Politechnika Wrocławska

FIELD OF SCIENCE: SCIENCE AND NATURAL SCIENCES

DISCIPLINE OF SCIENCE: MATHEMATICS

DOCTORAL DISSERTATION

Modeling Systemic Risk through Reduced-Form Dynamics of Bonds in Financial Networks

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Keywords: systemic risk, financial networks, cascading failure, crisis contagion, reduced-from models

WROCŁAW 2025

Contents

Abstract

How is a state likely to fail? The state is a cultural construct inherently tied to the very definition of civilization, and consequently to the concept of humanity itself. It represents a somewhat nebulous abstract notion, lacking a universally accepted academic definition, yet it significantly shapes the real lives of both societies and individuals. Thus, it is not surprising that the question of state survival has sparked considerable academic interest across time and space, from ancient Greece to pre-Columbian America and into contemporary Greece. An even more interesting problem lies in the sequences of failures experienced by multiple states, such as Ireland, Portugal, and Greece. The underlying causes of these phenomena stem from relational structures that are much older and more fundamental than the state itself, defined by the concept of money. Understanding how the activities of private institutions in America can trigger a wave of public institutional failures across the ocean can be achieved by examining these structures. Identifying the primary sources of risk enables the assessment of threats on a global scale.

These issues, both qualitatively and quantitatively, form the core of the discipline of systemic risk. It is defined as the threat of failure of the entire financial system or a significant part of it. Two main characteristics distinguish this discipline within the realm of financial mathematics: the endogeneity of risk and amplification mechanisms. The first characteristic indicates that the sources of failure are an integral part of the architecture of the systems themselves, allowing for the examination of these systems in terms of the likelihood and scale of the threats they pose. Amplification mechanisms describe the process of spreading and intensifying shocks when deteriorating institutional conditions mutually influence each other in a potentially infinite cycle of financial decay. However, the emphasis on these two characteristics has led to a widespread neglect of key factors shaping the price dynamics of financial instruments traded on global markets.

The aim of this dissertation is to develop new models of systemic risk that incorporate these factors, enabling a more realistic assessment of actual risk. Mathematically, this is achieved by integrating a rich class of reduced-form models used for pricing financial instruments with methods for calculating systemic risk. At the beginning of the thesis, it is demonstrated that that even a simple interest rate model can capture the fundamental mechanisms of amplification and the transition of the system into a crisis state. This model, derived from fundamental processes in the credit market, is further enhanced by transitioning to continuous time and adding random noise to represent unpredictable fluctuations. The resulting paths of interest rate evolution can exhibit calm behavior for a time, followed by turbulence in the long run, which presents an intriguing property in the context of market modeling and the challenges of recognizing risk of collapse.

Subsequently, a general framework for the pricing process of instruments in financial networks is presented. It facilitates the creation of a class of reduced-form network models, allowing for the construction of specific models that account for key factors in the pricing of debt instruments, such as creditworthiness and the dynamics of interest rates in the banking market. Notably, the latter played a fundamental role in the development of the 2008 crisis and its subsequent spread to the Eurozone. Its significance is confirmed by a wave of bankruptcies on a scale impossible to replicate without its inclusion in the model, in the case of simulations performed on American banks data from 2023.

Current network-based risk assessment methods primarily rely on subjecting the system to an initial shock and examining its evolution, which is then completely determined by the amplification mechanisms. However, in reality, the market may experience random fluctuations alongside the effects of systemic propagation at any given moment. A class of stochastic network models is constructed to capture this behavior. The analysis conducted using rigorous mathematical formalism yields highly useful results, enabling the calculation of key risk indicators used in determining reserve levels and investment strategies.

The culmination of this doctoral thesis is a model synthesizing the most important findings obtained throughout the dissertation. This model is built upon the class of stochastic network models from chapter five, integrating the deterministic, comprehensive network model from chapter four with the stochastic process of unstable dynamics in the credit market developed in chapters one and two. The combination of these components significantly impacts the calculations of fundamental risk indicators. This approach allows for a broad examination of risk aspects crucial to the dynamics of various crises, particularly the American banking crisis of 2023. The significance of research on detecting the vulnerabilities of financial systems is felt far beyond the circle of entities directly interested in market dynamics, as vividly illustrated by the example of Russia exploiting Europe's financial weaknesses and excessive energy exposure during its invasion of Ukraine.

Streszczenie

Jakie są szanse że państwo upadnie? Państwo jest tworem kultury zawartym w samej definicji cywilizacji, a w konsekwencji nierozerwalnie związanym z pojęciem ludzkości jako takiej. Stanowi ono dość mgliste pojęcie abstrakcyjne o definicji pozbawionej akademickiego konsensusu, niemniej w dużej mierze definiującej rzeczywiste życie tak społeczeństw, jak i pojedynczych osób. Nic zatem nadzwyczajnego w fakcie, że zagadnienie jego przetrwania wzbudza żywe zainteresowanie tych osób na przestrzeni wieków, od starożytnej Grecji, przez prekolumbijską Amerykę, aż do współczesnej Grecji. Jeszcze bardziej interesującym problemem są ciągi upadłości wielu państw, jak na przykład Irlandii, Portugalii, czy Grecji. U podstaw tego rodzaju zjawisk leżą struktury relacji o wiele starszych i bardziej fundamentalnych niż samo państwo, relacji definiowanych przez pieniądz. Zrozumienie w jaki sposób działalność instytucji prywatnych w Ameryce może przynieść falę zapaści instytucji publicznych za oceanem można uzyskać poprzez badanie tych struktur, a identyfikacja głównego źródła ryzyka umożliwia szacowanie zagrożeń o skali globalnej.

Zagadnienia te w ujęciu jakościowym i ilościowym stanowią oś zainteresowania dyscypliny ryzyka systemowego. Jest ono definiowane jako zagrożenie upadłości całego systemu finansowego bądź istotnej jego części. Dwie główne cechy charakterystyczne stanowią o odrębności tej dyscypliny w obszarze matematyki finansowej - endogeniczność ryzyka i mechanizmy amplifikacji. Pierwsza z nich oznacza, że źródła zapaści stanowią nieodłączny element architektury samych systemów, co daje możliwość badania tych ostatnich pod kątem szans realizacji i skali zagrożeń jakie ze sobą niosą. Mechanizmy amplifikacji opisują proces rozprzestrzeniania się i nasilania szoków w sytuacji, gdy pogarszające się stany instytucji wzajemnie na siebie wpływają w cyklu postepującego rozpadu finansowego. Niemniej, położenie nacisku na te dwie cechy doprowadziło do powszechnego zaniedbania kluczowych czynników kształtujących dynamikę cen instrumentów finansowych handlowanych na światowych rynkach.

Celem tej pracy jest opracowanie nowych modeli ryzyka systemowego, które uwzględnią te czynniki, co pozwoli na bardziej realistycznego szacowanie faktycznego ryzyka. Matematycznie jest to osiągane poprzez integrację bogatej klasy modeli formy zredukowanej, służących do wyceny instrumentów finansowych, z metodami kalkulacji ryzyka systemowego. Na początku zademonstrowane zostaje, że nawet prosty model stóp procentowych jest w stanie uchwycić podstawowe mechanizmy amplifikacji oraz tranzycji systemu do stanu kryzysowego. Model ten, wyprowadzony z fundamentalnych procesów rynku kredytowego, jest następnie rozwijany poprzez przejście do czasu ciągłego oraz dodanie szumu losowego, reprezentującego nieprzewidywalne fluktuacje. Uzyskane w ten sposób ścieżki ewolucji stóp mogą się zachowywać bardzo spokojnie przez pewien czas oraz burzliwie w dłuższej perspektywie, co stanowi interesującą własność w kontekście modelowania rynków oraz trudności w rozpoznawaniu ryzyka zapaści.

W dalszej kolejności wprowadzony zostaje ogólny szablon procesu wyceny instrumentów w sieciach finansowych, przy pomocy którego tworzona jest klasa sieciowych modeli formy

zredukowanej. Umożliwia to skonstruowanie konkretnych modeli, które uwzględniają kluczowe czynniki wyceny instrumentów dłużnych, takie jak wiarygodność kredytowa oraz dynamika stóp procentowych na rynku bankowym. W szczególności ta ostatnia miała fundamentalne znaczenie dla rozwoju kryzysu w 2008 roku i jego późniejszego rozprzestrzenienia się do strefy euro. Istotność stóp procentowych znajduje potwierdzenie w symulacjach wykonanych na danych banków amerykańskich z roku 2023, których wynikiem jest fala upadłości niemożliwa do uzyskania bez ich uwzględnienia w modelu.

Dotychczasowe metody sieciowe oceny ryzyka opierają się głównie na poddawaniu systemu początkowemu szokowi i badanie jego ewolucji, która jest następnie całkowicie determinowana przez mechanizmy amplifikacji. W rzeczywistości jednak rynek może podlegać wahaniom losowym równolegle z efektami rozprzestrzeniania systemowego w każdej chwili czasu. W celu uwzględnienia tych zachowań, skonstruowana zostaje klasa sieciowych modeli stochastycznych. Analiza wykonana przy użyciu ścisłego formalizmu matematycznego prowadzi do uzyskania wysoce użytecznych rezultatów, które umożliwiają obliczenie kluczowych w praktyce finansowej wskaźników ryzyka, używanych do ustalania wysokości rezerw i strategii inwestycyjnych.

Zwieńczeniem pracy doktorskiej jest model stanowiący syntezę najważniejszych wyników uzyskanych wcześniej. Jest on stworzony na bazie klasy sieciowych modeli stochastycznych z rozdziału piątego, przy pomocy której połączone zostają deterministyczny, kompleksowy model sieciowy z rozdziału czwartego oraz proces stochastyczny niestabilnej dynamiki rynku kredytowego zbudowany w rozdziałach pierwszym i drugim. Połączenie tych komponentów ma znaczący wpływ na wyniki kalkulacji fundamentalnych wskaźników ryzyka. Takie podejście pozwala na szerokie ujęcie aspektów ryzyka kluczowych dla dynamiki wielu kryzysów, w szczególności dla amerykańskiego kryzysu bankowego w 2023 roku. Znaczenie badań nad wykrywaniem wrażliwości systemów finansowych jest odczuwalne dalece poza kręgiem podmiotów ściśle zainteresowanych dynamiką rynków, jak dobitnie pokazał przykład Rosji wykorzystującej słabość finansową i nadmierną ekspozycję energetyczną Europy przy inwazji na Ukrainę.

Introduction

Silvergate Bank, Silicon Valley Bank, Signature Bank. On March 7, 2023, all of them were present on the registry of banks in the United States; on March 13, none remained. Together with the subsequent collapse of First Republic Bank, the U.S. financial system witnessed its second, third, and fourth largest bank failures in history, all occurring within a span of less than two months. Throughout this crisis, considerable uncertainty surrounded the status of these institutions. On March 12, after consulting with the President, three major financial regulatory agencies—the Department of the Treasury, the Federal Reserve System, and the Federal Deposit Insurance Corporation—issued a joint statement declaring a "systemic risk exception" as the justification for seizing two of these banks.

Systemic risk is the possibility of a widespread financial system failure, as opposed to collapses of its individual parts and components [1]. The joint definition of three international financial governing bodies, - International Monetary Fund, Bank for International Settlements, and Financial Stability Board - refers to the disruption of financial services caused by impairments within the financial system, which can have significant negative effects on the real economy [2]. An overview article from the Systemic Risk Center explains that systemic risk includes both the risk to the proper functioning of the financial system and the risks generated by the system itself, offering a formal definition based on the characteristics and states of the system [3].

This definition refers to two essential aspects of systemic risk: endogeneity of risk and amplification mechanisms. The endogeneity postulate claims that crises emerge from the internal dynamics of the system under investigation, rejecting the notion of external shocks as the main factor driving the crisis. It does not mean that external shocks cannot occur at all; they often serve as an impulse triggering a crash, but a vast majority of the loss should come from the system itself [4–6]. Different systems can be analyzed over time, particularly regarding how their dynamics translate into the evolution of risk [5–7]. At the model's limit point, even the smallest shocks are no longer absorbed, but lead to potential damage on an infinite scale [4,7]. This approach allows for the modeling, examination, and management of crises, including their risks, causes, and evolution [3] by not treating them as events that simply occur randomly beyond control and the capabilities of computational or analytical description. The amplification mechanisms arise from a broad range of relationships existing between entities of financial systems. The systemic collapse emerges as a result of crisis contagion and cascading failures [1,8], where devaluation of one or a few components leads to the failure of others, progressively worsening due to feedback loops created by various dependencies [9]. This contrasts with common modeling of dependencies with the use of correlations, which often implies that variables evolve nearly independently in extreme scenarios [10], while in reality financial instruments tend to fall together during crises [11]. Because of amplification mechanisms, even small shocks can result in significant losses and endanger the entire system [1,12], despite the system appearing stable according to other well-established measures [12,13]. Of particular importance here are critical points, which indicate crucial moments of abrupt qualitative changes, delineating scenarios of smooth system functioning from its complete collapse [14].

Understanding and managing systemic risk is crucial for maintaining economic stability and mitigating potential collapse consequences. Banks worldwide use systemic risk calculations to determine reserve requirements, which in turn influence interest rates and money supply. The fundamental importance of this research area to modern society was recently recognized by the 2022 Nobel Prize in Economics, awarded to Bernanke, Diamond, and Dybvig for their work on bank failures [15].

The context of research

Systemic risk research utilises complex mathematical tools [16] to analyze very improbable events and their probable consequences [14]. Its main challenge is the fact that the vast majority of such events will never occur, so their probability and potential impact must be assessed on the basis of the very sparse set of real-world observations and simulations of unrealised outcomes [17]. The tools include network models of financial contagion and phase transitions [18], self-fulfilling prophecies with possible switches between multiple equilibria [15], and complex computer calculations to simulate potential trajectories [19]. Network models are used to define the measures of systemic risk [20, 21] and the methods of their computation on real data [22, 23]. The measures are then utilised to propose methods of risk optimization in financial systems [24]. The solutions include the usage of financial instruments such as credit default swaps to effectively reorganize the network structure towards a safer one [25] and contingent convertible obligations to improve banks' solvency and survival ability [26]. The developed methodologies are, in turn, used to improve decision making on the management of financial systems in the real world [19]. They may also be applied to the analysis and valuation of the credit risk and contagion effects of reinsurance companies [27], as well as to the identification of systematically important entities in the credit network of the entire state [28].

From the historical perspective, the endogeneity postulate was part of the Financial Instability Hypothesis [29] formulated by Hyman Minsky, which constitutes a significant conceptual milestone in the emergence of systemic risk studies. It posits that financial crises are inherently embedded in the dynamics of financial systems, as periods of economic stability foster increased risk-taking among investors. The hypothesis outlines three stages of borrower risk—hedge, speculative, and Ponzi finance—demonstrating how excessive borrowing during optimistic times inevitably leads to instability. Minsky's theories have often been integrated into mathematical formalism to explain the emergence of financial crises [30]. The microeconomic foundations of balance sheet structures in credit markets frequently serve as the basis for deriving models of crisis contagion [31]. In particular, a positive feedback loop between the formation of successful entities and favorable market conditions has been explored in various studies [32–34]. A regime-switching model has been developed to describe different states of the economy, utilizing a specific definition of the investment function that varies across these states to capture the phenomenon of financial instability [35]. This model formalizes Minsky's distinction between hedge, speculative, and Ponzi companies through relationships among their financial characteristics, a methodology that has been further refined in subsequent research [6, 36, 37]. Building on this framework, the interactions among the banking system, debt, and interest rates have been incorporated into a broader economic model, which includes variables such as GDP growth, inflation, and labor share [35]. This model analytically derives equilibrium points based on the condition of expectation fulfillment and is quantitatively evaluated using stochastic simulations [35].

Similar research has demonstrated that integrating Minsky's concepts into Goodwin's economic modeling framework alters the system's behavior from stable and cyclical to chaotic, increasing the risk of divergent collapse [38]. A stochastic approach has also been employed to analyze the yield dynamics of European government bonds, utilizing a hybrid Heston model with a common stochastic volatility component to generate early-warning indicators for unstable phases [39]. Additionally, a family of power law models has been defined to describe different economic regimes, evaluating equilibrium points and their stability to distinguish between favorable and dangerous states of the economy [6].

Network models are particularly important in this discipline, as they enable to describe various types of relationships between financial entities, which can serve as a medium for distress contagion during crisis events. Historically, these models have been developed using structural financial frameworks, particularly to capture the networks of holdings and transactions among financial institutions. Starting with Merton's theoretical framework [40], structural asset pricing models utilize balance sheet data to evaluate company valuations. These models treat financial instruments issued by companies as derivatives of their balance sheet metrics, employing established techniques such as the Black-Scholes option pricing formula to assess company stocks [40]. This approach was naturally extended to networks of cross-holding companies through methods developed by Furfine [41], Eisenberg-Noe [42], and Bardoscia [4], as financial instruments held constitute a part of a company balance sheet data.

Although these established algorithms have provided foundational principles for understanding network interactions and crisis propagation, they primarily focus on either the equilibrium state resulting from crisis propagation following an initial shock [14, 21, 43, 44] or the value recovery process of defaulted entities [45, 46]. This approach has several limitations compared to other financial models and risk analysis processes. Firstly, it tends to overlook the complex aspects of valuing debt instruments, placing greater emphasis on the process of stress contagion. Key factors such as recovery rates, the time structure of debt, credit quality, and interest rates are often neglected, despite being fundamental components of debt valuation in financial mathematics [47–49]. These valuation aspects are crucial for understanding market evolution and can significantly impact the calculation of systemic risk, as shown in our research [50].

While the recovery rate has been integrated into several algorithms, it is frequently excluded from calculations [7, 44, 51]. It is a key factor in the models developed by Eisenberg-Noe [42] (EN), Suzuki [46], and Fischer [45]; however, these models primarily concentrate on the clearing process and the establishment of creditor claims during defaults, which tends to prioritize debt recovery over the dynamics of crisis contagion and systemic collapse [44]. Furfine's research [41] shares similar shortcomings, as it considers valuation dynamics only in the context of actual defaults. Nonetheless, recovery remains a vital aspect of credit assessment, especially for government-backed banks. Regulators typically step in to prevent the total loss of citizens' funds and to avert financial system collapse, as illustrated by the Swiss Finance Minister's comment on UBS's acquisition of Credit Suisse: "I myself am grateful as a customer that this worked out" [52].

Moreover, there is a notable lack of flexibility in modeling the diverse credit qualities, which are essential components that enable to valuate bonds across different companies based on their creditworthiness [47–49]. Fixed valuation rules do not adequately differentiate between varying scenarios; the largest bank in a robust economy may exhibit the same devaluation dynamics as a newly established local credit institution in a struggling financial environment.

Future cash flows are generally assessed using discount factors derived from interest rates [47, 53], which are significantly affected by the overall health of the banking system [54–56]. Interest rates, by their nature, are influenced by the pricing dynamics of debt in the interbank lending market [57, 58]. This interplay shapes the broader credit market and further impacts banks' lending policies. These interconnections form a critical mechanism that drives the dynamics of the financial system and can elucidate credit cycles marked by sudden downturns, as demonstrated in our previous work [59].

The maturity component captures the passage of time and enables differentiation across various tenors [47]. This aspect is notably lacking in earlier methods, including DebtRank [4, 7, 60], with the geometric Brownian motion model of external asset dynamics presented by Barucca et al. [44] and the multilayer network approach by Brummitt and Kobayashi [61] being significant exceptions. Moreover, in reality, a random shock can impact the system not just at a single point in time but throughout the entire evolution of the market, especially given that risk increases sharply as the time horizon extends [47, 62]. To highlight the importance of this factor, multiplying the remaining time to maturity by a constant a effectively applies the valuation function multiplier a times. Typical bond maturities can vary from three months to ten years, with some bonds maturing in as little as one day and others extending beyond thirty years [53]. Therefore, overlooking the evolution of time poses a significant limitation to the model and greatly restricts its applicability in areas such as risk metric derivation, reserves management, and strategic planning.

The goal and contents of the thesis

The aim of this thesis is to develop more realistic models of systemic risk that incorporate financial mathematics of dynamic debt valuation and solvency analysis. From a technical perspective, this is primarily achieved through the integration of reduced-form models [47, 49, 63] with methods of systemic risk. Chapter 1 is dedicated to the construction of a simple, deterministic model of the credit market dynamics, that satisfies the key postulates of systemic risk: the endogeneity of risk and the amplification mechanism [3]. It is illustrated that highly complicated models are not necessary to capture the very basic characteristics of systemic risk. Although such a model may be considered overly simplistic for providing a comprehensive description of the complex financial system, it is still deemed useful. Warning signals of impending crises can be provided and an analytical intuition can be developed to understand the behavior of the actual financial system, even at a high level of generality and simplification. In addition, it is regarded as an excellent starting point for the development of more complex models that are better suited for application to real, intricate financial systems [13].

The model developed in Chapter 1 serves as a basis to construct the stochastic, continuous-time, short-rate model in Chapter 2. A concise yet thorough introduction to the general theory of debt instrument valuation in financial mathematics is presented to embed the model in a broader context of the discipline. Although its construction may seem counterintuitive at first glance, it is analyzed in detail to reveal intriguing properties. Specifically, compared to similar models typically considered, it can behave in a more stable manner for a long time under appropriate parameter selection, while being ultimately unstable in the infinite time limit. This behavior reflects a crucial characteristic of financial systems: they can appear very stable for prolonged periods only to fail into the severe crises ultimately.

Following this, the standard tools for modeling systemic risk in complex financial systems, which consist of many interconnected entities, are introduced in Chapter 3. In particular, a highly useful and general framework known as the Network Valuation of Financial Systems (NEVA), developed by Barucca et al. [44], is presented. While this framework is indeed broad, simplistic cases are often employed for modeling and calculation purposes. Subsequently, the general framework for modeling debt instruments, introduced alongside the stochastic interest rates model, is integrated into NEVA in Chapter 4. This integration results in a model capable of incorporating critical factors of debt valuation into the dynamics of complex networks. It is shown that it has a significant influence on the security assessment of the real financial system of the United States.

In the network modeling of financial systems, the iterative procedure of network evolution is mainly a tool to calculate the point of equilibrium after the initial shock, which is not necessary realistic. To account for those deficiencies, a complex process of system dynamics is constructed in Chapter 5, with stochastic fluctuations incorporated into the network propagation at every recursive step of the model evolution. It is followed by the construction and analysis of the continuous equivalent of such a process, and in consequence by the derivation of powerful analytical tools for the calculation of key risk metrics applied in finance.

In Chapter 6, the most important results obtained up to this point are combined into a comprehensive model of systemic risk. The framework constructed in Chapter 5 allows to integrate the stochastic evolution of the credit market developed in chapters 1 and 2 with the reduced-form network model with interest rate feedback from Chapter 4, and to compute the metrics of financial risk on its basis. This approach enables the analysis of not only the influence of individual angles of systemic risk on the behavior of the financial system in the United States but, more importantly, the effects arising from their interactions.

Chapter 1

An unified framework for modeling credit cycles and systemic risk assessment

This chapter is based on the article written by the author of this thesis [59]. The Financial Instability Hypothesis [29] of Hyman Minsky represents a major conceptual breakthrough in the field of systemic risk studies. It states that crashes originate inherently from the dynamics of financial systems, as stable economic periods encourage increased risk attitude among investors. The hypothesis identifies three types of borrower risk—hedge, speculative, and Ponzi finance—showing how exaggerated, overoptimistic borrowing creates instability. Ideas of Minsky have been shaped many times into mathematical models of financial markets, which rich history has been excessively described in the Introduction to this thesis. However, existing models exhibit several limitations, mainly due to deficiencies in capturing the endogeneity of risk, particularly in the process of transitions between economic states. These limitations manifest in equations that are inconsistent across the cycle or rely on phenomena that are not part of those equations, such as stochastic noise or collective market behavior. Many approaches have in fact different models for different regimes. For example, Solomon [6] describes a phase of growth as a dynamic relationship between the volume of loans in the economy and the interest rate, and a phase of crisis as a dynamic relationship between the amount of Ponzi companies and the interest rate. Similarly, Ferri [35] distinguishes between growth and crisis phases using different equations for the investment function. This inconsistent approach can obscure the reasons and mechanisms of economic transitions. It fails to provide a clear, coherent, and comprehensive model capable of describing all economic phases within a single, unified methodology. Moreover, the existing models often rely on external sources to explain transitions between states. For instance, Hohnisch [64] models market sentiment swings as a result of influence dynamics within groups, an approach also applied by Solomon [6], who use the density threshold of Ponzi companies as a basic trigger of the crisis transition. The impact of collective behavior on the market is also considered by Cantono [65] and Biondi [66]. Another approach to modeling crashes involves stochastic noise, which can carry on the transition from the growth phase to the crisis on the back of the random effects. Such a method is employed in the works of Ferri [35] and Solomon [6], by the adaptation of ideas proposed in Refs. [5, 67, 68]. A notable exception here is the work of Keen [38], who handles stability transitions within a consistent model without relying on stochastic noise. However, Keen's model involves numerous economic variables and dependencies, including labor characteristics, wages, and investment dynamics. While computer simulations are conducted within a specific parameter space, analytical formulas characterizing different economic regimes are not derived. This limitation reduces the model's explanatory power, particularly in the broad context defined by a multidimensional set of variables. Furthermore, Keen's research does not address the risk of bubble formation, where falling interest rates and growing debt mutually amplify, potentially leading to a long-term unsustainable regime and significant collapse risk [69]. Indeed, entities that expanded their leverage without the coverage in increased earnings endangered themselves of becoming Ponzi at the minimal increase in the interest rate, which constituted a significant factor in the global financial crisis and a topic for regulatory reforms [70, 71].

This chapter aims to provide a coherent model of the credit cycle using a single system of equations. The model enable the depiction of the phases of stable growth, bubbles, and crises, as well as the transitions between them. The research builds on the work of Solomon and Golo [6], who constructed a family of power law models describing the dynamics of the interest rate formation process, with each model valid in different financial states. Compared to that approach, in the proposed model the transitions between states emerge from the mutually amplifying interactions of various components of the credit market. Moreover, risk measures are derived from the model to identify the system threats and assess potential collapse magnitudes. Hence, results obtained may be important in financial security research, as they provide a comprehensive mathematical, interpretable rationale for regime shifts, particularly during financial crises.

The chapter is structured as follows. Section 1.1 introduces the model with the mathematical analysis of its properties. Section 1.2 focuses on the topic of phase transitions and the derivation of system stability measures. Section 1.3 provides an example of a credit cycle generated from the model with details on the real-world interpretation of parameters and their economic consequences. A summary of the research results is presented in Section 1.4.

1.1 Models and methods

1.1.1 Original models of Solomon and Golo

A credit cycle represents the expansion and contraction of credit in the economy over time [72]. In order to describe the dynamics of the interest rate across its stages, schematically shown in Figure 1.1, Solomon and Golo use a family of power law models, which may be divided into two classes [6]. The first one is applicable before the Minsky moment (i.e., the onset of a crisis) of systemic collapse and states that the interest rate i charged for the credit at time t decreases along with the scale of the banking sector expansion, expressed as amount of loans in the economy N:

$$i(t+1) = i_0 N^{-\alpha}(t), \tag{1.1}$$

where α is the sensitivity of the interest rate to the volume of loans and i_0 is the initial interest rate at time $t = 0$.

The influence of the interest rate on the number of loans in the economy is a well-established paradigm, theoretically expressed in the Hansen-Hicks IS-LM model [73]. A decrease in interest rates typically stimulates borrowing, lowering the cost of financing investment projects, and thus increasing the demand for loans. Conversely, higher interest

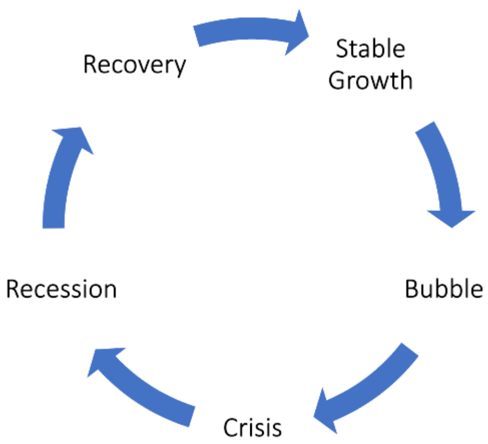


Figure 1.1: An example credit cycle with various phases.

rates may discourage borrowing, dampening investment activity. Hence, the following dependency of the amount of loans N on the interest rate i is assumed:

$$N(t) = \left(\frac{i(t)}{k}\right)^{-\mu}. \tag{1.2}$$

Here, μ describes the level of demand for credit and k is a technical scale parameter that can be used to tailor the model to specific situations. As a consequence, the equations defining the model of the credit market during the phase of loans expansion (the loans accelerator model) are given by

$$\begin{aligned} N(t) &= \left(\frac{i(t)}{k}\right)^{-\mu}, \\ i(t+1) &= i_0 N^{-\alpha}(t). \end{aligned} \tag{1.3}$$

This model describes the phases of recovery, stable growth, and excessive growth, where an increase in the money supply and price levels can lead to the formation of a speculative bubble, which is unsustainable in the long term and may result in a severe crash [74].

The model of the crisis phase is similar to the one of the expansion phase, but here the number of defaulted companies, D, primarily drives the interest rate. It is assumed that a company cannot borrow for the purpose of the debt repayment, and therefore defaulted and Ponzi companies are technically treated as being the same. Hence, the dynamics of the interest rate with an exponential parameter β is defined as follows:

$$i(t+1) = i_0 D^{\beta}(t). \tag{1.4}$$

It is natural to assume that the interest rate is proportional to the number of defaults, as banks would quote higher prices for credit to compensate for the increased risk of non-repayment [49]. On the other hand, the evolution of defaults is derived from the assumption of power law distribution of debt and earnings, which is supported empirically [75, 76]. Therefore it is assumed that companies with a resilience

$$r = \text{earnings}/\text{debt}, \tag{1.5}$$

are distributed in an economy according to a density

$$f(r) = \frac{\nu}{l}\left(\frac{r}{l}\right)^{\nu-1}, \tag{1.6}$$

where $r \in [0, l]$, $\nu > 0$ describes the level of defaults in relation to the interest rate and $l > 0$ is the scale parameter that can be used to calibrate the model to particular scenarios. As a result, the number of defaulted (Ponzi) companies with $r < i$ equals

$$D(t) = \left(\frac{i(t)}{l}\right)^{\nu}, \tag{1.7}$$

Finally, the equations defining the model of the credit market during the phase of rapid contraction (the crisis accelerator model) are given by

$$\begin{aligned} D(t) &= \left(\frac{i(t)}{l}\right)^{\nu}, \\ i(t+1) &= i_0 D^{\beta}(t). \end{aligned} \tag{1.8}$$

The work by Solomon and Golo may be seen as an attempt to integrate the Financial Instability Hypothesis [29] with a neoclassical market equilibrium methodology, focused on the determination of equilibrium points that define the state of the market and assessing their stability. It is referred to as Marshall-Walras Equilibrium (MWE) since it is built on the foundation of general economic equilibrium analysis, pioneered by Marshall and Walras [77]. Consequently, the model of Solomon and Golo will be called the MWE model throughout this work. It is designed to describe the dynamics of the loans accelerator phase and the crisis accelerator phase separately. Two presented subparts are not valid simultaneously, because the evolution of interest rate is defined differently in each of them, in order to model qualitatively very different cases of growth (Eq. (1.3)) and collapse (Eq. (1.8)). Moreover, the model does not factor in defaults and their influence on the interest rate during the loan accelerator phase (Eq. (1.3)), and, similarly, it does not factor in loans and their influence on the interest rate during the crisis accelerator phase (Eq. (1.8)). From the presented dynamics, it is not clear which subpart is valid at a given point in time t, and how the transition between them occurs. The systemic event of the transition (particularly the collapse of the financial system) is the main point of interest of this chapter.

1.1.2 Derivation of the unified model

The goal here is to unify the equations (1.3)-(1.8) into a single, consistent model (referred to as the unified MWE model, UMWE in short, throughout the rest of this work) that binds all variables of interest altogether and describes their relations at each point in time. In contrast to the original model, evolution of each variable is defined by only one type of equation valid throughout all the time. Such an approach allows to model the transitions between different phases of the credit cycle, particularly the collapse of the financial system. The proposed solution is as follows:

$$\begin{aligned} N(t) &= \left(\frac{i(t)}{k}\right)^{-\mu}, \\ D(t) &= \left(\frac{i(t)}{l}\right)^{\nu}, \\ i(t+1) &= \frac{D^{\beta}(t)}{N^{\alpha}(t)}, \\ i(0) &= i_0. \end{aligned} \tag{1.9}$$

where:

- t – a discrete moment in time;
- $N(t)$ – the volume of loans at time t;
- $D(t)$ – the volume of defaults occurred in a period $(t-t, t]$;
- $i(t)$ – the value of the interest rate at time t;
- μ – the level of demand for credit in relation to the interest rate;
- ν – the level of defaults in relation to the interest rate;
- α – the sensitivity of the interest rate to the volume of loans;
- β – the sensitivity of the interest rate to the volume of defaults;
- $k,\ l$ – scale parameters;
- i_0 – initial value of the interest rate.

Define the set of utilised model parametrization as $\Lambda := \{\alpha, \beta, \mu, \nu, k, l\} \in \mathbb{R}^6_+$. As k, l are technical scale parameters, they are not of big interest in the investigation. Therefore, throughout this chapter, Λ is considered primarily as $\Lambda = \ \{\alpha, \beta, \mu, \nu\}$.

The exponents in Λ describe various economic quantities of the real world. The parameter μ reflects the dependence of the volume of loans on the interest rate and therefore the demand for credit for different levels of the interest rate. On the other hand, ν captures the relation between the volume of defaults and the interest rate, so it can be interpreted as the indicator of the resilience of a particular economy: systems with different values of the parameter ν are characterized by distinct default tendencies and fragility. The parameters α and β describe the dynamics of the interest rate determination in response to the volumes of loans and defaults on the market.

The dynamics of loans and defaults are taken directly from the original model. The interest rate is modeled in a natural way often met in finance as the (exponentiated) ratio of defaults to loans. Consider the case

$$\alpha = \beta = 1, \tag{1.10}$$

which implies that

$$i(t+1) = \frac{D(t)}{N(t)}. \tag{1.11}$$

This is a very simple and intuitive parameterization, where the interest rate equals the estimated probability of default D_t/N_t. Moreover, it is also equal to the expected rate of return from the 'bare' credit with the same amount of loan and repayment, and without any premium for the lender. Indeed, consider a loan with the amount of M and the same amount M to repay at maturity T, without any interest. The expected rate of return from such a loan with the estimated probability of default D_t/N_t and the recovery rate of 0 equals

$$r(t+1) = \frac{1}{M}\left(M\left(1 - \frac{D(t)}{N(t)}\right) - M\right) = \left(1 - \frac{D(t)}{N(t)}\right) - 1 = -\frac{D(t)}{N(t)}. \tag{1.12}$$

Therefore, in order to not systematically lose money, banks charge the price for granted credit. Thus, the fair price of such a credit, namely the interest rate, could be considered as

$$i(t) = -r(t). \tag{1.13}$$

The addition of parameters α and β enhances the flexibility of the model and allows to reflect phenomena such as various market sentiments and premiums charged by banks for their offers.

It is worth noticing that if two of the power indices (either α and μ or β and ν) in Eq. (1.9) are set to 0, the result is effectively a two-equation model very similar to the original MWE. The rate i_0 is utilised in the original Equations (1.3)-(1.8) as a scale parameter that anchors the evolution dynamics of the interest rate to the starting point. This causes i_0 to prevail in the model dynamics at each point in time and significantly influences the interest rate evolution even in the distant future. The exclusion of i_0 from the iterative equation defining a single step dynamics of the interest rate ensures that the dynamics of newly proposed model (Eq. (1.9)) does not exhibit prevailing dependency on the arbitrarily chosen initial moment. More formally, the dynamics of the interest rate in the new model is defined by a recursive equation of the general form

$$i(t+1) = g_1(D(t), N(t)) = f_1(i(t)). \tag{1.14}$$

On the other hand, in previous approaches, the equations took the form

$$i(t+1) = g_2(D(t), i_0) = f_2(i(t), i_0) \tag{1.15}$$

or

$$i(t+1) = g_3(N(t), i_0) = f_3(i(t), i_0). \tag{1.16}$$

Here, $f_k, g_k,\ k \in \{1,2,3\}$, denote a functional dependency. Define a new process $\tilde{\imath}$ with the same specification of dynamics as the original i, but with a different choice of the initial moment $s \in \mathbb{N}_+$:

$$\begin{gathered} \tilde{\imath}(s) = i(s), \\ \tilde{\imath}(u+1) = f(i(u), i(s)), \end{gathered}$$

where $u \in \{s, s+1, s+2, ...\}$. While in the new model it is satisfied that

$$\tilde{\imath}(s+1) = f(i(s), i(s)) = f_1(i(u)) = i(s+1), \tag{1.17}$$

in the old approach if $i(s) \neq i_0$, then

$$\tilde{\imath}(s+1) = f(i(s), i(s)) \neq f_k(i(s), i_0) = i(s+1), \tag{1.18}$$

where $k \in \{2, 3\}$. Therefore, a trajectory of the interest rate in the old approach can change if a different moment s of this trajectory is assumed to be an initial one. As such, different trajectories can be obtained for each point of the model defined as a starting moment. Moreover, a procedure of resetting the initial point can be repeated iteratively for newly obtained trajectories, and generate subsequent different trajectories. Such a property can raise serious concerns about the reliability of the model. It becomes particularly problematic when one of the main points of interest are transitions between qualitatively different economic regimes. It would be very inconvenient to handle the state transitions with the long-lasting dependency of the model dynamics on the predefined initial moment. The exclusion of i_0 from the iterative equation defining the single step dynamics of the interest rate $i(t+1)$ in Eq. (1.9) ensures that the value of the variable of interest in the next period depends only on the current one in the history, thus allowing for a very natural introduction of the regime switches.

1.1.3 Mathematical analysis of the interest rate model

Inserting defaults and loans into the interest rate definition in Eq. (1.9) yields a recurrence formula for $i(t)$,

$$i\left(t+1\right) = \left(l^{-\beta\nu} k^{-\mu\alpha}\right) \left(i(t)\right)^{\alpha\mu+\beta\nu}. \tag{1.19}$$

The fixed point i_{fix} of the recurrence equation must satisfy

$$i(t+1) = i(t) = i_{fix}. \tag{1.20}$$

Hence, it follows that

$$i_{fix} = \left(k^{-\alpha\mu} l^{-\beta\nu}\right)^{\frac{1}{1-(\alpha\mu+\beta\nu)}}. \tag{1.21}$$

The fixed-point formula in the MWE model (Eq. (1.3)) reads

$$i_{fix}^{MWE} = \left(i_0 k^{-\alpha\mu}\right)^{\frac{1}{1-\alpha\mu}}. \tag{1.22}$$

Thus, the fixed point in the unified model does not feature the dependency on the initial value of the interest rate i_0. This is a noticeable enhancement, as it is natural to expect that the invariant point of the system depends on the parameters values Λ and should not be influenced by the initial value defined at an arbitrarily chosen point in time $t = 0$. It is hard to reasonably design transitions between different economic regimes if the model is persistently anchored to undefined moment in the past.

Let the power index parameter be defined as

$$a := \alpha\mu + \beta\nu. \tag{1.23}$$

It describes the combined strength of relationships existing within the economic system. From the recurrence Eq. (1.19), it follows that

$$i\left(t+1\right) = \left(l^{-\beta\nu} k^{-\mu\alpha}\right) i^{(\alpha\mu+\beta\nu)}(t) = i_{fix}^{(1-a)} i^{a}(t). \tag{1.24}$$

The logarithmic form is given by

$$\ln(i(t+1)) = (1-a)\ln(i_{fix}) + a\ln(i(t)). \tag{1.25}$$

Solving the last equation and returning to the nonlogarithmic form produces the compact formula for the value of the interest rate $i(t)$ at any given point in time t,

$$i(t) = i_{fix}\left(\frac{i_0}{i_{fix}}\right)^{a^t}. \tag{1.26}$$

1.2 Theoretical properties of the model

This section explores the theoretical properties of the model, examining foundational principles and the implications of its structure on dynamics. Particular emphasis is placed on critical points where the system undergoes significant changes in behavior, especially the transition to a crisis.

