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Report on the doctoral dissertation

Modeling Systemic Risk through Reduced-Form Dynamics of Bonds in Financial Networks

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submitted for the degree of Doctor of Philosophy

The doctoral dissertation by Kamil Fortuna is devoted to a study of systemic risk through network modeling of financial systems. Chapters 1 and 2 deal with a modeling of credit cycles in the discrete and continuous time, respectively, using either a deterministic recursion or its continuous-time counterpart represented by a stochastic differential equation. The main part of the thesis, Chapters 3-6, hinge on a recent paper by Barucca et al. [42] where the NEVA framework was proposed and studied, but are also closely related to the existing literature on systemic risk (most notably, the Furfine [38] algorithm and the DebtRank approach of Bardoscia et al. [40,41]). Since the primary topic of the dissertation is modeling of systemic risk I have put special emphasis in my report on Chapters 3–6 where the NEVA approach to the network modeling of financial systems is presented and further developed.

My overall assessment of the submitted work is based on the author's mathematical expertise, originality of results, evidence of critical ability, selection and synthesis of source material and, last but not least, clarity and style of presentation. As I argue below, the present version of the dissertations has some essential shortcomings in all respects and thus, in my opinion, it should be extensively amended before resubmission of the revised draft.

Comments on Chapter 1

Chapter 1 is devoted to modeling interest rates in discrete time through a combination of previously proposed by other authors models linking the level of interest to the volume of loans and number of defaults. No other factors are included in the unified model given by equation (1.9) in the thesis, but the deterministic dynamics of the interest rate $i(t)$ depends on four parameters with a plausible financial interpretation. Since the evolution of $i(t)$ reduces to a single recursive equation (1.14) its analysis does not present any mathematical difficulties but it is nevertheless interesting from a practical perspective.

The emphasis in the study of the model is put on interpretation of various parameters as elements of bank's policy although possible interventions of a central bank are also examined. The presentation of cycles (bubbles, crashes, stabilisation) refers to values of model parameters, which are here assumed to be driven by decisions of agents (commercial banks and a central bank) but there is no attempt to formalise the evolution of the interest rate and decisions of agents as an optimisation problem. The analysis of the model is conducted mainly at the level of intuition, frequently going beyond the mathematical features of the model and using instead informal arguments based on the real-world experience.

Comments on Chapter 2

The topic of Chapter 2 is and presents an analysis of a simple diffusion model of a risk factor, which is later used to specify the stochastic dynamics of a network. The author refers there to the *reduced-form approach* and introduces the hazard rate but, in fact, never directly defines the default time using the intensity-based approach. Frequent references to the Duffie-Singleton intensity-based model are not formally justified since the default time is defined as the first moment when the equity value falls below zero. Although reduced-form dynamics of bonds is emphasised in the title of the thesis, the actual method used by the author is much closer to the classical *structural approach* to credit risk where a default time is given in terms of the value of a firm.

Although the number of defaults is usually presented in numerical illustrations as the most important measure of systemic risk, the default event of a bank is never formally defined, except for page 57 where it is stated that *This relationship stems from the definition of default as the point in time* τ *when a company's assets* $A_i(\tau)$ *fail to cover its liabilities* $L_i(\tau)$*, leading to* $E_i(\tau) = A_i(\tau) - L_i(\tau)$. This is clearly the usual structural approach to credit risk and thus the claim on the preceding page that *the default intensity* λ_t *could be depicted analogously as* $\lambda(t) = \gamma_j\left(1 - \alpha_j \frac{E_j^+(t)}{E_j(0)}\right)$ is not consistent with the definition of the default event from page 59. Of course, the author is free to define the spread cs_t using equality (4.10), but the "default intensity" λ on page 58 is definitely not the actual default intensity under the (risk-neutral) probability, as implicitly defined through equation (2.12) where $\lambda(t)$ represents the classical intensity related to survival probability. It is worth noting that equality (2.12) for the case of zero recovery was obtained within the intensity-based approach to credit risk, and not postulated. More generally, the pricing formula for corporate bond was formally proven by Duffie and Singleton [67] for the case where default event is specified by its intensity, and not as the moment when a company cannot cover its liabilities. It is logically inconsistent to take the formula obtained within the intensity-based framework and use it is a structural model of default event.

To conclude, the pricing equations used by the author for valuation are based on the concept of the default intensity and hence on the notion of an unexpected default time, whereas default events are specified through the structural approach to credit risk modelling, which makes the approach presented in Chapters 3-6 internally inconsistent. It is really odd that the notions of default event, default time, direct defaults, default probability (either risk-neutral or statistical) and the intensity of default are never formally defined in the dissertation, although the author uses them throughout his work as important features of a systemic risk.

