Report on the Ph.D. thesis

DISCRETE FEYNMAN-KAC EVOLUTIONS

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The proposed Ph.D. thesis lies in the area of probabilistic potential theory for Markov chains. The thesis develops and studies a theory of discrete-time Feynman–Kac semigroups with general confining potentials for Markov chains defined on countably infinite discrete state spaces.

More precisely, let $(Y_n)_{n\geq 0}$ be a discrete-time Markov chain on a countable space X with transition kernel P, and let $V : X \to (0, \infty)$ be a potential function. The associated discrete-time Feynman–Kac semigroup $(\mathcal{U}_n)_{n\geq 0}$ is defined by

$$\mathcal{U}_0 f = f, \qquad \mathcal{U}_n f(x) = \mathbb{E}^x \left[\prod_{k=0}^{n-1} \frac{1}{V(Y_k)} f(X_k \right], \ n \geq 1.$$

This semigroup can be viewed as a discrete-time analogue of the classical and extensively studied continuous-time Feynman–Kac semigroup. The potential V is assumed to blow up at infinity, inducing a strong confining effect and leading to a killed Markov chain with increasing penalization at large distances. A central assumption on the underlying Markov chain is the *direct step property* (DSP): there exists a constant $C_* > 0$ such that

$$P_2(x, y) \leq C_* P(x, y), \quad x, y \in X,$$

i.e. the two-step transition probabilities are uniformly controlled by the one-step transition probabilities. Under this condition, together with additional natural assumptions, the thesis investigates harmonic functions, heat kernels, several notions of ultracontractivity, as well as ergodic and quasi-ergodic properties of the associated Feynman–Kac semigroups.

The thesis consists of six chapters. The first chapter is introductory; it explains the motivation, describes the main objects, and provides a summary of the principal results.

The second chapter recalls the notion of the nearest-neighbor random walk on a graph and continues with a study of the direct step property (DSP), illustrating the framework through general examples on metric spaces, including subordinate Markov chains. The third chapter is devoted to two-sided estimates of harmonic functions for Markov chains satisfying the

DSP property, leading to a uniform boundary Harnack inequality at infinity. These results are compared with corresponding results for the nearest-neighbor random walk. The results of these two chapters are based on the paper W. Cygan, K. Kaleta, and M. Śliwiński, Decay of harmonic functions for discrete-time Feynman–Kac operators with confining potentials, *ALEA, Lat. Am. J. Probab. Math. Stat.* **19**, 1071–1101 (2022).

Chapter 4 studies heat kernel estimates. Working under the direct step property (DSP), uniform laziness, and ultracontractivity of the one-step transition operator, the author derives sharp two-sided estimates for the integral kernels of the semigroup and its dual in terms of the ground state eigenfunctions. These estimates lead to a detailed analysis of intrinsic contractivity properties, including asymptotic intrinsic ultracontractivity (aIUC) and intrinsic hypercontractivity (IHC). A central contribution is the introduction and discrete-time development of progressive intrinsic ultracontractivity (pIUC), which captures situations where full intrinsic ultracontractivity fails but regularity improves with time. Necessary and sufficient conditions for aIUC are established and shown to be equivalent to intrinsic hypercontractivity of the ground-state transformed semigroup. The results reveal that sufficiently fast growth of the confining potential is essential for aIUC to hold. In contrast, for nearest-neighbour random walks on graphs with finite geometry, neither aIUC nor IHC typically occurs, regardless of the growth rate of the potential.

Chapter 5 investigates the ergodic properties of intrinsic (ground-state transformed) semigroup and the quasi-ergodic behaviour of the original Feynman–Kac semigroup. Building on the analytic results from earlier chapters, a general approach is developed that does not impose any assumptions on the growth rate of the potential. It is shown that the intrinsic semigroup is uniformly ergodic on $\ell^\infty(\nu)$ with a geometric rate of convergence. In contrast, quasi-ergodicity of the original Feynman–Kac semigroup requires finer conditions and is analysed in detail in the ℓ^1 setting. Using progressive intrinsic ultracontractivity, the chapter establishes uniform quasi-ergodicity of the Feynman–Kac semigroup and uniform ergodicity of the intrinsic semigroup along suitable exhaustions, with explicit convergence rates. A key outcome is the proof of an equivalence between ergodicity of the intrinsic semigroup and quasi-ergodicity of the original semigroup, including matching rates of convergence, a result that appears to be new even in comparison with the continuous-time literature.

The last chapter illustrates the results of Chapters 4 and 5 for concrete classes of discrete-time Markov chains and confining potentials, highlighting how intrinsic contractivity properties depend on the decay of the transition kernel and the growth of the potential. It also provides explicit examples of kernels satisfying the standing assumptions and supplies detailed proofs supporting the claims made for the various regimes of decay and growth.

Chapters 4-6 based on the paper W. Cygan, K. Kaleta, R. Schilling and M. Śliwiński, Heat kernels, intrinsic contractivity and ergodicity of discrete-time Markov chains killed by potentials, arXiv:2504.17879 (2025).

In conclusion, the thesis presents a collection of original and significant results that substantially advance the theory of discrete-time Feynman–Kac semigroups and their ergodic properties in the presence of confining potentials. The introduction of a systematic framework based on the direct step property, together with the discrete-time development of progressive intrinsic ultracontractivity and its applications to heat kernel estimates and quasi-ergodicity, constitutes a novel contribution that goes beyond existing results, which were previously available only for special models or in continuous time. The work demonstrates the candidate's deep understanding of probabilistic potential theory, functional-analytic techniques, and Markov chain theory, as well as a high level of technical skill and mathematical maturity. The results are obtained using modern methods and are presented in a clear, coherent, and well-structured manner, reflecting the candidate's independence and competence as a researcher. Overall, the thesis fully satisfies the standards and requirements for a successful Ph.D. dissertation, and the candidate is well qualified to be awarded the doctoral degree.

I strongly recommend to accept the submitted manuscript as a Ph.D. thesis.

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