1.2.1 Bifurcation analysis

The qualitative behavior of the model defined by Eq. (1.9) depends mainly on two factors: the value of the power index a and the position of the initial interest rate $i(0)$ relative to the fixed point i_{fix}. The conclusions presented below can be straightforwardly derived from Eq. (1.26) describing the interest rate dynamics over time. General overview of dynamical systems, fixed points, and their classification is presented in [78].

For $a < 1$ in Eq. (1.26), the fixed point is attractive (Fig. 1.2). This situation corresponds to a stable and healthy economic regime with interest rates that smoothly and monotonically converge to the fixed point i_{fix}. From the perspective of central bank seeking for stability, this is the regime that should be targeted.

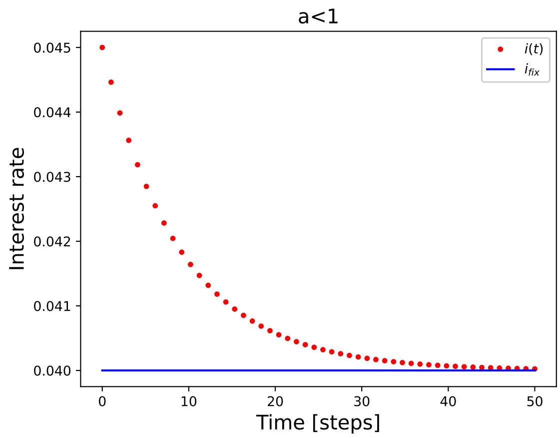


Figure 1.2: Stable regime with $a = 0.9,\ i_0 = 0.045,\ i_{fix} = 0.04$.

The point $a = 1$ corresponds to a transcritical bifurcation of the system which changes drastically around this point.

For $a = 1$, from Eq. (1.19) it follows that

$$\begin{aligned} i\,(t+1) &= \left(l^{-\beta\nu}k^{-\mu\alpha}\right) i^{a}(t) \\ &= \left(l^{-\beta\nu}k^{-\mu\alpha}\right) i(t) \\ &= \left(l^{-\beta\nu}k^{-\mu\alpha}\right)^{2} i(t-1) \\ &= \left(l^{-\beta\nu}k^{-\mu\alpha}\right)^{n} i(t-n+1) \\ &= \left(l^{-\beta\nu}k^{-\mu\alpha}\right)^{t+1} i_0 \\ &= c^{t+1} i_0, \qquad \text{where } c := \left(l^{-\beta\nu}k^{-\mu\alpha}\right). \end{aligned} \tag{1.27}$$

If $c = 1$, then

$$i\,(t+1) = \left(l^{-\beta\nu}k^{-\mu\alpha}\right) i(t) = c i(t) = i(t). \tag{1.28}$$

Therefore, the interest rate is constant over time and equals the initial value:

$$i\,(t) = i_0. \tag{1.29}$$

For $c \neq 1$, there is no fixed point of Eq. (1.27) (with the exception of 0, which is generally not considered a possible value of the interest rate in the analyzed model). If $c < 1$, interest rate converges to 0 and if $c > 1$, interest rate diverges to infinity.

For $a > 1$, the fixed point is repelling. This situation corresponds to a dangerous and unstable state of the economy, with the interest rate rapidly approaching 0 or infinity. Contrary to the stable regime $a < 1$, the behavior of the system is heavily dependent on the position of the initial interest rate i_0 relative to the fixed point i_{fix}. For $i_0 > i_{fix}$, it follows that

$$\begin{aligned} i_0 &> \left(k^{-\alpha\mu} l^{-\beta\nu}\right)^{\frac{1}{1-a}}, \\ \ln i_0 &> (-\alpha\mu \ln k - \beta\nu \ln l)/(1-a), \\ 0 &> \frac{1}{1-a}(-\alpha\mu \ln k - \beta\nu \ln l - (1-a)\ln i_0), \\ 0 &> \frac{1}{1-a}\left(\alpha\mu(\ln i_0 - \ln k) + \beta\nu(\ln i_0 - \ln l) - \ln i_0\right). \end{aligned} \tag{1.30}$$

The last form of Eq. (1.30) above formula will be referred to as the position inequality. It determines the qualitative behavior of the system in the unstable regime $a > 1$.

If $i_0 < i_{fix}$, $i(t)$ converges to 0 (Fig. 1.3) and the economy is in a bubble regime. This is an unsafe and unsustainable state. According to the model defined by Eq. (1.9), the volume of loans $N(t)$, and therefore the amount of money in the economy, is inversely proportional to the interest rate: $N(t) = (k/i(t))^{\mu}$. As interest rate approaches 0, the amount of money in the systems rapidly expands without constraints. Its exaggerated growth often translates into excessive inflation [79–81] and either the system collapses or the currency fails to fulfill a basic function of money: a stable measure of value [82].

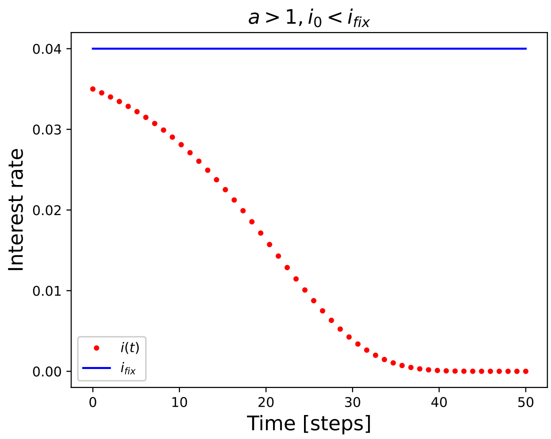


Figure 1.3: Bubble regime with $a = 1.1,\ i_0 = 0.035,\ i_{fix} = 0.04$.

If $i_0 > i_{fix}$, $i(t)$ diverges to infinity (Fig. 1.4). The economy is in a crash regime. The interest rate increases sharply, entering the feedback loop with vanishing credit and expanding defaults. The collapse of the banking system has very serious consequences beyond the limitation of the credit activity. In the modern economy, most of the money in circulation is in fact a credit created by banks [83]. Therefore, the collapse of the banking system is the collapse of money itself.

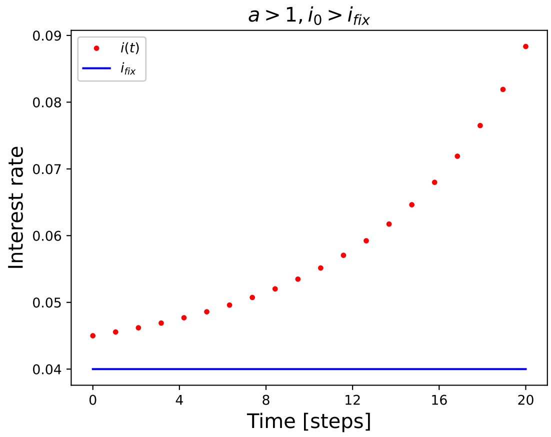


Figure 1.4: Crash regime with $a = 1.1,\ i_0 = 0.045,\ i_{fix} = 0.04$.

Overall, the UMWE model has features similar to those of the original model. However, it extends the results of the latter to a holistic framework, describing the comprehensive dynamics of the credit market with loans, defaults, and interest rates at each point in time. However, it is able to explain the stable growth, bubble, and crisis phases, depending on the values of the parameters. Therefore, to explain a credit cycle, the model has to allow for changes in the parameters in such a way that the transition occurs between different states of the system.

1.2.2 Critical parameters

In the dynamic regime, the parameter values depend on time, which can be more formally expressed as $\Lambda(t) := \{\alpha(t), \beta(t), \mu(t), \nu(t)\}$. However, for the sake of simplicity, their time dependence in the notation will be omitted below. The values of the parameters can be interpreted as being at a "current" or any arbitrary chosen point in time.

Recall that $a = \alpha\mu + \beta\nu$ and a transcritical bifurcation occurs at $a = 1$. Therefore, the critical value α_{crit} of α, for which the bifurcation occurs must satisfy

$$\alpha_{crit}\mu + \beta\nu = 1. \tag{1.31}$$

Thus, the value of the critical stability parameter α_{crit} equals

$$\alpha_{crit} = \frac{1 - \beta\nu}{\mu}. \tag{1.32}$$

The application of the same reasoning to the other parameters yields the following values of the critical stability parameters:

$$\begin{aligned} \alpha_{crit} &= \frac{1 - \beta\nu}{\mu}, \\ \beta_{crit} &= \frac{1 - \alpha\mu}{\nu}, \\ \mu_{crit} &= \frac{1 - \beta\nu}{\alpha}, \\ \nu_{crit} &= \frac{1 - \alpha\mu}{\beta}. \end{aligned} \tag{1.33}$$

At an aggregated level, the value Δ_{crit} of a scale parameter, for which a stability transition occurs, can be defined. Let Δ_{crit} be a multiplier for which a critical value $\alpha\mu + \beta\nu = 1$ is attained:

$$\Delta_{crit}\alpha\Delta_{crit}\mu + \Delta_{crit}\beta\Delta_{crit}\nu = 1. \tag{1.34}$$

Hence, it follows that Δ_{crit} satisfies

$$\begin{aligned} \Delta^2_{crit}\left(\alpha\mu + \beta\nu\right) &= 1 \\ \Delta_{crit} &= \frac{1}{\sqrt{a}}. \end{aligned} \tag{1.35}$$

On the other hand, consider the transition between bubble and crash regimes in the unstable state $a > 1$. The regime is determined by the relative positions of the current interest rate i_t and the fixed point i_{fix}. Therefore, the value of the critical direction parameter $\alpha_{crit}(i_t)$ for which there is a change of the sign in parentheses term in the position inequality (1.30) equals

$$\alpha_{crit}(i_t) = \frac{\ln i_t - \beta\nu(\ln i_t - \ln l)}{\mu(\ln i_t - \ln k)}. \tag{1.36}$$

The critical direction values of the other parameters can be derived analogously, yielding

$$\begin{aligned} \beta_{crit}(i_t) &= \frac{\ln i_t - \alpha\mu(\ln i_t - \ln k)}{\nu(\ln i_t - \ln l)}, \\ \mu_{crit}(i_t) &= \frac{\ln i_t - \beta\nu(\ln i_t - \ln l)}{\alpha(\ln i_t - \ln k)}, \\ \nu_{crit}(i_t) &= \frac{\ln i_t - \alpha\mu(\ln i_t - \ln k)}{\beta(\ln i_t - \ln l)}. \end{aligned} \tag{1.37}$$

Furthermore, at a general level, define the value $\Delta_{crit}(i_t)$ of a scale parameter for which a direction transition occurs. Let $\Delta_{crit}(i_t)$ be such that in the position inequality (1.30) 0 is attained:

$$\begin{aligned} 0 = {} & \Delta_{crit}(i_t)\alpha\Delta_{crit}(i_t)\mu(\ln i_t - \ln k) + \\ & + \Delta_{crit}(i_t)\beta\Delta_{crit}(i_t)\nu(\ln i_t - \ln l) - \ln i_t. \end{aligned} \tag{1.38}$$

Therefore

$$\begin{aligned} \Delta^2_{crit}(i_t) &= \frac{\ln i_t}{\alpha\mu(\ln i_t - \ln k) + \beta\nu(\ln i_t - \ln l)} \\ \Delta_{crit}(i_t) &= \sqrt{\frac{\ln i_t}{\alpha\mu(\ln \frac{i_t}{k}) + \beta\nu(\ln \frac{i_t}{l})}}. \end{aligned} \tag{1.39}$$

Remark 1.1. *Similar definitions can be introduced in the stable regime $a < 1$, but the position of i_{fix} relative to i_t is not so relevant there. In that regime, i_t converges to i_{fix} in all cases, so the qualitative behavior of the stable system does not change significantly depending on the location of the current interest rate (with the exception of the direction of convergence).*

For every critical direction parameter $\lambda_{crit}(i_t)$, the limit $\overline{\lambda}_{crit} := \lim_{t\to\infty} \lambda_{crit}(i_t)$ is called an asymptotic critical direction parameter. The relationship between the critical stability parameters and the asymptotic critical direction parameters is explained in the following result.

Theorem 1.1 (Equivalence of the Critical Parameters)**.** *Let $\Lambda = \{\alpha, \beta, \mu, \nu\}$ be the set of exponential parameters of the UMWE model* (1.9)*. For every parameter $\lambda \in \Lambda$ and every pair of the corresponding critical stability and asymptotic critical direction parameters $\left(\lambda_{crit}, \overline{\lambda}_{crit}\right)$ the following relation takes place:*

$$\lambda_{crit} = \overline{\lambda}_{crit}. \tag{1.40}$$

Furthermore, it holds that

$$\Delta_{crit} = \overline{\Delta}_{crit}. \tag{1.41}$$

Proof. The values of critical direction parameters are given by

$$\begin{aligned}
\alpha_{crit}(i_t) &= \frac{\ln i_t - \beta\nu(\ln i_t - \ln l)}{\mu(\ln i_t - \ln k)}, \\
\beta_{crit}(i_t) &= \frac{\ln i_t - \alpha\mu(\ln i_t - \ln k)}{\nu(\ln i_t - \ln l)}, \\
\mu_{crit}(i_t) &= \frac{\ln i_t - \beta\nu(\ln i_t - \ln l)}{\alpha(\ln i_t - \ln k)}, \\
\nu_{crit}(i_t) &= \frac{\ln i_t - \alpha\mu(\ln i_t - \ln k)}{\beta(\ln i_t - \ln l)}.
\end{aligned} \tag{1.42}$$

Each of them is a special case of the equation

$$\lambda_{crit}(i_t) = \frac{\ln i_t - bc(\ln i_t - d)}{g(\ln i_t - h)}, \tag{1.43}$$

where $b, c, g \in \mathbb{R}_+$ and $d, h \in \mathbb{R}$. As the system is in the unstable regime $a > 1$, in the infinite limit $t \longrightarrow \infty$ it holds that either $i_t \longrightarrow 0$ or $i_t \longrightarrow \infty$. Therefore

$$\lim_{t \longrightarrow \infty} \lambda_{crit}(i_t) = \frac{1 - bc}{g}. \tag{1.44}$$

Consequently, the limits of critical direction parameters can be expressed as

$$\begin{aligned}
\overline{\alpha}_{crit} &= \frac{1 - \beta\nu}{\mu}, \\
\overline{\beta}_{crit} &= \frac{1 - \alpha\mu}{\nu}, \\
\overline{\mu}_{crit} &= \frac{1 - \beta\nu}{\alpha}, \\
\overline{\nu}_{crit} &= \frac{1 - \alpha\mu}{\beta}.
\end{aligned} \tag{1.45}$$

Hence, for every critical stability parameter λ_{crit} and the corresponding asymptotic critical direction parameter $\overline{\lambda}_{crit}$ it holds that $\lambda_{crit} = \overline{\lambda}_{crit}$.

Similarly, it holds that

$$\Delta_{crit}(i_t) = \sqrt{\frac{\ln i_t}{\alpha\mu(\ln \frac{i_t}{k}) + \beta\nu(\ln \frac{i_t}{l})}}. \tag{1.46}$$

Consequently

$$\overline{\Delta}_{crit} = \lim_{t \longrightarrow \infty} \Delta_{crit}(i_t) = \sqrt{\frac{1}{\alpha\mu + \beta\nu}} = \frac{1}{\sqrt{a}} = \Delta_{crit}. \tag{1.47}$$

This completes the proof. □

The result of the Theorem 1.1 unifies asymptotic critical direction parameters with critical stability parameters, highlighting the latter as the most important indicators of the fragility of the analyzed UMWE model. Thus, the neighborhood of the point $a = 1$ constitutes indeed the critical area of the system, where the phase transitions (between stability and instability, as well as between a bubble and a crisis) ultimately occur.

1.2.3 Measures of systemic risk

Based on the critical (stability and direction) parameters, the measures of systemic risk for the unified MWE model can be constructed as the distances of the current values of the parameters to the critical ones. For every parameter $\lambda \in \Lambda$ and the corresponding critical stability parameter λ_{crit} define

$$\begin{aligned} \delta\lambda_{crit} &= \lambda_{crit} - \lambda, \\ \frac{\delta\lambda_{crit}}{\lambda} &= \frac{\lambda_{crit} - \lambda}{\lambda} = \frac{\lambda_{crit}}{\lambda} - 1. \end{aligned} \tag{1.48}$$

The exact formulas for all parameters are presented in Table 1.1. The above measures will be referred to as the (critical) stability distances: absolute stability distance $\delta\lambda_{crit}$ and relative one, $\frac{\delta\lambda_{crit}}{\lambda}$. According to Theorem 1.1, distances to critical stability parameters are equal to the distances to asymptotic critical direction parameters.

	α	β	μ	ν	Δ
$\delta\lambda_{crit}$	$\frac{1-a}{\mu}$	$\frac{1-a}{\nu}$	$\frac{1-a}{\alpha}$	$\frac{1-a}{\beta}$	$\frac{1}{\sqrt{a}} - 1$
$\delta\lambda_{crit}/\lambda$	$\frac{1-a}{\mu\alpha}$	$\frac{1-a}{\nu\beta}$	$\frac{1-a}{\mu\alpha}$	$\frac{1-a}{\nu\beta}$	$\frac{1}{\sqrt{a}} - 1$

Table 1.1: Critical stability distances.

For critical direction parameters, the distances can be defined analogously for $\lambda \in \Lambda$:

$$\begin{aligned} \delta\lambda_{crit}(i_t) &= \lambda_{crit}(i_t) - \lambda, \\ \frac{\delta\lambda_{crit}(i_t)}{\lambda} &= \frac{\lambda_{crit}(i_t) - \lambda}{\lambda} = \frac{\lambda_{crit}(i_t)}{\lambda} - 1. \end{aligned} \tag{1.49}$$

The exact formulas for all parameters are presented in Table 1.2. In that case, $\delta\lambda_{crit}(i_t)$ and $\frac{\delta\lambda_{crit}(i_t)}{\lambda}$ are referred to as absolute and relative (critical) distances, respectively. System

	α	β	μ
$\delta\lambda_{crit}\left(i_t\right)$	$-\frac{\beta\nu\ln\frac{i_0}{l} - \ln i_0}{\mu\ln\frac{i_0}{k}} - \alpha$	$-\frac{\alpha\mu\ln\frac{i_0}{k} - \ln i_0}{\nu\ln\frac{i_0}{l}} - \beta$	$-\frac{\beta\nu\ln\frac{i_0}{l} - \ln i_0}{\alpha\ln\frac{i_0}{k}} - \mu$
$\delta\lambda_{crit}\left(i_t\right)/\lambda$	$-\frac{\beta\nu\ln\frac{i_0}{l} - \ln i_0}{\alpha\mu\ln\frac{i_0}{k}} - 1$	$-\frac{\alpha\mu\ln\frac{i_0}{k} - \ln i_0}{\beta\nu\ln\frac{i_0}{l}} - 1$	$-\frac{\beta\nu\ln\frac{i_0}{l} - \ln i_0}{\alpha\mu\ln\frac{i_0}{k}} - 1$
	ν	Δ	
$\delta\lambda_{crit}\left(i_t\right)$	$-\frac{\alpha\mu\ln\frac{i_0}{k} - \ln i_0}{\beta\ln\frac{i_0}{l}} - \nu$	$\sqrt{\frac{\ln i_t}{\alpha\mu(\ln\frac{i_t}{k}) + \beta\nu(\ln\frac{i_t}{l})}}$	
$\delta\lambda_{crit}\left(i_t\right)/\lambda$	$-\frac{\alpha\mu\ln\frac{i_0}{k} - \ln i_0}{\beta\nu\ln\frac{i_0}{l}} - 1$	$\sqrt{\frac{\ln i_t}{\alpha\mu(\ln\frac{i_t}{k}) + \beta\nu(\ln\frac{i_t}{l})}}$	

Table 1.2: Critical direction distances.

fragility measures defined in Eq. (1.48) and (1.49) are of very practical use. Stability distances describe how far (from the perspective of values of parameters) the system is from the unstable (or stable) state. In a stable stage, this information can be utilised to keep a (arbitrary set) safe distance from the unstable phase. For example, during periods of increased risk, a central bank can raise interest rates to constrain the demand for loans and curb the credit expansion of banks. This action is expected to reduce banks' confidence in the market, which in the proposed model translates into a decrease in α. But, as is usually the case in finance, there could be a trade-off between risk and return [84]. In the case of a stable regime $a < 1$ and a base of power of the fixed interest rate formula $i_{fix} = \left(k^{-\alpha\mu} l^{-\beta\nu}\right)^{\frac{1}{1-(\alpha\mu+\beta\nu)}}$ being less than one ($k^{-\alpha\mu} l^{-\beta\nu} < 1$), scaling the exponential parameters α, β, μ, ν by the factor $d \in (0, 1)$ increases the distance to the unstable regime $\delta\Delta_{crit}$, thereby translating into greater financial security. However, this also increases the value of the fixed interest rate i_{fix}, which is generally associated with lower economic growth. In the case of the unstable phase $a > 1$, the stability distance can be utilized to steer the comeback to the stable regime. Since the parameters α and β represent the bank's policy of interest rate determination, $\delta\alpha_{crit}$ and $\delta\beta_{crit}$ describe the changes in policy needed to bring the financial system back to stability.

On the other hand, direction distance describes how far the system is from the crisis (or bubble) in the unstable regime $a > 1$. It can be helpful to determine how to escape the bubble with too much money present in the economy, causing inflation, instability, and phantom (nominal) growth without coverage in real terms. Based on this information, a central bank can estimate the appropriate window for raising interest rates to constrain financial expansion and avoid a crisis. It is worth to point out that it is not obvious if escaping bubble smoothly in the long period is a better solution than v-shaped crisis and recovery. Smooth bubble escape could mean long lasting, exhausting recession with prevailing consequences, often difficult to get rid off [85, 86]. In comparison, a rapid, severe, but quick shock, controlled to some extent by central bank supporting the financial situation, can often be followed by relatively fast recovery to the stable regime [87]. With enough knowledge regarding the fit of the presented model to reality, possibly the direction distance could be utilized to steer a more rapid recovery and then smooth transition to the stable regime. For these purposes, however, empirical validation of the model is required.

1.3 Derivation of a credit cycle from the model

1.3.1 Theoretical background

In the UMWE model (1.9), the parameters α and β reflect the policy of the banks regarding their market offer, expressed through the price charged for their products. Since the contribution of this chapter is dedicated mainly to be used by financial institutions, it is assumed that α and β could change their values throughout the credit cycle, as they represent the available instruments of the bank's financial policy and can be tailored to individual situations. The values of the parameters μ and ν are treated as external from the bank's perspective and therefore kept constant. The technical scale parameters k, l also do not vary over time in the proposed approach. The assumption that only α and β can change over time simplifies the analysis and makes it more clear. Nevertheless, it is worth noting that during the incorporation of the model into the real data analysis, the potential dynamics of the rest of the parameters should also be validated, possibly even with the relations between various parameters, as for example α and μ could both be driven by the common optimism in the credit market.

As the dynamics of the interest rate are given by the equation

$$i(t+1) = \frac{D^{\beta}(t)}{N^{\alpha}(t)}, \tag{1.50}$$

banks can determine their interest rate policy using the parameters α and β. Increased market confidence is associated with a relaxation of credit prices, which translates into an increase in α or a decrease in β. Conversely, banks' concerns result in the opposite dynamics of these parameters. In order to describe the full credit cycle, the key is not the exact functional form of this relationship, but its monotonic character. This constitutes an additional layer of the feedback loop within the framework, where banks, encouraged by economic growth, opt for more aggressive market expansion, thereby fueling further growth. This expansion might also be driven by the economies of scale described in Section 1.1.2, where an overly cautious policy poses an existential threat to a bank, risking displacement from the market. However, rapid expansion of the stocks of money and debt in the economy poses the risk of a system crash. In response to increasing risk, banks tighten their lending conditions at some point to escape the bubble, leading to a rapid collapse. This is an excellent example of another phenomenon characteristic for systemic risk discipline: the self-fulfilling prophecy [14]. This situation occurs when the anticipation of an event leads to its realization, constituting in fact another feedback loop, but inverted in time. Thus, the model is truly endogenous, as the evolution of the financial system stems from the interactions between borrowers, banks, and the dynamics of their policies in response to the market state.

1.3.2 Stable phase and the bubble

It is argued in [6] that the square root power law with an exponent $\mu = 0.5$ can describe the response of the economic demand for loans in relation to the interest rate. Based on that, the values of exponents that describe the dependence of loans and defaults on the interest rates in the system (1.9) are set to

$$\mu = \nu = 0.499. \tag{1.51}$$

The values are just below 0.5 in order to keep the system close to the edge of instability. For the exponent parameters in Eq. (1.9), the initial values are chosen as

$$\alpha = \beta = 1, \tag{1.52}$$

which implies that the interest rate equals the estimated probability of default

$$i_{t+1} = \frac{D_t}{N_t}. \tag{1.53}$$

The rationale for such a parameterization has been discussed in Section 1.1.2. The scale parameters are set to

$$\begin{aligned} k &= 105.5, \\ l &= 0.0096. \end{aligned} \tag{1.54}$$

Finally, the initial value of the interest rate is taken as

$$i_0 = 0.042. \tag{1.55}$$

The chosen specification of the parameters implies that

$$a = \alpha\mu + \beta\nu = 1 \times 0.499 + 1 \times 0.499 = 0.998 < 1. \tag{1.56}$$

It means that the default state of economy is a convergent one ($a < 1$), but it is very close to the instability border $a = 1$ at the same time. The value of the fixed interest rate of such parameterized system is close to 0.04186. Therefore, the evolution of the system starts at a stable and calm phase. The interest rate slowly and smoothly settles to its lower limit. The described situation is illustrated in Figure 1.5.

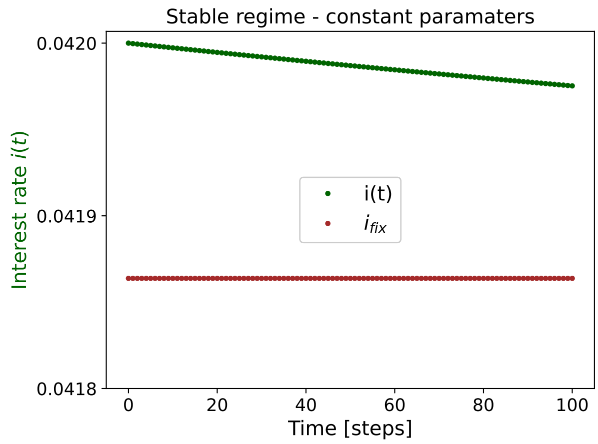


Figure 1.5: Stable phase of the cycle with $\alpha = \beta = 1,\ \mu = \nu = 0.499,\ k = 105.5, l = 0.0096,\ i_0 = 0.042,\ i_{fix} \approx 0.04186$. The evolution of the system starts at a stable and calm phase. The interest rate slowly and smoothly settles to its lower limit.

The prevailing period of prosperity amplifies the confidence and optimism of the banks. At some point in time, they become comfortable enough to start to relax the previous rules of determining the value of the interest rate in relation to loans and defaults. In the selected setup of the parameters Λ, the interest rates could be lowered by the increase of α or the decrease of β. Modification of the sensitivity to the amount of loans (money) in the system appears to be less risky than modification of the sensitivity to the number of defaults. Therefore, after a long period of stable economic growth, α increases. Our approach brings positive feedback from the economy, resulting in more beneficial dynamics of the credit market in comparison to the previous one. The favorable outcome fuels further optimism and encourages to increase α even more. The result is a positive feedback loop with the parameter α increasing at each step. Along with the rise of the value of α, the distance to instability $\delta\alpha_{crit}$ decreases and the system becomes more fragile. At some point in time, $\delta\alpha_{crit}$ passes through 0. The market enters a bubble state and continues to grow. The situation is presented in Figure 1.6.

Although it is generally assumed that lower interest rates fuel economic growth [88], too low levels of them are not healthy for the economy. Interest rates influence the supply of money [83, 89, 90], which is also reflected in the considered model. Interest rates impact the volume of loans and, in turn, the amount of money in the economy, by the relation given in Eq. (1.9):

$$N(t) = \left(\frac{i(t)}{k}\right)^{-\mu}. \tag{1.57}$$

Furthermore, excessive increase in the amount of money is associated with the increase in the level of prices [79–81, 91] (there are also direct relations between interest rates

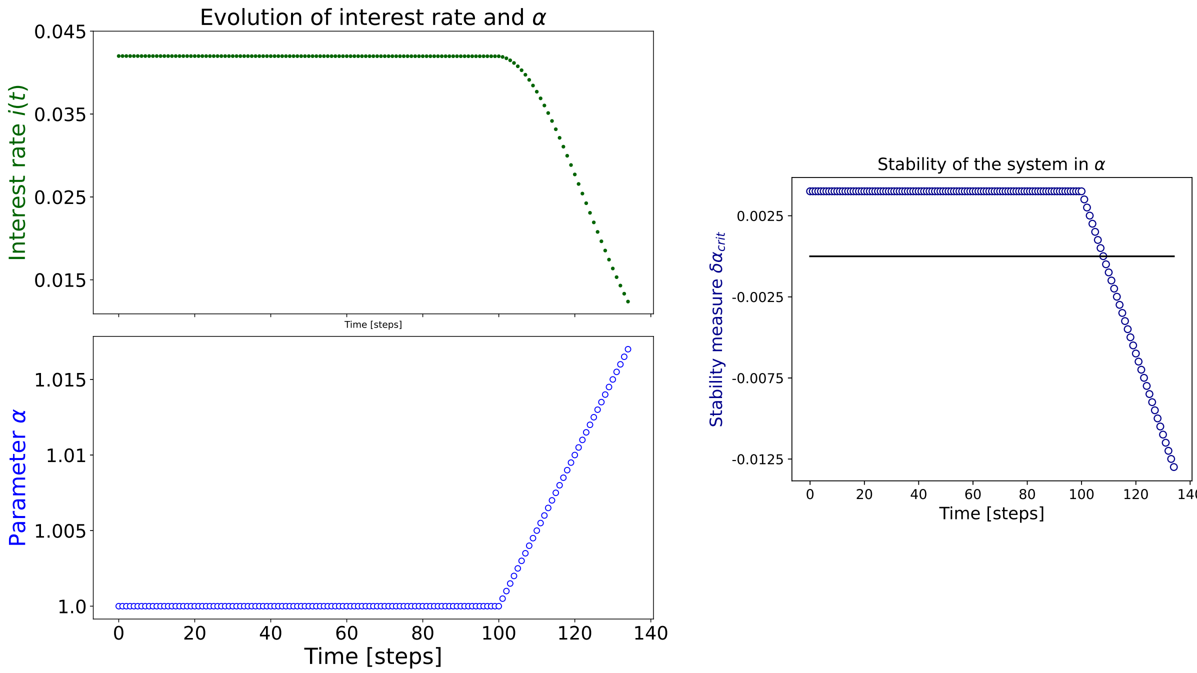


Figure 1.6: Bubble phase of the cycle with $\beta = 1,\ \mu = \nu = 0.499,\ k = 105.5, l = 0.0096$ and increasing α. Long period of stable growth improved market confidence, expressed in rising value of α (left plot). System stability decreases and at some point in time market enters the bubble phase (right plot).

and inflation derived in economic theory [92]). Exceeding the unhealthy threshold of the inflation level creates the risk of entering the vicious cycle of self-reinforcing increases in prices and wages [93]. Rising prices increase the value of collateral for credit, improving the credit quality of borrowers and amplifying the increase in credit volume in the economy [94]. This can create incentives for the formation of speculative bubbles, which amplifies the fragility of the financial system [95] and potentially leads to a crash, as was the case in 2007 [96]. When the interest rate reaches near zero territories in the bubble regime, the amount of money in the system and the level of prices expand uncontrollably without limitations; there are known cases of yearly inflation reaching levels of $10^{22}\%$ [97]. The currency no longer performs the function of a measure and a store of value, violating the definition of money itself [82].

1.3.3 Crisis and stabilization

As discussed in Section 1.3.2, there are practical limits on growth of the monetary base, and therefore the interest rate should not remain in the near-zero territory indefinitely (there are some exceptions even with negative interest rates [98–101], but they are beyond the scope of this study). Central banks are generally obliged to preserve the stable value of the currency and therefore maintain inflation at predefined acceptable levels [102]. Steering interest rates remains a key tool of central bank monetary policy implementation [103,104], therefore, interest rates are risen to bring inflation back to target levels. The lower bound on the interest rate offered by banks in the market is the rate of return on their deposits placed in the central bank accounts. This value, called the deposit rate, is established by the central bank [105]. Other limitations come from the Basel accords, placing capital and liquidity requirements, as well as the permissible leverage ratio [106,107].

All this leads to the conclusion that there are practical restrictions on the decrease of the interest rate. Therefore, it is assumed that it cannot fall below the minimum value of 0.0123. The situation discussed here is presented in Figure 1.7. When the interest rate

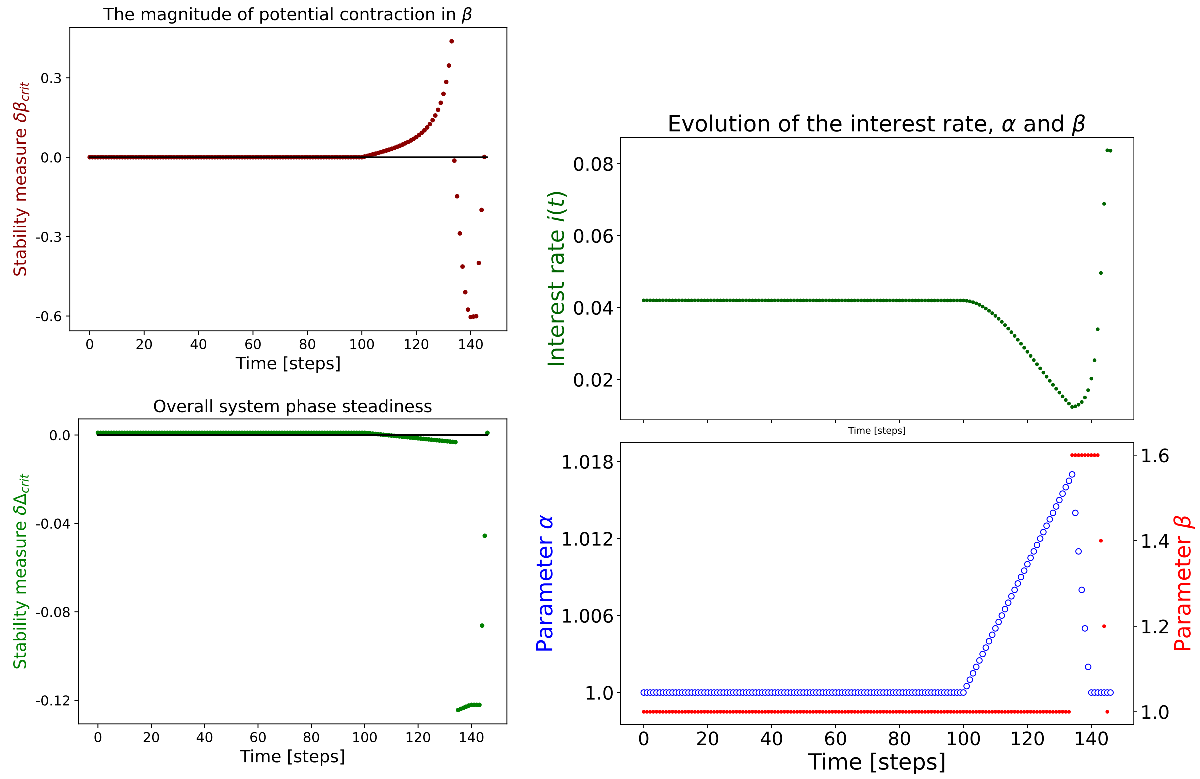


Figure 1.7: Crisis phase of the cycle with $\beta = 1.6,\ \mu = \nu = 0.499,\ k = 105.5, l = 0.0096$ and decreasing α. Left column: the top plot presents the severity of potential contraction of the system expressed in terms of value of critical direction distance for β. The bottom plot presents the overall system phase steadiness expressed in terms of critical direction distance $\delta\Delta_{crit}$. Right column: The values of α are put on the left vertical axis of the bottom plot, whereas the ones of β - on the right axis. The system comes close to the practical minimum value of interest rate, accumulating excessive financial material for the collapse. The result is the shock of β parameter and reverse in the direction of interest rate evolution. The overall system phase steadiness changes dynamically over time.

approaches the limit, banks start to fear. To remain safe, they increase the sensitivity to the number of defaults, β, in order to distance the interest rate from the limit level. Because β increases and $a = \alpha\mu + \beta\nu$, the market remains in the unstable regime $a > 1$. But to avoid danger, banks rise β to such a level that the distance from the minimum interest rate increases as well. This increase in conjunction with the presence of the unstable regime $a > 1$ implies a crash.

For the transition between bubble and crash to occur, the critical direction distance

$$\delta\beta_{crit} = \frac{\ln i_t - \alpha\mu(\ln i_t - \ln k)}{\nu(\ln i_t - \ln l)} - \beta \tag{1.58}$$

must fall below 0. The risk measure $\delta\beta_{crit}$ is a linear function of α with the slope of $\frac{-\mu(\ln i_t - \ln k)}{\nu(\ln i_t - \ln l)}$ and a hyperbola with respect to the current value of the logarithmic interest rate, $\ln i_t$. It is qualitatively dependent on values of $k,\ l$ parameters, as they determine the signs of appropriate fragments of the equation above and then the monotonicity of dependencies between the exponents of the model. This reflects an important feature of reality: the faster the bubble grows (greater α), the faster and severe the contraction, because $a = \alpha\mu + \beta\nu$ describes the velocity of divergence.

The transition to the crash regime results in a sharp increase in the volume of defaults and a significant contraction in the volume of loans. The crunch of the loan market amplifies the panic, causing banks to further reduce lending by decreasing the value of α, which ultimately falls to its initial level. The distance to stability Δ_{crit} starts to decrease in absolute terms, consequently driving the system closer to the stable state. Meanwhile, the central bank has time to intervene and calm down the situation. After bringing back the sensitivity α to territories known from the stable regime, banks start to reduce the shocked parameter β. Finally, Δ_{crit} passes through 0, parameters return to their initial level and the system returns to the stable state.

The first phase of a stabilization is a recessionary period with the level of interest rate elevated by the crisis (Figure 1.8). The interest rate consequently decreases towards the

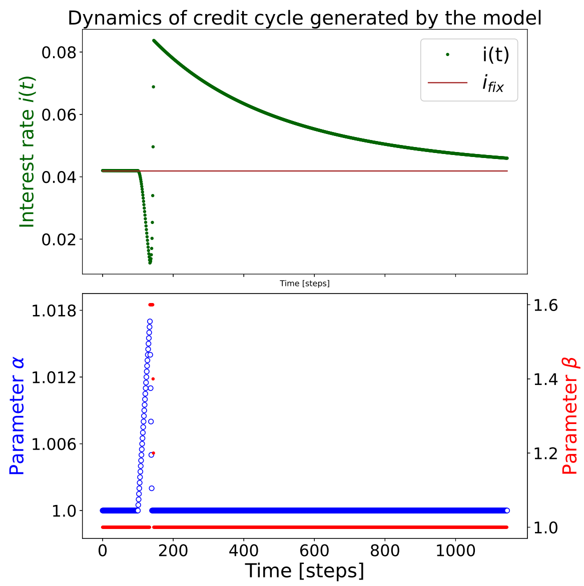


Figure 1.8: Recovery phase of the cycle with $\alpha = \beta = 1,\ \mu = \nu = 0.499,\ k = 105.5, l = 0.0096,\ i_0 = 0.042,\ i_{fix} \approx 0.04186$. The values of α are put on the left vertical axis of the bottom plot, whereas the ones of β - on the right axis. The system starts in a recession with increased value of interest rate. It decreases consequently towards a stability point the same as initially, smoothly entering the phase of profitable economic growth after some time. Finally the process of convergence brings the system back to the state nearly identical to the initial one, with the interest rate slowly and steadily approaching the fixed point. The cycle is closed.

fixed point 0.0419 and after some time enters the area of beneficial levels, which stimulates the growth of the economy. Further in time, the pace of change significantly reduces and the value of interest rate slowly and consequently settles on the fixed point. The cycle is closed.