The remaining part of Chapter 2 is devoted to an analysis of equation (2.12) for either a positive or a negative value of the reversion rate c. Unfortunately, the author makes no attempt to formalise the concept of the network's *stability*, though it is conjectured that the Ornstein-Uhlenbeck equation

$$dx_t = c(b - x_t)\,dt + \sigma dW_t$$

describes either *stable* (for $c > 0$) or *unstable* (when $c < 0$) dynamics, depending on whether the reversion rate c is positive or negative. Hence I understand that the latter convention can be taken as an informal definition of the system's stability. Mathematical study of the concept of stability is reduced to the analysis of relative values of some averaged quantities, such as the variance and the absolute mean displacement, for various values of the time parameter t and the volatility parameter σ. Computations leading to Theorems 2.1, 2.2 and 2.3 are elementary and correct, but conclusions from these theorems are not clear since precise definitions of the system's stability are missing. One can only guess that the asymptotic behaviour of the variance (or the absolute mean displacement) of the process x_t is considered to be a measure of stability for the purpose of Chapter 2.

In fact, on page 43 the author observes that for a high positive value of c ("stable" case) "*trajectories with a greater positive reversion are quickly drawn towards the direction of equilibrium point*" but also notes that "*Negative reversion paths follow an exponential trend that tends to dominate the noise effect as c decreases, whereas strongly positive reversion paths exhibit sharp oscillations around the equilibrium point 0. Consequently, negative reversion trajectories can be perceived to be significantly smoother and more stable*". He also mentions on the same page that "*Thus, a market can appear very stable for an extended period of time while being determined to abrupt changes eventually. Moreover, effects that enhance initial stability, such as a shorter distance to the fixed point* $|(b - x_0)|$ *and the reversion rate c closer to 0 lead to significantly greater long-term instability (greater asymptotic dispersion). This effect corresponds to the very important behavior of systemic risk: prolonged periods of seemingly calm and stable dynamics can translate into greater instability in the long run*". Once again, the concept of stability is not precisely defined and thus some comments are apparently self-contradictory.

To sum up, a study of the Ornstein-Uhlenbeck model is not convincing due to the fact that the notion of stability of a stochastic process was left largely unspecified. Therefore, it is impossible to make any precise statements about the system's stability since, in particular, a negative value of c may a priori equally well lead to a bull market or a crisis preceded by a stable state during the initial period. No attempt was made to fit the model to market data and assess whether a positive or negative value of c would better describe the actual dynamics of interest rate during either a financial crisis or a period of market stability.

Finally, the author seems to be unaware about a large body of existing literature on regime-switching models (both time series and continuous time models) and various attempts to calibrate such financial models to market data, though it is fair to say that they were only partially successful. The simplicity of the Ornstein-Uhlenbeck equation is by no means a sufficient argument to accept it as a valid model of systemic risk. One could develop a regime-switching version of the Ornstein-Uhlenbeck model by assuming that the parameter c is not constant but one deals with a hidden Markov model and hence it may take both negative and positive values depending on the prevailing state of the market or economy. This would allow to model future bubbles, crashes, as well as stable periods using a single stochastic model but, of course, other competing models of that kind were already studied in the existing literature.

Comments on Chapter 3

In very short (only six pages) Chapter 3, the author introduces the Network Valuation in Financial Systems (NEVA) framework. Although the author follows rather closely the paper by Barucca et al. [44], in Theorem 3.1, he presents a modified version of Theorem

3.1 in [44]. It should be made clear that the only problem examined in Chapters 3-6 is the computation of fixed points of the map Φ given by equation (3.5) and, as a result, identification of defaulted banks. In Chapters 3 and 4, the problem is static, though it can be considered at any fixed date t, whereas in Chapters 5 and 6 the author makes an attempt to introduce a dynamical version of Φ by introducing a stochastic process x_t in specifications of valuation functions.

One should observe that, at any fixed date t, the problem of finding fixed points of $\Phi = \Phi(x_t)$ is essentially the same as in Chapter 3 for almost every $\omega \in \Omega$. In principle, for every t and almost every $\omega \in \Omega$, one obtains a different set of fixed points. Since the minimal and maximal fixed points are always well defined, one can to study the stochastic processes given by them and this would be dynamic variant of the NEVE framework.

It is surprising that the author does not attempt to examine the set of all fixed points for particular instances of feasible valuation functions and presents only minor extensions of results obtained in [44]. The author insists on using the Picard iterations as a universal way of finding fixed points of Φ. However, the map Φ may have several fixed points and, most importantly, there is no reason to believe that the Picard iterations with an arbitrary initial value will converge to a fixed point of Φ in a monotonic fashion (although this holds for the minimal and maximal fixed point). In fact, it is plausible that for most initial values not lying below the minimal or above the maximal fixed point the iteration will not be monotonic since the postulated monotonicity of the valuation map does not entail the monotonicity iterations, in general. The author should make it clear whether he will try to identify some other nontrivial fixed points or only deal with the minimal and maximal ones. Valuation maps proposed in Chapter 4 are quite complex and the properties of their fixed points are not studied at all (not even by means of examples).

