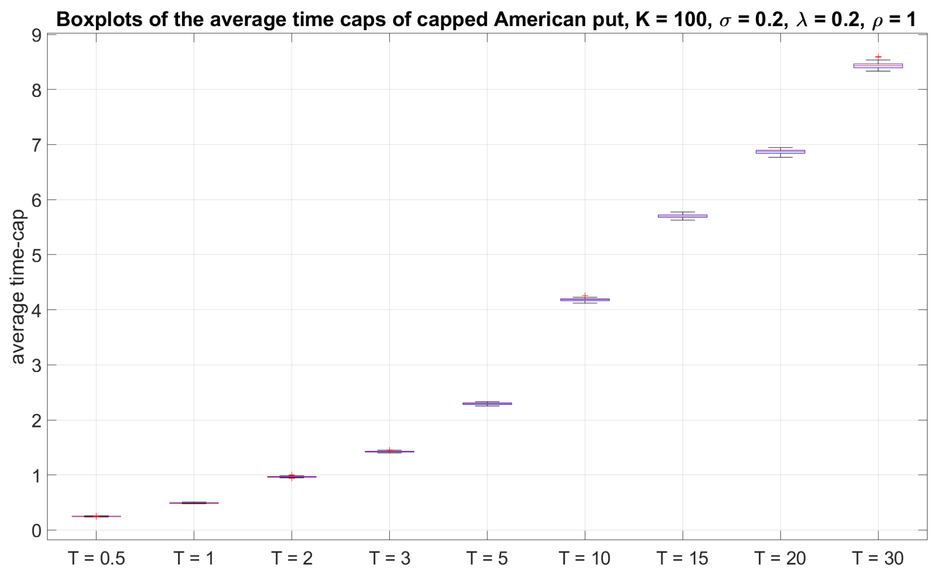

Figure 5.8: Average time-cap of the American put options capped by the exponential cap independent of asset price.

Bibliography

[1] Avram, F. Kyprianou, A. and Pistorius, M. (2004) *Exit problems for spectrally negative Lévy processes and applications to Russian, American and Canadized options*, Ann. Appl. Probab. **14**, 215–238.

[2] Björk, T. (1998) *Arbitrage Theory in Continuous Time*, Oxford: Oxford University Press,.

[3] Black, F. and Scholes, M. (1973) *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, Vol. 81, No. 3.

[4] Boyle, P.P. and Turnbull, S.M. (1989) *Pricing and Hedging Capped Options*, The Journal of Futures Markets, Vol. 9., No. 1.

[5] Broadie, M and Detemple, J. (1994) *American Capped Call Options on Dividend-Paying Assets*, CIRANO, CIRANO Working Papers.

[6] Clement, E. and Lamberton, D. and Protter, P. (2002) *An analysis of a least squares regression method for American option pricing.* Finance and Stochastics **6(4)**, 449–471.

[7] Cont R. and Tankov, P. (2003) *Financial modelling with jump processes.* Chapman and Hall, CRC Press.

[8] Crawford, G. Sen, B. (2008) *Derivatives for Decision Makers: Strategic Management Issues*, John Wiley & Sons, Inc.

[9] Deng D. and Peng, C. (2014) *New Methods with Capped Options for Pricing American Options*, Journal of Applied Mathematics.

[10] Egloff, D., Farkas, W. and Leippold, F. M. (2009) *American Options with Stopping Time Constraints.* SSRN Electronic Journal http://dx.doi.org/10.2139/ssrn.798124.

[11] Eisenbaum, N. and Kyprianou, A. (2008) *On the parabolic generator of a general one-dimensional Lévy process*, Elec. Comm. Probab., **(13)**, 198–208.

[12] Gapeev, P. (2007). *Perpetual barrier options in jump-diffusion models.* Stochastics, 79(1–2), 139–154. https://doi.org/10.1080/1744250060108638.

[13] Gapeev, P., Li, L. and Wu, Z. (2020) *Perpetual American Cancellable Standard Options in Models with Last Passage Times.* Algorithms **14**, 3.

[14] Gapeev, P. and Al Motairi, H. (2022) *Discounted optimal stopping problems in first-passage time models with random thresholds.* J. Appl. Probab. **59**, 714–33.