1.3.4 Further considerations regarding the credit cycle description

It is worth noting that the cycle has been explained only by the dynamics of the financial system, without referring to external shocks interpreted as abrupt and unexplained changes in μ, ν bringing crisis to the market. It is important because in reality the crisis can also be caused by the malfunctioning of the financial system itself [96] and modeling this phenomenon can help to avoid it by proper management of the system. Naturally, external shocks expressed in the dynamics of μ and ν can also be modeled by the analyzed system (1.9), providing insight into potential strategies to mitigate the crisis.

The proposed approach is just one example of the many possible dynamics that could be assumed within the framework. Other scenarios could include initially lowering β followed by a rapid increase in α, simultaneous evolution of both parameters throughout the cycle, or different non-linear, yet appropriately monotonic dynamics of evolution, such as parabolic, hyperbolic, exponential, and more. This demonstrates the flexibility and generality of the proposed framework, which can adapt to various market scenarios without adhering to a single exact specification, but rather to the crucial features of feedback loops existing within the financial system.

It is worth to point out that it is also possible to generate the transition from a bubble to a crisis only by increasing α, under some parameterizations Λ. Therefore, the analyzed model is also capable of explaining the situation where the bubble collapses because of the "excess optimism", without setting any effective constraints on the value of the interest rate (Figure 1.9).

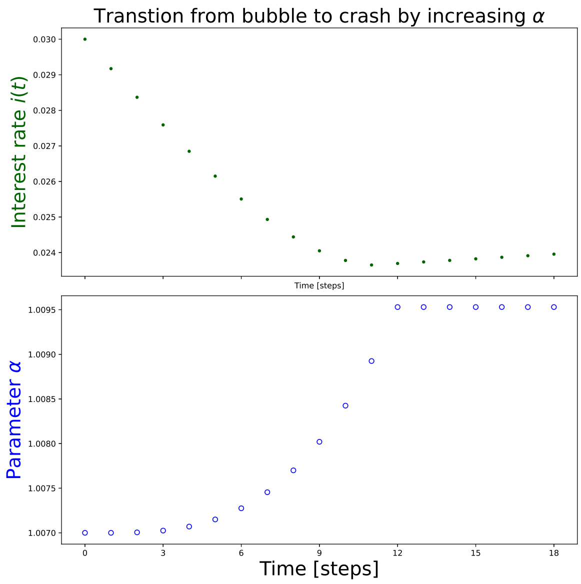


Figure 1.9: Transition from bubble to crash by constantly increasing α for α staring from 1.007, $\beta = 1$, $\mu = \nu = 0.499$, $k = 10^{-12}$, $l = 1.27 \cdot 10^{12}$, $i_0 = 0.03$.

Parameters α and β exhibit significantly different behaviors under the parameterization Λ proposed in Section 1.3.2. They are "countermonotonic" in the sense that optimism increases with α and decreases with β. On the other hand, the instability increases with both α and β (as $a = \alpha\nu + \beta\nu$ describes the instability and the pace of evolution). These characteristics can lead to very diverse consequences near the transcritical bifurcation point $a = 1$, as various approaches to increase/decrease optimism (by modifying α or modifying β) can bring very different results. In the neighbourhood of the transcritical bifurcation point, more optimistic (lower) β can help escape the unstable regime ($a < 1$), although it is rather hard to expect banks to bring β to significantly lower levels, as they should closely watch the number of defaults and express their optimism mainly in the sensitivity to the increasing amount of money in the system (α). On the other hand, more optimism expressed in α drives the system to more unstable territories. In the unstable regime, more pessimistic but careful choice of parameters (increase β, decrease α such that $a < 1$) can cause greater contraction in interest rate in one step (in comparison to the situation when α is not modified), but can also enable to avoid longer lasting crisis or recession regime.

Different possible outcomes for similar bank policies increase the uncertainty regarding the future dynamics of the system and hinder the decision-making, especially considering that there are many banks in the market and no one knows about the other's strategies. This situation is very similar to the coordination problem in bank run models [15, 108]. All the described effects can amplify uncertainty and deepen panic. There is also a possibility that banks react to some minor turbulence by increasing α even more and hoping that more money pumped into the system will help avoid the crisis (which can be true). Nevertheless, the continuation of that strategy can lead to the situation where the growth is artificially fueled by rapidly increasing amount of money in the system, and at some point is not sustainable any more. Even if a central bank and regulations allow it, market participants eventually switch to a stable currency to protect the value of their property [109, 110]. Moving the crisis further in time can make it more severe [5, 6].

1.4 Conclusion

In this chapter, an unified mathematical framework that explains the credit cycle in a coherent manner has been constructed. The Marshall-Walras equilibrium models for the bubble and crisis phases proposed by Solomon and Golo [6] have been consolidated into a single, consistent approach that describes all phases of the credit cycle at each point in time. The original models can be obtained as special cases of the proposed approach with parameters $\alpha = 0$ or $\beta = 0$. This framework not only enhances and generalizes existing results, but also mathematically describes transitions between different phases in an endogenous manner. Moreover, the dependency on the arbitrary choice of the initial moment $t = 0$ has been eliminated. Detailed mathematical analysis of the model has been performed to describe three very different economic regimes: stable state, bubble, and crisis, dependent on the values of its parameters. On the basis of these results, the measures of systemic risk has been constructed as distances to critical values of the parameters, for which the transition between different model regimes occurs. The theory has been applied to generate the interest rate evolution with features characteristic for a full credit cycle. For this purpose, the dynamics of the parameters has been utilized in the model, with the economic interpretation of casual relationships between them. The mathematical consequences of the model and their correspondence to the real-world phenomena has been analyzed as well.

Due to its relative mathematical simplicity, the model can be easily operated on and incorporated into the analyses and decision-making processes performed with the use of the standard financial models. It has natural economic interpretation of parameters and provides explanatory power regarding causes and effects of key market events. The standalone risk measures constructed in this chapter can provide information on the current market regime and possible dangers related to transitions to bubbles or crises. For example, the model can provide indicators of the presence of a stable market situation and help predict potential regime switches to turbulent periods. Based on this information, a financial instrument valuation or investment strategy can be modified to prepare for a possible crisis. Such topics also constitute very promising material for future research, along with the improvement of the model with the addition of inflation or several banking entities with interactions between them derived from empirical data.

Chapter 2

Extended Ornstein-Uhlenbeck process in financial modeling of debt instruments

This chapter focuses on the general theory of the valuation of financial instruments and the development of the extended Ornstein-Uhlenbeck process based on the UMWE model (1.9) constructed in Chapter 1. The fundamental apparatus of financial mathematics, particularly in the context of debt valuation, is briefly introduced in Section 2.1. This introduction includes the concepts of interest and hazard rates, as well as reduced-form models. Subsequently, an important family of short-rate models is presented in Section 2.2, along with their relationships to the unified framework for modeling credit cycles (1.9). This lays the foundation for the development of a new short-rate model, which can be viewed as a continuous, stochastic extension of the UMWE or as a generalization of the Ornstein-Uhlenbeck process. The latter one is a Wiener process embedded with a tendency to return towards a central location (mean reversion) which intensifies as the process moves further away from this center. The general strength of this tendency is determined by a value of the mean reversion rate parameter. Such properties are particularly useful in financial mathematics, especially in modeling interest rates. With this dynamics, a stable value for the interest rate can be defined around which the process oscillates, while also allowing for deviations caused by random factors. However, as discussed in Chapter 1, a tendency to converge towards a fixed point is well-suited for modeling stable financial regimes, but may not adequately capture the abrupt phases of bubbles and crises. Therefore, the Ornstein-Uhlenbeck process is extended in this chapter to allow the mean reversion rate parameter to take on negative values. Consequently, while previously reverting to the mean, the process can now diverge at an exponential rate toward positive or negative infinity. The formal analysis of this dynamics in the context of financial modeling is demonstrated in Section 2.3, followed by a summary in Section 2.4. Similar to the original UMWE model, it can describe very different economic regimes, including crashes. Moreover, the model can exhibit stable behavior for prolonged periods while ultimately displaying abrupt dynamics. Those features constitute fundamental aspects of the financial markets dynamics that many existing and commonly applied approaches struggle to reproduce [12, 13, 17].

2.1 Debt valuation and reduced-form models

One of the most important concepts in financial mathematics are interest rates, representing the cost of borrowing or the return on investment over time [53,111,112]. The instantaneous interest rate at time t is denoted as r_t or $r(t)$, with the notation used interchangeably depending on which is clearer in a given context. The rate r_t is assumed to follow a diffusion process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F}_t$ is a natural filtration. By definition, a diffusion process is a Markov process [113], which implies that for every $s < t$ it holds that [114]

$$\mathbb{P}(r_t \in A \mid \mathcal{F}_s) = \mathbb{P}(r_t \in A \mid r_s). \tag{2.1}$$

In other words, the future evolution of the process depends only on its current state and is independent of its past. Consequently, the expected value of a function of such a process satisfies

$$\mathbb{E}(\cdot \mid \mathcal{F}_t) = \mathbb{E}(\cdot \mid r_t). \tag{2.2}$$

To streamline the notation, the following definition is introduced:

$$\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot \mid r_t). \tag{2.3}$$

In particular, expectation at the moment $t = 0$ is denoted as

$$\mathbb{E}(\cdot) := \mathbb{E}_0(\cdot). \tag{2.4}$$

Interest rates reflect the time value of money, as a sum of money today is worth more than the same sum in the future due to its potential earning capacity [112]. This notion is formally captured in the concepts of present value (PV) and future value (FV) [115]. Present value refers to the current worth of a future sum of money or cash flows, discounted at a specific interest rate, while future value represents the amount of money that an investment will grow to over a specified period at a given interest rate. The relationship between these two can be articulated as [115]

$$PV = FV \cdot \mathbb{E}_t \left[\exp \left(- \int_t^T r(u)\, du \right) \right]. \tag{2.5}$$

The integral from t to T captures the accumulation of interest over the specified time period, reflecting the continuous compounding effect. Then the price at time t of a zero-coupon riskless bond maturing at time T with a payoff of 1 is given by

$$P(t,T) = \mathbb{E}_t \left[\exp \left(- \int_t^T r(u)\, du \right) \right]. \tag{2.6}$$

This allows to introduce the definition of an instantaneous forward rate

$$f(t,u) = -\frac{\partial}{\partial u} \ln(P(t,u)). \tag{2.7}$$

Thus, the price of the of a zero-coupon riskless bond $P(t,T)$ can be also expressed as

$$P(t,T) = \exp \left(- \int_t^T f(t,u)\, du \right). \tag{2.8}$$

It holds that

$$f(u,u) = r(u). \tag{2.9}$$

If the interest rate $r(u)$ is deterministic, then

$$P(t,T) = \exp\left(-\int_t^T f(t,u)\,du\right) = \exp\left(-\int_t^T r(u)\,du\right), \tag{2.10}$$

and for all t, u

$$f(t,u) = r(u). \tag{2.11}$$

Those relationships highlight how interest rates influence both the growth of investments and the valuation of future cash flows, making these exponential forms essential tools in financial analysis and decision-making.

The valuation of financial instruments is also inherently linked to the assessment of credit risk, with credit rates emerging as a crucial concept in this domain. Credit rates, or credit spreads, model the risk of default of the obligor, providing a quantitative measure that reflects the likelihood of default occurring over a specified time frame. Understanding and accurately estimating credit rates is essential for pricing credit-sensitive instruments and managing associated risks [47]. The credit rate, denoted by $\lambda(t)$ or λ_t, reflects the instantaneous rate of occurrence of the default event at time t, conditional on survival up to that point. Different notations are employed based on the context, particularly to emphasize the time dependency or to align with conventions specific to certain models. From the point of view of mathematical valuation of debt instruments, it is very similar to the concept of interest rate, but reflects the default factor in pricing, instead of the dynamic time value of money.

The concepts of interest and credit rates are fundamental for reduced-form models, which constitute a class of models widely used in financial mathematics, particularly in the context of credit risk assessment [48, 62]. These models are designed to capture the dynamics of default events by bypassing explicit modeling of the underlying economic factors that contribute to defaults. Rather than delving into the intricacies of these economic factors, these models concentrate on directly characterizing the timing and incidence of defaults [47, 49, 63]. Let $Z(t,T)$ denote the price of the zero-coupon bond with a nominal value to be repaid at maturity $Z(T,T)$, which is known from the very moment at which the instruments came to existence, and in particular at t. With the assumption of no debt recovery upon default, the general reduced-form dynamics of the price is defined as

$$Z(t,T) = Z(T,T)\mathbb{E}_t\left[\exp\left(-\int_t^T (r(u)+\lambda(u))\,du\right)\right], \tag{2.12}$$

Compared to the formula (2.6) that takes into account only the interest rate term structure $r(t)$, Eq. (2.12) also incorporates the credit risk component $\lambda(t)$ into the valuation. It can be also expressed in terms of forward rates (both interest and credit) as

$$Z(t,T) = Z(T,T)\exp\left(-\int_t^T (f(t,u)+h(t,u))\,du\right), \tag{2.13}$$

where

$$h(t,u) = -\frac{\partial}{\partial u}\ln(Z(t,u)) - f(t,u). \tag{2.14}$$

and

$$h(u,u) = \lambda(u). \tag{2.15}$$

Analogically to the case with the interest rates, if interest rate $r(u)$ and credit rate $\lambda(u)$ are deterministic, then

$$Z(t,T) = \exp\left(-\int_t^T \left(f(t,u) + h(t,u)\right) du\right) = \exp\left(-\int_t^T \left(r(u) + \lambda(u)\right) du\right), \qquad (2.16)$$

and for all t, u

$$h(t,u) = \lambda(u). \qquad (2.17)$$

One of the models incorporating the process of asset recovery is the Duffie-Singleton model [47]. It assumes that the amount of a company value that is recovered upon default is proportional to current market value of its bond $Z(t,T)$ by the rate of δ, and therefore equals $\delta Z(t,T)$. As a consequence, the price $Z(t,T)$ follows

$$Z(t,T) = Z(T,T)\mathbb{E}_t\left[\exp\left(-\int_t^T \left(r(u) + cs(u)\right) du\right)\right], \qquad (2.18)$$

where $cs(t) = \lambda(t)(1-\delta)$. In deterministic case, it takes the form

$$Z(t,T) = Z(T,T)\exp\left(-\int_t^T \left(r(u) + cs(u)\right) du\right). \qquad (2.19)$$

Under the assumption of time independence of default intensities $cs(t) \equiv cs$ and instantaneous rates $r(t) \equiv r$, a particularly simplified version of the formula (2.18) is attained:

$$Z(t,T) = Z(T,T)\exp\left(-\left(r + cs\right)\left(T-t\right)\right). \qquad (2.20)$$

This assumption proves advantageous when time t remains fixed and serves predominantly for discounting purposes, while at the same time the dynamic evolution of rates and intensities in time is not an important point of modeling interests. This simplification renders formulas more accessible and practical to manipulate, while preserving essential model aspects simultaneously. The inclusion of recovery rates in spreads serves to streamline the mathematical formulations, making the formulas clearer, more concise, and easier to use, especially compared to the other well-known reduced-form model of Jarrow and Turnbull [49].

2.2 Short-rate models and their relationship with UMWE

The notion of the short-term interest rate $r(u)$ introduced in Section 2.1 is in fact connected with a broad family of mathematical models called short-rate models [111]. Their key characteristic is the fact that they specify the dynamics of the short rate, which is the cost of borrowing money at time u for the infinitesimally small duration. The primary benefits of short-rate models are their inherent simplicity and the tendency to produce analytical formulas for bonds and related vanilla options. They are especially valuable in scenarios where numerous securities require valuation, and they still remain very popular [111]. Numerous short-rate models has been constructed over time [111]. One-factor models, in general, are defined as the unique strong solution of the SDE

$$dx_t = c_t(b_t - x_t)dt + \sigma(t, x_t)dW_t, \qquad (2.21)$$

where c_t is the speed of mean reversion, b_t is the long-term mean level, $\sigma(t, x_t)$ is the volatility function, and W_t is a Wiener process with respect to the probability measure $\mathbb{P}$. In general, these models are characterized by the reversion to the mean b_t at the rate c_t and the existence of random noise $\sigma(t, x_t)dW_t$, capturing the tendency of interest rates to fluctuate around a long-term average while being influenced by stochastic factors. One of the oldest and, in fact, pioneering models is the Vašiček model [116], which features constant parameters:

$$dr_t = c(b - r_t)dt + \sigma dW_t. \tag{2.22}$$

This model is, in fact, the Ornstein-Uhlenbeck process [117] applied to describe the dynamics of interest rates.

Consider the UMWE model (1.9) developed in Chapter 1. The discrete equation for the logarithm of the interest rate (1.25) is

$$\ln(i(t+1)) = (1-a)\ln(i_{fix}) + a\ln(i(t)). \tag{2.23}$$

From this, it follows that

$$\ln(i(t+1)) - \ln(i(t)) = (1-a)\left(\ln(i_{fix}) - \ln(i(t)\right). \tag{2.24}$$

This is the difference equation of the from

$$\Delta\ln(i(t)) = (1-a)\left(\ln(i_{fix}) - \ln(i(t)\right)\Delta t. \tag{2.25}$$

It corresponds to its continuous equivalent, the differential equation

$$dx_t = c(b - x_t)dt, \tag{2.26}$$

where $x_t \sim \ln(i(t)),\ c \sim (1-a),\ b \sim \ln(i_{fix})$. This analysis demonstrates that, aside from the random factor, the discrete interest rate model derived from a family of equations (1.9) that define the dynamics of the credit market aligns with the one-factor short rate model characterized by the Ornstein-Uhlenbeck process applied to the logarithm of the interest rate. A crucial aspect of this model is that it is rooted in a comprehensive set of equations that describe the dynamics of the general credit market, with detailed justifications and explanations regarding the real-world behavior of the market and its participants.

However, a significant distinction exists between the model (1.9) and the conventional Ornstein-Uhlenbeck process: the latter typically assumes a non-negative reversion rate c. This assumption ensures stable behavior, enforcing mean-reversion properties that pull the process back toward its asymptotic mean value and fixed point b. Consequently, abrupt changes are attributed to the random component σdW_t. This approach, however, does not align with the systemic risk paradigm of endogeneity, which posits that the dynamics of the system itself is the primary source of significant instability, rather than random external factors [3]. In the discrete model (1.9), the equivalent of c is not constrained to be positive. This critical assumption allows the model to effectively describe unstable states, such as bubbles and collapses. At first glance, it may seem unreasonable to posit $c < 0$ in (2.26), as this would imply that the model is not centered around b, but diverges towards $\pm\infty$ instead. However, there are important nuances to consider. In the UMWE framework derived from the model (1.9), regime switching is facilitated through the allowance of dynamic parameters, enabling transitions between three distinct states: stability, bubble, and economic collapse. With the introduction of the random noise component σdW_t, it is possible to construct a constant parameter model that can transit between bubble and collapse, but not between fundamentally stable and unstable states, defined as $c > 0$ and $c < 0$. Nevertheless, it is argued below that a fundamentally unstable regime ($c < 0$) can, under appropriate parameter selection, exhibit more stable behavior than a fundamentally stable regime ($c > 0$) for an arbitrarily long duration.

2.3 Extended Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is a type of stochastic process that is often used to model mean-reverting behavior in various fields such as finance [117], physics [118], and biology [119]. It is in fact a special case of the process (2.21) with constant parameters:

$$dx_t = c(b - x_t)dt + \sigma dW_t. \tag{2.27}$$

The existence of an unique strong solution of that stochastic differential equation is provided by the Lipschitz condition [120]. The latter states that for a SDE of the form

$$dx_t = h(x_t)dt + \sigma(x_t)dW_t \tag{2.28}$$

there exists an unique strong solution if there exists a constant $L > 0$ such that $\forall x, y \in \mathbb{R}$

$$|h(x) - h(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{2.29}$$

For an Ornstein-Uhlenbeck process given by Eq. (2.27), $h(x) = c(b - x)$, $\sigma(x) = \sigma$, and therefore

$$|h(x) - h(y)| + |\sigma(x) - \sigma(y)| = |c(b - x) - c(b - y)| + |\sigma - \sigma| = |c||x - y|. \tag{2.30}$$

Hence, the Lipschitz condition is satisfied for $L = |c|$. This result holds independently of the sign of the reversion parameter c.

In order to derive the solution, Eq. (2.27) can be rewritten as

$$dx_t + cx_t dt = cbdt + \sigma dW_t. \tag{2.31}$$

Multiplying the entire equation by e^{ct} yields

$$e^{ct}dx_t + ce^{ct}x_t dt = cbe^{ct}dt + \sigma e^{ct}dW_t. \tag{2.32}$$

Then, it follows that

$$d(e^{ct}x_t) = cbe^{ct}dt + \sigma e^{ct}dW_t \tag{2.33}$$

The integration of both sides leads to

$$\begin{aligned} e^{ct}x_t - x_0 &= \int_0^t cbe^{cs}ds + \int_0^t \sigma e^{cs}dW_s \\ e^{ct}x_t - x_0 &= b(e^{ct} - 1) + \sigma \int_0^t e^{cs}dW_s \\ e^{ct}x_t &= x_0 + b(e^{ct} - 1) + \sigma \int_0^t e^{cs}dW_s \end{aligned} \tag{2.34}$$

Thus, the solution to the Ornstein-Uhlenbeck process is:

$$x_t = b + (x_0 - b)e^{-ct} + \sigma e^{-ct} \int_0^t e^{cs}dW_s. \tag{2.35}$$

This expression shows that x_t reverts to the mean b over time, influenced by the initial condition x_0 and the stochastic integral term representing random fluctuations. The mean of x_t is calculated as follows:

$$\begin{aligned}\mathbb{E}[x_t] &= \mathbb{E}\left[b + (x_0 - b)e^{-ct} + \sigma e^{-ct}\int_0^t e^{cs}dW_s\right]\\ &= b + (x_0 - b)e^{-ct} + \sigma e^{-ct}\mathbb{E}\left[\int_0^t e^{cs}dW_s\right]\\ &= b + (x_0 - b)e^{-ct} + \sigma e^{-ct}\cdot 0\\ &= b + (x_0 - b)e^{-ct},\end{aligned} \tag{2.36}$$

as the mean $\mathbb{E}\left[\int_0^t f(s)dW_s\right] = 0$. The variance of x_t is given by

$$\begin{aligned}\mathrm{Var}(x_t) &= \mathrm{Var}\left(b + (x_0 - b)e^{-ct} + \sigma e^{-ct}\int_0^t e^{cs}dW_s\right)\\ &= \mathrm{Var}\left(\sigma e^{-ct}\int_0^t e^{cs}dW_s\right)\\ &= \sigma^2 e^{-2ct}\mathrm{Var}\left(\int_0^t e^{cs}dW_s\right)\\ &= \sigma^2 e^{-2ct}\int_0^t e^{2cs}ds\\ &= \sigma^2 e^{-2ct}\left[\frac{e^{2cs}}{2c}\right]_0^t\\ &= \sigma^2 e^{-2ct}\cdot\frac{e^{2ct}-1}{2c}\\ &= \frac{\sigma^2}{2c}(1 - e^{-2ct}).\end{aligned} \tag{2.37}$$

To summarize, it holds that

$$\mathbb{E}[x_t] = b + (x_0 - b)e^{-ct}, \tag{2.38}$$

$$\mathrm{Var}(x_t) = \frac{\sigma^2}{2c}(1 - e^{-2ct}). \tag{2.39}$$

Moreover, the mean displacement equals

$$\begin{aligned}\mathbb{E}[x_t - x_0] &= b + (x_0 - b)e^{-ct} - x_0 = b(1 - e^{-ct}) + x_0(e^{-ct} - 1) =\\ &= (b - x_0)(1 - e^{-ct}).\end{aligned} \tag{2.40}$$

The dispersion (coefficient of variation) $D(t)$ of the process at time t is the ratio of its standard deviation to its mean $\frac{\sqrt{\mathrm{Var}(x_t)}}{\mathbb{E}[x_t]}$. It measures the variability of the distribution, or, more precisely, the amount of mean values that is needed to attain the standard deviation. For the Ornstein-Uhlenbeck process x_t, the dispersion equals

$$D(t) = \frac{\sigma}{b + (x_0 - b)e^{-ct}}\sqrt{\frac{1 - e^{-2ct}}{2c}}. \tag{2.41}$$

The asymptotic dispersion $D = \lim_{t\to\infty} D(t)$ equals $D = \frac{\sigma}{b\sqrt{2c}}$ for a stable system ($c > 0$) and $D = \frac{\sigma}{(x_0-b)\sqrt{-2c}}$ for the unstable one ($c < 0$). Interestingly, in both cases, a small absolute reversion rate c results in significant asymptotic dispersion. This phenomenon is understandable, as it reflects the strength of the forces that push the system to its equilibrium points - b in the stable case and $\pm\infty$ in the unstable one. In the limit as $|c| \to \infty$, the system is 'immediately' driven to its equilibrium, resulting in zero dispersion. Additionally, in the stable case $c > 0$, the asymptotic dispersion follows a dispersion of a distribution with a mean of b and a standard deviation of $\frac{\sigma}{\sqrt{2c}}$. In the case of an unstable system $c < 0$, the initial deviation from the fixed point, $(x_0 - b)$, replaces b. This indicates that a system with a smaller initial mean displacement ultimately experiences greater relative variation in the long term.

Theorem 2.1. *Let $c, \tilde{c} > 0$ and let $x_t, \tilde{x}_t$ be two Ornstein-Uhlenbeck processes* (2.27) *with the same initial value $x_0 = \tilde{x}_0$ and equilibrium point $b = \tilde{b}$, but with positive and negative reversion rates c and $-\tilde{c}$, respectively. For all $0 < \tilde{c} < c$, there exists a time $T > 0$ such that for $0 \le t \le T$ the absolute mean displacement of the unstable system $\tilde{x}_t$ is lesser than that of the stable system x_t. Moreover, $T \ge \frac{\ln\left(\frac{c}{\tilde{c}}\right)}{\tilde{c}+c}$ and $T \to \infty$ for $\tilde{c} \to 0^+$, i.e., this time can be arbitrarily long for sufficiently small $\tilde{c}$.*

Proof. Define the difference

$$g(t) := |\mathbb{E}[\tilde{x}_t - \tilde{x_0}]| - |\mathbb{E}[x_t - x_0]|. \tag{2.42}$$

From the assumption $x_0 = \tilde{x}_0$, it follows that

$$g(t) = |(b - x_0)(1 - e^{\tilde{c}T})| - |(b - x_0)(1 - e^{-cT})|. \tag{2.43}$$

The absolute mean displacement of the unstable system $\tilde{x}_t$ is lesser (in a weak sense) than the absolute mean displacement of the stable system x_t if and only if

$$g(t) \le 0. \tag{2.44}$$

If $b - x_0 = 0$, the inequality is satisfied for all t. For $(b - x_0) \neq 0$ the inequality is equivalent to

$$|1 - e^{\tilde{c}t}| - |1 - e^{-ct}| \le 0.$$

Thus

$$e^{\tilde{c}t} - 1 - 1 - e^{-ct} \le 0$$

and

$$e^{\tilde{c}t} + e^{-ct} - 2 \le 0.$$

Introducing the function $f : \mathbb{R}_+ \to \mathbb{R}$, $f(t) := e^{\tilde{c}t} + e^{-ct} - 2$ allows to consider the problem of

$$f'(t) = 0. \tag{2.45}$$

It follows that

$$\begin{aligned}
\tilde{c}e^{\tilde{c}t} - ce^{-ct} &= 0, \\
\tilde{c}e^{\tilde{c}t} &= ce^{-ct}, \\
\tilde{c}e^{\tilde{c}t+ct} &= c, \\
\ln(\tilde{c}) + \tilde{c}t + ct &= \ln(c), \\
\tilde{c}t + ct &= \ln(c) - \ln(\tilde{c}), \\
(\tilde{c} + c)t &= \ln\left(\frac{c}{\tilde{c}}\right), \\
t &= \frac{\ln\left(\frac{c}{\tilde{c}}\right)}{\tilde{c} + c}.
\end{aligned}$$

Since f' is a continuous function of t, $f'(0) = \tilde{c} - c < 0$ and $\lim_{t\to\infty} f'(t) = \infty$, the function $f(t)$ attains its minimum at $T = \frac{\ln\left(\frac{c}{\tilde{c}}\right)}{\tilde{c}+c}$. Given that $f(0) = 0$, the condition $f(t) \le 0$ is satisfied for at least $t \in \left[0, \frac{\ln\left(\frac{c}{\tilde{c}}\right)}{\tilde{c}+c}\right]$. Moreover, as $\frac{\ln\left(\frac{c}{\tilde{c}}\right)}{\tilde{c}+c}$ approaches infinity when $\tilde{c}$ approaches 0, T can be made arbitrarily long for sufficiently small values of $\tilde{c}$. □

The absolute mean displacement $|E[x_t - x_0]| = |(b - x_0)(1 - e^{-ct})|$ is increasing as a function of t in the initial deviation from the fixed point $|x_0 - b|$. From this fact and from Theorem 2.1, it follows that the more 'stable' initially the system is ($-\tilde{c}$ and $(b - x_0)$ closer to 0), the more and longer it can effectively behave in a 'stable' manner, and therefore conceal the fundamental instability $-\tilde{c} < 0$.

Theorem 2.2. *Let $c, \tilde{c} > 0$ and let $x_t, \tilde{x}_t$ be two Ornstein-Uhlenbeck processes* (2.27) *with the same volatility parameter $\sigma = \tilde{\sigma}$, but with positive and negative reversion rates c and $-\tilde{c}$, respectively. Then the variance of the stable process Var(x_t) is always lesser than the variance of the unstable process Var$(\tilde{x}_t)$ at the same point in time t.*

Proof. Define the difference

$$g(t) := \mathrm{Var}(x_t) - \mathrm{Var}(\tilde{x}_t). \tag{2.46}$$

The variance of the stable system $\mathrm{Var}(x_t)$ is lesser (in a weak sense) than the variance of the unstable system $\mathrm{Var}(\tilde{x}_t)$ if and only if

$$g(t) \le 0. \tag{2.47}$$

It follows that

$$\frac{\sigma^2}{2c}(1 - e^{-2ct}) \le \frac{\sigma^2}{2\tilde{c}}(e^{2\tilde{c}t} - 1),$$

which is equivalent to

$$\tilde{c} + c \le ce^{2\tilde{c}t} + \tilde{c}e^{-2ct}.$$

Let the function $f : \mathbb{R}_+ \to \mathbb{R}$ be defined as

$$f(t) := ce^{2\tilde{c}t} + \tilde{c}e^{-2ct} - (\tilde{c} + c).$$

It holds that

$$f'(t) = 2c\tilde{c}(e^{2\tilde{c}t} - e^{-2ct}) \ge 0.$$

Therefore, $f(t)$ is an increasing function of t with an initial value $f(0) = 0$, hence $f(t) \ge 0$ for all $t \ge 0$. As a consequence, $g(t) = \mathrm{Var}(x_t) - \mathrm{Var}(\tilde{x}_t) \le 0$ for all $t \ge 0$, which completes the proof. □

Theorem 2.3. *Let $\hat{x}_t$ and x_t be two Ornstein-Uhlenbeck processes* (2.27) *with respective reversion rates $\hat{c}, c \in \mathbb{R}$, and such that the volatility parameter of the first process is lesser than that of the second one: $\hat{\sigma} < \sigma$. Then, there exist $T > 0$ such that Var$(\hat{x}_t) \le$ Var(x_t) for $t \in [0, T]$. Moreover, if $\hat{x}_t$ is an unstable process with $\hat{c} < 0$ and x_t is an stable process with $c > 0$, then $T \ge \frac{\ln(\sigma) - \ln(\hat{\sigma})}{(c - \hat{c})}$ and $T \to \infty$ as $\hat{\sigma} \to 0$.*

Proof. Consider a function

$$f(t) := \mathrm{Var}(\hat{x}_t) - \mathrm{Var}(x_t) = \frac{\hat{\sigma}^2}{2\hat{c}}(1 - e^{-2\hat{c}t}) - \frac{\sigma^2}{2c}(1 - e^{-2ct}). \tag{2.48}$$

Its derivative with respect to t equals

$$f'(t) = \hat{\sigma}^2 e^{-2\hat{c}t} - \sigma^2 e^{-2ct}. \tag{2.49}$$

The derivative f' is a continuous function of t, $f'(0) = \hat{\sigma} - \sigma < 0$ on the basis of the assumption, and $f(0) = 0$. Therefore, there exist $T > 0$ such that for $t \leq T$ $\mathrm{Var}(\hat{x}_t) - \mathrm{Var}(x_t) \leq 0$.

Moreover, the second derivative of f equals

$$f''(t) = -2\hat{c}\hat{\sigma}^2 e^{-2\hat{c}t} + 2c\sigma^2 e^{-2ct}. \tag{2.50}$$

If x_t is a stable system ($c > 0$) and $\hat{x}_t$ is unstable ($\hat{c} < 0$), then $f'' > 0$. In such a case, the derivative f' is an increasing function of t, and additionally $0 = f'(\frac{\ln(\sigma)-\ln(\hat{\sigma})}{(c-\hat{c})})$. Therefore, for $t \in [0, \frac{\ln(\sigma)-\ln(\hat{\sigma})}{(c-\hat{c})}]$, $f'(t) \leq 0$, and consequently $\mathrm{Var}(\hat{x}_t) \leq \mathrm{Var}(x_t)$ for $t \leq \frac{\ln(\sigma)-\ln(\hat{\sigma})}{(c-\hat{c})}$. As $\frac{\ln(\sigma)-\ln(\hat{\sigma})}{(c-\hat{c})} \to \infty$ when $\hat{\sigma} \to 0$, the time T can be arbitrarily long for sufficiently small $\hat{\sigma}$. □

The situation here is similar to that of the Theorem 2.1. The smaller the volatility σ of the unstable system, the greater the time for which it has lower variance than the stable one (it can be perceived as more stable for a longer time). The smaller the random noise in the system, the more and longer it can conceal the fundamental instability of the system with $c < 0$.

The trajectories of the extended Ornstein-Uhlenbeck process defined by Eq. (2.27) are shown in Figure 2.1. The absolute mean displacement is initially smaller for slightly negative values of the reversion rate compared to significantly positive ones, a finding formally established in Theorem 2.1. A minimally negative c indicates a weak force driving the trend, resulting in paths that remain relatively close to the initial value x_0, similarly to pure random noise (i.e., $c = 0$). In contrast, trajectories with a greater positive reversion rate are quickly drawn towards the direction of equilibrium point, leading to a substantially higher mean displacement. As stated by Theorem 2.3, trajectories with positive c and a larger parameter σ also exhibit greater variance during the initial period, particularly evident when comparing the plots for $(c, \sigma) = (-0.01, 1)$ and $(c, \sigma) = (0.7, 5)$. Negative reversion paths follow an exponential trend that tends to dominate the noise effect as c decreases, whereas strongly positive reversion paths exhibit sharp oscillations around the equilibrium point 0. Consequently, negative reversion trajectories can be perceived to be significantly smoother and more stable.

Overall, an unstable system with negative reversion rate $c < 0$ behaves more “stable” than a truly stable system with appropriately different parameters for some period of time. Moreover, the lower (closer to 0) the instability parameters c, $|b - x_0|$, σ, the more the system appears stable over an extended period. While this may seem intuitive for low degrees of instability, it represents a very interesting and somewhat paradoxical behavior. In fact, a fundamentally unstable system $c < 0$ can exhibit more stable behavior than a genuinely stable one $c > 0$ for an arbitrarily long time, given suitable parameter values for both systems. This highlights the challenge of recognizing instability before the system spirals out of control. Furthermore, the lower the degree of instability, the stronger and more prolonged this effect becomes, possibly extending to infinity in the limit. Thus, a market can appear very stable for an extended period of time while being determined to abrupt changes eventually. Moreover, effects that enhance initial stability, such as a smaller distance to the fixed point $|(b - x_0)|$ and reversion rate c closer to 0 lead to significantly

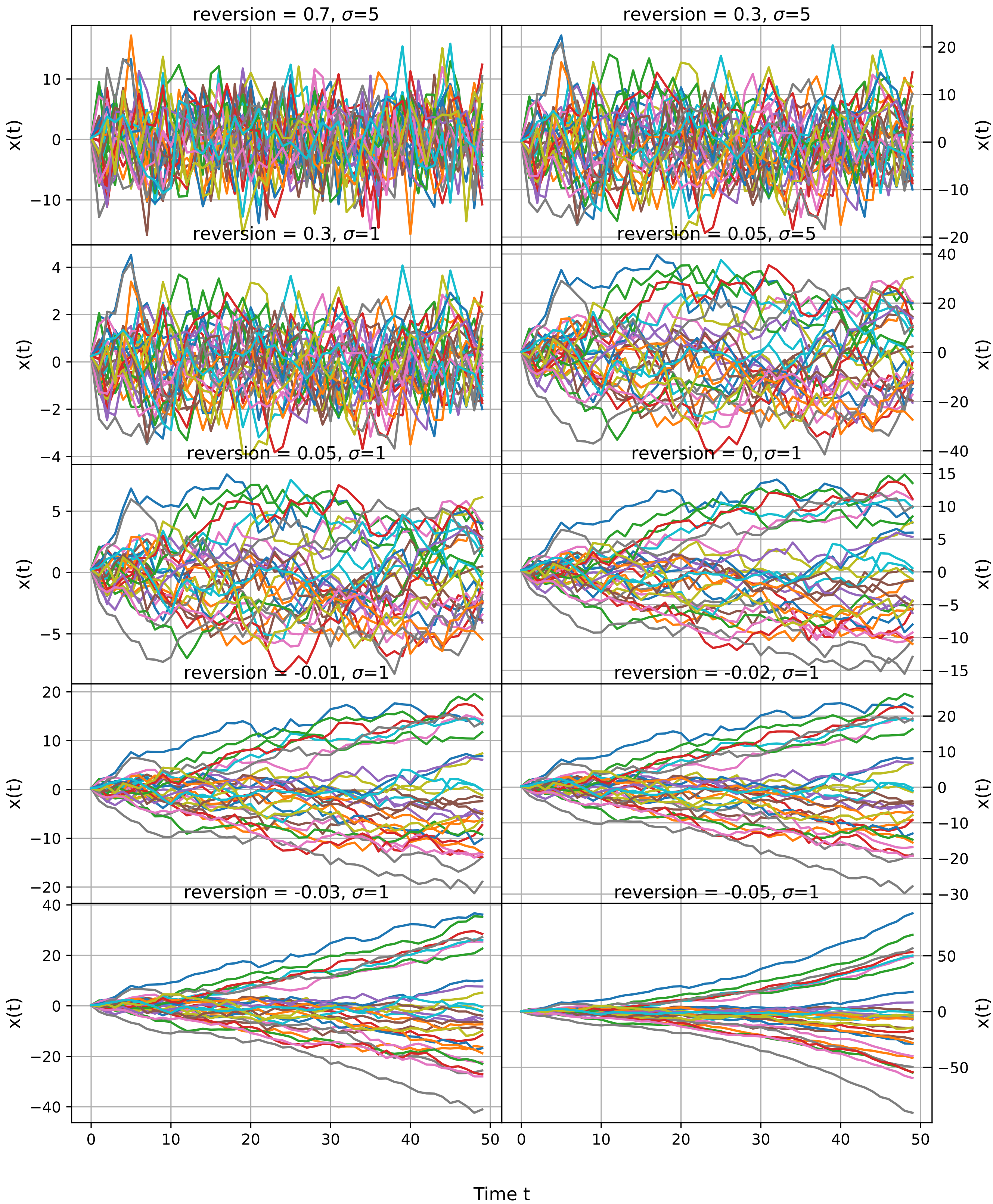


Figure 2.1: Trajectories of the extended Ornstein-Uhlenbeck process (2.27). The plots illustrate the influence of reversion c and variability σ on the dynamics of the extended Ornstein-Uhlenbeck process (2.27). The common horizontal axis represents the timesteps, while the vertical axes denote the values of the trajectories. The computations have been performed with a discrete version of the differential equation (2.27) with a timestep $dt = 1$, a fixed point $b = 0$, and an initial value $x_0 = 0.25$.

greater long-term instability (greater asymptotic dispersion). This effect corresponds to the very important behavior of systemic risk: prolonged periods of seemingly calm and stable dynamics can translate into greater instability in the long run.