In comments after Theorem 3.2, the author states that: "*Theorem 3.1 provides a practical approach for conducting stress test analysis, or, in other words, evaluating the influence of shocks on the system. Given the initial state of the system, the shock χ is applied to external assets, resulting in a dynamics as $A_i^e(t) \to (1-\chi)A_i^e(t)$, in a way such that the financial system indeed experiences a loss in the first iteration. Subsequently, employing the Picard iteration algorithm, the propagation of the crisis is computed, enabling assessment of the initial shocks effect. By the virtue of Theorem 3.1, for any specified precision ϵ, there exists $K(\epsilon)$ such that for all $k > K(\epsilon)$, it holds that $\|E^k(t) - E^*(t)\| < \epsilon$. However, it is worth to point out that ϵ is typically unknown beforehand. Hence, in practical scenarios, it is reasonable to estimate ϵ by examining the difference between consecutive terms of the sequence, ensuring it meets the predefined precision criteria, i.e., $\|E^{k+1}(t) - E^k(t)\| < \epsilon$*".

I do not feel that such comments are justified by Theorems 3.1 and 3.2. It is clear that as soon as the shock χ is applied to the values of external assets, so that $A_i^e(t)$ is replaced by $(1-\chi)A_i^e(t)$ in equation (3.5), the valuation operator Φ should be replaced by a modified operator, which I denote by Φ^χ. Then the claim that *the financial system indeed experiences a loss in the first iteration* is not clear since it depends on a choice of the initial point $E^0(t)$. It is known that the first iteration is monotonic if $E^0(t)$ is equal to either $m(t)$ or $M(t)$ but then only the minimal and maximal fixed points can be achieved, which is a limited choice from the practical perspective. As soon as Φ is replaced by Φ^χ one should also substitute $m(t)$ and $M(t)$ with $m^\chi(t)$ and $M^\chi(t)$, respectively. If the initial point is chosen arbitrarily (for instance, using the actual market data) and satisfies $m^\chi(t) \leq E^0(t) \leq M^\chi(t)$, then there is no reason to expect that the first iteration of Φ^χ is monotonic.

Furthermore, the statement that ϵ *is typically unknown beforehand* is misleading since, of course, ϵ should be chosen but $K(\epsilon)$ is not known, though in theory it exists. Hence the claim that, in practice, *it is reasonable to estimate* ϵ *by examining the difference between consecutive terms of the sequence* is not convincing. Unless some results on the speed of convergence are established, that argument hinges on wishful thinking.

It should be made clear in Theorem 3.3 that $E^1(t) = \Phi(E^0(t))$, as it was assumed in Theorem 3.1. On page 52 the author writes about Theorem 3.2: *This result is a powerful tool in practical programming applications of the NEVA framework. Calculations executed usually on a predetermined grid of initial shocks can now provide information not only for a finite set of points, but also for the whole areas in between them. This constitutes a wealth of insights regarding the models overall behavior, as well as for a broad range of specific inputs.*

Once again, I do not feel that these comments are justified and, in fact, they are not supported by any concrete examples in further chapters.

First, it should be noted that an initial shock is nothing else than a modification of the valuation operator. Instead of the original operator Φ, one wishes to study a family of operators Φ^χ with the parameter $\chi \in [0,1]$. If we focus on an examination of the map Φ, then the following question arises. Apart from the minimal and maximal fixed points, for which the analysis of convergence of iterations is rather clear from [44], does a given map Φ admits other fixed points and, if they do exist, whether they can be obtained by a monotonic iteration? Such fundamental questions about the properties of valuation maps are not addressed (or even mentioned) in the thesis but they are crucial in the context of Theorem 3.3. It may occur that Φ is such that Theorem 3.3 can be applied to the minimal and maximal fixed points but not to any other fixed point of Φ and, in fact, this seems to be a typical situation.

Second, regarding the concept of the initial shock, several questions need to be considered. It is clear that Theorem 3.3 can be applied to each operator Φ^χ for any fixed $\chi \in [0,1]$ but, in principle, the properties of Φ^χ are not dissimilar from the properties of Φ os, if the latter are unknown, one cannot expect to have a better information about Φ^χ. Hence any question formulated above for Φ is still valid when examining Φ^χ. In particular, under which conditions on the initial state $E^0(t)$ the first iteration of Φ^χ is known to be monotonic (either decreasing or increasing)?