[15] Ivanovs, I. (2021) *On scale funstions for Lévy processes with negative phase–type jumps*, Queueing Systems, **98**, 3–19.

[16] Jacod, J. and Shiryaev, A.N. (2003) *Limit Theorems for Stochastic Processes, Second Edition.* Berlin/Heidelberg: Springer.

[17] Kifer, Y. (2000) *Game options.* Finance and Stochastics **4**, 443–463.

[18] Kovacs, B. (2012) *American option pricing with LSM algorithm and analytic bias correction*, Thesis Paper.

[19] Kuznetsov, A., Kyprianou, A.E. and Rivero, V. (2012) *The Theory of Scale Functions for Spectrally Negative Lévy Processes.* In: Lévy Matters II. Lecture Notes in Mathematics, vol 2061. Springer, Berlin, Heidelberg.

[20] Kyprianou, A. E. (2013) Gerber-Shiu Risk Theory. *Cham: Springer.*

[21] Kyprianou, A. E. (2014) *Fluctuations of Lévy Processes with Applications.* Berlin/Heidelberg: Springer.

[22] Li, B., and Zhou, X. (2020) Local Times for Spectrally Negative Lévy Processes. *Potential Anal.* **52**, 689–711.

[23] Longstaff, F. A.and Schwartz, E. S. (2001) *Valuing American Options by Simulation: A Simple Least-Squares Approach.* The Review of Financial Studies **14(1)**.

[24] Meyer, G. H. (2016) *A PDE View of Games Options.* Research Paper Series **369**, Quantitative Finance Research Centre .

[25] Merton, R. (1976) *Option pricing when the underlying stock returns are discontinuous.* Journal of Financial Economics **3**, 125–144.

[26] Mijatović, A. and Pistorius M.R. (2012) *On the drawdown of completely asymmetric Lévy processes.* Stochastic Processes and their Applications, 122(11), Pages 3812-3836.

[27] Ott, C. (2013) *Optimal Stopping Problems for the Maximum Process*, PhD thesis, University of Bath.

[28] Palmowski, Z. and Stępniak, P. (2023) *Last-Passage American Cancelable Option in Lévy Models.* Journal of Risk and Financial Management **16(2)**.

[29] Palmowski, Z. and Stępniak, P. (2025) *Pricing time-capped American options using Least Squares Monte Carlo method.* Journal of Computational Finance, **28(3)**.

[30] Peskir, G. and Shiryaev, A.N. (2006) *Optimal Stopping and Free-Boundary Problems.* Birkhäuser: Basel, Switzerland.

[31] Peskir, G. (2007) *A Change-of-Variable Formula with Local Time on Surfaces.* Séminaire de Probabilités XL. Lecture Notes in Mathematics, vol 1899. Springer, Berlin, Heidelberg.

[32] Poitrars, G. (2008) *The Early History of Option Contracts*, Vinzenz Bronzin's Option Pricing Models: Exposition and Appraisal, Springer.

[33] Sato, K. (1999) *Lévy processes and infinitely divisible distributions.* Cambridge Univ. Press, Cambridge.

[34] Stentoft, L. (2004) *Convergence of the Least Squares Monte Carlo Approach to American. Option Valuation.* Management Science.

[35] Trabelsi, F. (2011) *Asymptotic Behavior of Random Maturity American Options.* IAENG International Journal of Applied Mathematics, **41**, 2.

[36] Weron, A and Weron, R. (2009) *Inżynieria finansowa*, Wydawnictwo Naukowe PWN.

[37] Wilmott, P. (2006.) *Paul Wilmott On Quantitative Finance.* John Wiley & Sons.

[38] Wu, Z. and Li, L. (2022) *The American Put Option with a Random Time Horizon.* arXiv:2211.13918.

[39] Wu, Z. and Li, L. (2022) *The Russian Option with A Random Time Horizon.* arXiv:2211.13917v1.

[40] Zaevski, T. (2020) *Discounted perpetual game call options.* Chaos, Solitons & Fractals **131**.

[41] Zaevski, T. (2020) *Discounted perpetual game put options.* Chaos, Solitons & Fractals **137**, 109858.

[42] Yamazaki, K. (2017) *Phase–Type Approximations of The Gerber–Shiu Function.* Journal of the Operations Research Society of Japan.