2.4 Summary

In this chapter, the short-rate process has been developed based on the UMWE model (1.9) of the credit market constructed in Chapter 1. The model can be viewed as both a continuous, stochastic enhancement of the UMWE, and an extension of the Ornstein-Uhlenbeck process [117], allowing for significant changes in its characteristics. It is capable of reproducing different economic regimes and long, seemingly stable periods that lead to abrupt dynamics, which are critical features not captured by a wide range of commonly used financial models [12, 13, 17]. Moreover, the model paves the way for further development in this area. Thus far, only a fixed value of c has been considered in the stochastic context to emphasize the impact of its negative value. However, a dynamic c has already been incorporated into the deterministic approach in Chapter 1. The possibly negative dynamic c_t can be introduced into the general formula for one-factor models (2.21). Notable examples include the Hull-White [62] and Black-Karasinski [121] variants. This approach can also be extended to multi-factor short-rate framework, such as the Longstaff–Schwartz [122] and Chen [123] models. It constitutes one of the foundational elements of the comprehensive systemic risk model developed in Chapter 6, as a component describing the evolution of risk in a general credit market.

The model offers unique insights into the dynamics of financial markets. As instability might be largely unrecognizable by well-established measures of mean displacement and variance, it puts into question the capabilities of commonly adopted methods for quantifying risk. These methods constitute the basis for determining bank reserves, pension schemes, and government financing policies worldwide [107, 124–126], therefore this outcome can have profound consequences for regulators, financial institutions, and individuals alike.

Chapter 3

Network modeling of financial systems: the NEVA framework

This chapter introduces and further develops the Network Valuation in Financial Systems (NEVA) framework, constructed and described in detail in Barucca et al. [44]. Main contributions of this chapter lie in Theorems 3.1 and 3.3. The first one is both a consolidation and extension of the original results into an additional range of cases, which was originally limited only to some boundary points. The second theorem offers even more comprehensive insights into the framework, allowing to infer properties satisfied by a countable set of points into broad areas. These results are significant from both a theoretical perspective and for the practical applications of the framework in computer simulations using real data.

NEVA is a versatile framework designed for valuation of asset embedded in the environment of crisis contagion. It takes into consideration the interconnectedness of balance sheets through interbank assets and addresses the uncertainty regarding the future values of external assets held by banks (non- interbank assets). One of the most significant contributions of NEVA to the discipline of the systemic risk research is a comprehensive mathematical formalism that analytically separates existing approaches into two key components. The first one establishes a general framework governing network behavior, forming the foundation for the process of shock propagation. This part is responsible for the modeling of crisis contagion mechanism, which remains a key interest point of the systemic risk research discipline, concerned about the states where a significant portion or a whole financial environment is seriously endangered by a collapse [14, 127]. The second component is a valuation function, a crucial element determining how the values of financial instruments are recalculated based on changing market conditions and variables. While the rules of the network transmission component are unambiguously determined within the framework, the valuation function provides an open field for assumptions regarding the market behavior. This flexibility allows for the adaptation of the framework to specific modeling environments and preferences. Many important methods of systemic risk calculation are special cases of NEVA, including algorithms of Furfine [41], Eisenberg-Noe [42], and linear DebtRank [43].

The framework describes balance sheets of n financial institutions and their evolution over time. Contracts mature at time T, and banks value their assets at time $t \leq T$. Balance sheets are divided into two parts: external and internal. For a given bank i, its internal assets $A_{ij}(t) \in \mathbb{R}_+$ and liabilities $L_{ij}(t) \in \mathbb{R}_+$ represents mutual contracts and financial instruments issued or possessed by other system participant j, respectively. They can consist of overnight credit in interbank lending market, longer-term debt, or derivatives instruments such as interest rates swaps. On the other hand, external assets $A_i^e(t) \in \mathbb{R}_+$ and liabilities $L_i^e(t) \in \mathbb{R}_+$ of a bank i constitutes a part of its balance sheet that is not included in balance sheets of any other institution of the considered financial system. Examples of external assets can encompass credit to retail customers and various kinds of financial instruments, like stocks, derivatives, government and corporate bonds, not issued by the other system participants. External liabilities may consist of retails deposits and debt not owned by other banks within the system.

At time t, banks perform a valuation of their respective interbank assets. In general, this valuation is based on available information and list of variables and parameters specified for that purpose, depending on a concrete method. This method is called a valuation function and denoted as $\mathbb{V}_{ij}(\cdot)$. It can depend on both i and j, reflecting the diversity in models used by different banks for valuing their interbank assets. Similarly the valuation of external assets is performed, with a valuation function denoted as $\mathbb{V}_i^e(\cdot)$. In this thesis, it is assumed that valuation functions are functions of a vector of equities of all institutions in a considered financial system, denoted as $\mathbf{E}(t) := (E_1(t), ..., E_n(t)) \in \mathbb{R}^n$. Consequently, for an institution i, the process of determining the values v_i^e and v_{ij} of its external and internal assets is specified as

$$\begin{aligned} v_i^e &= A_i^e(t)\mathbb{V}_i^e\left(\mathbf{E}(t)\right), && i \in \{1, \dots, n\}; \\ v_{ij} &= A_{ij}(t)\mathbb{V}_{ij}\left(\mathbf{E}(t)\right), && i \in \{1, \dots, n\},\ j \in \{1, \dots, n\}. \end{aligned} \tag{3.1}$$

While valuation functions may exhibit a general dependence on the equities of all banks, in many cases their dependency is solely on the equity of the debtor. Nonetheless, the ensuing results remain applicable to the broader context where valuation functions rely on the equities of any subset of banks.

In order to correspond with natural modeling environment and properly reflect the reality, constraints are imposed on admissible valuation functions. For that purpose, the definition of the order on a vector space is introduced first.

Definition 3.1. Let $q \in \mathbb{N}$ and $\mathbf{E}, \mathbf{E}' \in \mathbb{R}^d$. A partial order $\leq$ on the vector space $\mathbb{R}^q$ is defined as follows: $\mathbf{E} \leq \mathbf{E}'$ if and only if for every pair of respective coefficients E_k, E_k' of vectors $\mathbf{E}, \mathbf{E}'$, indexed by $k \in \{1, \dots, q\}$, it holds that $E_k \leq E_k'$.

Subsequently, the following definition of feasible valuation functions is introduced:

Definition 3.2. Given an integer $q \leq n$, where n is a number of institutions in a financial system, a function $\mathbb{V}\colon \mathbb{R}^q \to [0, 1]$ is called a feasible valuation function if and only if:

1. it is nondecreasing: $\mathbf{E} \leq \mathbf{E}' \Rightarrow \mathbb{V}(\mathbf{E}) \leq \mathbb{V}(\mathbf{E}'),\ \forall\ \mathbf{E}, \mathbf{E}' \in \mathbb{R}^q$

2. it is continuous.

For $\mathbf{E}, \mathbf{E}' \in \mathbb{R}^q$, $\mathbf{E} \leq \mathbf{E}'$ means that for every pair of respective coefficients E_k, E_k' indexed by $k \in \{1, \dots, q\}$, it holds that $E_k \leq E_k'$.

A feasible valuation function is constrained to range between zero and one. The first reason for that is the fact that the value of both internal and external assets should not exceed the amount of debt to repay. Moreover, assets may not be readily marketable without incurring losses relative to their intrinsic worth during stressed market conditions. Such an approach encapsulates to some extent liquidity problems and fire sales, reflecting broader scope of the systemic risk sources in the proposed model [14]. The first point of the Def. 3.2 emphasizes that a feasible valuation function is a nondecreasing function of equity. This constitutes a very natural and straightforward assumption, as, all else being equal, higher value of equity corresponds to more advantageous situation and increased level of creditworthiness. The second point is primarily a technical assumption.

Consequently, from the definition of balance sheet components, equity of a bank i can be expressed as

$$E_i(t) = A_i^e(t)\mathbb{V}_i^e\left(\mathbf{E}(t)\right) - L_i^e(t) + \sum_{j=1}^{n} A_{ij}(t)\mathbb{V}_{ij}\left(\mathbf{E}(t)\right) - \sum_{j=1}^{n} L_{ij}(t), \tag{3.2}$$

for all $i \in \{1, \dots, n\}$. Then, it follows that

$$m_i(t) \leq E_i(t) \leq M_i(t), \tag{3.3}$$

where

$$\begin{aligned} m_i(t) &:= -L_i^e(t) - \sum_{j=1}^{n} L_{ij}(t), \\ M_i(t) &:= A_i^e(t) - L_i^e(t) + \sum_{j=1}^{n} A_{ij}(t) - \sum_{j=1}^{n} L_{ij}(t). \end{aligned} \tag{3.4}$$

With the introduction of the map Φ

$$\begin{aligned} &\Phi : \prod_{i=1}^{n}[m_i(t), M_i(t)] \to \prod_{i=1}^{n}\left[m_i(t), M_i(t)\right], \quad \Phi = (\Phi_1, ..., \Phi_n), \\ &\Phi_i\left(\mathbf{E}(t)\right) : = E_i(t) \\ &\qquad = A_i^e(t)\mathbb{V}_i^e\left(\mathbf{E}(t)\right) - L_i^e + \sum_{j=1}^{n} A_{ij}(t)\mathbb{V}_{ij}\left(\mathbf{E}(t)\right) - \sum_{j=1}^{n} L_{ij}(t), \end{aligned} \tag{3.5}$$

the set of equations (3.2) can be expressed more concisely:

$$\mathbf{E}(t) = \Phi\left(\mathbf{E}(t)\right). \tag{3.6}$$

Consequently, the process of valuation reduces to the fixed-point problem for a map Φ. Moreover, there exist a correspondence between every solution $\mathbf{E}^*(t)$ of Eq. (3.6) and a fixed point of the iterative map

$$\mathbf{E}^{k+1}(t) = \Phi(\mathbf{E}^k(t)). \tag{3.7}$$

The iterative map in Equation (3.7) defines the Picard iteration algorithm, providing a method to determine the solutions of the fixed point problem (3.6) with arbitrarily specified precision, as shown in the following theorem.

Theorem 3.1. *Assume that all valuation functions in the map Φ are feasible, with $\mathbf{m}(t) \leq \mathbf{E}^0(t) \leq \mathbf{M}(t)$ for $\mathbf{m}(t) := (m_1(t), \dots, m_n(t))$ and $\mathbf{M}(t) := (M_1(t), \dots, M_n(t))$, and also $\mathbf{E}^0(t) R \mathbf{E}^1(t) = \Phi(\mathbf{E}^0(t))$, where $R \in \{\leq, \geq\}$. Then it holds that:*

1. *the sequence $\{\mathbf{E}^k(t)\}$ is R-monotonic;*
2. *the sequence $\{\mathbf{E}^k(t)\}$ is convergent to $\lim_{k\to\infty} \mathbf{E}^k(t) = \mathbf{E}^\infty(t)$;*

3. *$\mathbf{E}^{\infty}(t) = \mathbf{E}^{*}(t)$ is a solution of Eq (3.6).*

Proof. The proof is carried out using mathematical induction. The assumption of the theorem states that

$$\mathbf{E}^0(t) R \mathbf{E}^1(t) \ = \ \Phi(\mathbf{E}^0(t)). \tag{3.8}$$

Now, suppose that this property holds true for all $0 \leq m \leq n$. Then, from the fact that Φ is nondecreasing and by virtue of the induction assumption $\mathbf{E}^{m-1}(t) R \mathbf{E}^m(t)$, it follows that

$$\mathbf{E}^n(t) = \Phi(\mathbf{E}^{n-1}(t)) \ R \ \Phi(\mathbf{E}^n(t)) = \mathbf{E}^{n+1}(t).$$

Therefore, the sequence $\{\mathbf{E}^n(t)\}$ is monotonic and bounded by the definition (3.5) of the map Φ, and by the Monotone Convergence Theorem the limit $\lim_{n\to\infty} \mathbf{E}^n(t) = \mathbf{E}^*(t)$ exists. Moreover, from the continuity of Φ and definition of $\mathbf{E}^n(t)$, it follows that

$$\begin{aligned}\Phi(\mathbf{E}^*(t)) &= \Phi(\lim_{n\to\infty} \mathbf{E}^n(t)) = \lim_{n\to\infty} \Phi(\mathbf{E}^n(t)) \\ &= \lim_{n\to\infty} \mathbf{E}^{n+1}(t) = \mathbf{E}^*(t).\end{aligned}$$

This completes the proof. □

The following result is a consequence of Theorem 3.1.

Theorem 3.2. *Assume that all valuation functions in the map Φ are feasible, and either $\mathbf{E}^0(t) = \mathbf{m}(t)$ or $\mathbf{E}^0(t) = \mathbf{M}(t)$. Then it holds that:*

1. *the sequence $\{\mathbf{E}^k(t)\}$ is monotonic;*

2. *the sequence $\{\mathbf{E}^k(t)\}$ is convergent to $\lim_{k\to\infty} \mathbf{E}^k(t) = \mathbf{E}^*(t)$;*

3. *$\mathbf{E}^*(t)$ is a solution of Eq (3.6).*

Proof. It is sufficient to show that the conditions of Theorem 3.1 are satisfied. The feasibility of all valuation functions in the map Φ is provided by the assumption. Moreover, as either $\mathbf{E}^0(t) = \mathbf{m}(t)$ or $\mathbf{E}^0(t) = \mathbf{M}(t)$, it follows that

$$\mathbf{m}(t) \leq \ \mathbf{E}^0(t) \leq \ \mathbf{M}(t).$$

Furthermore, if $\mathbf{E}^0(t) = \mathbf{m}(t)$, then

$$\mathbf{E}^1(t) = \Phi(\mathbf{E}^0(t)) \geq \mathbf{E}^0(t) = \mathbf{m}(t),$$

as by the definition of the mapping Φ given in Equation (3.5), $\mathbf{m}(t) \leq \ \Phi \leq \ \mathbf{M}(t)$. Similarly, in the second case $\mathbf{E}^0(t) = \mathbf{M}(t)$, it holds that

$$\mathbf{E}^1(t) = \Phi(\mathbf{E}^0(t)) \leq \mathbf{E}^0(t) = \mathbf{M}(t).$$

Therefore, the conditions given in Theorem 3.1 are satisfied, which completes the proof. □

Theorem 3.1 is a consolidation of Theorems 3.2 and 3.3 of the original NEVA framework [44]. It assumes the continuity of the valuation function (which is included in the Definition 3.2 of the feasible valuation function proposed in this thesis), in contrast to original Theorems, which assume different one-sided continuity. It is a stronger assumption, which allows for the consolidation of the two Theorems into a single, more broad result. On the other hand, the condition of monotonicity of the initial iteration is presented in place of the assumptions of the starting point $\mathbf{E}^0(t)$ being equal to one of the boundary points of the mapping Φ, either $\mathbf{m}(t)$ or $\mathbf{M}(t)$. The initial monotonicity is in fact a weaker condition, as evidenced by Theorem 3.2 and its proof, derived on the basis of Theorem 3.1. In short, starting from the boundary points implies that the result of the first iteration is greater or lesser than the initial point, which is the assumption of Theorem 3.1. As such, the initial monotonicity covers both cases assumed in the original paper, while can also include more. Therefore, Theorem 3.1 provides greater flexibility, both from the theoretical perspective, as well as for the performance of computer simulations of financial networks. In particular, one does not need to assume some specific forms of the valuation function in a way that the system already starts from the upper boundary $\mathbf{M}(t)$. Moreover, it allows to examine the evolution of systems starting from any middle point in between. As real markets are not necessarily often located on the very extrema, it is reasonable to consider such scenarios as well. Theorem 3.1 allows to resolve the halting problem for some of such scenarios: if there is a monotonic iteration at any point of the algorithm execution, then it could be treated as an initial moment, subsequent iteration will be monotonic in the same direction, and the program will converge towards an equilibrium point. Therefore, from that point of view, it is also a result about the stability of states (or paths) of such financial systems.

Theorem 3.1 provides a practical approach for conducting stress test analysis, or, in other words, evaluating the influence of shocks on the system. Given the initial state of the system, the shock χ is applied to external assets, resulting in a dynamics as $A_i^e(t) \to (1-\chi)A_i^e(t)$, in a way such that the financial system indeed experiences a loss in the first iteration. Subsequently, employing the Picard algorithm, the propagation of the crisis is computed, enabling assessment of the initial shock's effect. By the virtue of Theorem 3.1, for any specified precision $\epsilon > 0$, there exists $K(\epsilon)$ such that for all $k > K(\epsilon)$, it holds that $||\mathbf{E}^k(t) - \mathbf{E}^*(t)|| < \epsilon$. However, it is worth to point out that that ϵ is typically unknown beforehand. Hence, in practical scenarios, it is reasonable to estimate ϵ by examining the difference between consecutive terms of the sequence, ensuring it meets the predefined precision criteria, i.e., $||\mathbf{E}^{k+1}(t) - \mathbf{E}^k(t)|| < \epsilon$. Further information about both such simulations, as well as the formal properties of the NEVA framework, is provided in the following theorem.

Theorem 3.3. *Assume that all valuation functions in the map Φ are feasible, and that $\mathbf{E}^0(t)$ satisfies conditions of Theorem 3.1: $\mathbf{m}(t) \leq \mathbf{E}^0(t) \leq \mathbf{M}(t)$ and $\mathbf{E}^0(t) R \mathbf{E}^1(t)$, $R \in \{\leq, \geq\}$. For any $p, q \in \mathbb{N} \cup \{\infty\}$, $p < q$, define*

$$\mathbb{B}_{p,q}(\mathbf{E}^0(t)) := \{\mathbf{E} \in \mathbb{R}^n : \mathbf{E}^p(t) \; R \; \mathbf{E} \; R \; \mathbf{E}^q(t)\}.$$

It holds that $\mathbb{B}_{p,q}(\mathbf{E}^0(t)) \subset \mathbb{B}_{0,\infty}(\mathbf{E}^0(t)) =: \mathbb{B}(\mathbf{E}^0(t))$, and for all $\mathbf{E} \in \mathbb{B}(\mathbf{E}^0(t))$, the sequence $\{\mathbf{E}^k\}$ defined iteratively by Eq. (3.7) *is convergent to a solution $\mathbf{E}^*(t)$ of Eq (3.6).*

Proof. Because $\mathbf{E}^0(t)$ satisfies the conditions of Theorem 3.1, the sequence $\{\mathbf{E}^k(t)\}$ is R-monotonic, and subsequently

$$\mathbf{E}^0(t) \; R \; \mathbf{E}^p(t) \; R \; \mathbf{E}^q(t) \; R \; \mathbf{E}^\infty(t) = \mathbf{E}^*(t), \tag{3.9}$$

for every $p, q \in \mathbb{N} \cup \{\infty\}$. It means that that $\mathbb{B}_{p,q}(\mathbf{E}^0(t)) \subset \mathbb{B}(\mathbf{E}^0(t))$. Furthermore, from the definition of $\mathbb{B}(\mathbf{E}^0(t))$, the assumption that $\mathbf{E} \in \mathbb{B}(\mathbf{E}^0(t))$ means $\mathbf{E}^0(t) \; R \; \mathbf{E} \; R \; \mathbf{E}^\infty(t)$. Consequently, because Φ is nondecreasing, it follows that

$$\Phi(\mathbf{E}^0(t)) \; R \; \Phi(\mathbf{E}) \; R \; \Phi(\mathbf{E}^\infty(t)) = \mathbf{E}^\infty(t).$$

The last equality is implied by the fact that the limit $\mathbf{E}^{\infty}(t)$ is a fixed point of the mapping Φ. Moreover, application of that procedure l times yields

$$\Phi^l(\mathbf{E}^0(t)) \; R \; \Phi^l(\mathbf{E}) \; R \; \mathbf{E}^{\infty}(t),$$

which from the definition of the iterative sequence given in Eq. (3.7) is the same as

$$\mathbf{E}^l(t) \; R \; \mathbf{E}^l \; R \; \mathbf{E}^{\infty}(t).$$

A sequence $\{\mathbf{E}^l(t)\}$ is convergent to the limit $\mathbf{E}^{\infty}(t) = \mathbf{E}^*(t)$. Therefore, by the virtue of squeeze theorem, the sequence $\{\mathbf{E}^l\}$ is also convergent to $\mathbf{E}^*(t)$. $\square$

In essence, while Theorems 3.1 and 3.2 are concerned about properties of sequences of financial network iterations and their convergence to a fixed point, Theorem 3.3 allows to determine areas of convergence. A sequence $\{\mathbf{E}^k(t)\}$ determines not only a countable set of points, but also an area $\mathbb{B}(\mathbf{E}^0(t)) = \{\mathbf{E} \in \mathbb{R}^n : \; \mathbf{E}^0(t) \; R \; \mathbf{E} \; R \; \mathbf{E}^{\infty}(t)\}$, such that for every $\mathbf{E} \in \mathbb{B}(\mathbf{E}^0(t))$ a sequence $\{\Phi^k(\mathbf{E})\}$ is convergent to a fixed point $\mathbf{E}^{\infty}(t) = \mathbf{E}^*(t)$. This result is a powerful tool in practical programming applications of the NEVA framework. Calculations executed usually on a predetermined grid of initial shocks can now provide information not only for a finite set of points, but also for the whole areas in between them. This constitutes a wealth of insights regarding the model's overall behavior, as well as for a broad range of specific inputs.

Particularly important method of systemic risk calculation is linear DebtRank [4]. The algorithm predates the establishment of the NEVA framework, yet in fact it is a special case of thereof [44], broadly applied for assessing security of financial systems [4, 21, 128]. Its valuation functions presents as follows:

$$\mathbb{V}_i^e(E_j(t)) = 1, \tag{3.10a}$$

$$\mathbb{V}_{ij}(E_j(t)) = \min\left[\frac{E_j^+(t)}{E_j(0)}, 1\right], \tag{3.10b}$$

where $E_j^+(t) := \max(E_j(t), 0)$.

Chapter 4

The role of debt valuation factors in systemic risk assessment

This chapter is based on an article written by the author of this thesis [50]. Its main motivation is the fact that existing methods of systemic risk calculation tend to focus mainly on the network structure and neglect various elements of debt valuation. Nevertheless, those elements are fundamental for the dynamics of financial markets [14, 21, 43, 44] and can significantly influence the outcomes of risk assessment. The research described in this chapter aims to integrate essential debt valuation factors into systemic risk analysis and assess their impact on the US financial system's stability. Particular focus is put on identifying critical points where cascading failures might occur due to system interdependencies. This is achieved by incorporating a robust class of reduced-form models [48, 49, 62, 63, 129] described in Chapter 2 into the comprehensive NEVA framework [44] (Chapter 3). In particular, the proposed approach relies on the following reduced-form model expression for debt valuation:

$$\exp(-[r_t + \lambda_t(1 - \delta_t)](T - t)), \tag{4.1}$$

where r_t, λ_t, δ_t, and T represent the interest rate, default intensity, recovery rate, and debt maturity, respectively. Based on this formula, specific parameterized forms of these components are introduced into systemic risk calculations and followed by the evaluation of their implications for financial system stability. The linear DebtRank [4] is used as a benchmark, given its significance within the field, evidenced by a substantial history of methods developed from its foundation [7, 60].

The proposed modeling approach incorporates the essential features of the debt valuation in systemic risk calculation on the one hand, and extends the debt pricing process to include a contagion mechanism on the other. It achieves a new level of precision in assessing financial stability, not only improving the robustness of systemic risk calculations, but also extending the applicability of debt valuation models to previously unaccounted situations.

As the construction of the new framework starts from the general reduced-form formula of debt valuation in Eq. (4.1), it is worth noting that the framework is opened to further extensions, particularly through integration with the existing broad family of models for valuation spreads λ_t and interest rates r_t. Since the reduced-form models are the industry standard for pricing fixed-income instruments [47, 53, 62, 63], this approach allows for the incorporation of systemic risk calculations into already established business processes.

This chapter is structured as follows. In Section 4.1, the general framework of the reduced-form network valuation is constructed. This provides the foundation to develop specific models in Section 4.2. They are subsequently employed in Section 4.3 to assess the security of the American financial system, with particular emphasis on their impact compared to existing methods. The results are discussed in Section 4.4.

4.1 Network valuation with reduced-form models

In this section, the essential components of debt valuation are incorporated into the network model of financial systems by integrating the reduced-form models (2.20) into the NEVA framework (3.2). Assume that the vector space $\mathbb{R}^q, q \in \mathbb{N}$, is equipped with a norm $\|\cdot\|_1$, where for $x = (x_1, x_2, ..., x_q) \in \mathbb{R}^q$ $\|x\|_1 := \sum_{k=1}^{n} |x_k|$. Moreover, let $B([0,T])$ denote a space of bounded, real-valued functions defined on $[0,T]$, and equipped with the supremum norm $\|\cdot\|_\infty$, where $\|f\|_\infty := \sup_{u\in[0,T]} |f(u)|$. Then, the reduced-form dynamics constitutes a feasible valuation function, as illustrated in the following Theorem.

Theorem 4.1. *Let $\mathbf{E} \in \mathbb{R}^q, q \in \mathbb{N}$, and let $y(\mathbf{E}; u) \geq 0$ be a non-increasing function of $\mathbf{E}$ that is continuous as a mapping from the metric space $(\mathbb{R}^q, \|\cdot\|_1)$ to $(B([0,T]), \|\cdot\|_\infty)$. Then, the function $\mathbb{V}\colon \mathbb{R}^q \to [0,1]$ defined as*

$$\mathbb{V}(\mathbf{E}) := e^{-\int_t^T y(\mathbf{E};u)\, du}$$

is a feasible valuation function.

Proof. For $T = t$, $\mathbb{V}(\mathbf{E}) = 1$ and the case is trivial. Then, assume $T > t$. Consider a function $g : \mathbb{R}^q \to \mathbb{R}$ defined as

$$g(\mathbf{E}) := \int_t^T y(\mathbf{E}; u)\, du. \tag{4.2}$$

It will be shown that g is continuous. It holds that

$$\begin{aligned} |g(\mathbf{E}) - g(\mathbf{E}')| &= \left| \int_t^T (y(\mathbf{E}; u) - y(\mathbf{E}'; u))\, du \right| \\ &\leq \int_t^T |y(\mathbf{E}; u) - y(\mathbf{E}'; u)|\, du \\ &\leq \int_t^T \|y(\mathbf{E}; u) - y(\mathbf{E}'; u)\|_\infty\, du. \end{aligned}$$

The last inequality follows directly from the definition of the norm $\|\cdot\|_\infty$ as a supremum and from the fact that the integral is a nondecreasing operator.

From the assumption that $y(\mathbf{E}; u)$ is continuous function of $\mathbf{E}$ as a mapping between the metric spaces $(\mathbb{R}^q, \|\cdot\|_1)$ and $(B([0,T]), \|\cdot\|_\infty)$, it follows that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\mathbf{E} - \mathbf{E}'\|_1 < \delta \implies \|y(\mathbf{E}; u) - y(\mathbf{E}'; u)\|_\infty < \frac{\varepsilon}{(T-t)}.$$

Hence,

$$\int_t^T \|y(\mathbf{E}; u) - y(\mathbf{E}'; u)\|_\infty\, du < \frac{\varepsilon}{(T-t)}(T-t) = \varepsilon.$$

In summary, combining these results implies that if $\|\mathbf{E} - \mathbf{E}'\|_1 < \delta$, the function g satisfies

$$|g(\mathbf{E}) - g(\mathbf{E}')| = \left| \int_t^T (y(\mathbf{E}; u) - y(\mathbf{E}'; u)) \, du \right| \leq \int_t^T \|y(\mathbf{E}; u) - y(\mathbf{E}'; u)\|_\infty \, du < \varepsilon.$$

Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\mathbf{E} - \mathbf{E}'\|_1 < \delta \implies |g(\mathbf{E}) - g(\mathbf{E}')| < \varepsilon. \tag{4.3}$$

Thus, g satisfies the definition of a continuous function.

Moreover, from the definition of the function $\mathbb{V}$, it follows that

$$\mathbb{V}(\mathbf{E}) = e^{-\int_t^T y(\mathbf{E}; u) \, du} = e^{-g(\mathbf{E})}.$$

The mapping $f(y) = e^{-y}$ is a continuous function of y. Consequently, the function $\mathbb{V}(\mathbf{E}) = f(g(\mathbf{E}))$ is a composition of continuous functions, and is therefore also continuous. Furthermore, $\mathbb{V}(\mathbf{E})$ is bounded in $[0, 1]$ since $y(\mathbf{E}; u) \geq 0$. It is also nondecreasing, as it is a decreasing function of the integral of a nonincreasing function. Therefore, it is a feasible valuation function, which completes the proof. $\square$

Reduced-form models are designed to capture the term structure of interest rates and default risk, but the function of the form $\mathbb{V}(\mathbf{E}) = e^{-\int_t^T y(\mathbf{E};u)du}$, while capable of modeling any maturity T, cannot incorporate various types of a single issuer's debt with different maturities at once. In order to accommodate this feature, the form of the valuation function proposed in Theorem 4.1 needs to be extended. This extension is facilitated below.

Definition 4.1 (Convex Combination of Functions)**.** Let $f_1, f_2, \ldots, f_n$ be functions defined on a convex set $C \subseteq \mathbb{R}^q$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-negative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then, the convex combination of these functions is defined as:

$$g(x) = \sum_{i=1}^n \alpha_i f_i(x),$$

for all $x \in C$.

Lemma 4.2. *Convex combination of feasible valuation functions is a feasible valuation function.*

Proof. Let $q, n \in \mathbb{N}$, $x \in \mathbb{R}^q$, and $V_1(x), V_2(x), \ldots, V_n(x)$ be feasible valuation functions as in Def. 3.2. Moreover, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-negative numbers such that $\sum_{i=1}^n \alpha_i = 1$. For all $x \in \mathbb{R}^q$, define

$$V(x) := \sum_{i=1}^n \alpha_i V_i(x). \tag{4.4}$$

Because $0 \leq V_i(x) \leq 1$ for $i \in \{1, 2, \ldots, n\}$ and for all $x \in \mathbb{R}^q$, it holds that

$$0 \leq \sum_{i=1}^n \alpha_i V_i(x) \leq \sum_{i=1}^n \alpha_i = 1,$$

which means that

$$0 \leq V(x) \leq 1$$

for all $x \in \mathbb{R}^q$. Furthermore, the convex combination of continuous and nondecreasing functions is a continuous and nondecreasing function. Therefore, $V(x)$ meets the conditions of the definition of the feasible valuation function. $\square$

Theorem 4.3. *Let $\mathbf{E} \in \mathbb{R}^q, q \in \mathbb{N}$. For every $i \in \{1,\ldots,n\}$, let $y_i(\mathbf{E};u) \geq 0$ be a nonincreasing function of $\mathbf{E}$ that is continuous as a mapping from the metric space $(\mathbb{R}^q, \|\cdot\|_1)$ to $(B([0,T]), \|\cdot\|_\infty)$. Let $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_n$ be functions of the form $\mathbb{V}_i(\mathbf{E}) = e^{-\int_t^{T_i} y_i(\mathbf{E};u)\,du}$, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-negative numbers such that $\sum_{i=1}^n \alpha_i = 1$. Then, the function $\mathbb{V}$ of the form*

$$\mathbb{V}(\mathbf{E}) = \sum_{i=1}^n \alpha_i \mathbb{V}_i(\mathbf{E})$$

is a feasible valuation function.

Proof. It is a consequence of Lemma 4.2, which states that a convex combination of feasible valuation functions is a feasible valuation function. □

Theorem 4.3 enables to construct a general definition of a network model with reduced-form valuation introduced in Eq. (2.18).

Definition 4.2. Let $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,l_i}$ and $\alpha_{ij,1}, \alpha_{ij,2}, \ldots, \alpha_{ij,n_{ij}}$ be non-negative numbers for all i, such that $\sum_{k=1}^{l_i} \alpha_{i,k} = 1$ and $\sum_{k=1}^{n_{ij}} \alpha_{ij,k} = 1$, representing the weights of the entity's i holdings of external and internal debt instruments with maturities $T_{i,k}$ and $T_{ij,k}$, accordingly. Moreover, let $\mathbf{E}_t \in \mathbb{R}^q, q \in \mathbb{N}$, and let $r(\mathbf{E}_t;u)$, $cs_j(\mathbf{E}_t;u)$ be a nonincreasing functions of $\mathbf{E}_t$ that are continuous as a mappings from the metric space $(\mathbb{R}^q, \|\cdot\|_1)$ to $(B([0,T]), \|\cdot\|_\infty)$ for all j. Then, the reduced-form network valuation model is defined as a NEVA framework (3.2) with valuation functions specified as

$$\begin{aligned}
\mathbb{V}_i^e(\mathbf{E}_t) &= \sum_{k=1}^{l_i} \alpha_{i,k} \exp\left(-\int_t^{T_{i,k}} r(\mathbf{E}_t;u)du\right), \\
\mathbb{V}_{ij}(\mathbf{E}_t) &= \sum_{k=1}^{n_{ij}} \alpha_{ij,k} \exp\left(-\int_t^{T_{ij,k}} \left(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u)\right)du\right).
\end{aligned}$$

Here, $\mathbf{E}_t$ is a vector of equities at time t, r is the interest rate, and cs_j is the valuation spread.

It is worth noting that the network valuation is defined in Eq. (3.6) as a fixed-point problem of an equity vector $\mathbf{E}(t)$. It is a deterministic problem, and as such the interest rate r and valuation spreads cs_j are deterministic in network valuation.

The framework proposed here is a very general approach for the valuation in financial systems, strictly aligned with the industry standards for debt valuation [47, 48, 53]. The notion of reduced-form models with interest rates and risk spreads is a crucial concept in financial mathematics, containing all information relevant in the broad process of cashflows valuation. In fact, it is a convenient language for financial markets, where actual financial instruments are often quoted by their implied risk spreads [47, 53]. With the approach proposed here, the market practices are open for integration with the complex network valuation of financial systems, which constitutes a significant advantage over methods proposed so far. Many of them are in fact very simple in comparison to existing market standards. Other types of methods are based on the mutlilayer networks for different term structures [46, 61, 92] with significantly complex valuation dynamics relying recursively on previous layers, which is analytically untracable and hard to reason about and operate on in practice. In comparison to them, the approach constructed here aligns perfectly with existing market standards [48, 53] and provides the interest rates and risk spread curves, containing all required information and convenient to process in practice at the same time.

4.2 Network models with the dynamics of debt valuation factors

This section builds upon the general Def. 4.2 of the reduced-form network valuation framework to incorporate essential components of debt valuation into the network model of financial systems. These components include the structure of debt over time, the recovery rate, credit quality, and interest rates. In order to retain the focus of this study on the dynamics of debt valuation factors in the financial network, the simplified version of the framework in Def. 4.2 is considered, with various possible debt maturities of a single entity and the term structure of valuation factors being omitted. While it is acceptable for the specified research purpose to focus on certain aspects of valuation, it is crucial to recognize that term structure and different maturities play a substantial role in real-world applications, as extensively discussed in the Introduction. The detailed analysis of the term structure influence in the real financial network constitutes a very promising matter for future research.

The assumption of a single tenor across all possible situation $T_{i,k} = T_{ij,k} \equiv T$ implies that the formula in Def. 4.2 can be written as

$$\begin{aligned}
\mathbb{V}_i^e(\mathbf{E}_t) &= \sum_{k=1}^{m_i} \alpha_{i,k} \exp\left(-\int_t^T r(\mathbf{E}_t;u)du\right), \\
\mathbb{V}_{ij}(\mathbf{E}_t) &= \sum_{k=1}^{n_{ij}} \alpha_{ij,k} \exp\left(-\int_t^T \left(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u)\right)du\right).
\end{aligned} \tag{4.5}$$

As the conditions in Def. 4.2 require that $\sum_{k=1}^{m_i} \alpha_{i,k} = 1$ and $\sum_{k=1}^{n_{ij}} \alpha_{ij,k} = 1$, it follows that

$$\begin{aligned}
\mathbb{V}_i^e(\mathbf{E}_t) &= \sum_{k=1}^{m_i} \alpha_{i,k} \exp\left(-\int_t^T r(\mathbf{E}_t;u)du\right) \\
&= \left(\sum_{k=1}^{m_i} \alpha_{i,k}\right) \exp\left(-\int_t^T r(\mathbf{E}_t;u)du\right) \\
&= \exp\left(-\int_t^T r(\mathbf{E}_t;u)du\right).
\end{aligned} \tag{4.6}$$

Analogically

$$\begin{aligned}
\mathbb{V}_{ij}(\mathbf{E}_t) &= \sum_{k=1}^{n_{ij}} \alpha_{ij,k} \exp\left(-\int_t^T \left(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u)\right)du\right) \\
&= \left(\sum_{k=1}^{n_{ij}} \alpha_{ij,k}\right) \exp\left(-\int_t^T \left(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u)\right)du\right) \\
&= \exp\left(-\int_t^T \left(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u)\right)du\right).
\end{aligned} \tag{4.7}$$

Moreover, it is assumed that the term structured is primarily ignored, and therefore $r(\mathbf{E}_t;u) = r(\mathbf{E}_t)$ and $cs_j(\mathbf{E}_t;u) = cs_j(\mathbf{E}_t)$. This leads to

$$\begin{aligned}
\mathbb{V}_i^e(\mathbf{E}_t) &= \exp\left(-\int_t^T r(\mathbf{E}_t)du\right) = \exp\left(-r(\mathbf{E}_t)(T-t)\right), \\
\mathbb{V}_{ij}(\mathbf{E}_t) &= \exp\left(-\int_t^T \left(r(\mathbf{E}_t) + cs_j(\mathbf{E}_t)\right)du\right) = \exp\left(\left(r(\mathbf{E}_t) + cs_j(\mathbf{E}_t)\right)(T-t)\right).
\end{aligned} \tag{4.8}$$

In order to simplify the notation, the valuation rule in this chapter will be denoted as

$$\begin{aligned}\mathbb{V}_i^e(E_j(t)) &= \exp\left(-r_t\left(T-t\right)\right), \\ \mathbb{V}_{ij}(E_j(t)) &= \exp\left(-\left(r_t+cs_t\right)\left(T-t\right)\right).\end{aligned} \tag{4.9}$$

In this approach, distinct models are characterized by the particular definitions of cs_t and r_t. Section 4.2.1 illustrates the construction of the cs_t, a crucial component allowing to connect the credit spread factors with crisis propagation. This model is enhanced in Section 4.2.2 by a specified dynamics of interest rates r_t, capturing the feedback loop relationships between banking system and general credit market. From this method, the linear approximation in Section 4.2.3 allows to derive a simplified model being a generalisation of linear DebtRank at the same time.