It is stated on page 62 that *The initial state, derived from market data, is presumed to be the point of equilibrium.* This important comment is not clear and hence it requires a detailed explanation and a reference to some mathematical results. Shall we understand that the initial state is taken from market quotes for equities and, if this is the case, can we really assume that it represents a point of equilibrium (for which map).

One may argue that it is "natural" to start from some fixed point of Φ but one needs to know all fixed points of Φ and is not clear which one should be chosen since, in principle, each of them can be used as $E^0(t)$. If we agree that the set of fixed points of Φ is usually not known, then one can propose to take the maximal fixed point of Φ as the initial state but it is still necessary to check whether it lies between $m^\chi(t)$ and $M^\chi(t)$ (otherwise, its choice would be inconsistent with the range of Φ^χ).

Finally, if the first iteration of Φ^χ happens to be decreasing, it would be good to know whether the successive iterations converge to either the maximal fixed point of Φ^χ (which is easy to identify anyway) or some less trivial fixed point. Even when dealing in Section 4.3 with explicit numerical examples the author does not provide any information about the choice of the initial state and does not describe the set of equilibria. Only graphical representation of the number of defaulted banks for alternative valuation functions and various values of the shock parameter χ are given

Since the idea of the initial shock is one of fundamental tools in a study of systemic risk I find it not acceptable that it is not analysed using strict mathematical results but introduced through informal comments.

Comments on Chapter 4

The goal of Chapter 4 is to propose some novel valuation functions referencing the interest rate and spreads. Valuation functions introduced by the author are reasonable and extend some valuation functions from the existing literature. They are plausible definitions but their properties, especially in the context of the issue of fixed points and Picard iterations, are not examined in detail so, as in Chapter 3, only the minimal and maximal solutions are well defined and they can be obtained through iterations with the initial states $m(t)$ and $M(t)$.

Regarding the proposed specifications of valuation functions, I have a few comments and questions.

First, the authors starts with a general representation of such functions in Sections 4.1 and 4.2, but ultimately he settles for much simpler representation (4.8) where the choice of functions $r(E_t)$ and $cs_j(E_t)$ defines a particular model. Apparently, the valuation is assumed to be performed at time t and relates to the time horizon T but in specific definitions of $r(E_t)$ and $cs_j(E_t)$ the quantities $E_j(0)$ and $A_j(0)$ are also used Hence the notation $r(E_t, E_0, A_t, A_0)$ and $cs_j(E_j(t), E_j(0), A_j(t), A_j(0))$ would be more appropriate. This additional feature of the valuation procedure should be explained, also in the context of implementations presented in Section 4.3 and Chapter 6. How the date 0 should be chosen and why it is used when assessing the state of the system at time t? In Chapter 5, the author introduces the notation t_0

Second, when introducing the recovery rate in equation (4.10), the author decided to ignore the size of liabilities and hence the expected recovery rate at time t is given as the ratio of the value at time t of assets $A_j(t)$ to the initial value. This is a formal specification of expected recovery rate not consistent with a more practical structural approach where the future ratio of assets to liabilities specifies the actual size of recovery payoff. The proposed specification of recovery rate ρ_j is artificial and there is no mention of its role at the default time of the bank. In fact, apart from the definition of default at time T on page 59, the author remains silent on an interpretation of recovery at default and its impact on the valuation map. It is not clear at all what happens with the assets and liabilities of banks when they default.

In the classical intensity-based approach to corporate debt elaborated by Duffie and Singleton the recovery rate is simply the proportion of the debts value just before default, which is delivered to bondholders at the event of default. Formally, it is related to the (risk-neutral) distribution of the default time, which is used in bond valuation. The proposed specification (4.24) of valuation functions looks like an ad hoc combination of two terms which are proxies for actual parameters. On the one hand, the first term does not represent the (risk-neutral) probability of default and the second one is not equal to one minus the recovery rate. On the other hand, however, it is a valid feasible valuation function though its specification is not based on some mathematical results but informal arguments and similarity to the Duffie and Singleton pricing formula.

It seems to me that the quantities $A_j(t)$ and $A_j(0)$ are never formally defined, except for equation (4.25). However, if we take equation (4.25) as the definition of $A_j(t)$ then one can observe that the right-hand side in (4.25) depends on $E(t)$ which is not known in advance but computed using a particular model and valuation functions, which in turn depend on $A_j(t)$ and thus one deals with a circular dependence. This circular dependence can be avoided when performing iterations but it is important when formally defining and analysing the set of fixed points of the valuation map.

In any case, equations (4.24) do not define a valuation function until they are combined with equality (4.25). But by inserting (4.25) into (4.24) we obtain the equation in which $\mathbb{V}_{ij}$ appears on both sides of equality (4.24), which means that it is not explicitly defined. Hence the following questions arises: if we take (4.24) and (4.25) as the implicit definition of the valuation functions, then is it true that that the function $\mathbb{V}_{ij}$ (as well as $\mathbb{V}_j^{\mathrm{e}}$) well defined (i.e., exists and is unique) and whether it satisfies the definition of a feasible valuation function?

On page 62, the author apparently makes the postulate, as in some other related works, that equalities (4.27) should be satisfied, which means that $E(0) = M(0)$, that is, the initial state at time 0 is simply $M(0)$. It is also stated that *The initial state, derived from the market data, is presumed to be at the point of equilibrium.* This supposedly means that $M(0)$ is a fixed point of the valuation map $\Phi(0)$ at time 0, which is is trivial since if equalities (4.27) hold then the map $\Phi(0)$ is constant. The right-hand side in (3.2) simply gives the book values of equities with no impact of the credit risk.

Next, the author writes *Subsequently, a shock is applied, and its propagation through financial dependencies, along with its ultimate impact on the system, is systematically assessed.* The concept of an initial shock was already briefly mentioned in Chapter 3 but now the following question arises: the static valuation at time t is done using equalities (4.17) and reduced values of external assets. Since, by definition, the valuation hinges on finding a fixed point, the iteration method at a fixed time t is a purely mathematical tool of finding the fixed point and hence it does not produce the dynamics (propagation) of defaults. One can imagine another way of calculating the fixed point thus yielding an alternative "dynamics" of the system. If, as I presume, direct defaults are defined as defaults occurring after the first iteration, then this does not make much sense since iterations at time t do not represent a real-world dynamics of the system but they are merely a tool to find its fixed point. This feature of the static NEVA framework is clearly acknowledged in Barucca et al. [44] (see page 1198 therein) who write *However, we stress that, since our valuations emerge as solution of a system of fixed-point equations, defaulting institutions do not default in a specific sequence, but default all at the same time. As a consequence, our approach is not well-suited to capture bilateral CVA (Brigo & Capponi, 2010; Gregory, 2009), which also accounts for the sequence in which defaults might occur.*

I believe that one cannot introduce the concept of direct and indirect defaults using only the definition of the valuation map and its fixed points at any given date. Hence the study of direct and indirect defaults in Section 4.2 is not justified by any theoretical results from Chapters 3 and 4 but hinges on the idea of mathematical iterations with no financial interpretation. If the author would like to stress the importance of direct and indirect defaults as one of the main features of a network then he would need to formalise these concepts in a dynamical variant of the network introduced in Chapter 5.

In Section 4.3, the author offers an analysis of stability of the financial system in the United States using his approach to valuation functions. The emphasis is put on the dependence of the number of defaulting banks on the size of the initial shock and values of relevant parameters. In my opinion, the presentation of implementations in Section 4.3 and Chapter 6 is not detailed enough.

First, the values of external assets, which are supposed to undergo the initial shock triggering defaults, is not even mentioned and only internal assets and liabilities are presented in Figure 4.1 (but no table with numerical values is included).

Second, it is not explained which state is chosen as the initial state of the banking system. I believe that the author takes to book value of all assets as the definition of the initial state or, alternatively, the reduced book value after the shock. In both cases, the valuations obtained through iterations would monotonically decrease to the maximal fixed point of the valuation map Φ^{χ}. It is not explained which entities are subject to either direct or indirect defaults and whether their identities are stable with respect to the choice of χ. The focus is on the proportion of direct and indirect defaults without apparently paying attention to the relative sizes of defaulted firms.

More generally, the following natural question arises: is it possible to prove that the valuation (maximal fixed point) is decreasing with the increase of the parameter χ? Of course, analogous questions arise with respect to other model parameters. Although the author attempts to draw conclusions from numerical experiments, they are solely based on a loose interpretation of graphs. In addition, it is impossible to form a well-founded opinion since the method of obtaining them is merely sketched. Numerical experiments in Section 4.3 are used to make tentative conclusions about the impact of parameters on the behaviour of the network but I would expect that at least some of them could be supported by mathematical proofs.

To sum up, the issue of credit contagion in financial systems was examined in numerous papers using various techniques and definitions of contagion so the author's approach can be seen as a practical contribution to that research area. However, as I argue above, there are some important points that are unclear and a theoretical analysis of his approach is missing. Furthermore, no hint is given how one could approach the issue of a model validation so that the interest of such abstract study is purely academic.