4.2.1 Reduced-form network valuation model

In this section, a model of risk spread cs_t from Eq. (4.9) will be constructed in the context of network valuation. The spread cs_t is represented by $\lambda_t(1-\delta_t)$, where λ_t denotes the default intensity at time t, and δ_t characterizes the recovery rate. It is often the case that the recovery rate is presumed to constitute a constant proportion. However, maintaining a constant recovery rate throughout the entire modeling period may not accurately capture the complexities inherent in real-world contexts. In fact, it is reasonable to posit that the expected recovery varies in accordance with the dynamic market conditions. During periods of prosperity, a higher recovery may be anticipated, given the overall health of a company. Conversely, when a company is facing significant stress, the prospect of recovery from default might be considerably diminished. Therefore, it will be modeled as $\delta_t = \beta\frac{A_j(t)}{A_j(0)}$. This representation highlights the recovery rate's dependence on the ratio of current assets to initial assets, offering a flexible and realistic depiction of its dynamics in the proposed model. The approach bears resemblance to recovery mechanisms in clearing algorithms [42, 46], where it is defined as the ratio of a company's assets to liabilities in the event of default. Thus, it reflects the cautious and recovery-oriented nature of clearing models to some extent, while also being tailored to the requirements of ex ante shock propagation algorithms.

Within the basic network propagation model of linear DebtRank, the default probability is expressed as $p_j(t) = 1 - \frac{E_j^+(t)}{E_j(0)}$ [60]. When fitting it into the framework of reduced-form models, the default intensity λ_t could be depicted analogously as $\lambda(t) = \gamma_j\left(1-\alpha_j\frac{E_j^+(t)}{E_j(0)}\right)$ for some j. Consequently, the formula for the risk spread cs_t presents as follows:

$$cs_t := \max\left[\gamma_j\left(1-\alpha_j\frac{\min\left[E_j^+(t),\ E_j(0)/\alpha_j\right]}{E_j(0)}\right)\left(1-\beta_j\frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right]. \tag{4.10}$$

Here, $\beta_j \in [0,1]$ is a recovery scale parameter, $\alpha_j \in [0,1]$ is responsible for the degree of dependence of the hazard rate on the equity, while $\gamma_j \in \mathbb{R}_+$ describes the overall rate of devaluation. Minima are introduced in order to formally ensure the feasibility conditions in Def. 3.2. Assuming no negative equity and assets, cs_t satisfies

$$0 \leq cs_t \leq \gamma_j. \tag{4.11}$$

for all $t \geq 0$. As a result, the minimum value of assets in the model equals $\mathbf{A}(0)e^{-\gamma_j(T-t)}$.

The parameter γ_j governs the rate of decline in asset value and establishes its minimum threshold. In the absence of the γ_j parameter ($\gamma_j \equiv 1$), the model has significantly limited capability to cover scenarios of actual defaults, which is also the case in DebtRank. The individual values γ_j assigned to each company j highlight the diverse credit qualities exhibited by various entities. This is a fundamental criterion for pricing debt securities in finance, enabling differentiation between bonds issued by different financial entities, each characterized by distinct risk profiles.

The term $\left(1 - \alpha_j \frac{E_j^+(t)}{E_j(0)}\right)$ captures the inherent dependence of a company's default intensity on its equity value. This relationship stems from the definition of default as the point in time τ when a company's assets $A_i(\tau)$ fail to cover its liabilities $L_i(\tau)$, leading to $E_i(\tau) := A_i(\tau) - L_i(\tau) \leq 0$. Consequently, the connection between the probability of a company's default and the current value of its equity is straightforward. It constitutes a fundamental aspect of systemic risk research. An initial decline in a company's equity value results in an elevated probability of default, precipitating the devaluation of its issued financial instruments. Consequently, holders of these instruments experience a decrease in asset value, equity, creditworthiness, and the value of their securities. This sets off a feedback loop, amplifying the crisis across interconnected entities within the market network. The iterative propagation mechanism is indispensable to the systemic risk discipline, distinguishing it from conventional financial mathematics, where market dependencies are typically represented as one-time correlation effects or shared exposures to a common factor like the market portfolio [84, 130]. In the model developed herein, the feedback loop is also ingrained in the dynamics of the recovery rate through its reliance on asset value. This integration not only facilitates the inclusion of debt recollection mechanisms in systemic risk calculations, but also enables it to reflect the behavior characteristic for crisis phenomena.

After the process described in Section 4.2.4, the spread dynamics is characterised by

$$cs_t = \max\left[\gamma_j \left(1 - \frac{\min\left[E_j^+(t),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j \frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right]. \quad (4.12)$$

4.2.2 Interest rates feedback model

In order to incorporate interest rates to the dynamics of financial networks, it is assumed that they are inherently integrated into the valuation from the outset and are factored into the calculation of initial assets $\mathbf{A}(0)$. Consequently, $r_t = r_0 + \Delta r_t$ is introduced, which leads to

$$\begin{aligned} \mathbb{V}_i^e(E_j(t)) &= \exp\left(-\Delta r_t \left(T - t\right)\right), \\ \mathbb{V}_{ij}(E_j(t)) &= \exp\left(-\left(\Delta r_t + cs_t\right)\left(T - t\right)\right). \end{aligned} \quad (4.13)$$

In the context of financial modeling, the simplistic approach of treating interest rates as exogenous variables may fall short in accurately representing the intricacies of real-world dynamics. Extensive empirical evidence underscores the existence of complex relationships between the banking system and interest rates [55, 56]. Detailed discussions regarding the motivations and rationale behind those connections are elaborated in Sec 4.3.2. Henceforth, it is assumed here that the interest rate may depend on the condition of the banking system, as represented by the value of the equity vector. This relationship is formally expressed by:

$$\Delta r_t = \Delta r_t \left(\mathbf{E}(t)\right). \tag{4.14}$$

Up to this point, the valuation function $\mathbb{V}_{ij}$ in various models has been tied solely to the equities $E_j(t)$ of counterparties j undergoing credit evaluation. However, the NEVA framework takes a broader approach through the formulation of the map Φ, as defined in Eq. (3.2). This definition facilitates a more general spectrum of dependencies of valuation functions $\mathbb{V}_{ij}$ on the entire vector of equities $\mathbf{E}(t)$. Consequently, all outcomes within the framework remain applicable in this extended context. Thus, under the modeling assumption presented in Equation (4.14), the framework's results hold true, provided that valuation functions remain feasible.

In order to elucidate the impact of banking system credibility on interest rates, the proposed dynamics is of the form

$$\Delta r_t = \widetilde{\gamma}\left(1 - \widetilde{\alpha}\frac{\|\mathbf{E}^+(t)\|_1}{\|\mathbf{E}(0)\|_1}\right)\left(1 - \widetilde{\beta}\frac{\|\mathbf{A}^+(t)\|_1}{\|\mathbf{A}(0)\|_1}\right). \tag{4.15}$$

Here, $\mathbf{u} = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ and $\|\mathbf{u}\|_1 = \sum_{k=1}^{n}|u_k|$. This proposition is motivated by the fact that fundamental basic interest rates reflect the overall borrowing costs within the interbank lending market [57, 58]. Eventually, the complete characterisation of the reduced-form model can by written as

$$\begin{aligned} cs_t &= \max\left[\gamma_j\left(1 - \alpha_j\frac{\min\left[E_j^+(t),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j\frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right] \\ \Delta r_t &= \max\left[\widetilde{\gamma}\left(1 - \widetilde{\alpha}\frac{\min\left\{\|\mathbf{E}^+(t)\|_1, \|\mathbf{E}(0)\|_1\right\}}{\|\mathbf{E}(0)\|_1}\right)\left(1 - \widetilde{\beta}\frac{\min\left\{\|\mathbf{A}^+(t)\|_1, \widetilde{\beta}^{-1}\|\mathbf{A}(0)\|_1\right\}}{\|\mathbf{A}(0)\|_1}\right), 0\right]. \end{aligned} \tag{4.16}$$

Following the process outlined in Section 4.2.4, the formula takes the form below:

$$\begin{aligned} cs_t &= \max\left[\gamma_j\left(1 - \frac{\min\left[E_j^+(t),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j\frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right] \\ \Delta r_t &= \max\left[\widetilde{\gamma}\left(1 - \frac{\min\left\{\|\mathbf{E}^+(t)\|_1, \|\mathbf{E}(0)\|_1\right\}}{\|\mathbf{E}(0)\|_1}\right)\left(1 - \widetilde{\beta}\frac{\min\left\{\|\mathbf{A}^+(t)\|_1, \widetilde{\beta}^{-1}\|\mathbf{A}(0)\|_1\right\}}{\|\mathbf{A}(0)\|_1}\right), 0\right]. \end{aligned} \tag{4.17}$$

4.2.3 Recovery DebtRank

In this section, a simplified model will be derived from Eq. (4.12), focusing specifically on the recovery rate impact. The result is in fact a model that incorporates a dynamic recovery mechanism into linear DebtRank, demonstrating that the latter can be obtained from the model (4.12) with reduced parameters.

As the emphasis of the model developed in this section is the recovery rate specifically, interest rates do not constitute an object of interest, and therefore will be disregarded from further consideration. Setting $\gamma_j = (T - t) = 1$ in Eq. (4.12) and employing the approximation $e^{-x} \approx 1 - x$ yields

$$\exp\left(-\left(1-\alpha_j\frac{E_j^+(t)}{E_j(0)}\right)\left(1-\beta_j\frac{A_j^+(t)}{A_j(0)}\right)\right) \tag{4.18}$$

$$\approx 1-\left(1-\alpha_j\frac{E_j^+(t)}{E_j(0)}\right)\left(1-\beta_j\frac{A_j^+(t)}{A_j(0)}\right) \tag{4.19}$$

$$\approx \alpha_j\frac{E_j^+(t)}{E_j(0)}+\beta_j\frac{A_j^+(t)}{A_j(0)}. \tag{4.20}$$

As the first-order linear approximation is considered here, the quadratic term $\alpha_j\frac{E_j^+(t)}{E_j(0)}\beta_j\frac{A_j^+(t)}{A_j(0)}$ is omitted. This leads to

$$\mathbb{V}_{ij}(E_j(t))=\left(\alpha_j\frac{E_j^+(t)}{E_j(0)}+\beta_j\frac{A_j^+(t)}{A_j(0)}\right). \tag{4.21}$$

Application of the technical process described below in Section 4.2.4 yields

$$\mathbb{V}_{ij}(E_j(t))=\left(\alpha_j\frac{E_j^+(t)}{E_j(0)}+(1-\alpha_j)\frac{A_j^+(t)}{A_j(0)}\right). \tag{4.22}$$

This effectively establishes a convex combination of two potential valuation rules. The first one is based on the probability of default, estimated by the proportion of the current equity relative to the initial one. The second one describes the recovery in case of default and utilizes the ratio of assets. Within this framework, the inclusion of a weight parameter α_j allows for the adjustment of the relative importance assigned to the default probability and recovery components. This flexibility enables the model to be tailored to various situations and application-specific needs. It is worth to point out that the linear DebtRank emerges as a specific case within this approach when α_j is set to 1. Another particularly simple parameterisation applies the same weights of importance to the default probability and recovery components, i.e. $\alpha_j=\frac{1}{2}$, yielding

$$\mathbb{V}_{ij}(E_j(t))=\frac{1}{2}\left(\frac{E_j^+(t)}{E_j(0)}+\frac{A_j^+(t)}{A_j(0)}\right). \tag{4.23}$$

This approach offers a direct compromise between the significance of model components, serving as a natural and effective starting point when there is a requirement to incorporate recovery rates into systemic risk calculations, traditionally conducted by the means of the linear DebtRank algorithm (3.10).

The complete model specification presents as follows:

$$\begin{aligned}&\mathbb{V}_i^e(E_j(t))=1,\\&\mathbb{V}_{ij}(E_j(t))=\min\left[\alpha_j\frac{E_j^+(t)}{E_j(0)}+(1-\alpha_j)\frac{A_j^+(t)}{A_j(0)},1\right].\end{aligned} \tag{4.24}$$

4.2.4 Towards model calibration

Starting from the assets identity equation

$$A_i(t)=A_i^e(t)\mathbb{V}_i^e\left(\mathbf{E}(t)\right)+\sum_{j=1}^{n}A_{ij}(t)\mathbb{V}_{ij}\left(\mathbf{E}(t)\right), \tag{4.25}$$

the unknown value of external assets is derived:

$$A_i^e(t)\mathbb{V}_i^e\left(\mathbf{E}(t)\right) = A_i(t) - \sum_{j=1}^{n} A_{ij}(t)\mathbb{V}_{ij}\left(\mathbf{E}(t)\right). \tag{4.26}$$

Furthermore, it is frequently observed, especially in the context of non-clearing contagion algorithms, that asset dynamics are modeled in relation to the initial state $A_{ij}(0)$, thereby assuming

$$\mathbb{V}_i^e(E_j(0)) = \mathbb{V}_{ij}(E_j(0)) = 1. \tag{4.27}$$

This holds true for linear and various versions of DebtRank [21,43,60], as well as Furfine [41] methods. Such an approach proves to be very natural and reasonable, especially in the context of stress test-focused research. The initial state, derived from market data, is presumed to be at the point of equilibrium. Subsequently, a shock is applied, and its propagation through financial dependencies, along with its ultimate impact on the system, is systematically assessed. This methodological framework will be employed in our work. However, it is important to note that the proposed solution is not exhaustive, and alternative approaches can be considered as well.

For linear DebtRank (Eq. (3.10)), the condition is given straightforward, as

$$\mathbb{V}_{ij}(E_j(0)) = \min\left[\frac{E_j^+(0)}{E_j(0)}, 1\right] = \min\left[1, 1\right] = 1. \tag{4.28}$$

In case of recovery DebtRank (Eq. (4.21)), it must hold that

$$\begin{aligned} 1 = \mathbb{V}_{ij}(E_j(0)) &= \min\left[\alpha_j \frac{E_j^+(0)}{E_j(0)} + \beta_j \frac{A_j^+(0)}{A_j(0)}, 1\right] \\ &= \min\left[\alpha_j + \beta_j, 1\right]. \end{aligned} \tag{4.29}$$

In order to satisfy the condition above, the recovery parameter is set to $\beta_j = 1 - \alpha_j$.

In case of valuation functions defined by the reduced-form models as in Eq. (4.13), the conditions in Eq. (4.27) can be expressed as

$$\Delta r_0 = cs_0 = 0. \tag{4.30}$$

Inserting this into the definition of risk spread cs_t and interest rates Δr_t given in Eq. (4.10) yields

$$0 = \max\left[\gamma_j \left(1 - \alpha_j \frac{\min\left[E_j^+(0),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j \frac{\min\left[A_j^+(0),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right] \tag{4.31}$$

$$0 = \max\left[\widetilde{\gamma}\left(1 - \widetilde{\alpha}\frac{\min\left\{\|\mathbf{E}^+(0)\|_1, \|\mathbf{E}(0)\|_1\right\}}{\|\mathbf{E}(0)\|_1}\right)\left(1 - \widetilde{\beta}\frac{\min\left\{\|\mathbf{A}^+(0)\|_1, \widetilde{\beta}^{-1}\|\mathbf{A}(0)\|_1\right\}}{\|\mathbf{A}(0)\|_1}\right), 0\right].$$

The first condition for cs_0 can be reduced to

$$0 = \max\left[-\gamma_j\left(1 - \alpha_j\right)\left(1 - \beta_j\right), 0\right]. \tag{4.32}$$

In order to meet the above criterion, the parameter α_j is fixed to $\alpha_j = 1$. The rationale behind the proposed approach is presented below.

In the terminology of credit valuation embedded in the Duffie-Singleton model (2.20), the approach proposed in Eq. (4.10) for the risk spread $cs(t) = \lambda(t)(1-\delta_t)$ can be expressed as follows:

$$\lambda(t)(1-\delta_t) = \gamma_j \left(1 - \alpha_j \frac{E_j^+(t)}{E_j(0)}\right)\left(1 - \beta_j \frac{A_j^+(t)}{A_j(0)}\right). \tag{4.33}$$

Here, the recovery rate δ_t is modeled dynamically by the term $\beta_j \frac{A_j^+(t)}{A_j(0)}$, while the hazard rate component $\lambda(t)$ is delineated by $\gamma_j \left(1 - \alpha_j \frac{E_j^+(t)}{E_j(0)}\right)$. The parameter α_j, constrained within the interval $[0,1]$, presents limited modeling prospects. It reflects the dependency of a company's default probability on its equity. However, as described in detail in Subsec. 4.3.2, the essential characteristics of default probability are predominantly governed by the γ_j parameter, which offers greater adaptability across diverse real-world scenarios compared to α_j. Particularly, γ_j is responsible for the rate of decline in valuation of company debt instruments, and provides full flexibility of valuation boundary up to 0 in the limit, a feature that cannot be achieved by the α_j (a thorough mathematical analysis is conducted in Subsec. 4.2.1). Consequently, it is sensible to calibrate the model with the use of the α_j parameter. This approach maintains the model's flexibility and comprehensive reflection of all pertinent aspects of the Duffie-Singleton framework, while also addressing the requirements of conducted research. Indeed, under the proposed parameterization, strength of the recovery effect is governed by $\beta_j \frac{A_j^+(t)}{A_j(0)}$, while the intensity of the hazard rate is regulated by γ_j. This delineation of responsibilities among model parameters aligns with real-world dynamics and facilitates clear interpretation. Alternative solutions that introduce interdependencies among parameters may not exhibit this clarity and coherence. The presented reasoning can be analogously extended to the case of the interest rate Δr_0, ultimately resulting in an aggregated condition $\widetilde{\alpha} = \alpha_j = 1$.

4.3 The risk assessment of the American financial system

4.3.1 Dataset

The proposed modeling framework will be employed to assess the stability of the financial system in the United States. The dataset encompasses the top 15 American banks in terms of assets, with the exclusion of Charles Schwab due to data availability limitation. Collectively, the subset represents approximately 58% of the assets of the domestic banking system and is assumed to be sufficiently representative of the American banking landscape for the intended research purposes. Additionally, the dataset includes information on five prominent mutual funds. The data is sourced from the Eikon Reuters analytical platform, providing comprehensive balance sheet details for the analyzed entities, including asset and liability values, as well as information on debt cross-holdings.

The internal assets and liabilities held by financial entities within the United States are illustrated in Figure 4.1. Bank of America and Morgan Stanley emerge as the predominant systemic borrowers, with Goldman Sachs and TD Bank following behind, collectively representing 61% of the total internal debt within the system. Consequently, they pose significant direct sources of systemic risk within the examined market. Conversely, Vanguard and Citigroup exhibit substantially lower absolute borrowing figures. Mutual

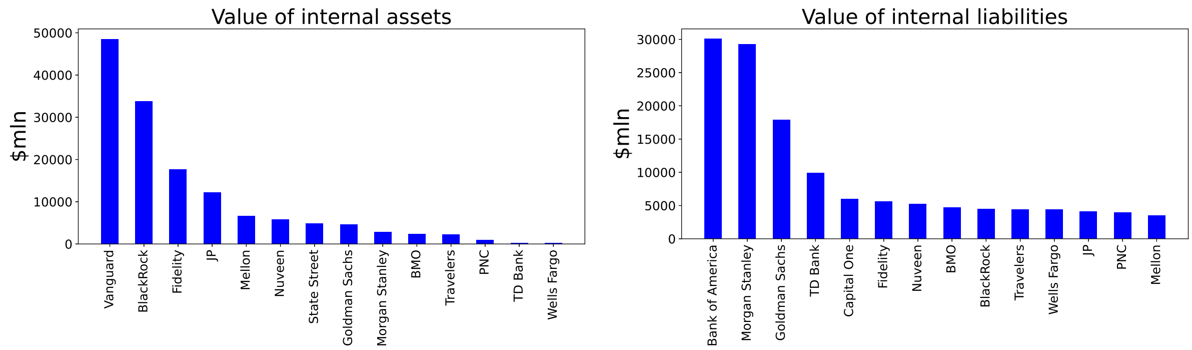


Figure 4.1: Breakdown of internal assets and internal liabilities among top banks and mutual funds in the United States. The left panel demonstrates a descending order of internal asset values, portraying a distribution akin to a typical power law curve. Mutual funds, particularly Vanguard and BlackRock, emerge as prominent stakeholders, holding a significant lead. The right panel showcases the hierarchy of internal liabilities among specific companies, revealing notably higher internal debt for the top four cases, gradually diminishing thereafter until the final two companies, which exhibit visibly lower internal indebtedness compared to their counterparts.

funds play a significant role among the holders of internal assets, collectively accounting for 75% of the system's total holdings. Among these, the three largest holders, Vanguard, BlackRock, and Fidelity, hold 70% of the whole system internal assets. This indicates that within the sample, mutual funds, especially the aforementioned three, are disproportionately exposed to systemic risk as measured by their internal asset holdings. As their portfolios decline sharply with worsening condition of the system, their devaluations will exacerbate each other, emphasizing the need for close monitoring. Excluding mutual funds from banking system analyses could lead to biases and overconfidence in system security, resulting in excessive risk-taking and potentially catastrophic outcomes. This mirrors the 2007 crisis, where failure to recognize the housing market's risks and the financial system's exposure, compounded by firms like Lehman Brothers and AIG, led to the worst global financial crisis since the Great Depression [131, 132].

While the provided data sample is sufficiently representative of the American banking landscape for the intended research purposes, it is important to address the calculation performed for real assessment of the full financial system's security. The first point is the issue of missing data, as exemplified by the case of Charles Schwab in this study. In practical calculations not intended for research purposes, overlooking a bank of such significant size is inadvisable. Various estimation methods can be employed in this context. For instance, the amount of internal assets can be estimated using the mean internal assets/total assets ratio within the system, which can then be allocated among financial instruments in proportion to the sizes of their issuers.

The second point concerns the execution of calculations across the extensive network of financial system participants. In reality, conducting straightforward calculations on large-scale networks is neither particularly feasible nor necessary. Financial systems often exhibit a significant degree of concentration, where a relatively small number of prominent institutions account for a substantial portion of the network under examination [133, 134]. This phenomenon is also illustrated in this study, where the selected subset of 14 banks represents approximately 58% of the total assets in the American banking system. Therefore, it is sufficient to perform precise calculations only for a specified subset of key institutions. Conversely, larger groups of similar, smaller institutions can be mapped to representative agents. This approach allows computations to be performed solely on a significantly reduced set of agents, which can then be appropriately scaled, rather than on the entire large network.

Even the straightforward methods discussed here for estimation of missing data and dimensionality reduction could prove effective in many cases. The system is monotonic in its parameters and network size, which allows for the evaluation of various well-defined unfavorable scenarios. However, the specific network structure can significantly influence the results [135, 136], so the viability of these methods should be carefully assessed before applying them in real calculations. In cases when simple approaches are not deemed plausible, more advanced techniques could be applied for network reconstruction [137, 138] and complexity reduction [139, 140].

4.3.2 Model behavior and its impact on financial system analysis

Recovery DebtRank

In this section, relevance of the recovery rate will be analyzed. The recovery DebtRank model, introduced in Eq. (4.24),

$$\mathbb{V}_{ij}(E_j(t)) = \left(\alpha_j \frac{E_j^+(t)}{E_j(0)} + (1 - \alpha_j) \frac{A_j^+(t)}{A_j(0)} \right), \tag{4.34}$$

will be used for that purpose, as it isolates the recovery rate $\rho_j = (1 - \alpha_j) \frac{A_j^+(t)}{A_j(0)}$ from other effects considered in this work. The results will be compared with the linear DebtRank due to its widespread adoption in the systemic risk community [4, 7, 28, 44]. As the general system dynamics is the primary focus of this study, α_j is assumed to be equal across all companies.

The influence of the recovery rate on the systemic risk assessment is illustrated in Figure 4.2. The impact of varying recovery rates on the outcomes is clearly evident. In the case of linear DebtRank (with no recovery, α_j=1) and lower recovery rate values, a noticeable shift in the series of defaults occurs near the 6% shock threshold, indicating that systemic risk effects have the potential to alter the boundary of a precarious regime. Around the critical point of 8%, where the number of direct defaults experiences a sharp increase, lower recovery rates result in a discernible network contagion effect, exacerbating the severity of the crisis. In contrast, when recovery rates are higher, the shifts of critical points and crisis amplification are notably less pronounced or virtually absent. Moreover, the price dynamics of debt, determined by valuation function in Eq. (4.34) with the absence of recovery ($\alpha_j = 1$), may exhibit an exaggeratedly pessimistic trend, as evidenced by the illustration provided at the bottom panel of Figure 4.2. It elucidates the evaluation of 1$ of debt from the “Big Four” banks, which collectively constitute 47% of the American banking market, when their assets are valued at 93% of their original worth, both with and without the prospect of recovery. The disparities between these scenarios are significant, with the ratio reaching 1:5. In the absence of recovery, precipitous decline to approximately 20% of original value appears implausible, whereas the results of calculations with $\alpha_j = 0.5$ appear far more reasonable, fluctuating around 60%. Furthermore, at the brink of default, assets equal liabilities, allowing for full debt repayment. Additionally, deposits are backed by state insurers like FDIC, and therefore rarely fall completely [141]. Therefore, the dynamics stemming from linear DebtRank might be inherently artificial, potentially leading to

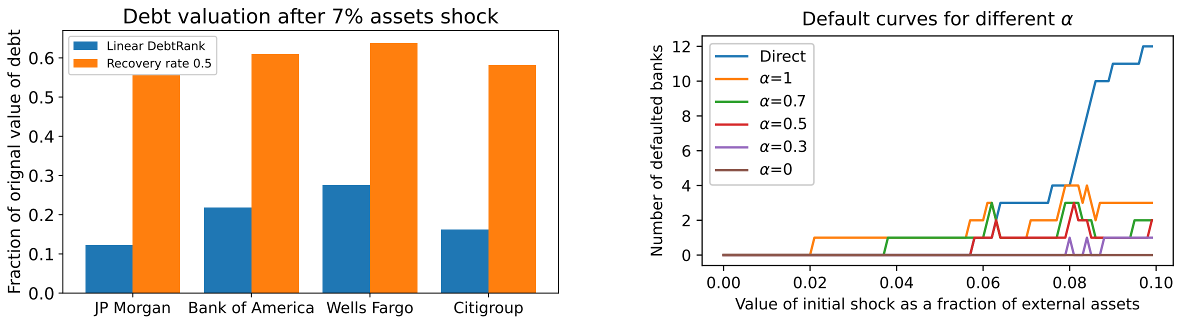


Figure 4.2: The impact of recovery rate on financial system dynamics. The left plot illustrates the valuation of debt instruments for leading American banks under a 7% asset shock, as determined by the DebtRank algorithm, considering a recovery rate with $\alpha_j = 0.5$. Calculations were conducted for the 'Big Four' banks, a prominent group in the American financial system, collectively representing 47% of the US bank market. The right plot depicts the relationship between the number of defaulted institutions and varying percentages of shock to external assets, calculated using a recovery DebtRank model (4.24). The blue curve represents direct defaults triggered by the shock, while distinct curves illustrate indirect defaults, each corresponding to a different value of the α_j parameter. Since the recovery parameter equals $1 - \alpha_j$, the number of defaults increases with α_j. For the sake of simplicity, the j subscript has been omitted in the plots since the same α_j was assumed for all companies in calculations.

inaccuracies in calculation results. Such inaccuracies, in turn, can distort risk perceptions and compromise the quality of decision-making processes. While failing to recognize the danger of a crisis can lead to catastrophic consequences, overly stringent regulation can also yield serious negative effects, particularly due to its prolonged influence and the accumulation of its consequences over time. Such regulations can impede individuals and businesses from accessing credit and financing, as banks may adopt more cautious lending practices, resulting in diminished availability of loans, especially for small businesses and individuals with imperfect credit histories [83]. Consequently, this can hamper economic growth, hinder job creation, reduce productivity, and dampen overall prosperity [142]. Moreover, strict regulations may exacerbate poverty levels, which in turn profoundly impacts people's health by constraining access to healthcare, nutritious food, safe drinking water, and adequate living conditions [143, 144].

Reduced-form: recovery and creditworthiness.

In this section, the framework of reduced-form models in network valuation will be applied to assess systemic risk within the U.S. financial landscape. To recall, the dynamics of the risk spread cs_t is given by Eq. (4.12):

$$cs_t = \max\left[\gamma_j\left(1 - \frac{\min\left[E_j^+(t),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j\frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right]. \quad (4.35)$$

The goal is to evaluate the impacts of recovery parameter β_j and credit quality parameter γ_j. Because in the absence of interest component r_t in the reduced-form model (4.9) the influence of maturity $(T - t)$ is qualitatively equivalent to γ_j, analysis

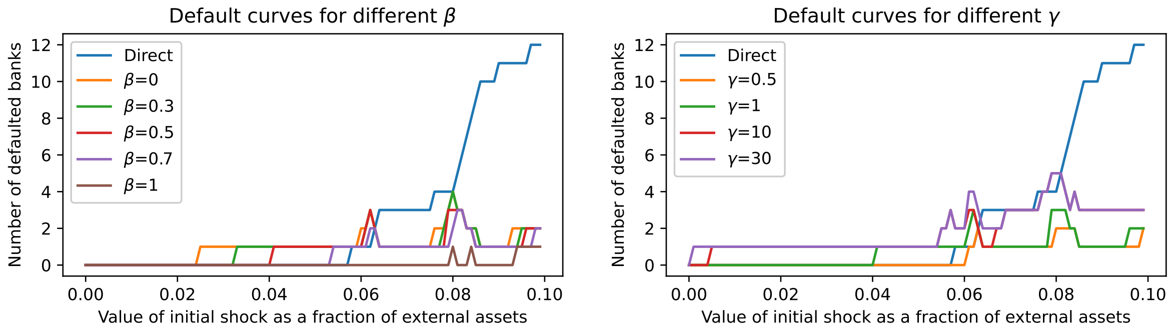


Figure 4.3: The relationship between the number of defaults and varying initial shocks to external assets. The figure illustrates the dependency between the number of defaulted institutions and different percentages of shock to external assets, computed using a reduced-form model (Eq. 4.12). The blue curve represents direct defaults triggered by the shock, while distinct curves depict indirect defaults, each corresponding to different parameter values. For the left plot, computations were performed with parameter values $\Delta r_t = 0$, $(T-t) = 1$, $\gamma_j = 1$, and various curves correspond to different values of the recovery parameter β_j. The quantity of defaults decreases with β_j as it determines the assets retrieved in case of default. Similarly, for the right plot, computations were conducted with parameter values $\Delta r_t = 0$, $(T-t) = 1$, $\beta_j = 0.5$, and various curves correspond to different values of the scale parameter γ_j. The amount of defaults increases with γ_j as it describes the pace of valuation decline. For the sake of simplicity, the j subscript has been omitted in the plots since the same parameter values were assumed for all companies.

is performed only for γ_j parameter. Moreover, parameters β_j, γ_j are assumed to be the same across all companies for the sake of simplicity.

The results are presented in Figure 4.3. It illustrates the relationship between the number of failed companies and the magnitude of external assets' shocks for different values of the model parameters. The influence of γ_j outweighs the impact of the recovery rate. In order to comprehensively assess the recovery rate, parameter values spanning the entire spectrum, including extreme values of 0 and 1, were examined. Given the fact that the number of defaults increase along with the β_j parameter, this choice offers a broad understanding of outcomes' dependence on recovery. The overall behavior observed in the analysed case is similar to that of the recovery DebtRank (Fig. 4.2). Conversely, the dynamics of γ_j exhibits a more substantial impact. In the case of recovery, systemic defaults primarily manifest as slight shifts around critical thresholds of 6% and 8%, resulting in immediate direct defaults. A minor amplification of the crisis, evidenced by two predominant indirect defaults, occurs within the range of shock magnitudes between 9% and 10%. This amplification occurs toward the end of the analyzed shock values, when the number of direct defaults reaches the point of 10, while up to that moment in the simulation there is only one prevailing indirect default. The effect magnitude is stronger in case of recovery DebtRank, with two prevailing indirect defaults from the point of 7% and three ones after the critical point of 8%, as depicted in Figure 4.2. Nevertheless, the impact stemming from variations in γ_j is more pronounced. Effects occurring around critical thresholds of 6% and 8% exhibit heightened severity, evidenced by a larger number of defaults and more substantial shifts. Moreover, following the critical threshold of 6%, the amount of indirect defaults remains consistently around 3, significantly contributing to the ongoing amplification of the crisis. In fact, from the very beginning of crisis around the 6% point, the network effect at least doubles the crisis magnitude most of the time up to the 8% threshold of sharp increase in direct defaults. This underscores the significance of the γ_j parameter and, by extension, the incorporation of reduced-form models into the network valuation framework.

System boundaries

The boundaries of the number of defaults in the analysed network structure are depicted in Figure 4.4. Notably, these boundaries closely align with the curve corresponding to the highest examined parameter value $\gamma_j = 30$ of the reduced-form model, as demonstrated in Figure 4.3. Regarding the structure of the financial system, network effects primarily induce shifts in crises and amplify pre-existing ones. A noticeable shift occurs around the point of 6% shock magnitude, important to consider when assessing system stability. However, this shift does not substantially alter system behavior. Based on these findings, there is insufficient evidence to posit that systemic effects serve as a primary source of severe crises; rather, they amplify or increase probability of occurrence for crises caused by direct shocks, without fundamentally altering the system's nature. Consequently, there is a basis to infer that network effects do not inherently pose a qualitative danger to the financial system of United States. It seems reasonable to state that, for instance, the incorporation of safety margins into critical point calculations conducted by standard non-network techniques is able to provide a sufficiently accurate assessment of secure area boundaries, without the actual need to resort to systemic risk methods. Nevertheless, financial systems are complex systems, with multiple layers of intertwining relationships.

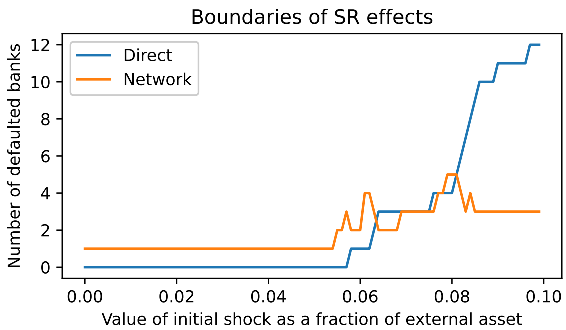


Figure 4.4: The boundaries of contagion effect. The figure illustrates the number of direct and boundary of indirect defaults for various initial shocks to external assets with static interest rates ($\Delta r_t \equiv 0$). Two prominent spikes in the default series are evident, occurring around shock magnitudes of 6% and 8%. At the 6% shock magnitude, systemic risk influence becomes apparent, with peaks indicating three systemic defaults in the case of zero direct failures and four in the case of one, respectively. The network effect diminishes as direct defaults surge sharply around the 6% threshold but regains momentum approaching the 8% shock magnitude, coinciding with another peak wherein four direct and five indirect defaults are observed. Subsequently, the number of systemic defaults begins to decline, stabilizing at three, while direct defaults progressively dominate, peaking at 12 at the 10% shock magnitude.

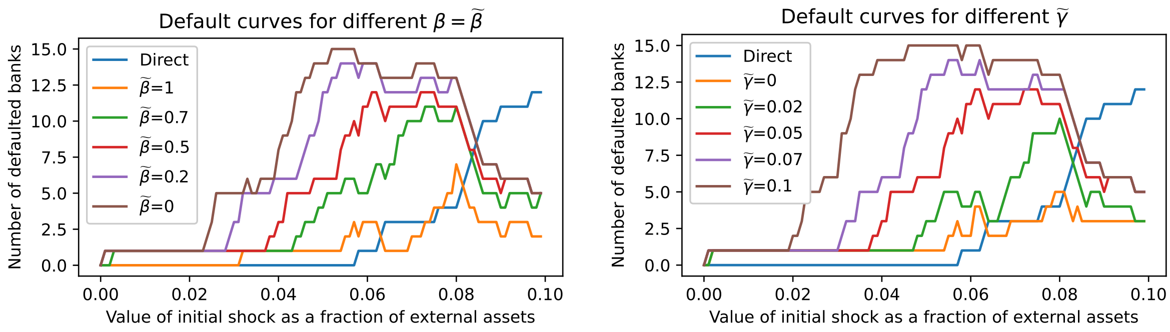


Figure 4.5: The impact of interest rate feedback on the number of defaults. The figure illustrates the relationship between the number of defaulted institutions and varying percentages of shock to external assets, computed using a reduced-form model incorporating interest rate feedback (Eq. (4.17)). The blue curve represents direct defaults triggered by the shock, while distinct curves depict indirect defaults, each corresponding to different parameter values. For the left plot, computations were performed with parameter values $(T-t) = 1$, $\gamma_j \equiv 20$, $\widetilde{\gamma} = 0.05$, and various curves corresponding to different values of the recovery parameter $\beta_j = \widetilde{\beta}$. The quantity of defaults decreases with β_j as it determines the assets retrieved in case of default. Similarly, for the right plot, computations were conducted with parameter values $(T-t) = 1$, $\gamma_j = 20$, $\beta_j = \widetilde{\beta} = 0.5$, with various curves corresponding to different values of the scale parameter $\widetilde{\gamma}$. The amount of defaults increases with $\widetilde{\gamma}$ as it describes the pace of valuation decline.

Interest rates feedback loop and cascading failure

The Secured Overnight Financing Rate (SOFR) is a pivotal benchmark interest rate that reflects the cost of borrowing cash overnight, secured by U.S. Treasury securities [145]. Developed by the Federal Reserve Bank of New York, SOFR is calculated as a volume-weighted median of interest in the repurchase agreement (repo) market transactions, where banks lend and borrow cash to manage their daily liquidity needs. In this market, when a bank has excess cash, it can lend it to another institution facing a temporary liquidity shortfall through a repo transaction. In this arrangement, the lending bank sells securities to the borrowing one with an agreement to repurchase them later, typically the next day. This mechanism allows banks to balance their cash positions amid fluctuating cash flows from customer transactions, loan disbursements, and interbank payments.

When banks can efficiently navigate this market, they are more likely to extend loans to businesses and individuals, fostering a stable lending environment. SOFR provides a transparent measure of short-term borrowing costs and serves as a reference rate for various financial products, including loans and mortgages. Fluctuations in SOFR directly affect the interest rates customers pay on their loans. When it rises, borrowers may face higher monthly payments, impacting household budgets, up to the point of inability to facilitate repayment of the remaining debt at all. Conversely, a decrease in SOFR can lead to lower interest rates, easing financial burdens for families.

The dynamics of the interbank lending market was a significant factor contributing to the scale and development of the 2008 subprime mortgage crisis and the subsequent spillover into the eurozone sovereign debt crisis [146]. As the first signs of stress began to emerge, banks started to restrict their activities and demand significantly higher prices for credit in the interbank lending market in anticipation of increased risk. This hindered access to financing further exacerbated the situation for weaker banks, amplifying panic and contributing to a further deterioration of lending conditions. The interbank lending market remains one of the most fundamental mechanisms in the operation of the global financial system.

The evolution of American banks according to the interest rates feedback model (Eq. (4.17)) is illustrated in Figure 4.5. It is notable that the parameter values, particularly $\widetilde{\gamma}$, are selected judiciously, indicating that the observed effect does not stem from an unreasonable exaggeration in magnitude, but rather arises from the inherent characteristics of the phenomenon under investigation. Analogously to the dynamics of a single institution credit spread cs_t in Eq. (4.11)), the change in the interest rate Δr_t is bounded from above by the value of parameter $\widetilde{\gamma}$, although it is important to mention that the system usually do not reach this extreme. In the examined case, the parameter values ranged between 0-0.1. This range is justified, as for example from the January 2022 to January 2024 the Federal Funds Rate varied from 0.0008 up to over 0.05, while historically reaching even nearly 0.2. The integration of interest rates into the model results in a significant alteration in the system's characteristics. In the absence of an interest rate feedback loop, as illustrated in Fig. 4.3, discernible yet constrained shifts and amplifications around critical points at 6% and 8% indicate a limited systemic impact. However, incorporating interest rate dependency from Eq. (4.15) leads to notable transpositions of these critical points. Specifically, the mild critical point transitions from 6% to a range spanning 2%-5%, while the severe crisis point shifts from 8% to a range of 3%-7%, attributable to extensive cascading failures induced by systemic feedback mechanisms. This difference underscores a fundamental discrepancy between models with and without interest rate dependencies on market conditions. The crunch of the financial system varies significantly, as the inclusion of interest rates precipitates a collapse driven by the common exposure to interest rates and indirectly by the interplay between interest rates and the banking environment. Consequently, neglecting interest rates and their interconnections with the banking system in modeling can obscure risk recognition, resulting in a substantial underestimation of actual risk and eventual system collapse due to decisions predicated on inadequate analysis.