The last comment applies to most papers on systemic risk and it is likely that this issue will remain open for a long time since, first, the valuation procedure such as the NEVA lacks a clear measure of its quality and, second, the real-world default events are extremely scarce and thus a model calibration is not possible. Hence the choice of the valuation function, as can be seen from its definition, is not based on any specific objective but it is arbitrarily chosen and, second, even when the valuation function is chosen in a "reasonable" way its parameters cannot be calibrated to market data. Of course, this is not the author's fault but I believe that a critical assessment of the area should be included and it should be acknowledged that, in essence, the method is based on going from nowhere (no objective) to nowhere (no validation).

This lack of clear mathematical objective of modeling and impossibility of model validation should be contrasted with other fields of financial mathematics, such as the portfolio optimisation and the pricing of financial derivatives where mathematical objectives are formally defined, e.g., maximisation of the expected return and risk minimisation, respectively. Then the mathematical problem is formally solved by proving results within a chosen framework and, crucially, mathematical results are validated through backtesting and calibration. The model validation step is feasible due to the fact that market data regarding trading are abundant and they are collected almost in continuous time. Nevertheless, the model risk is also acknowledged in robust finance and handled using, for instance, the recently developed machine learning techniques. From this perspective, a study of systemic risk using NEVA looks more like a guesswork since it lacks fundamental criteria and objectives.

Comments on Chapter 5

Chapter 5 is devoted to a dynamic extension of the framework examined in Chapters 3 nd 4. The crucial new feature introduced in Section 5 is the presence of a stochastic process x_t representing risk factors. The author proposes to incorporate this process into the valuation procedure and hence obtain its dynamic version. Unfortunately, the notation used in Chapter 5 is misleading on many occasions. First, $E(t_0)$ should not be used in equations (5.21) and (5.22) as a variable since on the next page $E(t_0)$ denotes the initial state. Next, the definition of a fixed point at time $t \leq t_0$ is not given and thus its interpretation is left to the reader. If the valuation problem hinges on finding all random vectors $E(t)$ such that $E(t) = \Phi(x_t, E(t))$ for any fixed $t \in [t_0, T]$ then an attempt of using iterations given by equation (5.24) is misguided since it would be more suitable to consider iterations for a fixed t.

Suppose that x_t is a stochastic process with values in $\mathbb{R}^K$ to avoid heavy notation. Suppose that the fixed point of Φ is defined as an $\mathcal{F}_t$-measurable, $\mathbb{R}^n$-valued random variable $E^*(t)$ such that $E^*(t) = \Phi(x_t; E^*(t))$. For any fixed t, one can also define the minimal and maximal fixed points (of course, random variables) and denote them by $\Phi^*(x_t; m(t_0))$ and $\Phi^*(x_t; M(t_0))$. Then it can be shown, as in the deterministic case, that any fixed point $E^*(t)$ satisfies $\Phi^*(x_t; m(t_0)) \leq E^*(t) \leq \Phi^*(x_t; M(t_0))$ where the inequalities hold almost surely.

Since the uniqueness of a fixed point $E^*(t)$ is practically excluded, a theoretical study of the set of all fixed points is challenging but, seemingly, the author asks the following question: if we start from some initial value $E(t_0)$ at time t_0 and define a sequence of random variables $E^{k+1}(t_0) = \Phi(x_{t_k}, E^k(t_0))$ then can we show that the iterations would converge to some fixed point $E^*(t)$ at time t, provided that we consider a normal sequence of partitions of the interval $[t_0, t]$? This seems to be feasible under suitable assumptions of the continuity of a process x_t and the mapping Φ complemented by the monotonicity of the first iteration (similar to Theorem 3.1).

However, the author decides to examine a on another question: can we find natural lower and upper bounds for the sequence $E^k(t_0)$? He proceeds to an analysis of the Picard iterations on a discrete-time grid, which are (perhaps wrongly) supposed to represent an approximation of the valuation process. In my opinion, the least convincing step is the replacement of the process x_t by its running maximum $\bar{x}_t$ since this step completely changes the formulation of the problem at hand, though it simplifies the proofs. The proofs in Chapter 5 rely on the postulate of monotonicity of a feasible valuation function, which is sufficient to obtain basic inequalities and hence also lower and upper bounds for the tentative valuation process, but it does not allow to establish more tangible results on stochastic dynamics of the valuation process, that is, the dynamics of the set of fixed points of stochastic valuation map. Perhaps some simulation-based studies of the original stochastic system (with no simplifications) would be more useful than an attempt to establish theoretical results, which cannot be achieved using only the continuity and monotonicity of Φ.

Theorems 5.6 and 5.7 require very strong assumptions, which are never substantiated, and do not offer any information about the dynamics of valuation, of course, unless the uniqueness of a fixed point is assumed. Their proofs rely on monotonicity of a generic feasible valuation function (but not on specification (5.22)) and thus they cannot be seen as an essential mathematical contribution. If assumptions of Theorem 5.6 are never satisfied (except for some trivial cases), then this result is of no interest. Theorem 5.5 does not require strong assumptions but it gives the lower and upper bounds for a_s that are almost trivial, as is also shown by the simplicity of the proof of Theorem 5.5.

Subsequently, in Section 5.4, the author first introduces in Definition 5.8 the map Ψ leading to the quantities $\Psi(x_t)$ and $\Psi(\bar{x}_t)$, which satisfy some natural inequalities. In essence, $\Psi(x_t)$ and $\Psi(\bar{x}_t)$ are the minimal fixed points of the valuation map $\Phi(x_t, E)$ and $\Phi(\bar{x}_t, E)$, respectively, and thus they can be obtained using the iterations $E^{k+1} = \Phi(x_t, E^k)$ and $E^{k+1} = \Phi(\bar{x}_t, E^k)$, respectively, with the initial state $m(t_0)$.

The author argues that the quantity $\Psi(\bar{x}_t)$ should be used in assessment of systemic risk since it is more conservative than $\Psi(x_t)$ or even $\inf_{s\leq t} \Psi(x_s)$ (see Lemma 5.11). So, finally, the author recommends to reduce an analysis of stochastic valuation given by equations (5.21) and (5.22) with an initial state $E(t_0)$ and stochastic process x_t to computations of the minimal fixed point $\Phi^*(\bar{x}_t, m)$ where $\bar{x}_t = \sup_{s\leq t} x_s$. The main argument is the relative simplicity of the latter problem and the fact that it is conservative so the risk cannot be underestimated, which could perhaps be a convincing argument provided that it shown through simulations that the solution to the reduced problem can be seen as a proxy for a solution to the original problem.

It should be noted that in Chapter 2 the author analyses the properties of the Ornstein-Uhlenbeck model x_t but he uses the process $\bar{x}_t = \sup_{s\leq t} x_s$ in numerical computations in Chapter 6 and the latter process has increasing sample paths with completely different properties. The dynamics of x_t may have some interpretation of fluctuations of the economy but the increasing process $\bar{x}_t$ represents a persistent gradual decline of the state of the economy and thus is not realistic. No cycles with bubbles and crises can be obtained using such approach and thus the connection to Chapter 2 is completely lost. In fact, the original valuation problem given by (6.3) is dynamic and leads to path-dependent iterations, in the sense that, $E^{k+1}(t_0) = E^{k+1}(x_{t_0}, x_{t_1}, \dots, x_{t_k}, E(t_0))$ for every k. Its reduced version represented by the map Ψ is either path-independent, if one considers $\Psi(x_t)$, or it depends on the running maximum of x_t, which is a one-dimensional dependence on the current value of the running maximum and not multi-dimensional dependence on the whole history of x_t. In addition, the map Ψ is always independent of the initial state $E(t_0)$ since $m(t_0)$ plays the role of the initial state.

The choice of $m(t_0)$ as the initial state is puzzling since it formally means that all banks defaulted. Since, for any fixed value a of $\bar{x}_t$ the sequence of iterations is increasing the number of defaults decreases to some limit, which is used to define the state $\Psi(a) := \Phi^*(a, m)$ (or its right-hand limit in a). On the one hand, this odd property emphasises the point that iterations are merely mathematical abstracts and they should not be used to represent the dynamics of the system. On the other hand, since $\Phi^*(a, m)$ is decreasing in a, the number of defaults given by $\Psi(\bar{x}_t) = \Phi^*(\bar{x}_t, m)$ is increasing (almost surely) when t increases. Unfortunately, even such basic, but important, properties of the original problem and its reduced variant are not discussed in the thesis.

Comments on Chapter 6

The author focuses on the algorithm outlined on page 89 and combined with equations (6.2)–(6.3). The first step in the algorithm is purely deterministic and hinges on finding $E^k(x_j, m)$, which is a proxy for $\Phi^*(x_j, m)$ and thus also $\Psi(x_j)$. The set $\{x_0, x_1, \ldots, x_K\}$ represents a selection of possible values of the random variable $\bar{x}_t$. To obtain the probability distribution for the number of defaults and hance also the ruin probability at a given level, it suffices to make use of the random variable $E^k(\bar{x}_t, m)$ for a sufficiently large k. Since only the process $\bar{x}_t$ is effectively employed in Chapter 6 its properties (in particular, the probability distribution of $\bar{x}_t$ for either a positive and negative value of c) should be analysed. The c.d.f. $F_{\bar{x}_t}$ is used in algorithm outlined on page 89 but it is not explained in Chapter 6 how it was computed in numerical examples.