4.3.3 Comments on practical implementation

The research conducted in this chapter focuses on the sophisticated dynamics within complex networks. This added complexity naturally raises questions regarding the practical implementability of the model by regulators and financial institutions, particularly in comparison to existing, much simpler dynamics like linear DebtRank. However, a key argument for introducing reduced-form models into network calculations is that they represent a well-established industry standard for the valuation of fixed-income instruments [47, 48, 53]. Therefore, from the perspective of practitioners, the technical implementation of this framework should not be significantly more challenging than that of the NEVA, which treats a general valuation function as an argument and can accept reduced-form models as inputs. In fact, practitioners are already using much more sophisticated models for the dynamics of recovery rates, credit spreads, and interest rates than those proposed here [47, 53]. Moreover, since the constructed framework may be understood as a network extension of the reduced-form models, often used by practitioners [47, 53], the proposed approach could be even more appealing for practical implementation than, for example, linear DebtRank, which offers a very simplistic treatment of valuation dynamics compared to existing standards that operate on spreads and rates models. For this reason, the method developed here can be perceived as an enhancement of already implemented models, while most network approaches proposed thus far would constitute a different branch, with their theoretical and practical relationship to existing practices being far from obvious.

4.4 Discussion

In this chapter, a structure of reduced-form models was incorporated into complex framework of Network Valuation in Financial Systems (NEVA), and then applied to evaluate the resilience of the United States' financial system. The significant role of the mutual funds was identified, illustrating the fact that a crucial component in the banking network is external to the set of banks. Furthermore, the substantial impact of debt valuation processes, particularly with recovery and credit quality parameters, was demonstrated, showcasing the adaptability of the proposed approach across diverse modeling environments, in contrast to the linear DebtRank framework. Finally, the complex relationship between the interest rates and the banking environment was integrated into the model dynamics, uncovering the potential for cascading failures and a collapse of financial system stemming from interdependencies among its participants, a phenomenon not captured by other models. These results underscore the complexity inherent in the study of systemic risk, where the exclusion of certain factors can lead to significant deviations in risk assessments, posing a danger of misjudging risks. Furthermore, they lay the foundations for integrating crucial industry processes such as credit quality assessment and stochastic projection of interest rates into systemic risk quantification, through the bond of reduced-form models.

Chapter 5

Stochastic network models

In systemic risk research, it is commonly assumed that the system experiences an initial shock, after which it converges 'instantaneously' to an equilibrium state [4, 7, 21, 44]. This equilibrium is defined as a solution to Eq. (3.6) and constitutes a fixed point of the mapping Φ defined in Eq. (3.5), which describes the network dynamics of the financial system within the NEVA framework outlined in Chapter 3. The solution is approximately calculated using the Picard algorithm detailed in Eq. (3.7), in which the map Φ is subjected to iterative dynamics. This method is also applied to the family of reduced-form valuation models and their specific realizations developed in Chapter 4.

In this approach, the recursive procedure is treated merely as a technical tool for computing the target value. However, this is not the only perspective that can be adopted. In the primer by Fischer [147], it is suggested that the iterative process following the shock could be viewed as market dynamics that remain persistently out of equilibrium under conditions of incomplete information. Nevertheless, this concept is only briefly introduced, lacking a thorough exploration of its formal and practical implications. Moreover, the model considers only the shock occurring at an arbitrarily chosen initial timestep. Although this framework is useful for a wide range of applications, particularly in predicting the risk and magnitude of failures in a stable environment prior to the shock, it imposes significant limitations on the model. Such restrictions are neither necessary nor feasible in a general context. In reality, it is questionable to assume that a shock can impact the market at a single moment in time, after which the evolution of the financial system is fully determined without room for stochasticity.

The stochastic process of the market governed by the random dynamics of risk and the iterative evolution of Φ is constructed in this chapter. Its properties and behavior over time are examined in detail, with particular emphasis on the boundaries derived at various levels of generality. Subsequently, the limit process occurring in continuous time is considered. The results obtained are highly valuable from a practical perspective in risk estimation, providing powerful analytical tools and a substantial reduction in computational complexity. They allow to derive analytically or semi-analytically key process characteristics without the necessity to simulate complex stochastic system, driven by sequentially interacting effects of the underlying random process x and the evolution rule governed by the mapping Φ.

Such an approach effectively introduces the time evolution to financial networks along with the random dynamics. It allows to account for the critical fact that risk and uncertainty increase significantly along with the prolonging time horizon. Consequently, the risk assessed over the next twenty years might be substantially higher than the one estimated only for the present moment [47, 62]. The model constructed here facilitates further development of concepts such as risk metrics and enhance decision-making processes, including the determination of desirable reserve amounts and portfolio allocation strategies. In fact, the foundations of financial mathematics are built on the dynamic development of available information over time, which is continuously integrated into price evolution, risk assessment, and strategy formulation [47, 48, 62]. In particular, key time-dimension metrics such as Value at Risk (VaR) and ruin probabilities can be derived, enabled by the consideration of shocks not only as random variables affecting the current state, but also as integral components of the market's evolution over a specified timeframe.

5.1 The notation and formal setup

The foundations of the instrument valuation in financial mathematics has been introduced in Chapter 2. In particular, the price of the basic credit instrument, the zero coupon bond, has been expressed in Eq. (2.19):

$$Z(t_0, T) = Z(T, T) \exp\left(-\int_{t_0}^{T} y(u)du\right), \tag{5.1}$$

where the valuation factor $y(u)$ consist of the interest rate $r(u)$ and the spread $cs(u)$ components:

$$y(u) = r(u) + cs(u). \tag{5.2}$$

Therefore, the $y(u)$ curve describes the dynamics of the value of money and the risk for a term u, and the process of price determination is based on the aggregation of those variables over the specified time period $[t_0,\ T]$. So far, $y(u)$ has been deterministic in the process of network valuation, defined in Eq. (3.6) as a fixed-point problem. In particular, it means that if the time of the valuation changes from t_0 to some t_1 in Eq. (5.1), a value $y(u)$ remains the same for every term u independently of this change. However, the valuation curve is subject of random fluctuations in time. As the stochastic dynamics of risk in financial networks is concerned in this chapter, a family of curves indexed by the time t is considered instead, as the shapes of curves $y(u)$ can differ depending on the time t. In order to distinguish current moment in time t and the term structure u, the valuation curve is denoted as $y_t(u)$. The random factor of the curve can arise from a broad range of different sources, originating from both interest rate and credit spread components. In this work, all these influences are aggregated into a single random process $x_t(u; \omega)$, resulting in the curve dynamics defined as

$$y_t(u; \omega) = r(u) + cs(u) + x_t(u; \omega). \tag{5.3}$$

In this approach, $y_t(u; \omega)$ denotes the forward curve specified in Eq. (2.13), which means that the value of y (and x) for every term u is known at every moment t (but can be different for different moments t, i.e it can hold that $y_t(u; \omega) \neq y_s(u; \omega)$ for $t \neq s$). As the model considers a family of curves for a set of institutions marked as $1, 2, \ldots, n$, the notation for every individual i is of the form

$$y_{i,t}(u; \omega) = r(u) + cs_i(u) + x_{i,t}(u; \omega). \tag{5.4}$$

In order to keep possibly clear and compact notation, $y_t(u; \omega)$ will denote a vector of functions from this moment. Moreover, it is assumed that the initial valuation moment is t_0

and the yield curve is evaluated up to some maximal tenor T. In order to guarantee sensible discounting, it is assumed that x_i are bounded. Therefore, from the formal perspective, $\{x_t : t \in [t_0, T]\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a natural filtration $\mathcal{F}_t$, such that

$$\exists \widetilde{\Omega} \subset \Omega : \mathbb{P}(\widetilde{\Omega}) = 1 \wedge \forall \omega \in \widetilde{\Omega}\ x_t(u; \omega) \in \prod_{i=1}^{K} B_+([t_0, T]^2).$$

Consequently, for all $t \in [t_0, T]$ $x_t \in \mathcal{F}_t$, i.e. at every moment t, the whole vectors of curves $x_t(u)$ are known, but vectors of curves can be different for various moments t. Additionally, it is assumed that curves x_t are measurable with respect to the Lebesgue measure with probability one for all $t \in [t_0, T]$.

Definition 5.1. Let $f : V \times W \to \mathbb{R}$ be a function defined on the product space $V \times W$, and let $\widetilde{V} \subseteq V$ be a subset of V. The infimum and supremum of f with respect to v over the subset $\widetilde{V}$ are functions $\inf_{v \in \widetilde{V}} f(v, w) : W \to \mathbb{R}$ and $\sup_{v \in \widetilde{V}} f(v, w) : W \to \mathbb{R}$ such that for every $w \in W$

$$\inf_{v \in \widetilde{V}} f(v, w) := \inf\{f(v, w) \mid v \in \widetilde{V}\}$$
$$\sup_{v \in \widetilde{V}} f(v, w) := \sup\{f(v, w) \mid v \in \widetilde{V}\}.$$

In particular, for a vector of functions $f(v) = (f_1(v), f_2(v), \ldots, f_n(v)) \in \mathbb{R}^n$, which can be represented as $f_w(v) = f(v, w)$ for $w \in \{1, \ldots, n\}$ in the notation from Def. 5.1, the infimum and supremum are expressed explicitly as vectors of infima and suprema:

$$\inf_{v \in V} f(v) = \left(\inf_{v \in V} f_1(v),\ \inf_{v \in V} f_2(v),\ \ldots,\ \inf_{v \in V} f_n(v)\right) \in \mathbb{R}^n, \tag{5.5}$$

$$\sup_{v \in V} f(v) = \left(\sup_{v \in V} f_1(v),\ \sup_{v \in V} f_2(v),\ \ldots,\ \sup_{v \in V} f_n(v)\right) \in \mathbb{R}^n. \tag{5.6}$$

Similarly, with the use of that terminology to a vector of random processes

$$x_t(u) = (x_{1,t}(u), \ldots, x_{K,t}(u)) \in \prod_{i=1}^{K} B_+([t_0, T]^2),$$

the following notation is introduced:

$$\underline{x}_t(u) := \inf_{s \leq t} x_s(u) = \left(\inf_{s \leq t} x_{i,s}(u)\right), \quad i \in \{1, \ldots, K\},\ u \in [t_0, T]; \tag{5.7}$$

$$\overline{x}_t(u) := \sup_{s \leq t} x_s(u) = \left(\sup_{s \leq t} x_{i,s}(u)\right), \quad i \in \{1, \ldots, K\},\ u \in [t_0, T]. \tag{5.8}$$

Here, K is a number of factors affecting the financial system. Moreover, let $B(t, \delta)$ denote the δ-neighbourhood of an element s of a metric space $(\mathbb{V}, || \cdot ||)$, i.e.

$$B(t, \delta) := \{v \in \mathbb{V} : ||t - v|| \leq \delta\}.$$

In an metric space equipped with an order $\leq$, the left-sided and right-sided δ-neighbourhoods are defined as

$$B^-(t, \delta) := \{v \in \mathbb{V} : v \leq t \wedge ||t - v|| \leq \delta\}, \tag{5.9}$$
$$B^+(t, \delta) := \{v \in \mathbb{V} : v \geq t \wedge ||t - v|| \leq \delta\}. \tag{5.10}$$

Then, the notation for local left-sided extrema is introduced as

$$\underline{x}_t^\delta(u) := \inf_{s\in B^-(t,\delta)} x_s(u) = \left(\inf_{s\in B^-(t,\delta)} x_{i,s}(u)\right), \quad i \in \{1,\dots,K\},\ u \in [t_0,T]; \tag{5.11}$$

$$\overline{x}_t^\delta(u) := \sup_{s\in B^-(t,\delta)} x_s(u) = \left(\sup_{s\in B^-(t,\delta)} x_{i,s}(u)\right), \quad i \in \{1,\dots,K\},\ u \in [t_0,T]. \tag{5.12}$$

Definition 5.2. Let $f : X \to V$ be a function between metric spaces $X,\ V$, where X is additionally equipped with an order $\leq$. The left-sided limit of f in $a \in X$ $\lim_{x\to a^-} f(x)$ is a $g \in V$ such that

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ x \in B^-(a,\delta) \setminus \{a\} \implies f(x) \in B(g,\varepsilon). \tag{5.13}$$

Analogically, the right-sided limit of f in $a \in X$ $\lim_{x\to a^+} f(x)$ is a $g \in V$ such that

$$\forall \varepsilon > 0\ \exists \delta > 0 :\ x \in B^+(a,\delta) \setminus \{a\} \implies f(x) \in B(g,\varepsilon). \tag{5.14}$$

Definition 5.3. Let $f : X \to V$ be a function between metric spaces $X,\ V$, where V is additionally equipped with a complete order $\leq$. The limit inferior $\liminf_{x\to a} f(x)$ and limit superior $\limsup_{x\to a} f(x)$ of f in $a \in X$ are defined as

$$\liminf_{x\to a} f(x) := \lim_{\varepsilon\to 0} \left(\inf\{f(x) : x \in B(a,\varepsilon) \setminus \{a\}\}\right); \tag{5.15}$$

$$\limsup_{x\to a} f(x) := \lim_{\varepsilon\to 0} \left(\sup\{f(x) : x \in B(a,\varepsilon) \setminus \{a\}\}\right). \tag{5.16}$$

If $f = (f_1, f_2, \dots, f_n) \in V^n$, then the limit inferior and limit superior are defined as

$$\liminf_{x\to a} f(x) = \left(\liminf_{x\to a} f_i(x)\right), \quad i \in \{1,\dots,n\}; \tag{5.17}$$

$$\limsup_{x\to a} f(x) = \left(\limsup_{x\to a} f_i(x)\right), \quad i \in \{1,\dots,n\}. \tag{5.18}$$

In this work, $x = (x_1(u),\dots,x_K(u))$ is an non-negative element of a space of bounded functions $\prod_{i=1}^K B_+([t_0,T])$, with a norm $\|\cdot\|_\infty$ defined as

$$\|x\|_\infty = \sup_{1\leq i\leq K} \left(\sup_u |x_i(u)|\right). \tag{5.19}$$

A partial order $\leq$ is defined on x_i straightforwardly: $x_i \leq \tilde{x}_i$ if and only if for all u $x_i(u) \leq \tilde{x}_i(u)$. This definition expands directly to vectors: $x \leq \tilde{x}$ if and only if $x_i \leq \tilde{x}_i$ for all $i \in \{1, 2, \dots, K\}$.

5.2 The discrete dynamic network model

In the NEVA framework, a valuation rule for the vector of equities in a financial system is defined by the mapping Φ in Eq. (3.5). Its concrete specifications are given through the valuation functions, which in reduced-form network models take the general form given by Def. 4.2:

$$\begin{aligned}
\mathbb{V}_i^e(\mathbf{E}(t_0)) &= \sum_{k=1}^{l_i} \alpha_{i,k} \exp\left(-\int_{t_0}^{T_{i,k}} r(\mathbf{E}(t_0);u)du\right),\\
\mathbb{V}_{ij}(\mathbf{E}(t_0)) &= \sum_{k=1}^{n_{ij}} \alpha_{ij,k} \exp\left(-\int_{t_0}^{T_{ij,k}} \left(r(\mathbf{E}(t_0);u) + cs_j(\mathbf{E}(t_0);u)\right) du\right).
\end{aligned} \tag{5.20}$$

The time dependence and the stochastic component have been introduced into the valuation curve in Eq. (5.3). Consequently, definitions needs to be appropriately extended.

Definition 5.4. Let $\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,l_i}$ and $\alpha_{ij,1}, \alpha_{ij,2}, \ldots, \alpha_{ij,n_{ij}}$ be non-negative numbers for all i, such that $\sum_{k=1}^{l_i} \alpha_{i,k} = 1$ and $\sum_{k=1}^{n_{ij}} \alpha_{ij,k} = 1$, representing the weights of the entity's i holdings of external and internal debt instruments with maturities $T_{i,k} \in \mathbb{R}_+$ and $T_{ij,k} \in \mathbb{R}_+$, accordingly. Moreover, let $\mathbf{E}(t_0) \in \mathbb{R}^n$, where $n \in \mathbb{N}$ is a number of institutions in a financial system under consideration, and let $r(\cdot; u)$, $cs_j(\cdot; u)$ be a nonincreasing functions of $\mathbf{E}(t_0)$ that are continuous as mappings from the metric space $(\mathbb{R}^n, \|\cdot\|_1)$ to $(B_+([t_0, T]), \|\cdot\|_\infty)$ for all j. Furthermore, let $T \in \mathbb{R}_+$ represent a maximal time horizon over which all valuations are concerned, $\mathbf{m}(t_0) := (m_1(t_0), ..., m_n(t_0)) \in \mathbb{R}^n$, $\mathbf{M}(t_0) := (M_1(t_0), ..., M_n(t_0)) \in \mathbb{R}^n$ denote the boundaries of the financial system, and let $\mathbf{m}(t_0) \leq \mathbf{E}(t_0) \leq \mathbf{M}(t_0)$. Then, the reduced-form, stochastic network valuation mapping Φ is defined as

$$\Phi : \prod_{i=1}^{K} B_+([t_0, T]) \times \prod_{i=1}^{n} [m_i(t_0), M_i(t_0)] \to \prod_{i=1}^{n} [m_i(t_0), M_i(t_0)], \quad \Phi = (\Phi_1, ..., \Phi_n),$$
$$\Phi_i\left(x_t; \mathbf{E}(t_0)\right) : = A_i^e(t_0)\mathbb{V}_i^e\left(x_t; \mathbf{E}(t_0)\right) - L_i^e(t_0) + \sum_{j=1}^{n} A_{ij}(t_0)\mathbb{V}_{ij}\left(x_t; \mathbf{E}(t_0)\right) - \sum_{j=1}^{n} L_{ij}(t_0). \tag{5.21}$$

Furthermore, valuation functions are defined as

$$\mathbb{V}_i^e(x_t; \mathbf{E}(t_0)) = \sum_{k=1}^{l_i} \alpha_{i,k}\mathbb{E}\left[\exp\left(-\int_{t_0}^{T_{i,k}} \left(r(\mathbf{E}(t_0); u) + x_{k,t}^{ext}(u)\right) du\right) \mid \mathcal{F}_t\right],$$
$$\mathbb{V}_{ij}(x_t; \mathbf{E}(t_0)) = \sum_{k=1}^{n_{ij}} \alpha_{ij,k}\mathbb{E}\left[\exp\left(-\int_{t_0}^{T_{ij,k}} \left(r(\mathbf{E}(t_0); u) + cs_j(\mathbf{E}(t_0); u) + x_{j,t}^{int}(u)\right) du\right) \mid \mathcal{F}_t\right]. \tag{5.22}$$

Here, $x_t^{ext} = (x_{k,t}^{ext})_{k \in \{1,...,J\}}$ is a vector of random valuation curves affecting external assets, $x_t^{int} = (x_{j,t}^{int})_{j \in \{1,...,n\}}$ is a vector of random valuation curves affecting internal assets, and $x_t = (x_t^{ext}, x_t^{int})$. Moreover, the only role of the expectation $\mathbb{E}[\cdot \mid \mathcal{F}_t]$ is to formally point out that all values under the expectation are known at the moment of valuation, which as a consequence is practically deterministic.

At this point, the definition of the mapping Φ is not much different than the previous Def. 4.2 of a reduced-form network model. The random factor x_t, known at t, can be in fact seen as a part of interest or credit components of the valuation function in a modified notation. The rationale behind such a modification is presented below.

The Picard algorithm used to compute the fixed point of the system Φ has been defined in Eq. (3.7) in the NEVA framework as

$$\mathbf{E}^{k+1}(t_0) = \Phi(\mathbf{E}^k(t_0)). \tag{5.23}$$

As elaborated in the introduction to this chapter, this iterative dynamics can be perceived not only as a method for the equilibrium calculation, but also as the process of the market development. In this view, random fluctuations introduced by the component x_t can occur along with the iterations of the algorithm, which is denoted as

$$\mathbf{E}^{k+1}(t_0) = \Phi(x_{t_k}; \mathbf{E}^k(t_0)). \tag{5.24}$$

It is worth noting that if $x_t(u; \omega) = x_s(u; \omega)$ for all t, s, u, ω, then the mapping $\Phi(\cdot)$ from the original definition of NEVA framework in Eq. (3.5) can be defined as $\Phi(\cdot) := \Phi(x_t; \cdot)$. It is fixed in iterations, and then the problem reduces to the previous case in Eq. (5.23). The concept of a fluctuating iteration leads to the following definition.

Definition 5.5. Let $(t_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of moments in time. Then, the process of stochastic network evolution is defined by the recursive equation

$$\begin{aligned} a_0 &= \Phi_0 := \mathbf{E}(t_0) \in \mathbb{R}^n, \\ a_{k+1} &= \Phi(x_{t_k}; a_k); \quad k \in \mathbb{N}. \end{aligned} \tag{5.25}$$

Here, a_k is used instead of $\mathbf{E}^k$ in the notation to clearly distinguish the current case of possibly varying iterations from the previous case of fixed iterations. The sequence of moments in time t_k corresponds to the iterations k in the previous approach, and as such, it evolves. Meanwhile, the moment t_0 (the lower boundary of the integrals in Eq. (5.22)) remains fixed. The main reason for this choice is that the valuation increases and approaches 1 as t_0 evolves toward the upper integral boundary, provided the valuation curves are bounded. While this property is natural and desirable when focusing on the valuation of a single debt instrument, it may not be the case when evaluating the security and resilience of financial systems. As elaborated in Chapter 1, a set of debt instruments is not constant over time—new debt is created at every moment, significantly affecting the state of the financial system. Rather than extending the already complex model with time dynamics of debt issuance, a different approach is adopted here. Specifically, it is assumed that the effects of the aforementioned increase in the valuation function over time and the issuance of new debt cancel each other out. Consequently, the debt structure remains constant across iterations, allowing only the iteration time t_k to evolve in the context of network valuation.

Moreover, since this research is primarily concerned with iterative systems, which can be naturally translated into computer algorithms for performing calculations based on real data, the dynamics of growing debt and liabilities can be conventionally integrated into such simulations (including different variants of flowing time in the model). It is also feasible to represent these phenomena through initial data adjustments, followed directly by the approach proposed in Def. 5.5. To remain conservative in calculations—favoring overestimations of risk over underestimations—a cap can be assumed for the possible debt expansion of the examined financial system within the specified time horizon. Then, both initial assets and liabilities should expand accordingly, along with internal debt holdings. Such an approach enables the estimation of systemic risk in real situations with a significant level of detail, while utilizing all previously obtained results centered on approximations and boundaries of these calculations. This could lead to a considerable reduction in implementational, analytical, and computational complexity, greatly enhancing the overall applicability of the model. These features are particularly important in the context of rapid developments in machine learning and artificial intelligence methods, which face serious challenges related to traceability and explainability of outputs, raising critical questions about the verifiability and reliability of the results.

A formal analysis of the properties of the discrete dynamic network process specified in Def. 5.5 is provided below.

Lemma 5.1. *For a process of stochastic network evolution $(a_k)_{k\in\mathbb{N}}$ and a mapping Φ in Def. 5.5, it holds that*

$$\Phi^k(\overline{x}_{t_k}; \Phi_0) \le a_k \le \Phi^k(\underline{x}_{t_k}; \Phi_0).$$

for all $k \in \mathbb{N}$.

Proof. The proof is based on the principle of mathematical induction. Because Φ specified through the functions in Eq. (5.22) is decreasing in x, for $k = 0$ it follows that

$$\Phi(\overline{x}_{t_0}; \Phi_0) \le \Phi(x_{t_0}; \Phi_0) \le \Phi(\underline{x}_{t_0}; \Phi_0).$$

Assume that the hypothesis is true for some $k \in \mathbb{N}$:

$$\Phi^k(\overline{x}_{t_k}; \Phi_0) \leq a_k \leq \Phi^k(\underline{x}_{t_k}; \Phi_0).$$

From this assumption and from the fact that $\Phi(x; a)$ is increasing in a, it follows that

$$\Phi(x_{t_k}; \Phi^k(\overline{x}_{t_k}; \Phi_0)) \leq \Phi(x_{t_k}; a_k) = a_{k+1} \leq \Phi(x_{t_k}; \Phi^k(\underline{x}_{t_k}; \Phi_0)).$$

Furthermore, because Φ is decreasing in x, it holds that

$$\Phi(\overline{x}_{t_k}; \Phi^k(\overline{x}_{t_k}; \Phi_0)) \leq \Phi(x_{t_k}; \Phi^k(\overline{x}_{t_k}; \Phi_0)) \leq a_{k+1} \leq \Phi(x_{t_k}; \Phi^k(\underline{x}_{t_k}; \Phi_0)) \leq \Phi(\underline{x}_{t_k}; \Phi^k(\underline{x}_{t_k}; \Phi_0)).$$

Considering only the outermost parts of this chain of inequalities allows to reduce the formula to

$$\Phi^{k+1}(\overline{x}_{t_k}; \Phi_0) \leq a_{k+1} \leq \Phi^{k+1}(\underline{x}_{t_k}; \Phi_0).$$

By the principle of mathematical induction the thesis is true for all $k \in \mathbb{N}$. □

Corollary 5.2. *Let $(a_k)_{k\in\mathbb{N}}$ be a process of stochastic network evolution given in Def. 5.5. Then, the following property is satisfied for all $k \in \mathbb{N}$:*

$$\Phi^k(\overline{x}; \Phi_0) \leq a_k \leq \Phi^k(\underline{x}; \Phi_0).$$

Proof. From Lemma 5.1, for all $k \in \mathbb{N}$ it holds that

$$\Phi^k(\overline{x}_{t_k}; \Phi_0) \leq a_k \leq \Phi^k(\underline{x}_{t_k}; \Phi_0).$$

$\Phi(x; a)$ is decreasing in x, therefore $\Phi^k(\overline{x}; \Phi_0) \leq \Phi^{k+1}(\overline{x}_{t_k}; \Phi_0)$ and $\Phi^{k+1}(\underline{x}_{t_k}; \Phi_0) \leq \Phi^k(\underline{x}; \Phi_0)$. Combining above facts results in

$$\Phi^k(\overline{x}; \Phi_0) \leq a_k \leq \Phi^k(\underline{x}; \Phi_0).$$

□

The methods of Lemma 5.1 and Corollary 5.2 are general and do not provide much possibilities to investigate the local behavior of the process around a given timepoint. A method that does it is presented below.

Lemma 5.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence as in Def. 5.5. Further, let $t_{m+k} \in B^-(s, \delta)$ for some $m, l \in \mathbb{N}$ and for all $k \in \{0, 1, ..., l\}$. Then it holds that*

$$\Phi^k(\overline{x}^\delta_s; \Phi^m(\overline{x}_{(s-\delta)}; \Phi_0)) \leq a_{m+k} \leq \Phi^k(\underline{x}^\delta_s; \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0)).$$

Proof. Lemma 5.1 implies that

$$\Phi^m(\overline{x}_{(s-\delta)}; \Phi_0) \leq a_m \leq \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0).$$

From the fact that $\Phi(x, a)$ is increasing in a, it follows that

$$\Phi(x_{t_m}; \Phi^m(\overline{x}_{(s-\delta)}; \Phi_0)) \leq \Phi(x_{t_m}, a_m) \leq \Phi(x_{t_m}; \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0)).$$

Since $\Phi(x, a)$ is decreasing in x, wider boundaries can be derived as

$$\Phi(\overline{x}^\delta_s; \Phi^m(\overline{x}_{(s-\delta)}; \Phi_0)) \leq \Phi(x_{t_m}, a_m) \leq \Phi(\underline{x}^\delta_s; \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0)).$$

Application of such a procedure k times result in

$$\Phi^k(\overline{x}^\delta_s; \Phi^m(\overline{x}_{(s-\delta)}; \Phi_0)) \leq \Phi\left(x_{t_{m+k-1}}; \Phi\left(x_{t_{m+k-2}}; ...; \Phi(x_{t_m}, a_m)...\right)\right) \leq \Phi^k(\underline{x}^\delta_s; \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0)),$$

which is the same as

$$\Phi^k(\overline{x}^\delta_s; \Phi^m(\overline{x}_{(s-\delta)}; \Phi_0)) \leq a_{k+m} \leq \Phi^k(\underline{x}^\delta_s; \Phi^m(\underline{x}_{(s-\delta)}; \Phi_0)).$$

This completes the proof. □

Corollary 5.2 presents straightforward rules for establishing the weakest boundaries for the process $(a_k)_{k\in\mathbb{N}}$. For the time-independent variables $\overline{x}$ and $\underline{x}$, the process evolves according to the mappings $\Phi(\overline{x};\cdot)$ or $\Phi(\underline{x};\cdot)$, which remain constant across iterations. Consequently, the boundaries can be calculated entirely using the existing methods outlined in Eq. (3.7). As in system stability research the primary focus is often on the system's minimum (or maximum loss) under predetermined conditions, calculating $\Phi^k(\overline{x};\Phi_0)$ is sufficient to establish a lower boundary for the entire process. Moreover, since the fixed point Φ^* is typically estimated numerically by Φ^k, it is adequate to compute up to some N to obtain an approximate boundary for the entire process. This approach is not only straightforward to implement and calculate, but also proves particularly useful when the variation in x is relatively small or when long-term boundaries of the entire process are of primary interest.

Lemma 5.1 offers a more detailed functionality by deriving boundaries for a specific a_k based on the extrema up to the kth time point t_k. In this approach, unlike the previous method outlined in Eq. (3.7), different mappings $\Phi(\overline{x}_{t_k};\cdot)$ and $\Phi(\underline{x}_{t_k};\cdot)$ are employed for various time points t_k. However, a single mapping, either $\Phi(\overline{x}_{t_k};\cdot)$ or $\Phi(\underline{x}_{t_k};\cdot)$, is still used to determine the boundary of the process at each specific time point. This methodology allows for the derivation of boundaries across different time horizons, enabling an analysis of how the time range influences process behavior.

Lemma 5.3 provides more localized boundaries near the moment s. It builds upon the results derived in Lemma 5.1 up to $s-\delta$ and incorporates local information about the extrema of the process x within the δ-neighborhood of s. This approach utilizes two mappings to derive a boundary: one defined by the extrema of x up to $s-\delta$ and the other by the extrema within the δ-neighborhood of s. This methodology allows for more detailed characterisation of the dynamics in the time range surrounding the specified point. Computationally, it involves applying the known fixed mapping algorithm twice. This approach proves particularly useful in longer time horizons, where the more general boundaries developed in previous methods may be unsatisfactory, especially when the local behavior around a given moment is of primary interest.

5.3 Pushing the model to the limit

So far, the analysis has been limited to the case when the process is defined on a fixed and predetermined discrete lattice of time points. Therefore, the natural question arises about the process defined on the continuous timeline and its relationships with the discrete equivalent. This section is dedicated to elaborate on this topic. As it is commonly the case in mathematics in the continuous approach, it turns out that it allows to 'smoothen' the model and in consequence obtain analytical results describing the modeled object of interest with high level of precision.

Below some formal definitions and notation are introduced which are needed in order to formally develop the continuous time model [148]. A partition $P(t)$ of an interval $[a,b]$ is a finite sequence of numbers of the form

$$a = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_n = b.$$

Each $[t_i, t_{i+1}]$ is called a sub-interval of the partition. The mesh or norm of a partition is defined to be the length of the longest sub-interval, that is,

$$\mathrm{mesh}(P) = \max\left(t_{i+1} - t_i\right), \quad i \in \{0, \ldots, n-1\}.$$

Suppose that two partitions $P(t)$ and $Q(s)$ are both partitions of the interval $[a, b]$. Then $Q(s)$ is called a refinement (or refining partition) of $P(t)$ if for each integer i, with $i \in \{0, \ldots, n\}$, there exists an integer $r(i)$ such that $t_i = s_{r(i)}$. That is, a refinement breaks up some of the sub-intervals "refining" the accuracy of the original partition. A natural order can be defined on partitions by stating that one partition is greater than or equal to another if the former is a refinement of the latter. If a partition is represented as a set of its points $P = \{t_0, t_1, ..., t_n\}$, then the partial order of the partitions is naturally defined as a well-known partial order of sets inclusion.

Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of refining partitions, i.e., such that $\forall k \in \mathbb{N}\ P_k \subset P_{k+1}$ and such that $\lim_{k\to\infty} \mathrm{mesh}(P) = 0$. If $s \in P_l$ for some $l \in \mathbb{N}$, then from the fact that the partitions are refining, it stems that s is also contained by every subsequent partition, i.e $s \in P_{l+k}$ for all $k \in \mathbb{N}$. Therefore, a notation a_s can be used independently of the partition up from some index l such that $s \in P_l$. Also, as per specification of stochastic network given in Def. 5.4, let Φ be bounded by

$$m \leq \Phi \leq M.$$

Time factor t_0 is omitted in order to keep possibly compact notation.

The introduction of the partitions and partitions sequence allows to consider process defined of sequences of densening lattices of timepoints and in consequence analyse the behavior of the process in the limit case, when the time granularity approaches the continuous line. The first result characterising the process in such a limit case is presented below.

Lemma 5.4. *Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of refining partitions of the interval $[t_0, T]$. Assume that there exists $l \in \mathbb{N}$ such that $s \in P_l$. Then, the following property is satisfied by the process of stochastic network evolution given in Def. 5.5 and evaluated at the moment s, denoted as a_s:*

$$\Phi^*(\overline{x}_s^\delta; m) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \Phi^*(\underline{x}_s^\delta; M).$$

Here, for a process $z_t = z$ that is constant in time and for a vector $a \in \mathbb{R}^n$, the limit $\Phi^*(z; a) := \lim_{k\to\infty} \Phi^k(z; a) \in \mathbb{R}^n$ is a fixed point of the mapping $\Phi(z; \cdot)$, that is, $\Phi(z; \Phi^*(z; a)) = \Phi^*(z; a)$.

Proof. From Lemma 5.3, it follows that

$$\Phi^k(\overline{x}_s^\delta; \Phi^l(\overline{x}_{(s-\delta)}; \Phi_0)) \leq a_s \leq \Phi^k(\underline{x}_s^\delta; \Phi^l(\underline{x}_{(s-\delta)}; \Phi_0)).$$

Moreover, as $\Phi(x; a)$ is increasing in a, this result can be extended to

$$\Phi^k(\overline{x}_s^\delta; m) \leq \Phi^k(\overline{x}_s^\delta; \Phi^l(\overline{x}_{(s-\delta)}; \Phi_0)) \leq a_s \leq \Phi^k(\underline{x}_s^\delta; \Phi^l(\underline{x}_{(s-\delta)}; \Phi_0)) \leq \Phi^k(\underline{x}_s^\delta; M).$$

which could be written more compact as

$$\Phi^k(\overline{x}_s^\delta; m) \leq a_s \leq \Phi^k(\underline{x}_s^\delta; M).$$

This leads to

$$\liminf_{k\to\infty} \Phi^k(\overline{x}_s^\delta; m) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \limsup_{k\to\infty} \Phi^k(\underline{x}_s^\delta; M).$$

If a process $z_t = z$ is constant in time, the mapping $\Phi(z;\cdot)$ is fixed in iterations, and reduces to the mapping $\Phi(\cdot) := \Phi(z;\cdot)$ from Eq. (3.5) in the original NEVA framework. Consequently, by the virtue of Theorem 3.2, for $a \in \{m, M\}$, $\Phi^k(z;a)$ is convergent to the equilibrium point $\Phi^*(z;a)$, and therefore

$$\Phi^*(\overline{x}_s^\delta; m) \le \liminf_{k\to\infty} a_s \le \limsup_{k\to\infty} a_s \le \Phi^*(\underline{x}_s^\delta; M),$$

which completes the proof. □

Lemma 5.4 provides the boundaries for the process around moment s when considering the continuous time limit. By the virtue of Theorem 3.1, in the limit case $k \to \infty$ iterative process Φ^k converges to the fixed point Φ^*. Therefore, continuous timeline allows to derive results that are independent of the arbitrarily chosen time granularity and number of iterations, providing significantly improved clarity and unambiguity of the obtained results. They can be extended even further, relying on the underlying risk process x in the limit precisely at the time point s, without the reference to the arbitrarily chosen neighbourhood δ, as presented in the subsequent theorems.

Theorem 5.5. *Let $(P_k)_{k\in\mathbb{N}}$ denote a sequence of refining partitions of the interval $[t_0, T]$. Suppose there exists $l \in \mathbb{N}$ such that $s \in P_l$. Then, the following property is satisfied by the stochastic network evolution process specified in Def. 5.5 and evaluated at the moment s, denoted by a_s:*

$$\liminf_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m) \le \liminf_{k\to\infty} a_s \le \limsup_{k\to\infty} a_s \le \limsup_{\delta\to 0} \Phi^*(\underline{x}_s^\delta; M).$$

Proof. From Lemma 5.4, it holds that

$$\Phi^*(\overline{x}_s^\delta; m) \le \liminf_{k\to\infty} a_s \le \limsup_{k\to\infty} a_s \le \Phi^*(\underline{x}_s^\delta; M).$$

From the fact that Φ^* is decreasing in x, it follows that the limits $\lim_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m)$ and $\lim_{\delta\to 0} \Phi^*(\underline{x}_s^\delta; M)$ exist and

$$\lim_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m) \le \liminf_{k\to\infty} a_s \le \limsup_{k\to\infty} a_s \le \lim_{\delta\to 0} \Phi^*(\underline{x}_s^\delta; M),$$

which completes the proof. □

Theorem 5.5 has fundamental significance, as it provides a rule for construction of the target process on the continuous timespace with a very general assumptions. In order to guarantee a significant degree of unambiguity of the limit, a series of much stronger conditions is required, which is illustrated in the following result.

Theorem 5.6. *Let x_t be continuous function of t with a metric determined by the norm $||x||_\infty$ defined in Eq. (5.19). If Φ^* is continuous in x_s, and there is an unique fixed point of Φ in x_s denoted as $\Phi^*(x_s)$, then the limit $\lim_{k\to\infty} a_s$ exists and*

$$\lim_{k\to\infty} a_s = \Phi^*(x_s).$$

Proof. From Theorem 5.5, it follows that

$$\lim_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m) \le \liminf_{k\to\infty} a_s \le \lim_{k\to\infty} a_s \le \limsup_{\delta\to 0} \Phi^*(\underline{x}_s^\delta; M).$$

The assumption of continuity of Φ^* in x_s and the fact that Φ is decreasing in x imply that

$$\Phi^*(\lim_{\delta\to 0} \overline{x}_s^\delta; m) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \Phi^*(\lim_{\delta\to 0} \underline{x}_s^\delta; M).$$

Similarly, the assumption of continuity of x_t in t implies that

$$\Phi^*(x_s; m) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \Phi^*(x_s; M).$$

Then, from the assumption that the equilibrium of Φ is unique in x_s it follows that $\Phi^*(x_s; m) = \Phi^*(x_s; M) = \Phi^*(x_s)$. Consequently

$$\Phi^*(x_s) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \Phi^*(x_s).$$

From the squeeze theorem, it follows that the limit $\lim_{k\to\infty} a_s$ exist and

$$\lim_{k\to\infty} a_s = \Phi^*(x_s),$$

which completes the proof. □

The subsequent theorem is just a conjunction of Theorems 5.5 and 5.6. Nevertheless, it is a consolidation of the most important results of this part of the thesis, with a clear, uniformly presented conditions under which various statements holds about the characteristics of the stochastic network valuation process. It is also central for subsequent content of this thesis, and as such is formally formulated below.