The author claims on page 92 that *The underlying risk factor x_t follows the extended Ornstein-Uhlenbeck process (2.27), in which the reversion rate $c \in \mathbb{R}$ is allowed to be negative and thus to describe the unstable behaviour of credit bubbles and crashes.* In fact, c is chosen to be negative in numerical experiments but since the process $\bar{x}_t = \sup_{s \leq t} x_s$ is used there and not x_t, the comment about stability, bubbles and crashes is meaningless since the dynamics of $\bar{x}_t$ are very different from the dynamics of x_t, which were examined in Chapter 2. How can you change completely the process and use still arguments based on the behaviour of another process?

Regarding the values of parameters used in numerical experiment, it is worth noting that external assets are supposed to be subject to shocks but their nature is not explained. Does the author means a negative shock or positive shock and of which size? The initial value for the process is taken to be $x_0 = \ln(0.03)$ which is clearly negative, though in Definition 5.4 the process x_t is assumed to be nonnegative. The mean-reverting level $b = x_0$ so it is negative as well. Is there any specific reason behind such assumptions or these values were selected randomly? Of course, as long as x_t has negative values, the values of external assets computed using equation (6.2) are increased. Was this intentional or just a coincidence? Finally, sample paths of the Ornstein-Uhlenbeck process x_t are presented in Figure 6.2, which is misleading since the process $\bar{x}_t = \sup_{s \leq t} x_s$ is used throughout Chapter 6 and, obviously, its sample paths behave in a completely different manner than the corresponding sample of x_t.

Since the author claims that the complex problem represented by equations (6.2)–(6.3) can be reduced to finding $\Psi(\sup_{s \leq t} x_s)$. I believe that such claim should be verified by comparing the direct solution to the original problem (of course, the original problem should be first precisely formulated) with the solution to the reduced problem. The author uses the term "estimation" in the context of Theorem 5.10 and Lemma 5.11 but, in fact, these results give some inequalities and they do not deal at all with the original problem since b_s is used in Theorem 5.10, whereas Lemma 5.11 is concerned with the properties of Ψ, which is the postulated "solution" to the reduced problem (see Definition 5.8).

It is natural to conjecture that the proposed reduction method is overly conservative since it uses the initial state $m(t_0)$ in combination with the process $\sup_{s\leq t} x_s$ so that in both cases the focus is on the worst-case scenario since, on the one hand, the valuation is decreasing in $\sup_{s\leq t} x_s$ and, on the other hand, the choice of $m(t_0)$ as a starting point leads to the minimal fixed point of Φ.

The dates t_0, t and T are not explicitly specified in Section 6.2 and no information is given about the initial state $m(t_0)$. In an analogous study of a static case in Section 4.3 the initial state is probably set to be $M(0)$ (since the first iteration is decreasing) and thus the choices of initial states in Sections 4.3 and 6.2 are inconsistent. I can be wrong here since the author does not explain how the initial states were chosen.

Since the setup in Chapter 6 is dynamical it makes sense to analyse the distinction between direct and indirect defaults. Unfortunately, these concepts are used in Chapter 6 without any formal definition so it is impossible to make any comments on graphs and tentative conclusions from numerical experiments presented in Figure 6.1. The author writes on page 92 *The impact of the network model without rates feedback appears to be minimal compared to the straightforward model of direct defaults* but does not explain what is the exact meaning of that statement. What the author means here by *the straightforward model of direct defaults*?

Finally, the loss function L_t on page 92 and the loss function also denoted by L_t on page 89 are completely different. The loss function given by equation (5.27) is strange since it uses the real-world initial state $E(t_0) = \Phi_0$ in denominator but the nominator in independent of E_0 since it is computed using an artificial initial state $m(t_0)$.

Conclusions

As I argue in my report, the level of various contributions presented in the doctoral dissertation by Kamil Fortuna is uneven and it is difficult to assess since even definitions of some important concepts are missing. The work contains some interesting practical contributions to the area of systemic financial risks, especially in Chapter 1. However, mathematical contributions are superficial and they are not based on any advanced mathematical techniques. Instead, the proofs hinge on elementary monotonicity properties of functions and sequences and thus they yield some basic inequalities of a minor relevance. As can be seen from my report, I strongly believe that Chapters 2-6 should be amended and the thesis resubmitted with extensive corrections. Furthermore, the clarity and precision of presentation is below the level expected from a doctoral dissertation and it should also be revised and amended. Therefore, I am not recommending Kamil Fortuna be awarded the degree of Doctor of Philosophy for the submitted doctoral dissertation entitled *Modeling Systemic Risk through Reduced-Form Dynamics of Bonds in Financial Networks.*

Sydney, 24.02.2026 [redacted]