Theorem 5.7. *Let x be an element of space of a bounded functions $x_t(u) \in \prod_{i=1}^K B_+([t_0, T]^2)$, with probability one. Let $(P_k)_{k\in\mathbb{N}}$ be a sequence of refining partitions of the interval $[t_0, T]$, i.e., such that $\forall\ k \in \mathbb{N}\ P_k \subset P_{k+1}$ and $\lim_{k\to\infty} \mathrm{mesh}(P) = 0$. Assume that there exists $l \in \mathbb{N}$ such that $s \in P_l$. Then, with probability one it holds that*

$$\lim_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m) \leq \liminf_{k\to\infty} a_s \leq \limsup_{k\to\infty} a_s \leq \lim_{\delta\to 0} \Phi^*(\underline{x}_s^\delta; M).$$

Furthermore, assume that with probability one x_t is continuous function of t with a metric determined by the norm $||x||_\infty$ defined in Eq. (5.19)*. If Φ^* is continuous in x_s and there is an unique fixed point of Φ in x_s denoted as $\Phi^*(x_s)$, then the limit $\lim_{k\to\infty} a_s$ exists and*

$$\lim_{k\to\infty} a_s = \Phi^*(x_s).$$

Proof. This theorem is a conjunction of Theorem 5.5 and Theorem 5.6. □

Assumptions required for the unambiguous convergence in the second part of the Theorem 5.7 are worth a detailed discussion. Among the three assumptions, the restriction of the underlying source of noise x_t from bounded functions to functions continuous in t is arguably the least questionable. However, this does not imply it is without doubt. While this restriction eliminates abrupt changes due to point jumps in the yield curve, it is still possible to define a sequence of continuous functions that converges pointwise at all points in the interval if a single jump occurs within it. Changes can still be arbitrarily large within any small timespan, potentially exceeding the finite precision limits of computers. Thus, from a computational perspective, this assumption does not appear overly restrictive. Moreover, systemic risk primarily examines the sources of abrupt changes arising from the dynamics of system evolution rather than from the underlying noise (the postulate of risk endogeneity [3]). Nevertheless, in analytical contexts, a continuous risk process facilitates the definition of strategies that enable continuous responses to infinitesimally small stepwise changes, which can be viewed as subdivisions of an "abrupt" change under the assumption of continuity. This has significant implications, as it theoretically allows for reactions and safeguards against shocks that, in reality, cannot be adequately protected against. Overall, while the assumption of continuity can be defended, there are substantial arguments against it, suggesting that it may not be warranted in a general environment.

The situation regarding the two remaining assumptions is clearer. Research on equilibrium nonuniqueness has evolved into an important thread within the systemic risk discipline [14], making the assumption of equilibrium uniqueness difficult to justify. Additionally, since equilibria are typically estimated through Φ iterations up to a predefined level of step error, variations in results based on starting points can be significant. The assumption of Φ^* continuity faces similar challenges. The existence and localization of critical points, where arbitrarily small perturbations can lead to disproportionately large changes, are characteristic features in systemic risk research, rendering this assumption unreasonable.

Consequently, it is difficult to justify the assertion that a continuous-time process can be constructed as $\Phi^*(x_s)$ based on unsubstantiated convergence conditions. This is particularly significant given that the minimum values or maximum ranges of losses in the process under consideration are of primary importance in the study of systemic risk [3, 14]. In constructing such a process, two of the restrictive assumptions from Theorem 5.6 pertain to the properties of Φ^* and can therefore be attributed to it. The third assumption, however, is specific to the underlying process x and cannot be ascribed to Φ^*, even though it influences the limit's value. One potential solution is to consider a modified process defined as $\tilde{x}_t = \max\{x_t, \ \limsup_{s \to t^-} x_s\}$. This is a reasonable approach, as from an interpretational standpoint, it effectively disregards anomalous behavior in the underlying risk process, replacing it with the local left-sided supremum of risk. However, it is argued here that this may not be ideal from a modeling perspective, as the value of loss at a specific moment in time is often not the primary point of interest; rather, it is the largest loss within a given time interval that holds greater significance. The details of this argument will be elaborated in the following section.

5.4 Risk analysis of the loss process

In the financial industry, the quantification and management of risk are crucial for the stability and success of financial institutions and investment strategies. Among the various methodologies employed to assess the risk of company's default, Value at Risk (VaR) and ruin probability stand out as critical tools [107,149,150]. VaR provides a statistical measure that estimates the potential loss in value of an asset or portfolio over a defined time horizon, given a specified confidence level [151]. This metric is instrumental in understanding the worst-case scenarios that investors may face, thereby enabling them to make informed decisions regarding capital allocation and risk exposure [152]. Ruin probability, on the other hand, offers insights into the likelihood of an entity facing insolvency or significant financial distress over a specified period [153]. By integrating ruin probability into risk assessments, financial professionals can better gauge the sustainability of their strategies and the adequacy of reserves needed to withstand adverse market conditions [154].

Together, VaR and ruin probability form a robust framework for evaluating risk, allowing institutions to establish appropriate reserves and develop market strategies that align with their risk appetite. The significance of these metrics extends beyond mere theoretical constructs; they are foundational in the decision-making processes of financial managers and investors [125]. By accurately calculating risk and reserves over a defined time horizon, organizations can devise strategies that not only mitigate potential losses but also capitalize on market opportunities. As such, the effective application of VaR and ruin probability is essential for navigating the complexities of financial markets, ultimately translating into sound decision-making that supports long-term financial health and stability.

In this section, the results regarding the continuous-time limit of the stochastic network process obtained in Section 5.3 are utilized to construct a model of market evolution. Further development of the theory focuses on the estimation of the minima of the network processes along specified paths. The key results of this section, Theorem 5.10 and Lemma 5.11, enable the mathematical separation of the underlying risk process x from the network dynamics Φ, providing powerful analytical tools for financial modeling by simplifying the originally complex stochastic network process defined in Def. 5.5, which involves iteratively convoluted networks and random effects. This separation enables a reduction of the original model to well-known components, leveraging their properties to estimate the distribution of losses within the system. Consequently, this approach provides a method for computing key metrics such as VaR and ruin probability for the given model.

Let L_t denote a loss function at time t.

Definition 5.6. The ruin probability $\chi_\alpha(L_t)$ at the level of $\alpha \in \mathbb{R}_+$ is the probability that the loss function at time t L_t exceeds α:

$$\chi_\alpha(L_t) := \mathbb{P}(L_t > \alpha).$$

Definition 5.7. Value at Risk at the level of $\alpha \in [0,1]$ is specified as

$$\mathrm{VaR}_\alpha(L_t) := \inf\{x \in \mathbb{R} : F_{L_t}(x) \geq 1-\alpha\} = F_{L_t}^{-1}(1-\alpha),$$

where F_Y denotes the distribution function of a random variable Y. It is the loss value that with probability $1-\alpha$ will not be exceeded.

In the field of the risk analysis, usually not value of the process at some specific point in time is of the main concern, but the infimum on some time interval. The following theorem addresses this concept.

Theorem 5.8. *Denote the lower boundary from Theorem 5.5 as* $b_s := \liminf_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m)$ *and let* $(t_1, t_2) \subset [t_0, T]$. *Then it holds that*

$$\Phi^*\left(\sup_{s\in(t_1,t_2)} x_s; m\right) \leq \inf_{s\in(t_1,t_2)} b_s.$$

Proof. Let $s \in (t_1, t_2)$ and $b_s := \liminf_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m)$. For $\delta \leq \min\{s-t_1, t_2-s\}$ it holds that $\sup_{s\in(t_1,t_2)} x_s \geq \overline{x}_s^\delta$, and therefore

$$\Phi^*\left(\sup_{s\in(t_1,t_2)} x_s; m\right) \leq b_s.$$

Taking the infimum over all $s \in (t_1, t_2)$ yields

$$\Phi^*\left(\sup_{s\in(t_1,t_2)} x_s; m\right) \leq \inf_{s\in(t_1,t_2)} b_s,$$

which completes the proof. □

Remark 5.1. *Theorem 5.8 also holds for all combinations of closed - opened intervals in place of the originally proposed* (t_1, t_2)*, if open* δ*-neighbourhoods of* $\overline{x}_s^\delta$ *in the definition of* b_s *are appropriately restricted to* $[t_1, t_2]$*.*

Theorem 5.8 states that the infimum of boundaries in an interval $\inf_{s\in(t_1,t_2)} b_s$ is bounded from below by $\Phi^*\left(\sup_{s\in(t_1,t_2)} x_s; m\right)$. In addition to that, the following result characterizes the mapping Φ^* itself.

Lemma 5.9. *Let a financial system attain its lower boundary* m *at the initial moment. Then an equilibrium point of a network evolution mapping* Φ *evaluated at a supremum of a risk process* $\sup_{s\le t} x_s$ *is lesser than an infimum of equilibrium points up to time* t*:*

$$\Phi^*(\sup_{s\le t} x_s; m) \le \inf_{s\le t} \Phi^*(x_s; m).$$

Proof. For all $\tau \le t$ it holds that

$$\sup_{s\le t} x_s \ge x_\tau.$$

From the fact that $\Phi^*(y; \cdot)$ is decreasing in y, it follows that

$$\Phi^*(\sup_{s\le t} x_s; m) \le \Phi^*(x_\tau; m).$$

Application of the infimum to both sides of the inequality yields

$$\inf_{\tau\le t} \Phi^*(\sup_{s\le t} x_s; m) \le \inf_{\tau\le t} \Phi^*(x_\tau; m).$$

The left side of the inequality does not exhibit the dependency on τ, therefore

$$\Phi^*(\sup_{s\le t} x_s; m) \le \inf_{s\le t} \Phi^*(x_s; m).$$

This completes the proof. □

Inequalities in Theorem 5.8 and Lemma 5.9 can be strict if x does not attain its supremum in (t_1, t_2) and Φ^* is not continuous there.

Theorem 5.8 and Lemma 5.9 provide very useful results. The first one facilitates the estimation of the supremal loss by leveraging the supremum of the underlying random process x, thereby eliminating the need to compute Φ^* across a wide range of arguments. This effectively transforms the analysis of $\inf_{s\in(t_1,t_2)} b_s$ into the evaluation of $\Phi^*\left(\sup_{s\in(t_1,t_2)} x_s; m\right)$. In a similar vein, Lemma 5.9 simplifies the problem of determining $\inf_{u\le t} \Phi^*(x_u; m)$ to the calculation of $\Phi^*(\sup_{u\le t} x_u; m)$. This duality not only reorients the original problem towards the analysis of Φ^*, but also outlines a method for conducting this analysis. Ultimately, both results converge on the computation of Φ^* at a single point: the supremum of x within a specified interval.

As discussed in relation to Theorem 5.7, the conditions for the existence of unambiguous limit $\lim_{k\to\infty} a_s$ are unjustified. In consequence, it is hardly justifiable to construct a continuous-time process as $\Phi^*(x_s)$ relying on groundless assumptions. Therefore, the lower bound $\lim_{\delta\to 0} \Phi^*(\overline{x}_s^\delta; m)$ from the Theorem 5.5 can be considered instead of $\Phi^*(x_s)$, thus not requiring the conditions of the unique equilibrium and continuity of Φ^*. From the fact that $\Phi^*(x; m)$ is bounded and decreasing in x, it follows that the right-sided limit $\lim_{y\to x^+} \Phi^*(y; m)$ exist and

$$\liminf_{y\to x} \Phi^*(y; m) = \lim_{y\to x^+} \Phi^*(y; m). \tag{5.26}$$

This leads to the following construction.

Definition 5.8. Define the mapping Ψ as

$$\Psi(x) := \lim_{y \to x^+} \Phi^*(y; m).$$

Then, the continuous systemic risk process with the underlying random process x_t is defined as $\Psi(x_t)$.

Since the mapping $\Phi^*(y; m)$ is decreasing in y, it follows straight from the definition that for all y, $\Psi(y) \leq \Phi^*(y; m)$. Furthermore, the properties of Φ^* established in Theorem 5.8 and Lemma 5.9 directly transfer to Ψ, as demonstrated below.

Theorem 5.10. *Denote the lower boundary from Theorem 5.5 as* $b_s := \liminf_{\delta \to 0} \Phi^*(\overline{x}_s^\delta; m)$ *and let* $(t_1, t_2) \subset [t_0, T]$. *Then it holds that*

$$\Psi\left(\sup_{s \in (t_1, t_2)} x_s\right) \leq \inf_{s \in (t_1, t_2)} b_s.$$

Proof. From the Def. 5.8 of the mapping Ψ and the fact that Φ^* is decreasing, it follows that

$$\Psi\left(\sup_{s \in (t_1, t_2)} x_s\right) = \lim_{y \to \sup^+_{s \in (t_1, t_2)} x_s} \Phi^*(y; m) \leq \Phi^*\left(\sup_{s \in (t_1, t_2)} x_s; m\right).$$

Moreover, Theorem 5.8 states that

$$\Phi^*\left(\sup_{s \in (t_1, t_2)} x_s; m\right) \leq \inf_{s \in (t_1, t_2)} b_s.$$

Combining those two facts yields

$$\Psi\left(\sup_{s \in (t_1, t_2)} x_s\right) \leq \inf_{s \in (t_1, t_2)} b_s,$$

which completes the proof. □

Lemma 5.11. *The financial system defined through the mapping* Ψ *evaluated at a supremum of a risk process up to time* t $\sup_{s \leq t} x_s$ *is lesser than an infimum of the system up to time* t*:*

$$\Psi(\sup_{s \leq t} x_s) \leq \inf_{s \leq t} \Psi(x_s).$$

Proof. For every τ such that $\tau \leq t$, the following inequality holds:

$$\sup_{s \leq t} x_s \geq x_\tau.$$

Since the function $\Psi(y; \cdot)$ is decreasing in y, it follows that

$$\Psi(\sup_{s \leq t} x_s) \leq \Psi(x_\tau).$$

Taking the infimum over all $\tau \leq t$ on both sides of the inequality results in

$$\inf_{\tau \leq t} \Psi(\sup_{s \leq t} x_s) \leq \inf_{\tau \leq t} \Psi(x_\tau).$$

The left-hand side of the inequality does not depend on τ, allowing for simplification to

$$\Psi(\sup_{s \leq t} x_s) \leq \inf_{s \leq t} \Psi(x_s).$$

This completes the proof. □

Therefore, $\Psi(x_t)$ is considered as the continuous time equivalent of the discrete process $(a_k)_{k\in\mathbb{N}}$ from Def. 5.5. As discussed, it is not grounded to expect the conditions for the existence of the limit

$$\lim_{\delta\to 0}\lim_{k\to\infty} a_s = \Phi^*(x_s)$$

from Theorem 5.7 to be satisfied. Instead, the solution $\Psi(x_t)$ corresponding to the lower limit from this theorem is proposed. This definition could be in fact considered a more general; when the assumptions for a unique limit are met, it follows that

$$\Psi(x) = \lim_{y\to x^+} \Phi^*(y;m) = \Phi^*(x;m) = \Phi^*(x).$$

Thus, $\Psi(x_t)$ can be viewed as an extension of $\Phi^*(x_t)$ for cases where unique convergence of the discrete process does not occur.

The results obtained here are interesting from both theoretical and practical perspectives. The originally very complex process given in Def. 5.5, characterized by convoluted and iteratively overlapping effects arising from the dynamics of the system governed by the mapping Φ and the random process x_t, has been significantly streamlined. It is now reduced to the task of finding the infimum of a process $\Psi(x_t)$, which can, in turn, be practically simplified to the calculation of the supremum of x_t. This transformation greatly enhances the analysis, allowing for the separate examination of the exposure function Ψ across the various processes responsible for the randomness. As a result, insights applicable to broad ranges of different scenarios can be conveniently obtained.

The practical application of the developed theory is presented below. The supremal percentage loss up to the time t can be calculated as $\sup_{s\le t}\left(1 - \frac{||\Psi(x_t)||_1}{||\Phi_0||_1}\right)$. From Lemma 5.11, $\Psi(\sup_{s\le t} x_t) \le \inf_{s\le t}\Psi(x_t)$, and therefore $1 - \frac{||\Psi(\sup_{s\le t} x_t)||_1}{||\Phi_0||_1} \ge \sup_{s\le t}\left(1 - \frac{||\Psi(x_t)||_1}{||\Phi_0||_1}\right)$. As already discussed, this formulation is practically useful as it allows for a clear separation and utilization of knowledge regarding the distribution of the underlying random process x_t. However, in cases where x_t is multidimensional, applying the supremum for each coordinate separately may lead to significant overestimation of loss. Nevertheless, when focusing on aggregate loss, it is standard practice to assume a single, common, one-dimensional risk factor that represents the overall market environment. In this context, it is assumed that the risks associated with particular entities can be represented as a sum of a common risk source and individual risks for all entities. Notable examples include the Nobel Prize-winning Markowitz model [84], which is used for optimizing risk and return in portfolio construction, and the credit risk model developed by Vašiček [130], which served as the foundation for the international regulatory framework of Basel II, determining the reserve requirements that banks worldwide must maintain [107, 155]. Furthermore, even when idiosyncratic risks must be considered, projecting them individually is not always required, particularly when the primary modeling concern is the aggregate loss of the system. The common risk source can be adjusted to reflect the overall distribution of individual risk factors, thereby eliminating the need for extensive calculations and analyses to model their dynamics separately. This approach can prevent the model from becoming overly complex and difficult to trace, understand, and apply in various decision-making scenarios. Consequently, in the practical application recipe being developed here, it is assumed that the process x_t is driven by a single, one-dimensional random factor $z_t \in \mathbb{R}$, while idiosyncratic factors are disregarded for the purposes of this analysis. Notably, z_t may still take a form encompassing multiple risk factors, such as $z_t = v_{1,t} + v_{2,t} + \ldots + v_{m,t}$ or more generally $z = f : v_t \to \mathbb{R}$ for some vector of risk factors v_t. The critical aspect is that $v_t \in \mathbb{R}$ ensures a complete ordering over the states of the market. For example, the following dynamics of the risk source can be defined:

$$x_{i,t}(u) = z_t + f_i(u) \quad \text{or} \quad x_{i,t}(u) = z_t \times f_i(u),$$

where for a curve $i \in \{1, ..., K\}$, f_i is a deterministic function that specifies the shape of its spread curve. Ultimately, the loss function is defined as

$$L_t(x) = 1 - \frac{||\Psi(\sup_{s\leq t} x_t)||_1}{||\Phi_0||_1}. \tag{5.27}$$

Remark 5.2. *There exist other possibilities to define the loss function. For instance, even for a single-dimensional risk factor, X_t does not always attain its supremum, therefore $\inf_{s\leq t}\Psi(x_t)$ instead of $\Psi(\sup_{s\leq t} x_t)$ can be considered. Similarly, based on Theorem 5.8, $\Phi^*(x, m)$ may replace its right-sided limit $\Psi(x)$ in the definition. However, both those approaches share a common drawback. Notable differences between $\inf_{s\leq t}\Psi(x_t)$ and $\Psi(\sup_{s\leq t} x_t)$, or between $\Phi^*(x, m)$ and $\Psi(x)$, indicate that even a small increase in the risk factor could lead to a substantial increase in the loss. Therefore, using $\Psi(\sup_{s\leq t} x_t)$ serves as a safety margin in loss calculations; without this margin, results may become highly sensitive to input variations that fall below the precision of the computer.*

As previously discussed, the loss function defined in Eq. (5.27) establishes a clear and practical distinction between the source of randomness x_t and the exposure Ψ. Consequently, the methods for calculating ruin probability and VaR defined in Def. 5.6 and Def. 5.7, can be simplified to the computation of Ψ at specific points, which are determined based on the distribution of the underlying random process $\overline{x}_t := \sup_{s\leq t} x_t$. The exact specification is provided below.

1. Calculate $\Phi^l(x_{j+1}; m)$ on points x_j of a sufficiently dense lattice $A = \{x_0, \ldots, x_K\}$ sorted in ascending order.
2. Approximate $\Psi(x_j)$ by $\widetilde{\Psi}(x_j) = \Phi^l(x_{j+1})$ and $L(x_j)$ by $\widetilde{L}(x_j) = \left(1 - \frac{||\Phi^l(x_{j+1})||_1}{||\Phi_0||_1}\right)$.
3. Estimate the ruin probability as
$$\chi_\alpha(L_t) = \mathbb{P}(L_t(x) > \alpha) \approx F_{\overline{x}_t}(y),$$
where $y = \operatorname{argmin}_{x_i \in A:\ \widetilde{L}(x_i)\leq\alpha} |\widetilde{L}(x_i) - \alpha|$.
4. Estimate the Value at Risk as
$$\mathrm{VaR}_\alpha(L_t(x)) \approx \widetilde{L}\left(\min\{x_i \in A : F_{\overline{x}_t}(x_i) \geq 1-\alpha\}\right).$$

The approach proposed here facilitates straightforward computation of results across a diverse array of distributions, thereby providing a robust method for financial analysis. It enables the assessment of a wide range of assumptions and scenarios, offering a rich and valuable foundation for the evaluation of risk. The separation between the source of randomness x_t and the exposure Ψ significantly streamlines the analysis, yielding a clearer understanding of the model's behavior.

Chapter 6

A comprehensive systemic risk model

In this chapter, a comprehensive model of systemic risk is constructed, constituting the synthesis of all the key results obtained in this thesis. It incorporates the systemic stochastic dynamics of the credit market (Chapter 1 and Chapter 2) into the reduced-form network model with interest rate feedback (Chapter 4). Based on the theory constructed in Chapter 5, the model is applied to the calculation of Value at Risk and the probability of ruin of prominent financial institutions of the United States, also considered in Chapter 4. The impact of the subsequent elements of the model on its dynamics is quantified and analyzed in detail.

6.1 The model

According to Chapter 5, the general dynamics of the financial system is defined as the vector of equities of the entities in the system $\Psi(x_t)$ depending on the underlying risk process x_t. The evolution of the system is governed by the reduced-form valuation functions in Eq. (5.22):

$$\begin{aligned}
\mathbb{V}_i^e(x_t;\mathbf{E}_t) &= \sum_{k=1}^{m_i} \alpha_{i,k} \exp\left(-\int_t^{T_{i,k}} r(\mathbf{E}_t;u) + x_{k,t}^{ext}(u;\omega)du\right), \\
\mathbb{V}_{ij}(x_t;\mathbf{E}_t) &= \sum_{k=1}^{n_{ij}} \alpha_{ij,k} \exp\left(-\int_t^{T_{ij,k}} \Big(r(\mathbf{E}_t;u) + cs_j(\mathbf{E}_t;u) + x_{j,t}^{int}(u;\omega)\Big)\, du\right).
\end{aligned} \tag{6.1}$$

Particular models are obtained through the specification of interest rate r, valuation spread cs and the random noise x_t. In the study of systemic risk, a well-established practice involves treating internal assets as derivatives of external assets and their network dependencies [44–46]. Consequently, only external assets are subject to shocks, while the valuation of internal assets is determined based on this framework [4, 21, 44]. To adhere to this approach and ensure straightforward comparability with existing results, it is assumed here that random fluctuations represented by the process x_t directly affect only external assets, while internal assets are valued only on the basis of network dynamics. Moreover, the term structure is principally ignored in this study. Such an approach allows to simplify the analysis and the computation, while still maintaining the focus on the main factors of interest. Taking into account all the discussed components, the general formula of valuation functions reduces to

$$\begin{aligned}
\mathbb{V}_i^e(x_t;\mathbf{E}_t) &= \exp\left(-\left(r(\mathbf{E}_t) + x_t\right)(T-t)\right), \\
\mathbb{V}_{ij}(x_t;\mathbf{E}_t) &= \exp\left(-\left(r(\mathbf{E}_t) + cs_j(\mathbf{E}_t)\right)(T-t)\right).
\end{aligned} \tag{6.2}$$

Derivation of this result is analogical to the one presented in Section 4.2.

In the model proposed here, the network dynamics is governed by the interest rate feedback model (4.17). The underlying risk factor x_t follows the extended Ornstein-Uhlenbeck process (2.27), in which the reversion rate $c \in \mathbb{R}$ is allowed to be negative and thus to describe the unstable behavior of credit bubbles and crashes:

$$\begin{aligned}
&dx_t = c(b - x_t)dt + \sigma dW_t, \\
&cs_j(\mathbf{E}_t) = \max\left[\gamma_j\left(1 - \frac{\min\left[E_j^+(t),\ E_j(0)\right]}{E_j(0)}\right)\left(1 - \beta_j\frac{\min\left[A_j^+(t),\ A_j(0)/\beta_j\right]}{A_j(0)}\right), 0\right], \qquad (6.3)\\
&r(\mathbf{E}_t) = \max\left[\widetilde{\gamma}\left(1 - \frac{\min\left\{\|\mathbf{E}^+(t)\|_1, \|\mathbf{E}(0)\|_1\right\}}{\|\mathbf{E}(0)\|_1}\right)\left(1 - \widetilde{\beta}\frac{\min\left\{\|\mathbf{A}^+(t)\|_1, \widetilde{\beta}^{-1}\|\mathbf{A}(0)\|_1\right\}}{\|\mathbf{A}(0)\|_1}\right), 0\right].
\end{aligned}$$

6.2 Results

In this section, the comprehensive systemic risk model is evaluated in the context of its influence on assessing the security of the American financial system. The dataset, described in detail in Section 4.3.1, encompasses the top 15 American banks by assets excluding Charles Schwab, and five prominent mutual funds. The system is represented by the reduced-form stochastic network in Eq. (6.2), with dynamics outlined in Eq. (6.3). As presented in Section 5.4, this process can be modeled using the mapping Ψ (Def. 5.8) applied to the underlying random noise x_t. Furthermore, Theorem 5.10 and Lemma 5.11 enable the estimation of the infimum of the stochastic network process over the given timeframe, using the value of Ψ at the supremum of the risk x_t during this period. The distribution of the supremum determines the points at which the system is evaluated with the algorithm (3.7). This approach, constructed in Chapter 5, effectively reduces the complex problem of stochastic network analysis of the convoluted sequence $(a_k)_{k\in\mathbb{N}}$ to the already existing and well-established method of computing Φ^k [44–46], by the use of the mapping Ψ. VaR and ruin probability, introduced in Def. 5.6 and 5.7 are utilized as risk metrics for the evaluated financial network. These definitions are based on the loss function L_t, which quantitatively describes the unfavorable behavior of the system over the specified timeframe. It is defined as the fraction of defaulted institutions, which can be formally expressed as

$$L_t = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}_{\{z<0\}}\left(\Psi_i\left(\sup_{s\le t} x_t\right)\right). \qquad (6.4)$$

The results of the calculations are illustrated in Figure 6.1. The impact of the network model without rates feedback appears to be minimal compared to the straightforward model of direct defaults. While there occur some increases relative to the direct defaults model, these are largely reproduced by neighboring evaluation points of the latter. A notable discrepancy between the two models is observed in the VaR for positive values of the rate reversion parameter c. However, based on these findings, it seems unreasonable to conclude that the introduction of the network model without the interest feedback effect significantly affects the risk metrics.

Conversely, the influence of rates feedback incorporation on the results is substantial, even for the Ornstein-Uhlenbeck dynamics with a positive reversion rate as the foundational

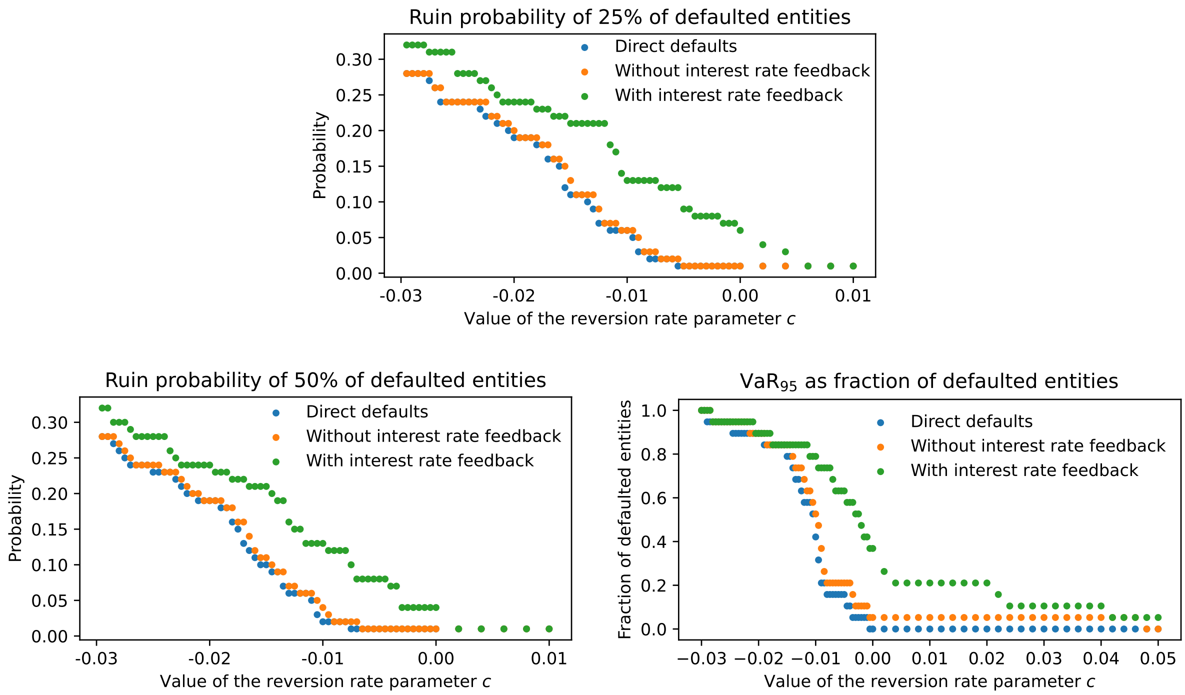


Figure 6.1: Risk metrics of comprehensive systemic risk model (6.3) and its components. The plots illustrate the influence of the reversion parameter c in the model (6.3), depicted on the x axes, on the probabilities of ruin at the level of 25% (top left panel) and 50% of defaulted institutions (top right), as well as the Value at Risk at the level of 95% (bottom). The blue color marks the model of defaults coming directly from the shock, orange - the network model without the interest feedback effect (r_t in Eq. (6.3) equal to zero), and green - the entire comprehensive model with the feedback effect (6.3). The computations has been performed with parameter values of $b = x_0 = \ln(0.03)$, $\sigma = b/100$ and $\beta = \widetilde{\beta} = 0.5, \gamma = 1, \widetilde{\gamma} = 0.05$.

risk process. The probabilities of ruin escalate from virtually nonexistent levels to ranges of 4% and 5% at $c = 0$. This effect is even more pronounced for the Value at Risk, which increases from 5% to 37% of the institutions that default in the system at $c = 0$.

The underlying risk process with a negative reversion rate ($c < 0$) exhibits a strong influence, pushing the maximum levels of the examined indices close to 0.3 for probabilities and 1 for VaR, even within the direct defaults model. Nevertheless, the disparity between the interest feedback model and the others remains significant. For Value at Risk, this difference reaches 58% of the entities in the financial system under examination at $c = -0.008$, although it diminishes consequently thereafter. The difference in the probability of ruin reaches 11-14 percentage points, but remains more persistent for more negative values of c, stabilizing around 4-5 percentage points near the end of the examined range at $c = -0.03$.

6.3 Model analysis in the context of real crisis

The emergence of the coronavirus pandemic in early 2020 triggered unprecedented actions by governments and central banks worldwide, resulting in significant interest rate decreases (Fig. 6.2a) and unparalleled economic aid packages. The fundamental interest rate in the American financial system, Secured Overnight Financing Rate, plummeted from 1.64% on 3 March 2020, to 0.01% by 24 March. Meanwhile, the CARES Act emerged as the largest stimulus initiative in U.S. history, accounting for 10% of the nation's GDP [156]. Those events significantly affected the financial conditions of the entire country. The credit default swap (CDS) premium for the United States, which reflects the cost of insurance against default, surged from 15.4bps on 3 February 2020 to 22.14bps on 16 March (Fig. 6.2c). The influx of funding and the relaxation of lending conditions, combined with pandemic-related restrictions, led to a sharp increase in the money supply within the financial system. The monetary base rose by 45.7%, from \$3.5 trillion in January to \$5.1 trillion in May 2020, and by 82.9% up to December 2021, reaching \$6.4 trillion (Fig. 6.2b). This dramatic surge was followed by a significant rise in inflation (Fig. 6.2b). Although it initially decreased from 2.5% in January 2020 to 0.1% in May due to the initial economic shutdown, it began to trend upwards thereafter. In response, the Federal Reserve was forced to raise interest rates preserve financial stability and value of the national currency. This series of rate increases commenced in March 2022 (Fig. 6.2a), when inflation reached 8.5%, peaking at 9.1% before beginning a steady decline.

While banks generally held conservative portfolios, primarily consisting of mortgage-backed securities and U.S. Treasuries—assets typically viewed as stable until maturity—they remained highly susceptible to interest rate fluctuations [157]. Such changes can significantly affect the present value of future cash flows from these assets. Consequently, the rise in interest rates led to a sharp decline in the market values of these securities, diminishing the capital reserves of banks that held them. To maintain liquidity, some banks had to sell their assets, incurring substantial losses [157]. In a matter of days, from March 8 to March 12 2023, Silvergate Bank, Silicon Valley Bank, and Signature Bank collapsed. Following these events, intense scrutiny and pressure were applied to First Republic Bank (FRB), which shares plummeted by 62% on 13 March [158, 159]. Facing substantial liquidity issues, FRB received a help package of \$30 billion from major U.S. banks and a \$70 billion from JPMorgan Chase [160]. Its credit rating was downgraded by Moody's on 14 March, along with several other regional banks and the outlook on the U.S banking system as a whole [161]. Consequently, SOFR increased by 3 basis points to

4.58% on 15 March, up from a relatively stable level of 4.55% (Fig. 6.2a). It indicated heightened market uncertainty due to financial instability, leading banks to demand higher credit prices in response to perceived risks.

As the United States plays a crucial role in the global financial system, established on the principles of the Bretton Woods Agreement, any disturbances tend to reverberate worldwide, affecting foreign institutions that issue instruments traded on American stock markets and held by U.S. financial firms. On 19 March, the downgraded FRB's credit rating to junk status by S&P Global came along with the acquisition of Credit Suisse, Switzerland's second largest bank, in an emergency arrangement facilitated by the Swiss government to prevent its collapse [162, 163]. On 23 March, the Federal Reserve System, tasked with maintaining the stable value of the currency, opted to raise federal rates, causing SOFR to rise from 4.55% to 4.8% (Fig. 6.2a). This increase in borrowing costs triggered another wave of instability in the global financial system. On 24 March the market saw a spike in insurance costs against the potential failure of Deutsche Bank, the largest bank in Europe, with the price of the 5-year credit default swap for its debt soaring by 70% [164]. International banking indices experienced significant declines simultaneously [165]. On the next trading day, 27 March, financial systems entered a phase of chaotic fluctuations, with SOFR sharply oscillating and reaching peaks of 4.84% and 4.87% on 28 and 31 March, respectively (Fig. 6.2a). The latter spike marked a notable deviation from the previous plateau, which had been primarily around 4.80-4.81%. This deviation signals an alarming scenario, underscoring the substantial challenges faced by the central bank in maintaining stability within the banking system while adhering to targeted interest rate levels.

On 25 April, First Republic Bank (FRB) reported a drop of over $100 billion in deposits, leading to a further decline in its share price [166]. This event triggered massive spillovers that threatened the solvency of the United States and shook the very foundations of the global financial system. The premium of the U.S. credit default swap rose from 32.26bps on 25 April to 45.68bps the following day, reaching 58.92bps by 28 April (Fig. 6.2c), representing a relative increase of 83% over the course of three days. On 1 May, First Republic Bank collapsed, with its assets sold to JPMorgan for $10.6 billion [167]. This was followed by a further increase in the U.S. CDS to 63.44bps on 2 May. It peaked at 69.76bps on 19 May, more than double the value just before the publication of the FRB quarterly report and over four and a half times higher than its pre-pandemic level. After that, the situation stabilized, with the CDS spread returning to levels seen before the FRB report announcement. Occurring within a span of less than two months, the collapses of First Republic Bank, Silicon Valley Bank, and Signature Bank represented the second, third, and fourth largest bank failures in U.S. history [168].

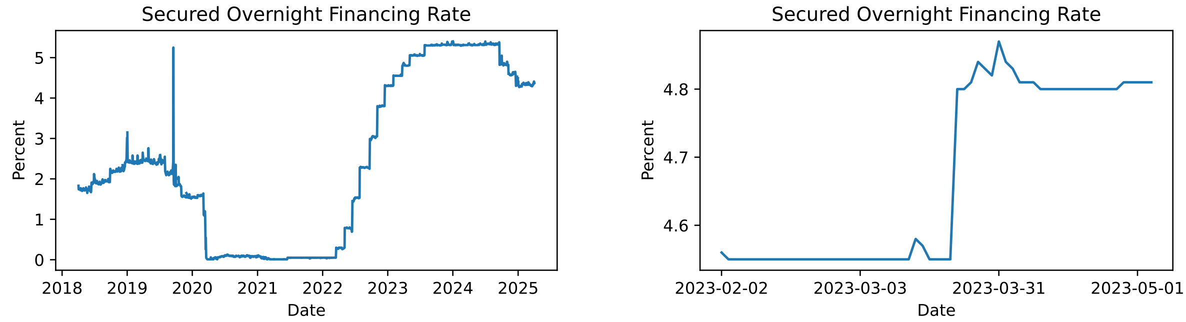
The comprehensive model has been constructed in this chapter as a dynamic, stochastic network representation of the financial system, designed to capture the time-sensitive nature of risk embedded in and amplified by its complex environment. In the model specification (6.3), the dynamics of valuation within network interactions is represented by the credit risk cs and the interest rate r components. The first one captures the fact that default risk increases in response to deteriorating balance sheet conditions of the affected entity, while the latter reflects the process of general lending cost determination in the interbank market [50]. On the other hand, the rate process x_t included in the model (6.3) captures the formation of bubbles and crises. As shown in Chapter 1, this process is based on several assumptions about credit market behavior, including credit creation, defaults, interest rate formation, and the distribution of income and wealth. Therefore, while the model does not directly account for central bank actions or inflation - the key driver of rising interest rate policies - it is rooted in fundamental financial aspects that are closely related to these phenomena, particularly credit creation. In fact, the determination of basic interest rates levels in the financial system remains a fundamental monetary policy instrument used by central banks to realize their basic mandate: the stability of prices. By determining the rates, central banks influence the scale of credit creation process, thereby affecting the money supply and, consequently, price levels.

In the model of x_t, significant decreases in interest rates (the bubble regime) is related to a substantial increase in the money supply within the financial system, which is associated with inflation. This situation corresponds to the low interest rate policies implemented during the pandemic, which were further amplified by government spending aimed at sustaining the economy while production contracted due to restrictions. The relationship between the model and inflation is described in detail in Chapter 1. When c is negative, the expected values of interest rates can diverge towards zero or infinity (as the logarithm of interest rates is modeled), enabling the trajectories to conveniently exhibit sharp, persistent movements in interest rates in the direction opposite to the equilibrium level (Fig. 6.2d) This behavior contrasts significantly with the conventional Ornstein-Uhlenbeck process, where trajectories tend to stabilize around a mean value rather than diverging from it. The degree of negativity of the parameter c indicates the rate at which bubbles and crises can intensify, reflecting the risk level of the system under examination. As such, strongly negative c aligns with aggressive money creation policies during the pandemic and the subsequent sharp contraction caused by excessive money growth and central bank interventions to control inflation. The transition between regimes is captured in the model by a random noise factor that allows the trajectories to pass through the equilibrium point b and therefore switch between the divergence to plus and minus infinity. Therefore, the constructed model incorporates phenomena fundamental to the banking crisis of 2023.

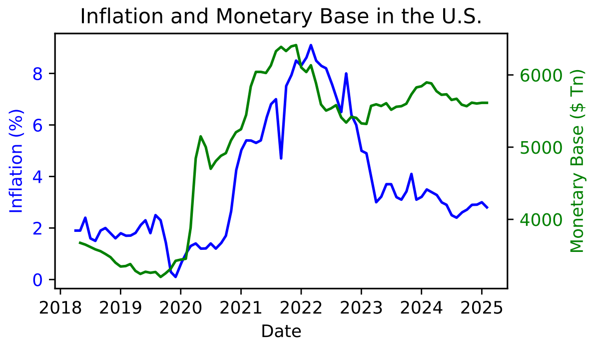
The influence of these effects on the stability of the financial system is evaluated through risk metrics, as discussed in detail in the previous section (Fig. 6.1). As outlined in Section 6.2, the extension of the Ornstein-Uhlenbeck process to include negative reversion, along with the incorporation of the SOFR formation process within the banking network, significantly impacts the results of the calculations. Therefore, the application of the comprehensive model can enable to detect substantial risks that may be overlooked by standard approaches. The results suggest that the range of losses that could occur appears more probable when using the comprehensive model, as opposed to scenarios where either the benchmark models or individual components of the comprehensive model are analyzed in isolation. For instance, when relying on either a non-network model or a network model without rates feedback (represented by the blue and orange lines), the perceived threat may seem less likely to materialize. Although the Value at Risk for these models increases upon entering the negative reversion model, this effect is not significant for ruin probabilities, which remain relatively stable up to a threshold of $c = -0.01$. A similar situation arises with the rate feedback model that does not incorporate negative reversion (illustrated by the green line in the region where $c \geq 0$). The effects observed in terms of VaR and the probability of ruin are noticeable in this range compared to other models considered, but are evidently limited when examining the area where $c < 0$. Consequently, these results suggest that severe crises may appear less likely when using any of the analyzed models instead of the comprehensive one. Therefore, employing the latter could have enabled the banks that defaulted during the 2023 crisis to better detect risks. As a result, they might have adjusted their financial strategies to prepare for such scenarios and potentially avoided defaults.

6.4 Conclusions

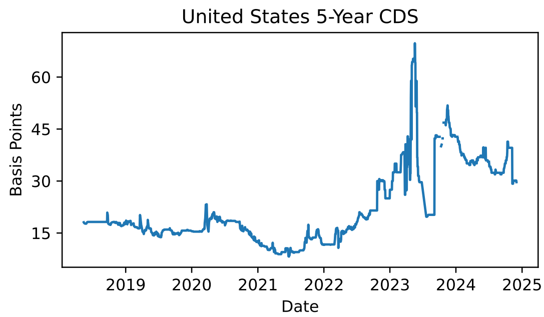
Particular components of the comprehensive model (6.3), namely the rates feedback loop in the financial network (4.17) and the Ornstein-Uhlenbeck process with negative reversion (2.27), significantly influenced the calculation of metrics characterizing the stability of the United States financial system. However, it is the integration of various factors that is often crucial in the discipline of systemic risk, leading to shock amplification and possible collapse [3]. Various aspects of systemic risk that has been of substantial importance for the 2023 banking crisis in the United States are holistically captured in the synthesized model (6.3). The contagion arises from amplification mechanisms stemming from interactions between its components. Neither those components nor other methods considered in isolation reveal risks that the comprehensive model does, as shown by the results of the risk metrics calculations presented in Figure 6.1. Therefore, its application can potentially prevent bankruptcies, and even the occurrence of a financial crisis as a consequence.



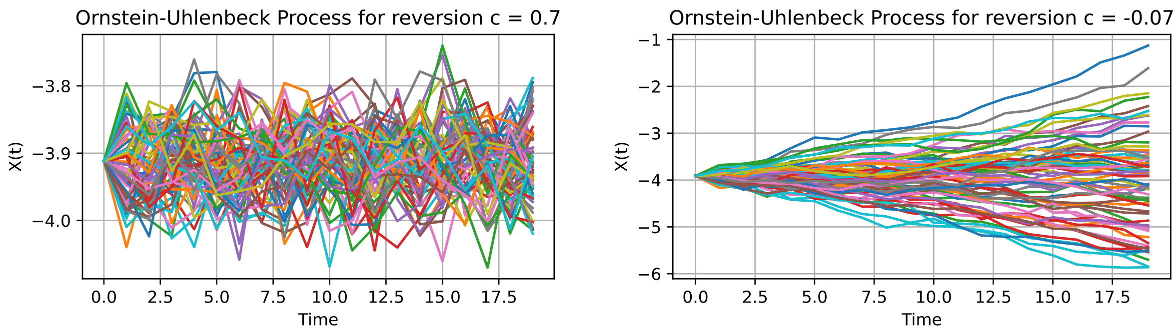
(a) The evolution of the Secured Overnight Financing Rate over time. The left plot shows the rates decrease at the beginning of pandemic in 2020, as well as a series of increases that started in March 2020 to tame escalating inflation. The right plot shows significant turbulences in SOFR during the series of American banks collapses and resulting following internation spillovers.



(b) Inflation and monetary base in the United States. The plot shows the substantial increase of the monetary base caused by rates decrease in pandemic and followed by significant rise of inflation.



(c) The CDS premium of U.S. The plot shows a significant increase in the risk of the United States default during the pandemic and a massive spillover related to collapses of American banks in 2023.



(d) The Ornstein-Uhlenbeck process with positive (left) and negative (right) reversion. The process with positive reversion fluctuates around the fixed point, while the negative reversion paths tends to sharply diverge from it.

Figure 6.2: The American financial system over time. This figure shows complex relationships between key components such as SOFR, monetary base, inflation, and national CDS spread. The dynamics of the U.S. banking system constitute a core for the functioning of the country that determines the global order. The extreme SOFR dynamics after the start of the pandemic (a) is hardly obtainable from the mean-centered standard Ornstein-Uhlenbeck process, but possible sharp movements in both directions corresponds well to the paths of the negative reversion process (d).

Summary

The goal of this thesis was to develop more realistic methods for systemic risk assessment that incorporate financial theory and the practice of debt valuation and solvency analysis. From a technical perspective, this was primarily achieved through the integration of reduced-form models with network dynamics. The way this integration is approached is often critical. In the models developed in Chapters 1 (Eq. (1.9)) and 2 (Eq. (2.27)), it allowed for the description of the crucial process of system transition into a crash, thereby facilitating the risk assessment of such an event within the dynamics of the credit market. Moreover, Theorems 2.1 and 2.3 illustrated how enormous risk can be hidden by arbitrarily long, apparently stable periods. In Chapter 3, the mathematical formalism of the NEVA framework, which is fundamental in a broad area of systemic risk research, was extended to assess model behavior in previously unaccounted situations. This includes points located in the interior of a financial system (Theorem 3.1), as well as entire areas (Theorem 3.3). In Chapter 4, the model incorporating relationships between the financial network, credit spreads, and interest rate dynamics was introduced (Definition 4.2 and Equation (4.17)), enabling to derive cascading failures that would otherwise be unattainable given the network structure of the real financial system. In Chapter 5, a complex model with stochastic fluctuations incorporated into the recursive network dynamics was constructed (Definitions 5.4 and 5.8) using a series of building blocks of mathematical formalism (Definition 5.5, Lemmas 5.1, 5.3, 5.4, 5.11, Theorems 5.5, 5.6, 5.7, 5.8, 5.10). This approach allows for a realistic representation of the risk processes associated with market evolution over time. In this model, shocks are not confined to occur only at arbitrarily chosen time step; rather, they can materialize throughout the entire process of system evolution. This flexibility allows for risk evaluation over any given time horizon, rather than solely on an instantaneous basis, which is crucial for portfolio management and reserve calculations. Ultimately, the comprehensive model presented in Chapter 6 synthesized all these results. The model specified in Equation (6.3) incorporates the stochastic dynamics of the fixed income market (Eq. (2.27)) alongside the reduced-form network model with interest rate feedback (Equation (4.17))) using continuous stochastic network framework (Definitions 5.4 and 5.8). Financial risk metrics were derived from this foundation, resulting in a holistic assessment of systemic risk. This synthesis demonstrated that it is the combination of various models developed throughout this thesis that identifies points of fragility that individual models fail to capture.

These results underscore the complexity inherent in the study of systemic risk, where the exclusion of certain factors can lead to significant deviations in risk assessments, posing a danger of risk misjudgment. Furthermore, they establish a foundation for further integration of essential industry processes of financial analysis and solvency assessment with systemic risk evaluation through the bond of reduced-form models, which represent a well-established industry standard for valuing fixed-income instruments [47,48,53]. In fact, practitioners are already employing more sophisticated models for the dynamics of recovery rates, credit spreads, and interest rates than those developed in this thesis [47, 53]. This indicates promising directions for future research, specifically the further incorporation of these advanced models into the reduced-form network modeling framework established in Chapters 4 and 5. Such an approach will enable a closer reproduction of the real dynamics of financial markets in network models, while simultaneously enhancing existing methods with critical aspects of modeling financial collapses.

Systemic risk research should be continued despite all the challenges related to modeling very rare events. Critical points, feedback loops, and complex systems constitute promising tools for studying this topic. Its significance has been clearly demonstrated by recent events, especially with the rise of systemic risk weaponization phenomenon [169]. Examples can be found all around the globe, from Taiwanese factories that constitute a critical node in semiconductor trade [170], through Eurasian gas pipelines igniting the Ukrainian fields, [171], to the digital banking infrastructure used by Western nations to attack their adversaries [172]. Understanding the dynamics and fragilities of the global financial system can enable to induce loss on a scale that far exceeds the attacker's resources, as demonstrated by the actions of Houthi militias in the Red Sea [173]. It is essential to deepen our knowledge of these phenomena, as a stable and secure financial system serves as the backbone of modern economies, governments, and societies, which may struggle to function effectively without it. Continued exploration in this field is necessary to protect global security, as threats extend beyond economic repercussions to physical dangers through the interconnected dependencies that characterize systemic risk.

Bibliography

[1] Thomas Ilin and Liz Varga. The uncertainty of systemic risk. *Risk Management*, 17(4):240–275, November 2015.

[2] IMF, BIS, and FSB. Guidance to Assess the Systemic Importance of Financial Institutions, Markets and Instruments: Initial Considerations. October 2009.

[3] Jean-Pierre Zigrand. Systems and Systemic Risk in Finance and Economics. *Systemic Risk Centre*, January 2014.

[4] Marco Bardoscia, Stefano Battiston, Fabio Caccioli, and Guido Caldarelli. DebtRank: A microscopic foundation for shock propagation. *PLOS ONE*, 10(6):e0130406, June 2015.

[5] Moshe Levy, Haim Levy, and Sorin Solomon. Microscopic Simulation of the Stock Market: the Effect of Microscopic Diversity. *Journal de Physique I*, 5(8):1087–1107, August 1995.

[6] Sorin Solomon and Natasa Golo. Minsky Financial Instability, Interscale Feedback, Percolation and Marshall-Walras Disequilibrium. *Accounting, Economics and Law*, 3(3), January 2013.

[7] Marco Bardoscia, Stefano Battiston, Fabio Caccioli, and Guido Caldarelli. Pathways towards instability in financial networks. *Nature Communications*, 8(1):14416, February 2017.

[8] Rebekka Burkholz and Frank Schweitzer. Framework for cascade size calculations on random networks. *Physical Review E*, 97(4):042312, April 2018.

[9] Matthew Elliott, Benjamin Golub, and Matthew O. Jackson. Financial Networks and Contagion. *SSRN Electronic Journal*, 2012.

[10] Gerrit Draisma, Holger Drees, Ana Ferreira, and Laurens De Haan. Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli*, 10(2), April 2004.

[11] Barbel Finkenstadt and Holger Rootzen, editors. *Extreme Values in Finance, Telecommunications, and the Environment*. Chapman and Hall/CRC, July 2003.

[12] José Moran and Jean-Philippe Bouchaud. May's instability in large economies. *Physical Review E*, 100(3):032307, September 2019.

[13] Davide Valenti, Giorgio Fazio, and Bernardo Spagnolo. Stabilizing effect of volatility in financial markets. *Physical Review E*, 97(6):062307, June 2018.

[14] Matthew O. Jackson and Agathe Pernoud. Systemic Risk in Financial Networks: A Survey. *Annual Review of Economics*, 13(1):171–202, August 2021.

[15] Douglas W. Diamond and Philip H. Dybvig. Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy*, 91(3):401–419, June 1983.

[16] Alessandro Doldi and Marco Frittelli. Conditional Systemic Risk Measures. *SIAM Journal on Financial Mathematics*, 12(4):1459–1507, January 2021.

[17] Nassim Nicholas Taleb. *The black swan: the impact of the highly improbable*. Random House, New York, 2007.

[18] Matthew Elliott, Benjamin Golub, and Matthew O. Jackson. Financial Networks and Contagion. *American Economic Review*, 104(10):3115–3153, October 2014.

[19] Amit Gill, Madegedara Lalith, Sebastian Poledna, Muneo Hori, Kohei Fujita, and Tsuyoshi Ichimura. High-Performance Computing Implementations of Agent-Based Economic Models for Realizing 1:1 Scale Simulations of Large Economies. *IEEE Transactions on Parallel and Distributed Systems*, 32(8):2101–2114, August 2021.

[20] Paolo Bartesaghi, Michele Benzi, Gian Paolo Clemente, Rosanna Grassi, and Ernesto Estrada. Risk-Dependent Centrality in Economic and Financial Networks. *SIAM Journal on Financial Mathematics*, 11(2):526–565, January 2020.

[21] Stefano Battiston, Michelangelo Puliga, Rahul Kaushik, Paolo Tasca, and Guido Caldarelli. DebtRank: Too Central to Fail? Financial Networks, the FED and Systemic Risk. *Scientific Reports*, 2(1):541, August 2012.

[22] Florian Bourgey, Emmanuel Gobet, and Clément Rey. Metamodel of a Large Credit Risk Portfolio in the Gaussian Copula Model. *SIAM Journal on Financial Mathematics*, 11(4):1098–1136, January 2020.

[23] Sebastian Poledna, Serafín Martínez-Jaramillo, Fabio Caccioli, and Stefan Thurner. Quantification of systemic risk from overlapping portfolios in the financial system. *Journal of Financial Stability*, 52:100808, February 2021.

[24] Anton Pichler, Sebastian Poledna, and Stefan Thurner. Systemic risk-efficient asset allocations: Minimization of systemic risk as a network optimization problem. *Journal of Financial Stability*, 52:100809, February 2021.

[25] Matt Leduc, Sebastian Poledna, and Stefan Thurner. Systemic risk management in financial networks with credit default swaps. *The Journal of Network Theory in Finance*, 2017.

[26] Zachary Feinstein and Thomas R. Hurd. Contingent Convertible Obligations and Financial Stability. *SIAM Journal on Financial Mathematics*, 14(1):158–187, March 2023.

[27] Claudia Ceci, Katia Colaneri, Rüdiger Frey, and Verena Köck. Value Adjustments and Dynamic Hedging of Reinsurance Counterparty Risk. *SIAM Journal on Financial Mathematics*, 11(3):788–814, January 2020.

[28] Sebastian Poledna, Abraham Hinteregger, and Stefan Thurner. Identifying Systemically Important Companies by Using the Credit Network of an Entire Nation. *Entropy*, 20(10):792, October 2018.

[29] Hyman P. Minsky. The Financial Instability Hypothesis: An Interpretation of Keynes and an Alternative to Standard Theory. *Challenge*, 20(1):20–27, 1977.

[30] Gabriele Tedeschi, Fabio Caccioli, and Maria Cristina Recchioni. Taming financial systemic risk: models, instruments and early warning indicators. *Journal of Economic Interaction and Coordination*, 15(1):1–7, January 2020.

[31] Ruggero Grilli, Giulia Iori, Niccolò Stamboglis, and Gabriele Tedeschi. A Networked Economy: A Survey on the Effect of Interaction in Credit Markets. In *Introduction to Agent-Based Economics*, pages 229–252. Elsevier, Berlin, August 2017.

[32] Yaniv Dover, Sonia Moulet, Sorin Solomon, and Gur Yaari. Do all economies grow equally fast? *Risk and Decision Analysis*, 1(3):171–185, 2009.

[33] Ofer Biham, Ofer Malcai, Moshe Levy, and Sorin Solomon. Generic emergence of power law distributions and Lévy-stable intermittent fluctuations in discrete logistic systems. *Physical Review E*, 58(2):1352–1358, August 1998.

[34] Gur Yaari, Andrzej Nowak, Kamil Rakocy, and Sorin Solomon. Microscopic study reveals the singular origins of growth. *The European Physical Journal B*, 62(4):505–513, April 2008.

[35] Piero Ferri and AnnaMaria Variato. Financial Fragility, the Minskian Triad, and Economic Dynamics. *International Journal of Political Economy*, 39(2):70–82, July 2010.

[36] Yuri Biondi. Hyman Minsky's Financial Instability Hypothesis and the Accounting Structure of Economy. *Accounting, Economics and Law*, 3(3), January 2013.

[37] Natasa Golo, David S. Bree, Guy Kelman, Leanne Usher, Marco Lamieri, and Sorin Solomon. Too dynamic to fail. Empirical support for an autocatalytic model of Minsky's financial instability hypothesis. *Journal of Economic Interaction and Coordination*, 11(2):247–271, October 2016.

[38] Steve Keen. Finance and Economic Breakdown: Modeling Minsky's "Financial Instability Hypothesis". *Journal of Post Keynesian Economics*, 17(4):607–635, July 1995.

[39] Maria Cristina Recchioni and Gabriele Tedeschi. From bond yield to macroeconomic instability: A parsimonious affine model. *European Journal of Operational Research*, 262(3):1116–1135, November 2017.

[40] Robert C. Merton. On the pricing of corporate debt: the risk structure of interest rates. *The Journal of Finance*, 29(2):449–470, May 1974.

[41] Craig Furfine. Interbank Exposures: Quantifying the Risk of Contagion. *SSRN Electronic Journal*, 1999.

[42] Larry Eisenberg and Thomas H. Noe. Systemic Risk in Financial Systems. *Management Science*, 47(2):236–249, February 2001.

[43] Marco Bardoscia, Stefano Battiston, Fabio Caccioli, and Guido Caldarelli. Correction: DebtRank: A Microscopic Foundation for Shock Propagation. *PLOS ONE*, 10(7):e0134888, July 2015.

[44] Paolo Barucca, Marco Bardoscia, Fabio Caccioli, Marco D'Errico, Gabriele Visentin, Guido Caldarelli, and Stefano Battiston. Network valuation in financial systems. *Mathematical Finance*, 30(4):1181–1204, October 2020.

[45] Tom Fischer. No-arbitrage pricing under systemic risk: accounting for cross-ownership. *Mathematical Finance*, 24(1):97–124, January 2014.

[46] Teruyoshi Suzuki. Valuing corporate debt : the effect of cross-holdings of stock and debt. *Journal of the Operations Research Society of Japan*, 45(2):123–144, 2002.

[47] Tomasz R. Bielecki and Marek Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[48] Steven Allen and Steven L. Allen. *Financial risk management*. Wiley Finance Series. Wiley, Hoboken, N.J, 2013.

[49] Robert A. Jarrow and Stuart M. Turnbull. Pricing Derivatives on Financial Securities Subject to Credit Risk. *The Journal of Finance*, 50(1):53–85, March 1995.

[50] Kamil Fortuna and Janusz Szwabiński. Role of debt valuation factors in systemic risk assessment. *Physical Review E*, 111(5), May 2025.

[51] Alan Roncoroni, Stefano Battiston, Marco D'Errico, Grzegorz Hałaj, and Christoffer Kok. Interconnected banks and systemically important exposures. *Journal of Economic Dynamics and Control*, 133:104266, December 2021.

[52] Jamey Keaten and Courtney Bonnel. Credit Suisse customers feel mix of anger, relief after sale. *Associated Press News*, March 2023.

[53] Bruce Tuckman and Angel Serrat. *Fixed income securities*. Wiley Finance Series. Wiley, Hoboken, New Jersey, 2022.

[54] Daron Acemoglu, Asuman Ozdaglar, James Siderius, and Alireza Tahbaz-Salehi. Systemic credit freezes in financial lending networks. *Mathematics and Financial Economics*, 15(1):185–232, January 2021.

[55] Anne-Caroline Hüser, Caterina Lepore, and Luitgard Anna Maria Veraart. How do secured funding markets behave under stress? Evidence from the gilt repo market. *Bank of England Working Papers*, 910, 2021.

[56] David Bowman, Chiara Scotti, and Cindy M. Vojtech. How Correlated is LIBOR with Bank Funding Costs? *FEDS Notes*, 2020(2539), June 2020.

[57] Bruce Tuckman. Short-Term Rate Benchmarks: The Post-LIBOR Regime. *Annual Review of Financial Economics*, 15(1):473–491, November 2023.

[58] Sven Klingler and Olav Syrstad. *Burying Libor.* Number 2019, 13 in Working papers of Norges Bank. Norges Bank, Oslo, 2019.

[59] Kamil Fortuna and Janusz Szwabiński. An unified framework for modeling credit cycles and systemic risk assessment. *Journal of Economic Interaction and Coordination*, January 2025.

[60] Marco Bardoscia, Fabio Caccioli, Juan Ignacio Perotti, Gianna Vivaldo, and Guido Caldarelli. Distress Propagation in Complex Networks: The Case of Non-Linear DebtRank. *PLOS ONE*, 11(10):e0163825, October 2016.

[61] Charles D. Brummitt and Teruyoshi Kobayashi. Cascades in multiplex financial networks with debts of different seniority. *Physical Review E*, 91(6):062813, June 2015.

[62] John Hull. *Options, futures, and other derivatives.* Pearson, New York, NY, 2018.

[63] John C. Cox, Jonathan E. Ingersoll, and Stephen A. Ross. A Theory of the Term Structure of Interest Rates. *Econometrica*, 53(2):385, March 1985.

[64] Martin Hohnisch, Sabine Pittnauer, Sorin Solomon, and Dietrich Stauffer. Socioeconomic interaction and swings in business confidence indicators. *Physica A: Statistical Mechanics and its Applications*, 345(3-4):646–656, January 2005.

[65] Sebastian Cantono and Sorin Solomon. When the collective acts on its components: economic crisis autocatalytic percolation. *New Journal of Physics*, 12(7):075038, July 2010.

[66] Yuri Biondi. The Firm as an Entity: Management, Organization, Accounting. *SSRN Electronic Journal*, 2005.

[67] Moshe Levy, Haim Levy, and Sorin Solomon. A microscopic model of the stock market. *Economics Letters*, 45(1):103–111, May 1994.

[68] Haim Levy, Moshe Levy, and Sorin Solomon. *Microscopic simulation of financial markets: from investor behavior to market phenomena.* Academic Press, San Diego, Calif., 2000.

[69] Hyman P. Minsky. The Financial Instability Hypothesis. *SSRN Electronic Journal*, 1999.

[70] Larry Randall Wray. Global Financial Crisis: A Minskyan Interpretation of the Causes, the Fed's Bailout, and the Future. *SSRN Electronic Journal*, 2012.

[71] Larry Randall Wray. What Do Banks Do? What Should Banks Do? *SSRN Electronic Journal*, 2010.

[72] Chester Phillips, Thomas McManus, and Richard Nelson. Banking and the Business Cycle: a study of the Great Depression in the United States. *International Affairs*, July 1938.

[73] John Hicks. Mr. Keynes and the "Classics"; A Suggested Interpretation. *Econometrica*, 5(2):147, April 1937.

[74] Joseph Benning. *Trading strategies for capital markets. Chapter 16, The credit cycle.* McGraw-Hill, New York, 2007.

[75] Hideaki Aoyama, Wataru Souma, Yuichi Nagahara, Mitsuhiro P. Okazaki, Hideki Takayasu, and Misako Takayasu. Pareto's law for income of individuals and debt of bankrupt companies. *Fractals*, 08(03):293–300, September 2000.

[76] Oren S. Klass, Ofer Biham, Moshe Levy, Ofer Malcai, and Sorin Solomon. The Forbes 400 and the Pareto wealth distribution. *Economics Letters*, 90(2):290–295, February 2006.

[77] Oded Galor. *Discrete Dynamical Systems.* Springer Berlin Heidelberg, Berlin, Heidelberg, 2007.

[78] Steven Strogatz. *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering.* Studies in nonlinearity. Westview Press, Cambridge, Mass, repr. edition, 2007.

[79] Dan Fuller and Doris Geide-Stevenson. Consensus Among Economists: Revisited. *The Journal of Economic Education*, 34(4):369–387, January 2003.

[80] Dan Fuller and Doris Geide-Stevenson. Consensus Among Economists—An Update. *The Journal of Economic Education*, 45(2):131–146, April 2014.

[81] Milton Friedman. Quantity Theory of Money. In John Eatwell, Murray Milgate, and Peter Newman, editors, *Money*, pages 1–40. Palgrave Macmillan UK, London, 1989.

[82] William Stanley Jevons. Money and the mechanism of exchange. In *General Equilibrium Models of Monetary Economies*, pages 55–65. Elsevier, Berlin, Heidelberg, 1989.

[83] Michael McLeay and Amar Radia. Money creation in the modern economy. *Bank of England Quarterly Bulletin*, 2014.

[84] Harry Markowitz. Portfolio Selection. *The Journal of Finance*, 7(1):77, March 1952.

[85] Edward J. Kane. What Lessons Should Japan Learn from the U.S. Deposit-Insurance Mess? *Journal of the Japanese and International Economies*, 7(4):329–355, December 1993.

[86] Müge Adalet McGowan, Dan Andrews, and Valentine Millot. The walking dead? Zombie firms and productivity performance in OECD countries. *Economic Policy*, 33(96):685–736, October 2018.

[87] Milton Friedman and Anna Jacobson Schwartz. *A monetary history of the United States 1867-1960.* Number 12 in Princeton paperbacks. Princeton University Press, Princeton, 9th pbk. printing, 22nd printing edition, 2008.

[88] John Maynard Keynes. *The General Theory of Employment, Interest, and Money.* Springer International Publishing, Cham, 2018.

[89] Paul R. Krugman and Robin Wells. *Macroeconomics.* Worth Publishers, New York, NY, 2009.

[90] N. Gregory Mankiw. *Principles of macroeconomics.* South-Western, Cengage Learning, Mason, OH, 2009.

[91] Michael D. Bordo. Equation of Exchange. In Palgrave Macmillan, editor, *The New Palgrave Dictionary of Economics*, pages 1–5. Palgrave Macmillan UK, London, 2008.

[92] Irving Fisher. *The nature of capital and income.* Martino Pub, Mansfield Center, 2009.

[93] N. Gregory Mankiw. *Brief principles of macroeconomics.* South-Western Cengage Learning, Mason, OH, 2008.

[94] Ben Bernanke, Mark Gertler, and Simon Gilchrist. The Financial Accelerator and the Flight to Quality. *The Review of Economics and Statistics*, 78(1):1, February 1996.

[95] Nobuhiro Kiyotaki and John Moore. Credit Cycles. *Journal of Political Economy*, 105(2):211–248, April 1997.

[96] Arvind Krishnamurthy. Amplification Mechanisms in Liquidity Crises. *American Economic Journal: Macroeconomics*, 2(3):1–30, July 2010.

[97] Steve H Hanke and Alex K F Kwok. On the Measurement of Zimbabwe's Hyperinflation. *Cato Journal*, 29(2), June 2009.

[98] Carlo Altavilla, Lorenzo Burlon, Mariassunta Giannetti, and Sarah Holton. Is there a zero lower bound? The effects of negative policy rates on banks and firms. *Journal of Financial Economics*, 144(3):885–907, June 2022.

[99] Olivier Blanchard. Public Debt and Low Interest Rates. *American Economic Review*, 109(4):1197–1229, April 2019.

[100] Alan S. Blinder. Revisiting Monetary Policy in a Low-Inflation and Low-Utilization Environment. *Journal of Money, Credit and Banking*, 44:141–146, February 2012.

[101] Florian Heider, Farzad Saidi, and Glenn Schepens. Life below Zero: Bank Lending under Negative Policy Rates. *The Review of Financial Studies*, 32(10):3728–3761, October 2019.

[102] Eduardo Levy Yeyati and Federico Sturzenegger. Monetary and Exchange Rate Policies. In *Handbook of Development Economics*, volume 5, pages 4215–4281. Elsevier, Berlin, Heidelberg, 2010.

[103] Hollis Burnley Chenery, T. N. Srinivasan, Jere R. Behrman, Dani Rodrik, and Mark R. Rosenzweig. *Handbook of development economics.* Number 9 in Handbooks in economics. North-Holland; Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co, Amsterdam; New York: New York, N.Y., U.S.A, 1988.

[104] Neil J. Smelser and Paul B. Baltes. *International encyclopedia of the social & behavioral sciences.* Elsevier, Amsterdam ; New York, 1st ed edition, 2001.

[105] Ben S. Bernanke. The New Tools of Monetary Policy. *American Economic Review*, 110(4):943–983, April 2020.

[106] Peter Grundke and André Kühn. The impact of the Basel III liquidity ratios on banks: Evidence from a simulation study. *The Quarterly Review of Economics and Finance*, 75:167–190, February 2020.

[107] BCBS. The Basel Framework. *https://www.bis.org/basel_framework/*, January 2023.

[108] Hubert J. Kiss, Ismael Rodriguez-Lara, and Alfonso Rosa-Garcia. Preventing (panic) bank runs. *Journal of Behavioral and Experimental Finance*, 35:100697, September 2022.

[109] Mete Bumin and Mehmet Ozcalici. Predicting the direction of financial dollarization movement with genetic algorithm and machine learning algorithms: The case of Turkey. *Expert Systems with Applications*, 213:119301, March 2023.

[110] Louis-Philippe Rochon and Mario Seccareccia. *Dollarization: lessons from Europe and the Americas*. Number 22 in Routledge international studies in money and banking. Routledge, London; New York, 2003. OCLC: ocm51172267.

[111] Daragh McInerney and Tomasz Zastawniak. *Stochastic Interest Rates*. Mastering Mathematical Finance. Cambridge University Press, Cambridge, 2015.

[112] Charles Moyer and Charles Moyer, editors. *Contemporary financial management*. South-Western, Cengage Learning, Mason, 2012.

[113] Oliver C. Ibe. Diffusion Processes. In *Markov Processes for Stochastic Modeling*, pages 295–327. Elsevier, 2013.

[114] Richard Durrett. *Probability: theory and examples*. Number 31 in Cambridge series in statistical and probabilistic mathematics. Cambridge Univ. Press, Cambridge, 4. ed edition, 2010.

[115] Samuel A. Broverman. *Mathematics of investment and credit*. ACTEX academic series. ACTEX Publ., Inc, Winsted, 2010.

[116] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, November 1977.

[117] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*. Number 113 in Graduate texts in mathematics. Springer, New York, 2019.

[118] Simon F. Nørrelykke and Henrik Flyvbjerg. Harmonic oscillator in heat bath: Exact simulation of time-lapse-recorded data and exact analytical benchmark statistics. *Physical Review E*, 83(4):041103, April 2011.

[119] Emilia P. Martins. Estimating the Rate of Phenotypic Evolution from Comparative Data. *The American Naturalist*, 144(2):193–209, August 1994.

[120] Bernt Karsten Øksendal. *Stochastic differential equations: an introduction with applications*. Universitext. Springer, Berlin, 2005.

[121] Fischer Black and Piotr Karasinski. Bond and Option Pricing when Short Rates are Lognormal. *Financial Analysts Journal*, 47(4):52–59, July 1991.

[122] Francis A. Longstaff and Eduardo S. Schwartz. Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model. *The Journal of Finance*, 47(4):1259–1282, September 1992.

[123] Lin Chen. *Interest rate dynamics, derivatives pricing, and risk management.* Number 435 in Lecture notes in economics and mathematical systems. Springer, Berlin ; New York, 1996.

[124] Olivier Blanchard and David R. Johnson. *Macroeconomics.* Always learning. Pearson, Boston Munich, 2013.

[125] Michel Crouhy, Dan Galai, and Robert Mark. *Risk management.* McGraw-Hill, New York, 2001.

[126] Joseph E. Stiglitz and Jay K. Rosengard. *Economics of the public sector.* W. W. Norton & Company, New York, 2015.

[127] Marco Bardoscia, Paolo Barucca, Stefano Battiston, Fabio Caccioli, Giulio Cimini, Diego Garlaschelli, Fabio Saracco, Tiziano Squartini, and Guido Caldarelli. The physics of financial networks. *Nature Reviews Physics*, 3(7):490–507, June 2021.

[128] European Central Bank. Interconnected banks and systemically important exposures. 2019.

[129] David Heath, Robert Jarrow, and Andrew Morton. Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60(1):77, January 1992.

[130] Oldrich A Vasicek. *Finance, Economics and Mathematics.* Wiley, 1 edition, December 2015.

[131] Mark T. Williams. Uncontrolled risk: the lessons of Lehman Brothers and how systemic risk can still bring down the world financial system. *McGraw-Hill*, 2010.

[132] Jerry M. Rosenberg. *The concise encyclopedia of the Great Recession.* Scarecrow Press, Lanham, Md., 2012.

[133] Sadamori Kojaku, Giulio Cimini, Guido Caldarelli, and Naoki Masuda. Structural changes in the interbank market across the financial crisis from multiple core-periphery analysis. *The Journal of Network Theory in Finance*, 4(3):33–51, 2018.

[134] Agostino Capponi, Felix Corell, and Joseph E. Stiglitz. Optimal bailouts and the doom loop with a financial network. *Journal of Monetary Economics*, 128:35–50, May 2022.

[135] Luka Klinčić, Vinko Zlatić, Guido Caldarelli, and Hrvoje Štefančić. Systemic risk measured by the resiliency of the system to initial shocks. *Physical Review E*, 108(4):044303, October 2023.

[136] Rebekka Burkholz, Antonios Garas, and Frank Schweitzer. How damage diversification can reduce systemic risk. *Physical Review E*, 93(4):042313, April 2016.

[137] Giulio Cimini, Tiziano Squartini, Andrea Gabrielli, and Diego Garlaschelli. Estimating topological properties of weighted networks from limited information. *Physical Review E*, 92(4):040802, October 2015.

[138] Tiziano Squartini, Assaf Almog, Guido Caldarelli, Iman Van Lelyveld, Diego Garlaschelli, and Giulio Cimini. Enhanced capital-asset pricing model for the reconstruction of bipartite financial networks. *Physical Review E*, 96(3):032315, September 2017.

[139] Prosenjit Kundu, Hiroshi Kori, and Naoki Masuda. Accuracy of a one-dimensional reduction of dynamical systems on networks. *Physical Review E*, 105(2):024305, February 2022.

[140] Jakob J. Kolb, Finn Müller-Hansen, Jürgen Kurths, and Jobst Heitzig. Macroscopic approximation methods for the analysis of adaptive networked agent-based models: Example of a two-sector investment model. *Physical Review E*, 102(4):042311, October 2020.

[141] Donald Resseguie and Barry S. Zisman. *Banks and thrifts: government enforcement and receivership.* LexisNexis, New York, NY, 2020.

[142] Silvia Angelina and Nugi Mohammad Nugraha. Effects of Monetary Policy on Inflation and National Economy Based on Analysis of Bank Indonesia Annual Report. *Technium Social Sciences Journal*, 10:423–435, July 2020.

[143] Mathieu J. P. Poirier. Systematic comparison of household income, consumption, and assets to measure health inequalities in low- and middle-income countries. *Scientific Reports*, 14(1):3851, February 2024.

[144] Robson Bruniera De Oliveira, Felipe Alves Rubio, Rodrigo Anderle, Mauro Sanchez, Luis Eugenio De Souza, James Macinko, Ines Dourado, and Davide Rasella. Incorporating social determinants of health into the mathematical modeling of HIV/AIDS. *Scientific Reports*, 12(1):20541, November 2022.

[145] Federal Reserve Bank of New York. Secured Overnight Financing Rate Data.

[146] Puriya Abbassi, Falk Bräuning, Falko Fecht, and José-Luis Peydró. Cross-border interbank liquidity, crises, and monetary policy. *Journal of International Economics*, 139:103657, November 2022.

[147] Tom Fischer. A primer on reflexivity and price dynamics under systemic risk, January 2013. arXiv:1301.6415 [q-fin].

[148] Steven G. Krantz. *Real analysis and foundations.* Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, Fla., 2005.

[149] Pierre Devolder and Adrien Lebègue. Risk measures versus ruin theory for thecalculation of solvency capital for long-termlife insurances. *Dependence Modeling*, 4(1):000010151520160018, December 2016.

[150] David C. M. Dickson. *Insurance risk and ruin.* International series on actuarial science. Cambridge University Press, Cambridge, 2017.

[151] Philippe Jorion. *Value at risk: the new benchmark for managing financial risk.* McGraw-Hill, New York, 2007.

[152] Kevin Dowd. *Measuring market risk.* John Wiley & Sons, Chichester, England Hoboken, NJ, 2005.

[153] Søren Asmussen and Hansjörg Albrecher. *Ruin Probabilities.* Advanced Series on Statistical Science & Applied Probability. World Scientific, September 2010.

[154] Hans U. Gerber and Elias S.W. Shiu. On the Time Value of Ruin. *North American Actuarial Journal*, 2(1):48–72, January 1998.

[155] Koen P P Munniksma. Credit Risk Measurement under Basel II.

[156] Sandhya Kambhampati. The coronavirus stimulus package versus the Recovery Act. *The Los Angeles Times*, March 2020.

[157] Telis Demos. Silvergate's Story Is About Fundamentals, Not Just Crypto. *The Wall Street Journal*, March 2023.

[158] Hugh Son. Why regulators seized Signature Bank in third-biggest bank failure in U.S. history. *CNBC*, March 2023.

[159] Jennifer Hughes. First Republic and other US regional banks tumble on fears of deposit flight. *Financial Times*, March 2023.

[160] Pete Schroeder. Major US banks inject $30 billion to rescue First Republic Bank. *Reuters*, March 2023.

[161] Chris Price. Entire US banking system downgraded after Silicon Valley Bank chaos. *The Daily Telegraph*, March 2023.

[162] Mike Segar. S&P cuts First Republic deeper into junk, says $30 billion infusion may not solve problems. *CNBC*, March 2023.

[163] Katrina Bishop and Ashley Capoot. UBS buys Credit Suisse for $3.2 billion as regulators look to shore up the global banking system, March 2023.

[164] Martin Baccardax. Deutsche Bank Tumbles, Default Protection Costs Spike as U.S. Bank Crisis Spreads to Europe. *The Street*, March 2023.

[165] Anna Isaac. Deutsche Bank leads sell-off in European bank shares. *The Guardian*, March 2023.

[166] Mehnaz Yasmin. First Republic Bank deposits tumble more than $100 billion as it explores options. *Reuters*, April 2023.

[167] Scott Murdoch. JPMorgan lands as winner in U.S. regulator auction of First Republic. *Reuters*, May 2023.

[168] Rachel Ensign. First Republic Bank Is Seized, Sold to JPMorgan in Second-Largest U.S. Bank Failure. *The Wall Street Journal*, May 2023.

[169] Lucia Quaglia and Amy Verdun. Weaponisation of finance: the role of European central banks and financial sanctions against Russia. *West European Politics*, 46(5):872–895, July 2023.

[170] Celian Colon and Stefan Hochrainer-Stigler. Systemic risks in supply chains: a need for system-level governance. *Supply Chain Management: An International Journal*, 28(4):682–694, April 2023.

[171] En Zhou and Xinyu Wang. Dynamics of systemic risk in European gas and oil markets under the Russia–Ukraine conflict: A quantile regression neural network approach. *Energy Reports*, 9:3956–3966, December 2023.

[172] Anum Qureshi, Muhammad Suhail Rizwan, Ghufran Ahmad, and Dawood Ashraf. Russia–Ukraine war and systemic risk: Who is taking the heat? *Finance Research Letters*, 48:103036, August 2022.

[173] Hadad Elroi, Davinder Malhotra, and Evangelos Vasileiou. Risk Spillovers and Optimal Hedging in Commodity ETFs: A TVP-VAR Approach. *Finance Research Letters*, October 2024.